

Quantitative Convergence and Stability of Seismic Inverse Problems.

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Reconstruction Methods for Inverse Problems
Indam Workshop, Roma, Italy,
May 28th – June 1st, 2018.



Istituto Nazionale di Alta Matematica

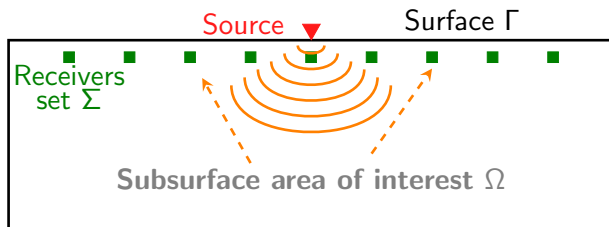


¹Inria Bordeaux Sud-Ouest, Project-Team Magique-3D.

²Total E& P

Seismic inverse problem

Reconstruction of subsurface Earth properties from seismic campaign: collection of **wave** propagation data at the surface.



- ▶ Reflection (back-scattered) partial data,
- ▶ only from the surface of the (large) domain,

nonlinear, ill-posed inverse problem.

Overview

- 1 Time-Harmonic Inverse Problem, FWI
- 2 Quantitative stability and convergence of FWI
 - Finite Curvature/Limited Deflection problem
 - Numerical convergence estimates
 - Numerical stability estimates
- 3 Numerical experiments
- 4 Conclusion

Plan

1 Time-Harmonic Inverse Problem, FWI

Time-harmonic wave equation

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$$-\rho\omega^2 \mathbf{u} - \nabla(\lambda \nabla \cdot \mathbf{u}) - \nabla \cdot (\mu [\nabla \mathbf{u} + (\nabla \mathbf{u})^T]) = 0.$$

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- ▶ elastic isotropic (λ, μ, ρ)

$$-\rho\omega^2 \mathbf{u} - \nabla (\lambda \nabla \cdot \mathbf{u}) - \nabla \cdot (\mu [\nabla \mathbf{u} + (\nabla \mathbf{u})^T]) = 0.$$

- ▶ anisotropy (stiffness tensor, ρ)

$$-\omega^2 \rho \mathbf{u} - \nabla \cdot (\underline{\underline{C}} \epsilon(\mathbf{u})) = 0.$$

Viscous behavior are considered with complex coefficients.

Full Waveform Inversion (FWI)

FWI provides a quantitative reconstruction of the subsurface by solving a minimization problem,

$$\min_{m \in \mathcal{M}} \mathcal{J}(m) = \frac{1}{2} \|F(m) - d\|^2.$$

- ▶ d are the observed data,
- ▶ $F(m)$ represents the simulation using an initial model m :

$$F : m \rightarrow \{p(\mathbf{x}_1), p(\mathbf{x}_2), \dots, p(\mathbf{x}_{n_{rcv}})\}.$$



P. Lailly

The seismic inverse problem as a sequence of before stack migrations
[Conference on Inverse Scattering: Theory and Application, SIAM, 1983](#)



A. Tarantola

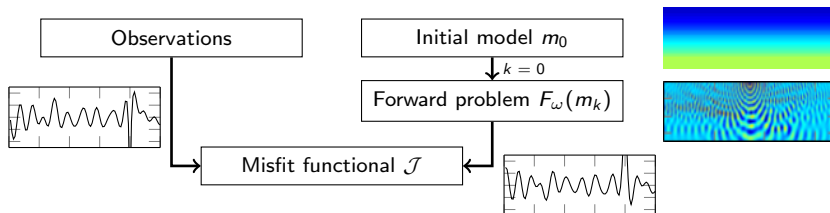
Inversion of seismic reflection data in the acoustic approximation
[Geophysics, 1984](#)



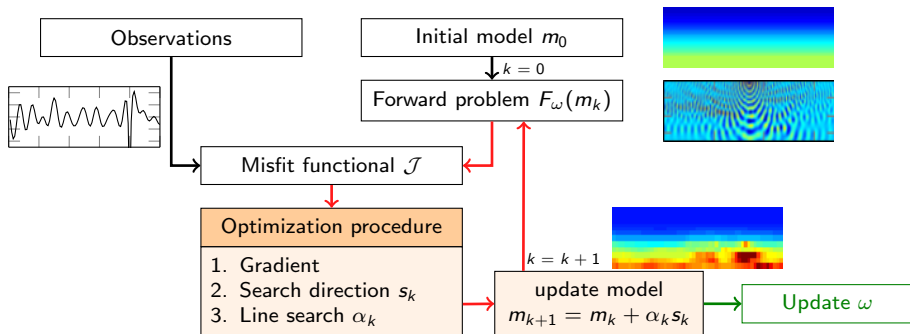
A. Tarantola

Inversion of travel times and seismic waveforms
[Seismic tomography, 1987](#)

