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Roxana Dumitrescu, Miryana Grigorova, Marie-Claire Quenez, Agnès Sulem. BSDEs with default jump. Elena Celledoni; Giulia Di Nunno; Kurusch Ebrahimi-Fard; Hans Munthe-Kaas. Computation and Combinatorics in Dynamics, Stochastics and Control - The Abel Symposium, Rosendal, Norway August 2016, 13, Springer, 2018, The Abel Symposia book series, 10.1007/978-3-030-01593-0 . hal-01799335

**HAL Id: hal-01799335**

**<https://inria.hal.science/hal-01799335v1>**

Submitted on 24 May 2018

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# BSDEs with default jump

Roxana Dumitrescu, Miryana Grigorova, Marie-Claire Quenez, Agnès Sulem

**Abstract** We study (nonlinear) Backward Stochastic Differential Equations (BSDEs) driven by a Brownian motion and a martingale attached to a default jump with intensity process  $\lambda = (\lambda_t)$ . The driver of the BSDEs can be of a generalized form involving a singular *optional* finite variation process. In particular, we provide a comparison theorem and a strict comparison theorem. In the special case of a generalized  $\lambda$ -linear driver, we show an explicit representation of the solution, involving conditional expectation and an adjoint exponential semimartingale; for this representation, we distinguish the case where the singular component of the driver is predictable and the case where it is only optional. We apply our results to the problem of (nonlinear) pricing of European contingent claims in an imperfect market with default. We also study the case of claims generating intermediate cashflows, in particular at the default time, which are modeled by a singular *optional* process. We give an illustrating example when the seller of the European option is a large investor whose portfolio strategy can influence the probability of default.

## 1 Introduction

The aim of the present paper is to study BSDEs driven by a Brownian motion and a compensated default jump process with intensity process  $\lambda = (\lambda_t)$ . The applications we have in mind are the pricing and hedging of contingent claims in an imperfect financial market with default. The theory of BSDEs driven by a Brownian motion and a Poisson random measure has been developed extensively by several authors (cf., e.g., Barles, Buckdahn and Pardoux [2], Royer [22], Quenez and Sulem [21], Delong [10]). Several of the arguments used in the present paper are similar to those used in the previous literature. Nevertheless, it should be noted that BSDEs with a default jump do not correspond to a particular case of BSDEs with Poisson random measure. The treatment of BSDEs with a default jump requires some specific arguments and we present here a complete analysis of these

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M. Grigorova acknowledges financial support from the German Science Foundation through SFB1283.

