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Manipulating Opinion Dynamics in Social Networks in Two Phases

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Abstract

We propose a setting for two-phase opinion dynamics in social networks, where the final opinion of a node in the first phase acts as its initial biased opinion in the second phase. In this setting, we study the problem of two camps aiming to maximize adoption of their respective opinions by strategically investing on nodes, where the effectiveness of a camp’s investment on a node depends on the node’s initial bias. We propose an extension of Friedkin-Johnsen model for our setting, and hence formulate the utility functions of the camps. For the non-competitive case where only one camp invests, we present a polynomial time algorithm for determining an optimal way to split the camp’s budget between the two phases. For the case of competing camps, we show the existence of Nash equilibria under reasonable assumptions, and that they can be computed in polynomial time. Our main conclusion is that, if nodes attribute high weightage to their initial biases, it is advantageous to have a high investment in the first phase, so as to exploit the manipulated biases in the second phase.

1 Introduction

Studying opinion dynamics in a society is important to understand and influence elections, viral marketing, propagation of ideas and behaviors, etc. In this paper, we consider two camps who aim to maximize the adoption of their respective opinions in a social network. We consider a strict competition in the space of real-valued opinions, where one camp aims to drive the overall opinion of the network towards being positive while the other camp aims to drive it towards negative; we refer to them as good and bad camps respectively. We consider a well-accepted quantification of the overall opinion of a network: the average or equivalently, sum of opinion values of nodes [Gionis \textit{et al.}, 2013; Grabisch \textit{et al.}, 2018]. Hence the good and bad camps simultaneously aim to respectively maximize and minimize this sum.

The average or sum of opinion values is of relevance in several applications, e.g., fund collection where the magnitude of a node’s opinion value corresponds to the amount and the sign indicates the camp towards which it is willing to contribute. Another example is when nodes are reporting agents on social media, where a node’s opinion corresponds to its reported intensity of an event based on the information it receives; this information could be influenced by the camps. In such scenarios, the opinion value of a node would be a real-valued number, and the sum or average of opinion values would be the overall opinion of the network.

Social networks play a prime role in determining the opinions of constituent nodes, since nodes usually update their opinions based on the opinions of their connections [Easley and Kleinberg, 2010; Acemoglu and Ozdaglar, 2011]. This fact is exploited by camps; they determine influential nodes to invest on in the form of money, free products, convincing discussions, etc., so as to drive these nodes’ opinions in their favor. Thus given a budget constraint, the strategy of a camp comprises of how much to invest on each node, in presence of a competing camp who also would invest strategically.

1.1 Motivation

In the popular model by Friedkin and Johnsen [1990, 1997], every node holds an initial bias in opinion. It could have formed owing to the node’s fundamental views, experiences, mass media exposure, opinion dynamics in the past, etc. This initial bias plays an important role in determining a node’s final opinion, and consequently the opinions of its neighbors and hence that of its neighbors’ neighbors and so on. If nodes give significant weightage to their biases, the camps would want to manipulate these biases. This could be achieved by campaigning in multiple phases, wherein the opinion at the conclusion of a phase would act as the initial biased opinion for the next phase. Such campaigning is often used during elections and marketing, to gradually drive nodes’ opinions.

The initial bias of a node often impacts the effectiveness of camps’ investments on that node. For instance, if the initial bias of a node is positive, the investment made by the good camp is likely to be more effective on it than that made by the bad camp. The reasoning is similar to that of the bounded confidence model [Krause, 2000], where a node pays more attention to opinions that do not differ too much from its own opinion. Since a camp’s effectiveness depends on the nodes’ biases, its investment in the first phase not only manipulates the biases for getting a head start in the second phase, but also the effectiveness of its investment in the second phase.

With the possibility of campaigning in two phases, a camp could not only decide which nodes to invest on, but also how to split its available budget between the two phases.
1.2 Related Work

Several models of opinion dynamics have been proposed in the literature [Acemoglu and Ozdaglar, 2011; Lorenz, 2007], some noteworthy ones being DeGroot [DeGroot, 1974], Voter [Holley and Liggett, 1975], Friedkin-Johnsen [Friedkin and Johnsen, 1990, 1997], bounded confidence [Krause, 2000], etc. In Friedkin-Johnsen model, each node updates its opinion using a weighted combination of its initial bias and its neighbors’ opinions. We generalize this model to multiple phases, while also accounting for the camps’ investments.

Problems related to determining influential nodes in social networks have been extensively studied [Guille et al., 2013; Easley and Kleinberg, 2010; Kempe et al., 2003; Gionis et al., 2013]. For instance, Yildiz et al. [2013] study the problem of optimal placement of stubborn nodes (whose opinion values stay unchanged) in the discrete binary opinions setting. There also have been game theoretic studies [Ghaderi and Srikant, 2014; Anagnostopoulos et al., 2015; Bharathi et al., 2007]. Specific to analytically tractable models such as DeGroot, there have been studies in the competitive setting to identify influential nodes and the amounts to be invested on them [Dubey et al., 2006; Bimpikis et al., 2016; Grabisch et al., 2018]. Our work extends these studies to two phases, by identifying influential nodes in the two phases and how much they should be invested on in each phase.