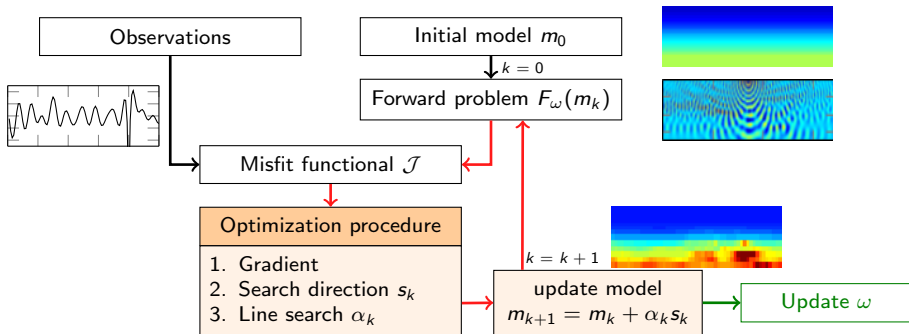
FWI, iterative minimization



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Numerical methods

- ▶ Forward problem resolution with Discontinuous Galerkin methods,
 - ▶ parallel computation, HPC, large-scale optimization.
-
- ▶ Multi-frequency algorithm, stability and convergence.

Plan

- 2 Quantitative stability and convergence of FWI
 - Finite Curvature/Limited Deflection problem
 - Numerical convergence estimates
 - Numerical stability estimates

Stability & Convergence

Helmholtz inverse problem from back-scattered **partial** data,
quantitative reconstruction with iterative optimization.

$$\min_{m \in \mathcal{M}} \mathcal{J}(m) = \frac{1}{2} \|\mathcal{F}(m) - d\|^2.$$

Convergence radius

- ▶ initial model needs to be within the radius of convergence.

Stability

- ▶ ensures reconstruction accuracy.

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Numerical estimates to guide the procedure

- ▶ frequency;
- ▶ parametrization;
- ▶ geometry;
- ▶ forward problem, ...

Convergence (Finite Curvature/Limited Deflection)

1/ Convergence radius

- ▶ initial model needs to be within the radius of convergence.

Least squares minimization problem

$$\min_{m \in \mathcal{M}} \mathcal{J}(m) = \frac{1}{2} \|\mathcal{F}(m) - d\|^2.$$



M. V. de Hoop, L. Qiu, O. Scherzer

An analysis of a multi-level projected steepest descent iteration for nonlinear inverse problems in Banach spaces subject to stability constraints

[Numerische Mathematik 2015](#)



G. Chavent

Nonlinear least squares for inverse problems: theoretical foundations and step-by-step guide for applications.

[Springer 2010](#)



G. Chavent and K. Kunisch

On weakly nonlinear inverse problems.

[SIAM Journal on Applied Mathematics 1996](#)

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Finite Curvature/Limited Deflection guarantees **uniqueness** of the solution and **unimodality**: no local minimum.



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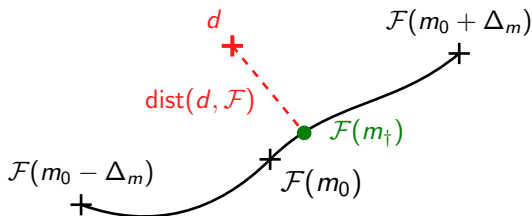


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Condition 1/2: Finite Curvature problem



$$\forall m_0, \Delta_m \in \mathcal{M}, \quad P : t \in [0, 1] \rightarrow \mathcal{F}(m_0 + (2t - 1)\Delta_m).$$

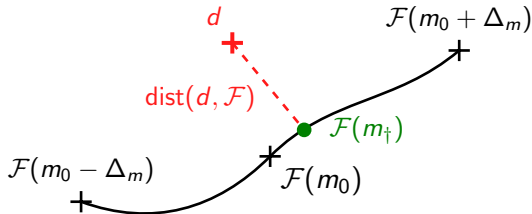


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Finite Curvature controls the distance between data and attainable set,

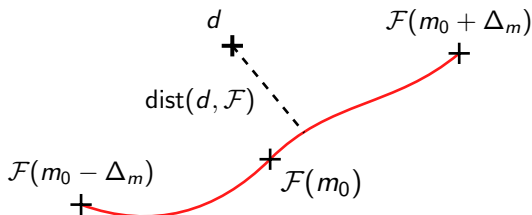
$$\text{dist}(d, \mathcal{F}) < \frac{\|P'(t)\|^2}{\|P''(t)\|}.$$



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Nonlinear least squares for inverse problems: theoretical foundations and step-by-step guide for applications.
Springer 2010