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BSDEs, which is particularly useful in default risk modeling in finance. To our knowledge, there are few works on nonlinear BSDEs with default jump. The papers [6] and [1] are concerned only with the existence and the uniqueness of the solution, which are established under different assumptions. In this paper, we first provide some a priori estimates, from which the existence and uniqueness result directly follows. Moreover, we allow the driver of the BSDEs to have a singular component, in the sense that the driver is allowed to be of the generalized form  $g(t, y, z, k)dt + dD_t$ , where  $D$  is an optional (not necessarily predictable) right-continuous left-limited (RCLL) process with finite variation. We stress that the case of a singular *optional* process  $D$  has not been considered in the literature on BSDEs, even when the filtration is associated with a Brownian motion and a Poisson random measure. Moreover, these BSDEs are useful to study the nonlinear pricing problem in imperfect markets with default. Indeed, in this type of markets, the contingent claims often generate intermediate cashflows - in particular at the default time - which can be modeled via an optional singular process  $D$  (see e.g. [4, 3, 8, 7]). We introduce the definition of a  $\lambda$ -linear driver, where  $\lambda$  refers to the intensity of the jump process, which generalizes the notion of a linear driver given in the literature on BSDEs to the case of BSDEs with default jump and *generalized driver*. When  $g$  is  $\lambda$ -linear, we provide an explicit solution of the BSDE associated with the *generalized  $\lambda$ -linear driver*  $g(t, y, z, k)dt + dD_t$  in terms of a conditional expectation and an adjoint exponential semimartingale. We note that this representation formula depends on whether the singular process  $D$  is predictable or just optional. Under some suitable assumptions on  $g$ , we establish a comparison theorem, as well as a strict comparison theorem. We emphasize that these comparison results are shown for optional (not necessarily) predictable singular processes, which requires some specific arguments. We then give an application in mathematical finance. We consider a financial market with a defaultable risky asset and we study the problems of pricing and hedging of a European option paying a payoff  $\xi$  at maturity  $T$  and intermediate dividends (or cashflows) modeled by a singular process  $D$ . The option possibly generates a cashflow at the default time, which implies that the dividend process  $D$  is not necessarily predictable. We study the case of a market with imperfections which are expressed via the nonlinearity of the wealth dynamics. Our framework includes the case of different borrowing and lending treasury rates (see e.g. [17] and [8]) and "repo rates"<sup>1</sup>, which is usual for contracts with intermediate dividends subjected to default (see [7]). We show that the price of the option is given by  $X_{\cdot, T}^g(\xi, D)$ , where  $X_{\cdot, T}^g(\xi, D)$  is the solution of the nonlinear BSDE with default jump (solved under the primitive probability measure  $P$ ) with *generalized driver*  $g(t, y, z, k)dt + dD_t$ , terminal time  $T$  and terminal condition  $\xi$ . This leads to a non linear pricing system  $(\xi, D) \mapsto X_{\cdot, T}^g(\xi, D)$ , for which we establish some properties. We emphasize that the monotonicity property (resp. no arbitrage property) requires some specific assumptions on the driver  $g$ , which are due to the presence of the default. Furthermore, for each driver  $g$  and each (fixed) singular process  $D$ , we define the  $(g, D)$ -conditional expectation by  $\mathcal{E}_{t, T}^{g, D}(\xi) := X_{t, T}^g(\xi, D)$ , for  $\xi \in L^2(\mathcal{G}_T)$ . In the case where  $D = 0$ , it reduces to the  $g$ -conditional expectation  $\mathcal{E}^g$  (in the case of default). We also introduce the notion of  $\mathcal{E}^{g, D}$ -martingale, which is a useful tool in the study of nonlinear pricing problems: more specifically, those of American options and game options with intermediate dividends (cf. [13], [14]).

The paper is organized as follows: in Section 2, we present the properties of BSDEs with default jump and *generalized driver*. More precisely, in Section 2.1, we present the mathematical setup. In Section 2.2, we state some a priori estimates, from which we derive the existence and the uniqueness of the solution. In Section 2.3, we show the representation property of the solution of the BSDE associated with the *generalized driver*  $g(t, y, z, k)dt + dD_t$  in the particular case when  $g$  is  $\lambda$ -linear. We distinguish the two cases: the case when the singular process  $D$  is predictable and the case when it is just optional. In Section 2.4, we establish the comparison theorem and the strict comparison theorem. Section 3 is devoted to the application to the nonlinear pricing of European options with dividends in an imperfect market with default. The properties of the nonlinear pricing system as well as those of the  $(g, D)$ -conditional expectation are also studied in this section. As an illustrative example of market imperfections, we consider the case when the seller of the option is a large investor whose hedging strategy (in particular the cost of this strategy) has impact on the default probability.

<sup>1</sup> which can be seen as securities lending or borrowing rates in a "repo market" (cf. [7]).

## 2 BSDEs with default jump

### 2.1 Probability setup

Let  $(\Omega, \mathcal{G}, \mathbb{P})$  be a complete probability space equipped with two stochastic processes: a unidimensional standard Brownian motion  $W$  and a jump process  $N$  defined by  $N_t = \mathbf{1}_{\vartheta \leq t}$  for any  $t \in [0, T]$ , where  $\vartheta$  is a random variable which models a default time. We assume that this default can appear after any fixed time, that is  $P(\vartheta \geq t) > 0$  for any  $t \geq 0$ . We denote by  $\mathbb{G} = \{\mathcal{G}_t, t \geq 0\}$  the *augmented filtration* generated by  $W$  and  $N$  (in the sense of [9, IV-48]). In the following,  $\mathcal{P}$  denotes the  $\mathbb{G}$ -predictable  $\sigma$ -algebra on  $\Omega \times [0, T]$ . We suppose that  $W$  is a  $\mathbb{G}$ -Brownian motion.