There have been a few studies on adaptive selection of influential nodes for opinion diffusion in multiple phases [Seeman and Singer, 2013; Rubinstein et al., 2015; Horel and Singer, 2015; Correa et al., 2015; Badanidiyuru et al., 2016; Tong et al., 2016; Yuan and Tang, 2017]. Singer [2016] presents a survey of such adaptive methodologies. Dhamal et al. [2016] empirically study the problem of optimally splitting the available budget between two phases, which has been extended to multiple phases by Dhamal [2018]. While the reasoning behind using multiple phases in these studies is adaptation of strategy based on previous observations, we aim to use multiple phases for manipulating the initial biases of nodes, which requires a very different analytical treatment.

To the best of our knowledge, there has not been an analytical study on a rich model such as Friedkin-Johnsen, for opinion dynamics in two phases (not even for single camp). The most relevant to this study is our work in [Dhamal et al., 2018] where, however, a camp’s influence on a node is assumed to be independent of the node’s bias.

1.3 Our Contributions

- We formulate the two-phase objective function under Friedkin-Johnsen model, where a node’s final opinion in the first phase acts as its initial bias for the second phase, and the effectiveness of a camp’s influence on the node depends on this initial bias.

- For the non-competitive case, we develop a polynomial time algorithm for determining an optimal way to split a camp’s budget between the two phases.

- For the case of two competing camps, we show the existence of Nash equilibria under reasonable assumptions, and that they can be computed in polynomial time.

2 Our Model

Given a social network, let \( N \) be the set of nodes and \( E \) be the set of weighted directed edges. Let \( n = |N| \). The weights could hold any sign, since the influences could be positive or negative [Altafini, 2013]. Our model can be viewed as a multiphase extension of [Dhamal et al., 2017].

2.1 Parameters

As our opinion dynamics runs in two phases, most parameters have two values, one for each phase. For such parameters, we denote its value corresponding to phase \( p \) using superscript \((p)\), where \( p = 1, 2 \) for first and second phase, respectively.

Initial Bias. Every node \( i \) holds an initial bias in opinion prior to the opinion dynamics process; we denote it by \( v_i^0 \in \mathbb{R} \). In the multiphase setting, this acts as the initial bias for the first phase. We denote the opinion value of node \( i \) at the conclusion of phase 1 by \( v_i^{(1)} \). This acts as its initial bias for phase 2.

So by convention, we have \( v_i^{(0)} = v_i^0 \). We denote the weightage that node \( i \) attributes to its initial bias by \( w_i^{(0)} \).

Network Effect. Let \( w_{ij} \) be the weightage attributed by node \( i \) to the opinion of its connection \( j \). Consistent with Friedkin-Johnsen model, we assume the influence on node \( i \) owing to node \( j \) in phase \( p \), to be \( w_{ij} v_j^{(p)} \). So the net influence on node \( i \) owing to all of its connections is \( \sum_{j \in N} w_{ij} v_j^{(p)} \).

Weightage to Campaigning. We denote the weightage that node \( i \) attributes to the good and bad campaigning in phase \( p \) by \( w_{ig}^{(p)} \) and \( w_{ib}^{(p)} \), respectively. Since we consider that the initial bias of a node impacts the effectiveness of camps’ investments, \( v_j^{(p-1)} > 0 \) would likely mean \( w_{ig}^{(p)} > w_{ib}^{(p)} \). Note that \( w_i^{(0)} \) would also play a role since it quantifies the weightage given by node \( i \) to its initial bias. We hence propose a model on this line, wherein \( w_{ig}^{(p)} \) is a monotone non-decreasing function of \( w_{ii}^{0} v_i^{(p-1)} \), and \( w_{ib}^{(p)} \) is a monotone non-increasing function of \( w_{ii}^{0} v_i^{(p-1)} \). Let node \( i \) attribute a total of \( \theta_i \) to the influence weights of the camps, that is, \( w_{ig}^{(p)} + w_{ib}^{(p)} = \theta_i \).

We propose the following natural model:

\[
\begin{align*}
 w_{ig}^{(p)} &= \theta_i \left( 1 + \frac{w_{ii}^{0} v_i^{(p-1)}}{2} \right), \\
 w_{ib}^{(p)} &= \theta_i \left( 1 - \frac{w_{ii}^{0} v_i^{(p-1)}}{2} \right) \quad (1)
\end{align*}
\]