Condition 2/2: Limited Deflection problem



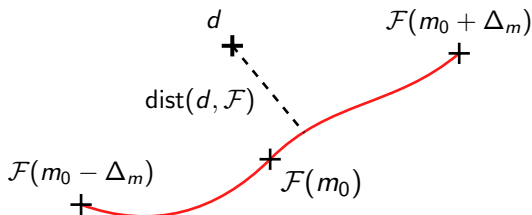
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$$\forall m_0, \Delta_m \in \mathcal{M}, \quad P : t \in [0, 1] \rightarrow \mathcal{F}(m_0 + (2t - 1)\Delta_m).$$

Limited Deflection property controls the attainable set and model space,

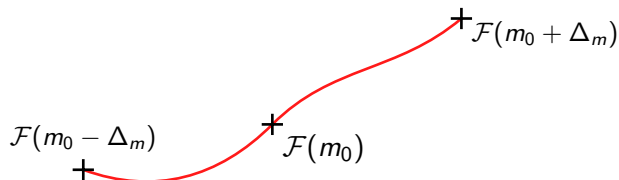
$$\Theta(P) \leq \int_0^1 \frac{\|P''(t)\|}{\|P'(t)\|} dt. \quad \text{FC/LD requires } \boxed{\Theta(P) \leq \frac{\pi}{2}}.$$



G. Chavent

Nonlinear least squares for inverse problems: theoretical foundations and step-by-step guide for applications.
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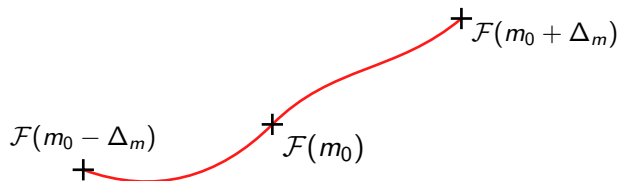
Model space size estimation via Limited Deflection



For a given m_0 , we estimate the maximal distance Δ_m that still verifies the Limited Deflection property $\Theta(P) \leq \frac{\pi}{2}$.

Larger size are required when missing a priori info.

Model space size estimation via Limited Deflection



For a given m_0 , we estimate the maximal distance Δ_m that still verifies the Limited Deflection property $\Theta(P) \leq \frac{\pi}{2}$.

- Δ_m is estimated in the direction δ_k ,

$$\|\Delta_m^{\delta_k}\| \leq \frac{\pi}{4} \frac{\|\mathcal{F}'(m_0)(\delta_k)\|}{\|\mathcal{F}''(m_0)(\delta_k, \delta_k)\|}.$$

Larger size are required when missing a priori info.

Estimation of the basin of attraction

Numerical estimate of the size $\Delta_m^{\delta_k}$

- 1 with direction δ_k (the geometry of the unknown),
- 2 with the frequency ω ,
- 3 with the parametrization.

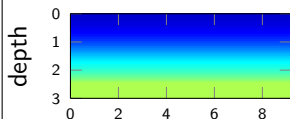
Estimation of the basin of attraction

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context: Helmholtz equation with back-scattered data

$$(-\omega^2 m - \Delta)p = 0,$$



m_0 is a smooth velocity

- ▶ reflection data (top free surface),
- ▶ absorbing conditions on the sides,
- ▶ piecewise constant m .

Estimation of the basin of attraction

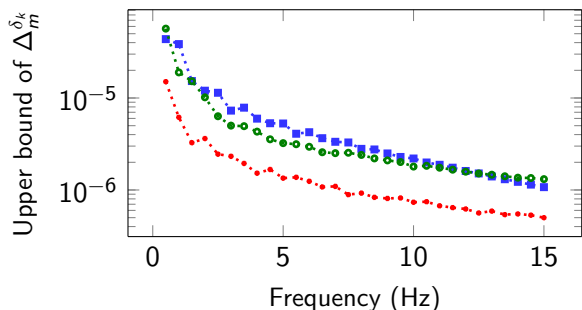
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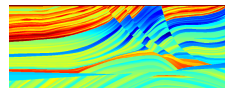
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(a) direction δ_1



(b) direction δ_2

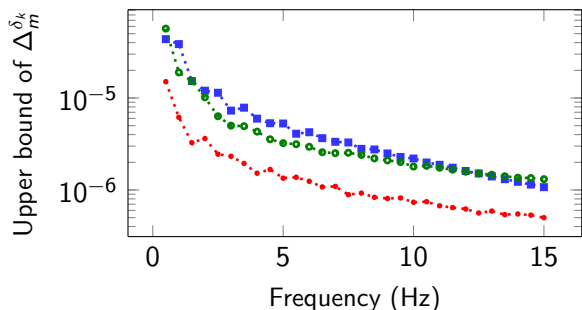


(c) direction δ_3

Estimation of the basin of attraction

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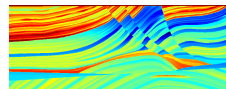


Low frequencies increase the size.