Let  $(\Lambda_t)$  be the  $\mathbb{G}$ -predictable compensator of the non decreasing process  $(N_t)$ . Note that  $(\Lambda_{t \wedge \vartheta})$  is then the  $\mathbb{G}$ -predictable compensator of  $(N_{t \wedge \vartheta}) = (N_t)$ . By uniqueness of the  $\mathbb{G}$ -predictable compensator,  $\Lambda_{t \wedge \vartheta} = \Lambda_t$ ,  $t \geq 0$  a.s. We assume that  $\Lambda$  is absolutely continuous w.r.t. Lebesgue's measure, so that there exists a nonnegative  $\mathbb{G}$ -predictable process  $(\lambda_t)$ , called the intensity process, such that  $\Lambda_t = \int_0^t \lambda_s ds$ ,  $t \geq 0$ . Since  $\Lambda_{t \wedge \vartheta} = \Lambda_t$ , the process  $\lambda$  vanishes after  $\vartheta$ . We denote by  $M$  the  $\mathbb{G}$ -compensated martingale given by

$$M_t = N_t - \int_0^t \lambda_s ds. \quad (1)$$

Let  $T > 0$  be the finite horizon. We introduce the following sets:

- $\mathcal{S}_T^2$  (also denoted by  $\mathcal{S}^2$ ) is the set of  $\mathbb{G}$ -adapted right-continuous left-limited (RCLL) processes  $\varphi$  such that  $\mathbb{E}[\sup_{0 \leq t \leq T} |\varphi_t|^2] < +\infty$ .
- $\mathcal{A}_T^2$  (also denoted by  $\mathcal{A}^2$ ) is the set of real-valued finite variational RCLL  $\mathbb{G}$ -adapted (thus *optional*) processes  $A$  with square integrable total variation process and such that  $A_0 = 0$ .
- $\mathcal{A}_{p,T}^2$  (also denoted by  $\mathcal{A}_p^2$ ) is the set of *predictable* processes belonging to  $\mathcal{A}^2$ .
- $\mathbb{H}_T^2$  (also denoted by  $\mathbb{H}^2$ ) is the set of  $\mathbb{G}$ -predictable processes with  $\|Z\|^2 := \mathbb{E}\left[\int_0^T |Z_t|^2 dt\right] < \infty$ .
- $\mathbb{H}_{\lambda,T}^2 := L^2(\Omega \times [0, T], \mathcal{P}, \lambda_t dP \otimes dt)$  (also denoted by  $\mathbb{H}_{\lambda}^2$ ), equipped with scalar product  $\langle U, V \rangle_{\lambda} := \mathbb{E}\left[\int_0^T U_t V_t \lambda_t dt\right]$ , for all  $U, V$  in  $\mathbb{H}_{\lambda}^2$ . For all  $U \in \mathbb{H}_{\lambda}^2$ , we set  $\|U\|_{\lambda}^2 := \mathbb{E}\left[\int_0^T |U_t|^2 \lambda_t dt\right]$

For each  $U \in \mathbb{H}_{\lambda}^2$ , we have  $\|U\|_{\lambda}^2 = \mathbb{E}\left[\int_0^{T \wedge \vartheta} |U_t|^2 \lambda_t dt\right]$  because the  $\mathbb{G}$ -intensity  $\lambda$  vanishes after  $\vartheta$ .

Note that, without loss of generality, we may assume that  $U$  vanishes after  $\vartheta^2$ .

Moreover,  $\mathcal{T}$  is the set of stopping times  $\tau$  such that  $\tau \in [0, T]$  a.s. and for each  $S$  in  $\mathcal{T}$ ,  $\mathcal{T}_S$  is the set of stopping times  $\tau$  such that  $S \leq \tau \leq T$  a.s.