Camp Investments. The good and bad camps attempt to directly influence the nodes so that their opinions are pulled towards being positive and negative respectively. We denote the investments made by the good and bad camps on node \( i \) in phase \( p \) by \( x_i^{(p)} \) and \( y_i^{(p)} \) respectively (\( x_i^{(p)}, y_i^{(p)} \geq 0, \forall i \in N \) for \( p = 1, 2 \)). Since the influence of good camp (+ve opinion) on node \( i \) in phase \( p \) would be an increasing function of both \( x_i^{(p)} \) and \( w_{ig}^{(p)} \), we assume the influence to be \( +w_{ig}^{(p)} x_i^{(p)} \) (maintaining the linearity of Friedkin-Johnsen model). Similarly, \( -w_{ib}^{(p)} y_i^{(p)} \) is the influence of bad camp (−ve opinion) on node \( i \) in phase \( p \). Let \( k_g \) and \( k_b \) be the respective budgets of the good and bad camps. Hence the camps should invest in the two phases such that \( \sum_{i \in N} (x_i^{(1)} + x_i^{(2)}) \leq k_g \) and \( \sum_{i \in N} (y_i^{(1)} + y_i^{(2)}) \leq k_b \).
Weight Constraints. Friedkin-Johnsen model has the following condition on influence weights:
\[ \forall i \in N : |w_{0i}^0| + \sum_{j \in N} |w_{ij}| \leq 1 \]
Since the model follows an opinion update rule (as we see later), convergence is an important factor. A standard assumption for guaranteeing convergence is
\[ \sum_{j \in N} |w_{ij}| < 1 \]
This condition is usually enforced to ensure convergence. In our setting, this condition is natural and well suited, since we would have non-zero weights attributed to bias \((w_{0i}^0)\).

Matrix Forms. Let \( W \) be the matrix consisting of weights \( w_{ij} \) for each pair of nodes \((i, j)\). Let \( v^0, w^0, \Theta, w^p_g, w^p_b, x^p, y^p \) be the vectors consisting of elements \( v_{i0}, w_{i0}, \Theta_i, w^p_g, w^p_b, x^j_i, y^j_i, v_{i0}^p \) respectively. Let operation \( \circ \) denote Hadamard product vector, i.e., \((a \circ b)_i = a_i b_i\).

2.2 Opinion Update Rule
The update rule in Friedkin-Johnsen model is \( \forall i \in N : v_i \leftarrow u_{ii}^0 v_i^0 + \sum_{j \in N} u_{ij} v_j \). Extending to multiple phases, the update rule in the \( p \)th phase is \( \forall i \in N : v_i^p \leftarrow u_{ii}^p v_i^p + \sum_{j \in N} u_{ij}^p v_j^p \). Accounting for camps' investments, we get
\[ \forall i \in N : v_i^p \leftarrow u_{ii}^p v_i^{p-1} + \sum_{j \in N} u_{ij}^p w_i^j + w_i^p x_i^p - w_i^p y_i^p \]
\[ \Leftarrow v_i^p \leftarrow v_i^{p-1} + \sum_{j \in N} u_{ij}^p (w_i^j + w_i^p x_i^p - w_i^p y_i^p) \quad (2) \]

In phase \( p \), the vectors \( x^p, y^p \) stay unchanged; the weights \( w^p_g, w^p_b \) (which depend on \( v^{p-1} \)) and \( v^0 \) also stay unchanged; while \( v^p \) gets updated. Hence, writing the update rule as recursion (with iterating integer \( \tau \geq 0 \)):
\[ v_{(\tau)}^p = W v_{(\tau-1)} + w^0 \circ v_{(\tau-1)} + w^p_g \circ x^p - w_b^p \circ y^p \]

It can be simplified as
\[ v_{(\tau)}^p = W^\tau v_0^p + \left( \sum_{\eta=0}^{\tau-1} W^\eta \right) (w^0 \circ v_{(\tau-1)} + w^p_g \circ x^p - w^p_b \circ y^p) \]

Now, the initial bias for phase \( p \) is \( v_{(0)}^p = v^{p-1} \). Also, \( W \) is strictly substochastic (sum of each row strictly less than 1); its spectral radius is less than 1. So when \( \tau \rightarrow \infty \), we have \( \lim_{\tau \rightarrow \infty} W^\tau = 0 \) and \( \lim_{\tau \rightarrow \infty} \sum_{\eta=0}^{\tau-1} W^\eta = (I - W)^{-1} \) [Grabisch et al., 2018]. Hence,
\[ \lim_{\tau \rightarrow \infty} v_{(\tau)}^p = (I - W)^{-1} \left( w^0 \circ v_{(\tau-1)} + w^p_g \circ x^p - w^p_b \circ y^p \right) \]
which is a constant vector. So the dynamics in phase \( p \) converges to the steady state
\[ v^p = (I - W)^{-1} (w^0 \circ v^{p-1} + w^p_g \circ x^p - w^p_b \circ y^p) \quad (3) \]

2.3 Formulation of Two-Phase Objective Function
We now derive \( \sum_{i \in N} v_i^{(p)} \), the sum of nodes' opinions at the end of phase \( p \in \{1, 2\} \). Premultiplying (3) by \( I^T \) gives
\[ I^T v^p = I^T (I - W)^{-1} (w^0 \circ v^{p-1} + w^p_g \circ x^p - w^p_b \circ y^p) \]