Reflecting objects complicate the procedure.



(a) direction δ_1



(b) direction δ_2



(c) direction δ_3

Complex frequency

$$(-\omega^2 c^{-2} - \Delta)p = 0$$

- ▶ $-\omega^2 = (s + 2i\pi f)^2$
- ▶ $s = 0$: Fourier domain $\omega^2 = (2\pi f)^2$,
- ▶ $f = 0$: Laplace domain $-\omega^2 = s^2$.



C. Shin and Y. H. Cha

Waveform inversion in the laplace domain ; Waveform inversion in the laplace fourier domain
Geophysical Journal International 2008–2009



W. Ha, S. Pyun, J. Yoo and C. Shin

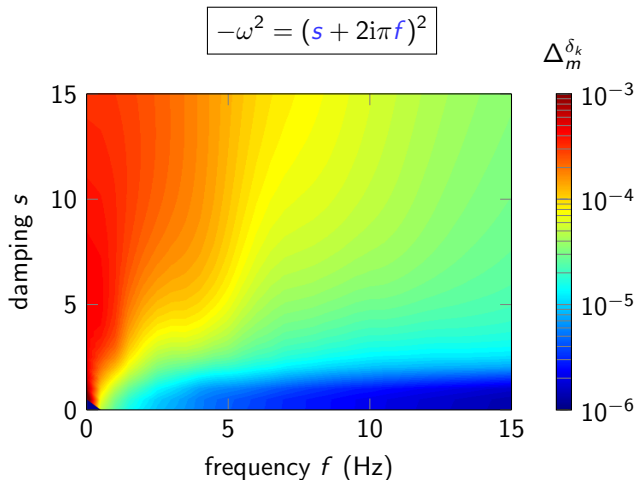
Acoustic full waveform inversion of synthetic land and marine data in the laplace domain ;
Geophysical Prospecting 2010



P. V. Petrov and G. A. Newman

Three-dimensional inverse modelling of damped elastic wave propagation in the fourier domain.
Geophysical Journal International 2014

Complex frequency



Selection of complex frequency.

Stability with frequency

2/ Stability

- ensures reconstruction accuracy.

The stability indicates how minimizing the data recovers the model

$$\|m_1 - m_2\| \leq \mathcal{C} \|\mathcal{F}(m_1) - \mathcal{F}(m_2)\|.$$

The stability constant \mathcal{C} depends on the frequency and the number of unknowns.



G. Alessandrini

Stable determination of conductivity by boundary measurement
[Applicable Analysis 1988](#)



E. Beretta, M. V. de Hoop, F. and O. Scherzer

Inverse boundary value problem for the Helmholtz equation: quantitative conditional Lipschitz stability estimates.
[SIAM J. Math. Analysis 2016](#)



G. Alessandrini, M.V. de Hoop, R. Gaburro and E. Sincich

Lipschitz stability for a piecewise linear Schrödinger potential from local Cauchy data
[arXiv:1702.04222, 2017](#)

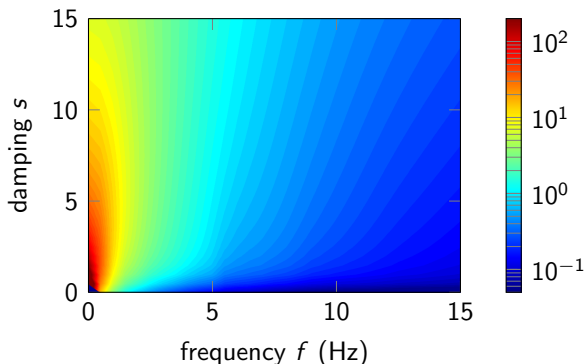
Stability with frequency

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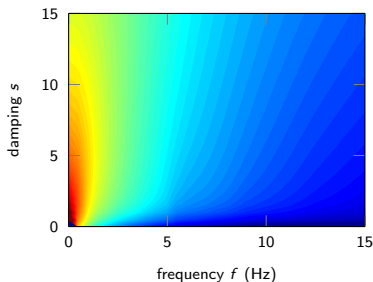
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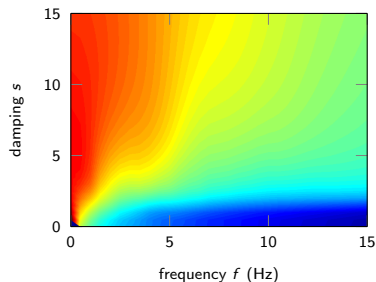


Stability & Convergence estimates

From our estimates, we see different conditions for stability and convergence.



(a) stability



(b) convergence

Quantitative estimates provide an initial, comprehensive, relation to guide the iterative procedure.