We recall the martingale representation theorem in this framework (see [18]):

**Lemma 1 (Martingale representation).** *Let  $m = (m_t)_{0 \leq t \leq T}$  be a  $\mathbb{G}$ -local martingale. There exists a unique pair of  $\mathbb{G}$ -predictable processes  $(z_t, l_t)$ <sup>3</sup> such that*

$$m_t = m_0 + \int_0^t z_s dW_s + \int_0^t l_s dM_s, \quad \forall t \in [0, T] \quad a.s. \quad (2)$$

*If  $m$  is a square integrable martingale, then  $z \in \mathbb{H}^2$  and  $l \in \mathbb{H}_{\lambda}^2$ .*

We now introduce the following definitions.

**Definition 1 (Driver,  $\lambda$ -admissible driver).**

- A function  $g$  is said to be a *driver* if  $g : \Omega \times [0, T] \times \mathbf{R}^3 \rightarrow \mathbf{R}$ ;  $(\omega, t, y, z, k) \mapsto g(\omega, t, y, z, k)$  is  $\mathcal{P} \otimes \mathcal{B}(\mathbf{R}^3)$ -measurable, and such that  $g(\cdot, 0, 0, 0) \in \mathbb{H}^2$ .

<sup>2</sup> Indeed, each  $U$  in  $\mathbb{H}_{\lambda}^2 (= L^2(\Omega \times [0, T], \mathcal{P}, \lambda_t dP \otimes dt))$  can be identified with  $U \mathbf{1}_{t \leq \vartheta}$ , since  $U \mathbf{1}_{t \leq \vartheta}$  is a  $\mathbb{G}$ -predictable process satisfying  $U_t \mathbf{1}_{t \leq \vartheta} = U_t \lambda_t dP \otimes dt$ -a.s.

<sup>3</sup> such that the stochastic integrals in (2) are well defined.

- A driver  $g$  is called a  $\lambda$ -admissible driver if moreover there exists a constant  $C \geq 0$  such that for  $dP \otimes dt$ -almost every  $(\omega, t)$ , for all  $(y_1, z_1, k_1), (y_2, z_2, k_2)$ ,

$$|g(\omega, t, y_1, z_1, k_1) - g(\omega, t, y_2, z_2, k_2)| \leq C(|y_1 - y_2| + |z_1 - z_2| + \sqrt{\lambda_t(\omega)}|k_1 - k_2|). \quad (3)$$

A non negative constant  $C$  such that (3) holds is called a  $\lambda$ -constant associated with driver  $g$ .

Note that condition (3) implies that for each  $(y, z, k)$ , we have  $g(t, y, z, k) = g(t, y, z, 0), t > \vartheta$   $dP \otimes dt$ -a.e. Indeed, on the set  $\{t > \vartheta\}$ ,  $g$  does not depend on  $k$ , since  $\lambda_t = 0$ .

*Remark 1.* Note that a driver  $g$  supposed to be Lipschitz with respect to  $(y, z, k)$  is not generally  $\lambda$ -admissible. Moreover, a driver  $g$  supposed to be  $\lambda$ -admissible is not generally Lipschitz with respect to  $(y, z, k)$  since the process  $(\lambda_t)$  is not necessarily bounded.

**Definition 2.** Let  $g$  be a  $\lambda$ -admissible driver, let  $\xi \in L^2(\mathcal{G}_T)$ .

- A process  $(Y, Z, K)$  in  $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}_\lambda^2$  is said to be a solution of the BSDE with default jump associated with terminal time  $T$ , driver  $g$  and terminal condition  $\xi$  if it satisfies:

$$-dY_t = g(t, Y_t, Z_t, K_t)dt - Z_t dW_t - K_t dM_t; \quad Y_T = \xi. \quad (4)$$

- Let  $D \in \mathcal{A}^2$ . A process  $(Y, Z, K)$  in  $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}_\lambda^2$  is said to be a solution of the BSDE with default jump associated with terminal time  $T$ , generalized  $\lambda$ -admissible driver  $g(t, y, z, k)dt + dD_t$  and terminal condition  $\xi$  if it satisfies:

$$-dY_t = g(t, Y_t, Z_t, K_t)dt + dD_t - Z_t dW_t - K_t dM_t; \quad Y_T = \xi. \quad (5)$$