Let \( \Delta = (I - W)^{-1} \) and \( r^T = I^T (I - W)^{-1} \), that is, \( r_i = \sum_{j \in N} \Delta_{ji} \). Since \( \Delta = \sum_{\eta=0}^{\infty} W^\eta \), we have that \( \Delta_{ji} \) is the influence that \( j \) receives from \( i \) through walks of all possible lengths. So \( r_i = \sum_{j \in N} \Delta_{ji} \) can be viewed as overall influencing power of \( i \). Substituting these in above equation,
\[ \sum_{i \in N} v_i^{(p)} = \sum_{i \in N} \sum_{j \in N} \Delta_{ji} \left( -w_{ib}^p v_i^{(p-1)} + v_{ig}^p x_i^p - w_{ib} y_i^p \right) \]

When \( p = 1 \), this is the sum of opinions at the end of phase 1:
\[ \sum_{i \in N} v_i^{(1)} = \sum_{i \in N} \sum_{j \in N} \Delta_{ji} \left( -w_{ib}^p v_i^{(0)} + v_{ig}^p x_i^p - w_{ib} y_i^p \right) \]

Similarly, the sum of opinion values at the end of phase 2 is
\[ \sum_{j \in N} v_j^{(2)} = \sum_{j \in N} \sum_{i \in N} \Delta_{ij} \left( 1 + \theta_2^j (x_i^1 + y_i^1) - \theta_2^j y_i^2 \right) \]

The first term \( \sum_{i \in N} v_i^{(1)} \) equals
\[ \sum_{j \in N} \sum_{i \in N} \left( w_{i0}^0 \left( 1 + \theta_2^j x_i^1 + \theta_2^j y_i^1 \right) + \theta_2^j (x_i^1 - y_i^1) \right) \]
\[ \cdot \left( r_j \theta_2^j \Delta_{ij} \left( 1 + \theta_2^j x_i^2 + \theta_2^j y_i^2 \right) \right) + \sum_{j \in N} r_j \theta_2^j (x_i^2 - y_i^2) \quad (5) \]

Here \( (x^1, x^2) \) is the strategy of the good camp for the two phases, and \( (y^1, y^2) \) is the strategy of the bad camp. Given an investment strategy profile \( (x^1, x^2), (y^1, y^2) \), let \( u_g((x^1, x^2), (y^1, y^2)) \) be the camp's utility and \( u_b((x^1, x^2), (y^1, y^2)) \) be the bad camp's utility. The good camp aims to maximize the value of (5), while the bad camp aims to minimize it. So,
\[ u_g((x^1, x^2), (y^1, y^2)) = \sum_{i \in N} v_i^{(2)} \]
and \( u_b((x^1, x^2), (y^1, y^2)) = -\sum_{i \in N} v_i^{(2)} \quad (6) \)

with the following constraints on the investment strategies:
\[ \sum_{i \in N} (x_i^{(1)} + x_i^{(2)}) \leq k_g, \quad \sum_{i \in N} (y_i^{(1)} + y_i^{(2)}) \leq k_b \]
\[ \forall i \in N : x_i^{(1)}, x_i^{(2)}, y_i^{(1)}, y_i^{(2)} \geq 0 \]

The game can thus be viewed as a two-player zero-sum game, where the players determine their investment strategies \((x^1, x^2)\) and \((y^1, y^2)\); the good camp invests as
Proof. Given any $x^{(1)}$ in the first phase and as per $x^{(2)}$ in the second phase, and the bad camp invests as per $x^{(1)}$ in the first phase and as per $y^{(2)}$ in the second phase. Our objective essentially is to find the Nash equilibrium strategies of the two camps.

First we consider a simplified yet interesting (and not-yet-studied-in-literature) case where budget of one of the camps is 0 (say $b_0 = 0$); so effectively we have only the good camp.

3 The Non-Competitive Case

For notational simplicity, let $b_{ij} = r_j w_j^{(i)} \Delta_{ij}$, $c_i = w_i^{(i)} v_0$. We saw that $r_i = \sum_{j \in N} \Delta_{ij}$ indicates the influencing power of node $i$. Now, $b_{ij} = \Delta_{ij} w_j^{(i)} r_j$ quantifies the overall influence $\Delta_{ij}$ of $i$ on $j$, which would give weightage $w_j^{(i)}$ to its initial opinion in the next phase, and have an influencing power of $r_j$ in the next phase. Hence $b_{ij}$ can be interpreted as the influence of node $i$ on the network through node $j$, looking one phase ahead. So $\sum_{j \in N} b_{ij}$ can be viewed as the overall influencing power of node $i$, looking one phase ahead.

For non-competitive case, we have $y_{ij} = y_{ij}^{(1)} = 0$, $\forall i \in N$ in (5). Using above notation, we get that $\sum_{i \in N} v_i^{(1)}$ equals

$$\sum_{i \in N} \sum_{j \in N} \left( c_i + \theta_i x_i^{(1)} + 1 \right) + \sum_{j \in N} r_j x_j^{(2)}$$

$$= \sum_{i \in N} \left( c_i + \theta_i x_i^{(1)} + 1 \right) + \sum_{i \in N} \sum_{j \in N} c_i b_{ij} \left( 1 + \theta_{ij} x_{ij}^{(2)} \right)$$

$$= \sum_{j \in N} \left( \theta_j x_j^{(2)} \sum_{i \in N} b_{ij} \left( c_i + \theta_i x_i^{(1)} + 1 \right) + \theta_j x_j^{(2)} \right)$$

Deducing from Equations (8) and (9) that $\sum_{i \in N} v_i^{(2)}$ is a bilinear function in $x^{(1)}$ and $x^{(2)}$, we can prove our next result.