Plan

3 Numerical experiments

Multi-level algorithm: stability and convergence

0./ No initial information on the subsurface

largest radius of convergence,

1./ Start with low frequency



high stability constant.

Multi-level algorithm: stability and convergence

0./ No initial information on the subsurface

largest radius of convergence,

1./ Start with low frequency



high stability constant.

improve stability, *i.e.* resolution,

2./ Increase frequency

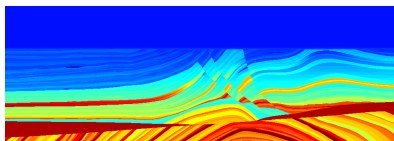


reduced radius of convergence.

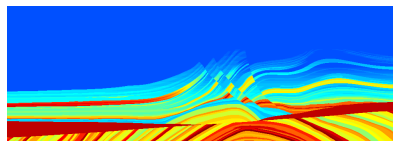
Elastic Marmousi model

Multi-parameters inversion $17 \times 3.5\text{km}$, true models.

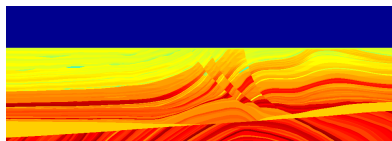
$$-\rho\omega^2\mathbf{u} - \nabla(\lambda\nabla\cdot\mathbf{u}) - \nabla\cdot(\mu[\nabla\mathbf{u} + (\nabla\mathbf{u})^T]) = 0.$$



(a) P-wave speed $\sqrt{\frac{\lambda+2\mu}{\rho}}$



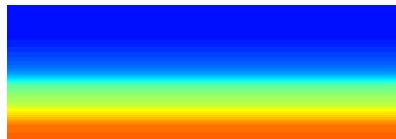
(b) S-wave speed $\sqrt{\frac{\mu}{\rho}}$



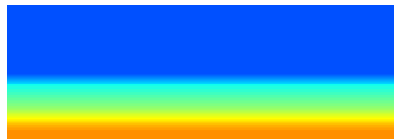
(c) Density

Elastic Marmousi model

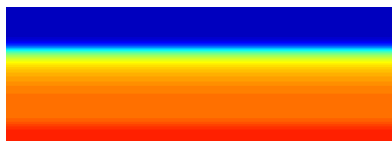
Multi-parameters inversion $17 \times 3.5\text{km}$, starting models.



(a) P-wave speed



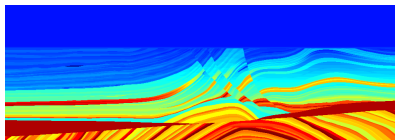
(b) S-wave speed



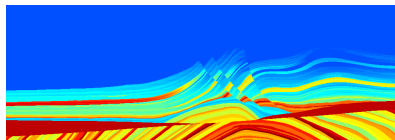
(c) Density

Elastic Marmousi reconstruction $17 \times 3.5\text{km}$

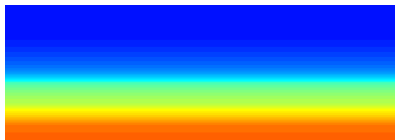
Frequency from 1 to 10Hz, unknown density, multi-level algorithm.



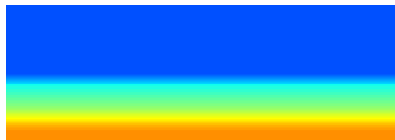
(a) True P-wave speed



(b) True S-wave speed



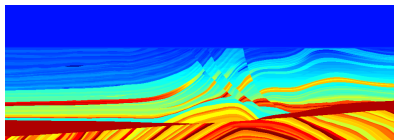
(c) Initial P-wave speed



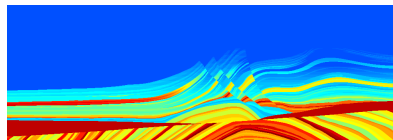
(d) Initial S-wave speed

Elastic Marmousi reconstruction $17 \times 3.5\text{km}$

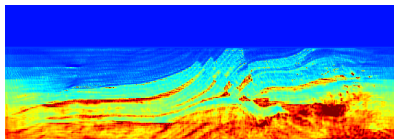
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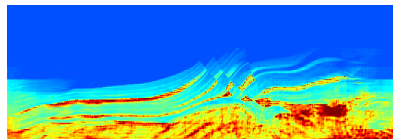
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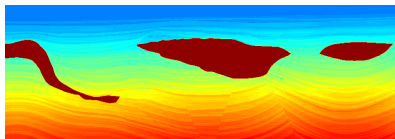
(c) 10Hz P-wave speed



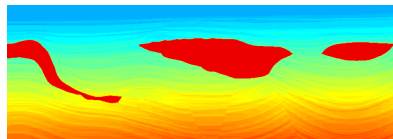
(d) 10Hz S-wave speed

Elastic Pluto models

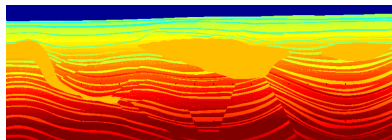
Multi-parameters inversion $31 \times 7\text{km}$, true models.