*Remark 2.* Let  $D = (D_t)_{0 \leq t \leq T}$  be a finite variational RCLL adapted process such that  $D_0 = 0$ , and with integrable total variation. We recall that  $D$  admits at most a countable number of jumps. We also recall that the process  $D$  has the following (unique) canonical decomposition:  $D = A - A'$ , where  $A$  and  $A'$  are integrable non decreasing RCLL adapted processes with  $A_0 = A'_0 = 0$ , and such that  $dA_t$  and  $dA'_t$  are mutually singular (cf. Proposition A.7 in [11]). If  $D$  is predictable, then  $A$  and  $A'$  are predictable.

Moreover, by a property given in [14], for each  $D \in \mathcal{A}^2$ , there exist a unique (predictable) process  $D'$  belonging to  $\mathcal{A}_p^2$  and a unique (predictable) process  $\eta$  belonging to  $\mathbb{H}_\lambda^2$  such that for all  $t \in [0, T]$ ,

$$D_t = D'_t + \int_0^t \eta_s dN_s \quad \text{a.s.}$$

If  $D$  is non decreasing, then  $D'$  is non decreasing and  $\eta_\vartheta \geq 0$  a.s on  $\{\vartheta \leq T\}$ .

*Remark 3.* By Remark 2 and equation (5), the process  $Y$  admits at most a countable number of jumps. It follows that  $Y_t = Y_{t-}, 0 \leq t \leq T$   $dP \otimes dt$ -a.e. Moreover, we have  $g(t, Y_t, Z_t, K_t) = g(t, Y_{t-}, Z_t, K_t), 0 \leq t \leq T$   $dP \otimes dt$ -a.e.

## 2.2 Properties of BSDEs with default jump

We first show some a priori estimates for BSDEs with a default jump, from which we derive the existence and uniqueness of the solution. For  $\beta > 0$ ,  $\phi \in \mathbb{H}^2$ , and  $l \in \mathbb{H}_\lambda^2$ , we introduce the norms  $\|\phi\|_\beta^2 := \mathbb{E}[\int_0^T e^{\beta s} \phi_s^2 ds]$ , and  $\|l\|_{\lambda, \beta}^2 := \mathbb{E}[\int_0^T e^{\beta s} k_s^2 \lambda_s ds]$ .

### 2.2.1 A priori estimates for BSDEs with default jump

**Proposition 1.** Let  $\xi^1, \xi^2 \in L^2(\mathcal{G}_T)$ . Let  $g^1$  and  $g^2$  be two  $\lambda$ -admissible drivers. Let  $C$  be a  $\lambda$ -constant associated with  $g^1$ . Let  $D$  be an (optional) process belonging to  $\mathcal{A}^2$ .

For  $i = 1, 2$ , let  $(Y^i, Z^i, K^i)$  be a solution of the BSDE associated with terminal time  $T$ , generalized driver  $g^i(t, y, z, k)dt + dD_t$  and terminal condition  $\xi^i$ . Let  $\bar{\xi} := \xi^1 - \xi^2$ . For  $s$  in  $[0, T]$ , denote  $\bar{Y}_s := Y_s^1 - Y_s^2$ ,  $\bar{Z}_s := Z_s^1 - Z_s^2$  and  $\bar{K}_s := K_s^1 - K_s^2$ . Let  $\eta, \beta > 0$  be such that  $\beta \geq \frac{3}{\eta} + 2C$  and  $\eta \leq \frac{1}{C^2}$ . For each  $t \in [0, T]$ , we have

$$e^{\beta t} (\bar{Y}_t)^2 \leq \mathbb{E}[e^{\beta T} \bar{\xi}^2 | \mathcal{G}_t] + \eta \mathbb{E}\left[\int_t^T e^{\beta s} \bar{g}(s)^2 ds | \mathcal{G}_t\right] \quad \text{a.s.}, \quad (6)$$

where  $\bar{g}(s) := g^1(s, Y_s^2, Z_s^2, K_s^2) - g^2(s, Y_s^2, Z_s^2, K_s^2)$ . Moreover,

$$\|\bar{Y}\|_{\beta}^2 \leq T[e^{\beta T} \mathbb{E}[\bar{\xi}^2] + \eta \|\bar{g}\|_{\beta}^2]. \quad (7)$$