Let the budget $k_g$ be split such that $k_g^{(1)}$ and $k_g^{(2)}$ are the first and second phase investments, respectively. Our method of finding optimal $k_g^{(1)}$ and $k_g^{(2)}$ is to first search for them in the search space $k_g^{(1)} + k_g^{(2)} \in (0, k_g)$ and then compare the thus obtained value of $\sum_{i \in N} v_i^{(2)}$ with that corresponding to $k_g^{(1)} = k_g^{(2)} = 0$, to get the optimal value of $\sum_{i \in N} v_i^{(2)}$.

Proposition 1. In the search space $k_g^{(1)} + k_g^{(2)} \in (0, k_g)$, it is optimal for good camp to exhaust entire budget $k_g^{(1)} + k_g^{(2)} = k_g$, and to invest on at most one node in each phase.

Proof. Given any $x^{(2)}$, Expression (8) can be maximized with respect to $x^{(1)}$ by allocating $k_g - \sum_{j \in N} x_j^{(2)}$ to a single node $i$ that maximizes $\frac{\theta_i}{2} c_i + 1 + \theta_i x_i^{(2)}$, if this value is positive. In case of multiple such nodes, one node can be chosen at random. If this value is non-positive for all nodes, it is optimal to have $x^{(1)} = 0$. When $x^{(1)} = 0$, Expression (9) now implies that it is optimal to allocate the entire budget $k_g$ to the first phase to a single node $j$ that maximizes $\frac{\theta_j}{2} c_j + 1 + \theta_j x_j^{(2)}$, if this value is positive. If this value is non-positive for all nodes, it is optimal to have $x^{(2)} = 0$. This is the case where starting with an $x^{(2)}$, we conclude that it is optimal to either invest $k_g - \sum_{j \in N} x_j^{(2)}$ on a single node in first phase, or invest the entire budget $k_g$ on a single node in second phase, or invest in neither phase.

Similarly using (9), starting with a given $x^{(1)}$, we can conclude that it is optimal to either invest $k_g - \sum_{j \in N} x_j^{(1)}$ on a single node in the second phase, or invest the entire $k_g$ on a single node in the first phase, or invest in neither phase.

So starting from any $x^{(1)}$ or $x^{(2)}$, we can iteratively improve (need not be strictly) on the value of (7) by investing on at most one node in a given phase. Furthermore, it is sub-optimal to have $k_g^{(1)} + k_g^{(2)} < k_g$ unless $k_g^{(1)} = k_g^{(2)} = 0$. So there exist optimal vectors $x^{(1)}$ and $x^{(2)}$ that maximize (7), such that $x_i^{(1)} = x_j^{(2)} = k_g$, $\forall i \neq j$, $x_i^{(1)} = x_j^{(2)} = 0$. Now the next step is to find nodes $\alpha$ and $\beta$ that maximize (7).

By incorporating $\alpha$ and $\beta$ in (7), we have $\sum_{i \in N} v_i^{(2)}$ equals

$$\sum_{i \in N} \sum_{j \in N} c_i b_{ij} (c_i + \theta_i x_i^{(1)} + 1) + \sum_{j \in N} r_j x_j^{(2)}$$

$$= \sum_{i \in N} \sum_{j \in N} c_i b_{ij} \left( c_i + \theta_i x_i^{(1)} + 1 + \theta_j x_j^{(2)} \right)$$

Equating its first derivative w.r.t. $k_g^{(1)}$ to zero, we get

$$k_g^{(1)} = \frac{k_g}{2} + \sum_{i \in N} c_i b_{ij} - \frac{1}{\theta_i} \sum_{i \in N} c_i b_{ij} c_i + + r_j \frac{\theta_j x_j^{(2)}}{\sum_{j \in N} b_{ij}}$$

A valid value of $k_g^{(1)}$ can be obtained only if the denominators in above expression are non-zero. However, a zero denominator would mean that Expression (10) is linear, resulting in only two possibilities of $k_g^{(1)}$, namely, 0 or $k_g$. Also, if the second derivative with respect to $k_g^{(1)}$ is positive, that is, $-\theta_i b_{ij} r_j w_j^{(i)} \Delta_{ij} (c_i v_0 + 1) > 0$, optimal $k_g^{(1)}$ is either 0 or $k_g$. If the second derivative with respect to $k_g^{(1)}$ is negative: $-\theta_i b_{ij} r_j w_j^{(i)} \Delta_{ij} (c_i v_0 + 1) < 0$, and since $k_g^{(1)}$ is bounded in $[0, k_g]$, optimal $k_g^{(1)}$ for pair $(\alpha, \beta)$ is (since $c_i = w_j^{(i)} v_0$, $b_{ij} = r_j w_j^{(i)} \Delta_{ij}$, $s_i = \sum_{i \in N} b_{ij}$):
When we assumed $k_g^{(1)}$ and $k_g^{(2)}$ to be fixed, we had to iterate through all $(\alpha, \beta)$ pairs to determine the one that gives the optimal value of Expression (10). Now, whenever we look at an $(\alpha, \beta)$ pair, we can determine the corresponding optimal values of $k_g^{(1)}$ and $k_g^{(2)}$ using (11) and (12), and hence determine the value of Expression (10) by plugging in the optimal $k_g^{(1)}$ and $k_g^{(2)}$ and that $(\alpha, \beta)$ pair. The optimal pair $(\alpha, \beta)$ can thus be obtained as the pair that maximizes (10).