(a) P-wave speed



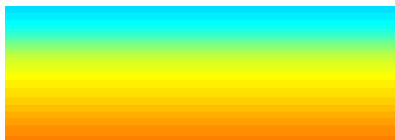
(b) S-wave speed



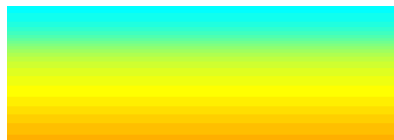
(c) Density

Elastic Pluto models

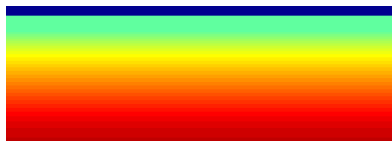
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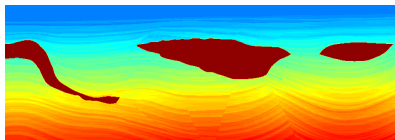
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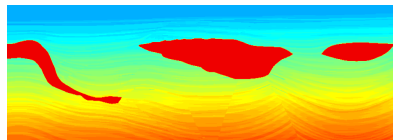
(c) Density

Elastic Pluto reconstruction $31 \times 7\text{km}$

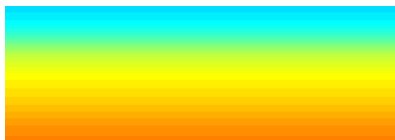
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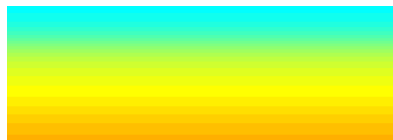
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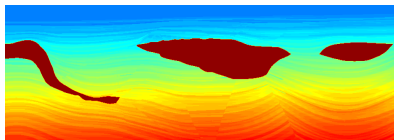
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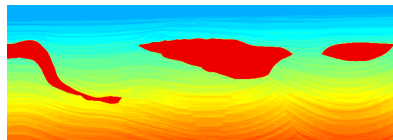
(d) Initial S-wave speed

Elastic Pluto reconstruction $31 \times 7\text{km}$

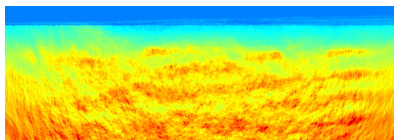
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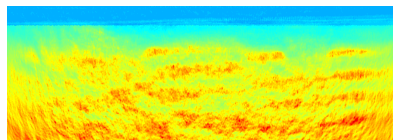
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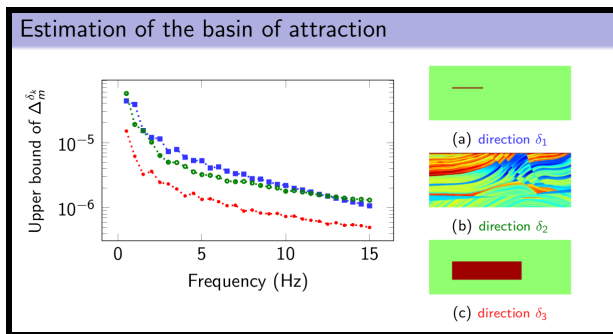
(c) 10Hz P-wave speed



(d) 10Hz S-wave speed

Elastic Pluto reconstruction $31 \times 7\text{km}$

Reminder from the convergence analysis.



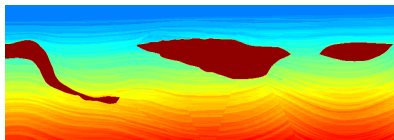
- Salt domes reduce the size of the radius of convergence.

Solution

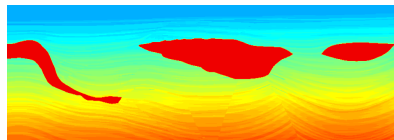
- Start with low or complex frequency.

Elastic Pluto reconstruction $31 \times 7\text{km}$

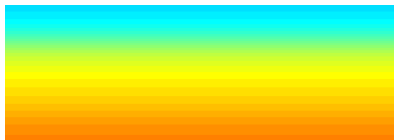
Using complex frequencies, unknown density, multi-level algorithm.



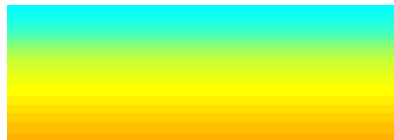
(a) True P-wave speed



(b) True S-wave speed



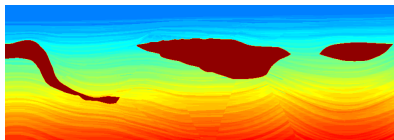
(c) Initial P-wave speed



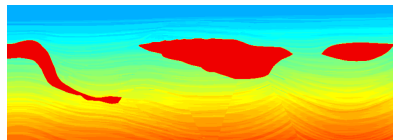
(d) Initial S-wave speed

Elastic Pluto reconstruction $31 \times 7\text{km}$

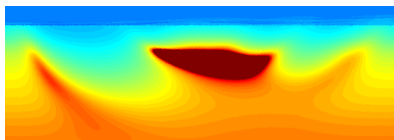
Using complex frequencies, unknown density, multi-level algorithm.



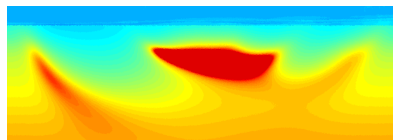
(a) True P-wave speed



(b) True S-wave speed



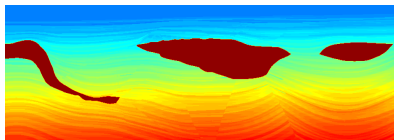
(c) (0Hz,10) P-wave speed



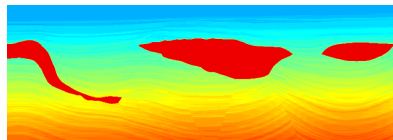
(d) (0Hz,10) S-wave speed

Elastic Pluto reconstruction $31 \times 7\text{km}$

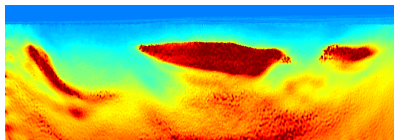
Using complex frequencies, unknown density, multi-level algorithm.



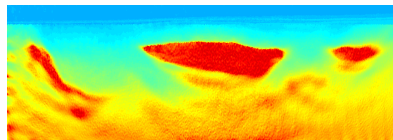
(a) True P-wave speed



(b) True S-wave speed



(c) 10Hz P-wave speed



(d) 10Hz S-wave speed

Plan

4 Conclusion

Conclusion

Comprehensive FWI

- ▶ Quantitative stability and convergence analysis:
 - ▶ depending on the frequency and geometry,
 - ▶ **depending on the methods.**
- ▶ We can also quantify the noise robustness.

Perspectives

- ▶ Analytical relation to develop between the two components,
- ▶ obtain conditional progression in frequency from stability and convergence estimates

Quantitative reconstruction method for inverse problem

- ▶ Discontinuous Galerkin discretization in HPC framework,
- ▶ acoustic, elastic, anisotropy, viscosity; 2D, 3D, dual-sensors data.

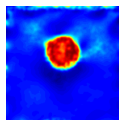
Applications and ongoing investigations:

Quantitative reconstruction method for inverse problem

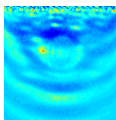
- ▶ Discontinuous Galerkin discretization in HPC framework,
- ▶ acoustic, elastic, anisotropy, viscosity; 2D, 3D, dual-sensors data.

Applications and ongoing investigations:

- ▶ **Inverse scattering** using obstacles positions,
- ▶ **visco-elastic** reconstruction: five unknowns,

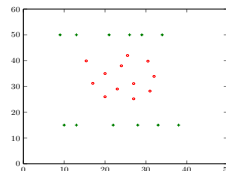


(a) λ



(b) Q_λ

- ▶ $(\lambda, \mu, \rho, Q_\lambda, Q_\mu)$,
- ▶ unknown Q does not prevent λ recovery,
- ▶ procedure to recover attenuation parameter?



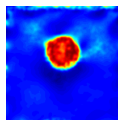
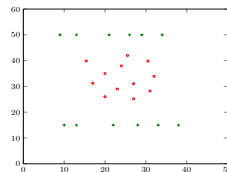
- ▶ **helioseismology**; **anisotropy**; model **parametrization**.

Quantitative reconstruction method for inverse problem

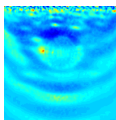
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- ▶ **Inverse scattering** using obstacles positions,
- ▶ **visco-elastic** reconstruction: five unknowns,



(a) λ



(b) Q_λ

- ▶ $(\lambda, \mu, \rho, Q_\lambda, Q_\mu)$,
- ▶ unknown Q does not prevent λ recovery,
- ▶ procedure to recover attenuation parameter?

- ▶ **helioseismology**; **anisotropy**; model **parametrization**.

THANK YOU

APPENDIX

Quantitative convergence for MBTT

Numerical estimate of the size $\Delta_m^{\delta_k}$ the problem components

- 1 the frequency ω ,
- 2 with direction δ_k (the geometry of the unknown),
- 3 with the parametrization.