Above analysis holds when $k_g^{(1)} + k_g^{(2)} = k_g$. From Proposition 1, we need to consider one more possibility that $k_g^{(1)} = k_g^{(2)} = 0$, which gives a constant value $\sum_{i \in N} \sum_{j \in N} c_i b_{ji}$ for Expression (10). Let $(0,0)$ correspond to this additional possibility. It is hence optimal to invest $k_g^{(1)}$ (obtained using (11)) on node $\alpha$ in the first phase and $k_g^{(2)}$ (obtained using (12)) on node $\beta$ in the second phase, subject to it giving a value greater than $\sum_{i \in N} \sum_{j \in N} c_i b_{ji}$ to Expression (10).

Since we iterate through $(n^2 + 1)$ possibilities (namely, $(\alpha, \beta) \in N \times N \cup \{(0,0)\}$, the above procedure gives a polynomial time algorithm for determining the optimal budget split and the optimal investments on nodes in two phases.

**Remark 1.** For non-negative values of parameters, (11) indicates that for a given $(\alpha, \beta)$ pair, the good camp would want to invest more in the first phase for a higher $s_\alpha$. This is intuitive from our understanding of $s_\alpha$ being viewed as the influencing power of node $\alpha$ looking one phase ahead. Similarly, (12) indicates that it would want to invest more in second phase for a higher $r_\beta$, since $r_\beta$ can be viewed as the influencing power of node $\beta$ in the immediate phase. Also, (11) and (12) indicate that a higher $\theta_\alpha$ drives the camp to invest in first phase and a higher $\theta_\beta$ drives it to invest in second phase. Since $w_{i\alpha}$ is an increasing function of $\theta_\alpha$, this implicitly means that a node with a higher $\theta_\beta$ drives the good camp to invest in the phase in which that node is selected. Further, we illustrate the role of $w_{i\alpha}$ using simulations in Section 5.

### 4 The Case of Competing Camps

Similar to Proposition 1, we can show that $\sum_{i \in N} v_{i}^{(2)}$ is a multilinear function in this case also, since it can be written as a linear function in $x^{(1)}, y^{(1)}, x^{(2)}, y^{(2)}$ separately. The following proposition can hence be proved on similar lines.

**Proposition 2.** In the search space $k_g^{(1)} + k_g^{(2)} \in (0, k_g]$, it is optimal for good camp to have $k_g^{(1)} + k_g^{(2)} = k_g$, and invest on at most one node in each phase. In the search space $k_g^{(1)} + k_g^{(2)} \in (0, k_g)$, it is optimal for bad camp to have $k_g^{(1)} + k_g^{(2)} = k_g$, and invest on at most one node in each phase.

Denoting $w_{i\alpha}^{(1)} = c_i$, $r_{j\beta} w_{j\beta}^{(1)} \Delta_{j\beta} = b_{ji}$, $s_i = \sum_{j \in N} b_{ji}$ as before, from Expression (5), $\sum_{i \in N} v_{i}^{(2)}$ can be expanded as

$$\sum_{i \in N} \sum_{j \in N} c_i b_{ji} + \sum_{i \in N} x_i^{(1)} \frac{\theta_\alpha}{2} (\sum_{j \in N} c_i b_{ji} + r_i) + \sum_{j \in N} y_j^{(1)} \frac{\theta_\beta}{2} (\sum_{i \in N} c_i b_{ji} - r_j) + \sum_{i \in N} x_i^{(1)} \frac{\theta_\alpha}{2} (1 + c_i) \left( s_i + \sum_{j \in N} x_j^{(1)} \frac{\theta_\alpha}{2} b_{ij} + \sum_{j \in N} y_j^{(1)} \frac{\theta_\beta}{2} b_{ij} \right) - \sum_{i \in N} w_{i\alpha}^{(1)} \frac{\theta_\alpha}{2} (1 - c_i) \left( s_i + \sum_{j \in N} x_j^{(1)} \frac{\theta_\alpha}{2} b_{ij} + \sum_{j \in N} y_j^{(1)} \frac{\theta_\beta}{2} b_{ij} \right).$$

Expression (13) for $\sum_{i \in N} v_{i}^{(2)}$ simplifies to

$$\sum_{i \in N} \sum_{j \in N} c_i b_{ji} + k_g^{(1)} \frac{\theta_\alpha}{2} \left( \sum_{i \in N} c_i b_{ji} + r_i \right) + k_g^{(2)} \frac{\theta_\beta}{2} \left( \sum_{i \in N} c_i b_{ji} - r_j \right) + \frac{k_g^{(1)} \theta_\alpha}{2} (1 + c_i) \left( s_i + k_g^{(2)} \frac{\theta_\beta}{2} b_{\alpha \beta} + k_g^{(2)} \frac{\theta_\beta}{2} b_{\alpha \beta} \right) + \frac{r_i}{2} \left( k_g^{(1)} \theta_\alpha + k_g^{(2)} \frac{\theta_\beta}{2} \right) - \frac{r_j}{2} \left( k_g^{(1)} \theta_\alpha + k_g^{(2)} \frac{\theta_\beta}{2} \right).$$

From Proposition 2, there exist optimal vectors $x^{(1)}, x^{(2)}$ for good camp and optimal vectors $y^{(1)}, y^{(2)}$ for bad camp, such that $x_i^{(1)} = k_g^{(1)}$, $x_j^{(2)} = k_g^{(2)}$, $y_j^{(1)} = k_g^{(1)}$, $y_j^{(2)} = k_g^{(2)}$, and $x_{i,j}^{(1)} = x_j^{(2)} = y_{i,j}^{(1)} = y_{i,j}^{(2)} = 0$. Assuming such profile of nodes $((\alpha, \beta), (\gamma, \delta))$, we find $(x_i^{(1)}, x_j^{(2)}), (y_j^{(1)}, y_j^{(2)})$, or equivalently, the optimal $((k_g^{(1)}, k_g^{(2)}), (k_g^{(1)}, k_g^{(2)}))$ corresponding to such a profile. By incorporating $((\alpha, \beta), (\gamma, \delta))$.

First, we consider the case when $k_g^{(1)} + k_g^{(2)} = k_g$ and $k_g^{(1)} + k_g^{(2)} = k_g$. Now, for a given profile of nodes $((\alpha, \beta), (\gamma, \delta))$, we will find the optimal values of $k_g^{(1)}, k_g^{(2)}, k_g^{(1)}, k_g^{(2)}$. In this case, we have $k_g^{(2)} = k_g^{(1)}$ and $k_g^{(2)} = k_g^{(1)}$. Substituting this in (14) and simultaneously solving $\sum_{i \in N} v_{i}^{(2)} = 0$ and $\sum_{i \in N} v_{i}^{(2)} = 0$, we get $k_g^{(1)}$ equal to (letting $A = \theta_\alpha \theta_\beta (1 - c_i) b_{ji}, B = \frac{1}{2} (\theta_\alpha \theta_\beta (1 + c_i) b_{ji} - \theta_\alpha \theta_\beta (1 + c_i) b_{ji})$: )

$$\frac{1}{B^2 + \theta_\alpha \theta_\beta (1 + c_i) b_{ji}} \left[ s_{i\alpha} \theta_\alpha (1 + c_i) A - r_{i\alpha} \theta_\beta A - s_{i\beta} (1 - c_i) B + r_{i\beta} B \right] + k_g^{(1)} \frac{\theta_\alpha \theta_\beta (1 + c_i) b_{ji} A - \theta_\alpha \theta_\beta (1 - c_i) b_{ji} B}{2} - k_g^{(1)} \frac{\theta_\alpha \theta_\beta (1 + c_i) b_{ji} A - \theta_\alpha \theta_\beta (1 - c_i) b_{ji} B}{2} + \frac{\theta_\alpha \theta_\beta (1 + c_i) b_{ji} A - \theta_\alpha \theta_\beta (1 - c_i) b_{ji} B}{2} - \theta_\alpha \theta_\beta \sum_{i \in N} c_i b_{ji} - \theta_\alpha \theta_\beta B \sum_{i \in N} c_i b_{ji}.$$
the bad camp can change
sign this value
of values we consider for
the good camp can change
run on networks larger than a few hundred nodes. It is hence
results for the non-competitive case. We point that though we
et al., 2003; Chen
experimental justifications in the literature on opinion diffusion
5 Simulations and Results

Thus using above technique, we obtain \( u_g((\alpha, \beta), (\gamma, \delta)) \)
for all profiles of nodes \((\alpha, \beta), (\gamma, \delta)\) when \(k_g^{(1)} + k_b^{(2)} = k_g \)
and \(k_b^{(1)} + k_b^{(2)} = k_b\). From Proposition 2, the only other cases
to consider are \(k_b^{(1)} = k_b^{(2)} = 0\) and \(k_g^{(1)} = k_g^{(2)} = 0\). Let the
profile \((\alpha, \beta), (0, 0)\) correspond to \(k_b^{(1)} = k_b^{(2)} = 0\). Note
that when \(k_b^{(1)} = k_b^{(2)} = 0\), it reduces to non-competitive case
with only good camp (Section 3); the value of \(\sum v_{ij}^{(2)}\) for
an \((\alpha, \beta)\) pair hence be assigned to \(u_g((\alpha, \beta), (0, 0))\).
Thus we can obtain \(u_g((\alpha, \beta), (\gamma, \delta))\) for all profiles of nodes
\((\alpha, \beta), (0, 0)\). Similarly, we can obtain \(u_g((\alpha, \beta), (\gamma, \delta))\)
for all profiles of nodes \((0, 0), (\gamma, \delta)\). And from Equation
(14), \(u_g((0, 0), (0, 0)) = \sum_{i \in I} \sum_{j \in J} c_{ij}^2\).