The estimates allow a comparison of methods, in particular we want to compare the dependency of the optimization on the low frequencies.

Quantitative convergence for MBTT

Numerical estimate of the size $\Delta_m^{\delta_k}$ the problem components

- 1 the frequency ω ,
- 2 with direction δ_k (the geometry of the unknown),
- 3 with the parametrization.

the MBTT method decomposes the model with a smooth part (propagator) and the reflectors:

$$c^{-2} = p + r = p + D\mathcal{F}^*(p)s.$$



F. Clément and G. Chavent

Waveform inversion through MBTT formulation – 1992



F. Clément, G. Chavent and S. Gómez

Migration-based traveltime waveform inversion of 2-d simple structures: A synthetic example
Geophysics 2001



G. Chavent, K. Gadylyshin and V. Tcheverda

Reflection fwi in mbtt formulation
EAGE 2015

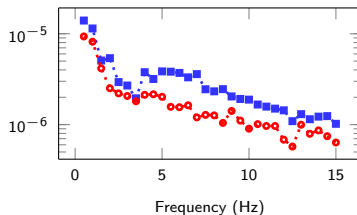
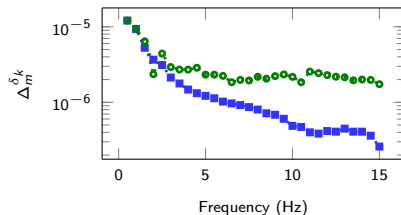
Quantitative convergence for MBTT

Numerical estimate of the size $\Delta_m^{\delta_k}$ the problem components

- 1 the frequency ω ,
- 2 with direction δ_k (the geometry of the unknown),
- 3 with the parametrization.

Comparison of convergence with parametrization,

$$c^{-2} = p + D\mathcal{F}^*(p)s.$$



Quantitative convergence estimates with frequency

Frequency progression for iterative algorithm, $-\omega^2 = (s + 2i\pi f)^2$.

Other application

- ▶ Single frequency gives high radius than range of frequencies,
- ▶ can be apply for other minimization problem,
- ▶ ...

Finite curvature indicates robustness to noise

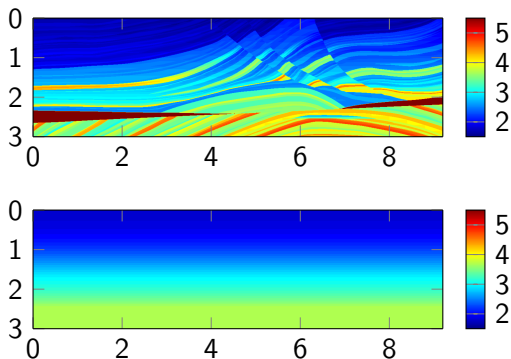
- ▶ Low and complex frequencies more affected.

Stability

- ▶ Indicates the accuracy of the reconstruction.

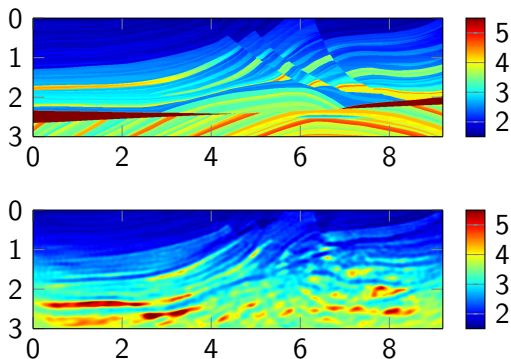
Acoustic Marmousi reconstruction $9.2 \times 3\text{km}$

Starting available frequency is 4Hz.



Acoustic Marmousi reconstruction $9.2 \times 3\text{km}$

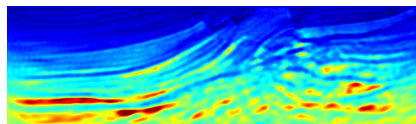
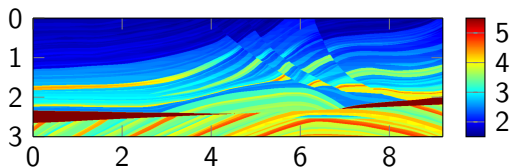
Starting available frequency is 4Hz.



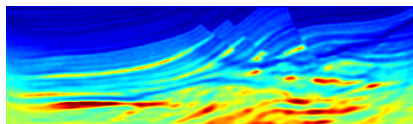
Using Fourier frequencies only: from 4 to 10Hz

Acoustic Marmousi reconstruction $9.2 \times 3\text{km}$

Starting available frequency is 4Hz.



(a) Using Fourier frequencies only



(b) Using Complex frequencies from (4Hz, 7)