So we have that the good camp has \((n^2 + 1)\) possible pure
strategies to choose from, namely, \((\alpha, \beta) \in N \times N \cup \{(0, 0)\}\).
Similarly, the bad camp has \((n^2 + 1)\) possible pure strategies
to choose from, namely, \((\gamma, \delta) \in N \times N \cup \{(0, 0)\}\). We
thus have a two-player zero-sum game, for which the utilities
of the players can be computed for each strategy profile
\((\alpha, \beta), (\gamma, \delta)\) as explained above. Though we cannot ensure
the existence of a pure Nash equilibrium, the finiteness
of the number of strategies ensures the existence of a
mixed strategy Nash equilibrium. Further, owing to it being a
two-player zero-sum game, the Nash equilibria can be found
efficiently by solving a linear program [Osborne, 2004].

Summarizing, under practically reasonable assumptions
\(w_{ij} \geq 0, \forall (i, j)\) and \(w_{ii}^0 \geq 0, \theta_i \geq 0, \forall v_i^0 \in [-1, 1], \forall v_i \in N\),
we transformed the problem into a two-player zero-sum game
with each player having \((n^2 + 1)\) pure strategies, and showed
how the players’ utilities can be computed for each strategy
profile. We thus deduced the existence of Nash equilibria and
that they can be found efficiently using linear programming.

5 Simulations and Results

For determining implications of our analytical results on
real-world networks, we conducted simulations on NetHEPT
(15,233 nodes, 31,376 edges): a dataset widely used for
experimental justifications in the literature on opinion diffusion
[Kempe et al., 2003; Chen et al., 2009, 2010]. We present our results
for the non-competitive case. We point that though we presented a polynomial time algorithm for two camps, which
is of theoretical interest, it is computationally expensive to
run on networks larger than a few hundred nodes. It is hence
worth exploring the possibility of a more efficient algorithm.

Setup. We assume the value of \(w_{ii}^0\) to be same for all
nodes, in order to study the effect of this value. The range
of values we consider for \(w_{ii}^0\) is \([0, 0.05, \ldots, 0.95]\). Let
\(N(i)\) be the set of \(i\)'s neighbors. We model the influence

\[
\theta_i \theta_j \Delta \theta_i \Delta \theta_j (1 - w_{ij}^0 v_i^\gamma v_j^\gamma) \geq 0.
\]

That is, \(w_{ij}^0\), \(v_i^\gamma\), \(v_j^\gamma\) and \(\theta_i\), \(\theta_j\) should be
positive or \(1 - w_{ij}^0 v_i^\gamma v_j^\gamma\) should be negative.

\[
\theta_i \theta_j \Delta \theta_i \Delta \theta_j (1 - w_{ij}^0 v_i^\gamma v_j^\gamma) \geq 0.
\]

That is, \(w_{ij}^0\), \(v_i^\gamma\), \(v_j^\gamma\) and \(\theta_i\), \(\theta_j\) should be
positive or \(1 - w_{ij}^0 v_i^\gamma v_j^\gamma\) should be negative.

\[
\theta_i \theta_j \Delta \theta_i \Delta \theta_j (1 - w_{ij}^0 v_i^\gamma v_j^\gamma) \geq 0.
\]

That is, \(w_{ij}^0\), \(v_i^\gamma\), \(v_j^\gamma\) and \(\theta_i\), \(\theta_j\) should be
positive or \(1 - w_{ij}^0 v_i^\gamma v_j^\gamma\) should be negative.

\[
\theta_i \theta_j \Delta \theta_i \Delta \theta_j (1 - w_{ij}^0 v_i^\gamma v_j^\gamma) \geq 0.
\]

That is, \(w_{ij}^0\), \(v_i^\gamma\), \(v_j^\gamma\) and \(\theta_i\), \(\theta_j\) should be
positive or \(1 - w_{ij}^0 v_i^\gamma v_j^\gamma\) should be negative.

\[
\theta_i \theta_j \Delta \theta_i \Delta \theta_j (1 - w_{ij}^0 v_i^\gamma v_j^\gamma) \geq 0.
\]

That is, \(w_{ij}^0\), \(v_i^\gamma\), \(v_j^\gamma\) and \(\theta_i\), \(\theta_j\) should be
positive or \(1 - w_{ij}^0 v_i^\gamma v_j^\gamma\) should be negative.

\[
\theta_i \theta_j \Delta \theta_i \Delta \theta_j (1 - w_{ij}^0 v_i^\gamma v_j^\gamma) \geq 0.
\]

That is, \(w_{ij}^0\), \(v_i^\gamma\), \(v_j^\gamma\) and \(\theta_i\), \(\theta_j\) should be
positive or \(1 - w_{ij}^0 v_i^\gamma v_j^\gamma\) should be negative.
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References