If  $\eta < \frac{1}{C^2}$ , we have

$$\|\bar{Z}\|_{\beta}^2 + \|\bar{K}\|_{\lambda, \beta}^2 \leq \frac{1}{1 - \eta C^2} [e^{\beta T} \mathbb{E}[\bar{\xi}^2] + \eta \|\bar{g}\|_{\beta}^2]. \quad (8)$$

*Remark 4.* If  $C = 0$ , then (6) and (7) hold for all  $\eta, \beta > 0$  such that  $\beta \geq \frac{3}{\eta}$ , and (8) holds (with  $C = 0$ ) for all  $\eta > 0$ .

*Proof.* By Itô's formula applied to the semimartingale  $(e^{\beta s} \bar{Y}_s^2)$  between  $t$  and  $T$ , we get

$$\begin{aligned} & e^{\beta t} \bar{Y}_t^2 + \beta \int_t^T e^{\beta s} \bar{Y}_s^2 ds + \int_t^T e^{\beta s} \bar{Z}_s^2 ds + \int_t^T e^{\beta s} \bar{K}_s^2 \lambda_s ds \\ &= e^{\beta T} \bar{Y}_T^2 + 2 \int_t^T e^{\beta s} \bar{Y}_s (g^1(s, Y_s^1, Z_s^1, K_s^1) - g^2(s, Y_s^2, Z_s^2, K_s^2)) ds \\ & \quad - 2 \int_t^T e^{\beta s} \bar{Y}_s \bar{Z}_s dW_s - \int_t^T e^{\beta s} (2\bar{Y}_s \bar{K}_s + \bar{K}_s^2) dM_s. \end{aligned} \quad (9)$$

Taking the conditional expectation given  $\mathcal{G}_t$ , we obtain

$$\begin{aligned} & e^{\beta t} \bar{Y}_t^2 + \mathbb{E}\left[\beta \int_t^T e^{\beta s} \bar{Y}_s^2 ds + \int_t^T e^{\beta s} (\bar{Z}_s^2 + \bar{K}_s^2 \lambda_s) ds | \mathcal{G}_t\right] \\ & \leq \mathbb{E}\left[e^{\beta T} \bar{Y}_T^2 | \mathcal{G}_t\right] + 2\mathbb{E}\left[\int_t^T e^{\beta s} \bar{Y}_s (g^1(s, Y_s^1, Z_s^1, K_s^1) - g^2(s, Y_s^2, Z_s^2, K_s^2)) ds | \mathcal{G}_t\right]. \end{aligned} \quad (10)$$

Now,  $g^1(s, Y_s^1, Z_s^1, K_s^1) - g^2(s, Y_s^2, Z_s^2, K_s^2) = g^1(s, Y_s^1, Z_s^1, K_s^1) - g^1(s, Y_s^2, Z_s^2, K_s^2) + \bar{g}_s$ . Since  $g^1$  satisfies condition (3), we derive that

$$|g^1(s, Y_s^1, Z_s^1, K_s^1) - g^1(s, Y_s^2, Z_s^2, K_s^2)| \leq C|\bar{Y}_s| + C|\bar{Z}_s| + C|\bar{K}_s| \sqrt{\lambda_s} + |\bar{g}_s|.$$

Note that, for all non negative numbers  $\lambda, y, z, k, g$  and  $\varepsilon > 0$ , we have

$$2y(Cz + Ck\sqrt{\lambda} + g) \leq \frac{y^2}{\varepsilon^2} + \varepsilon^2(Cz + Ck\sqrt{\lambda} + g)^2 \leq \frac{y^2}{\varepsilon^2} + 3\varepsilon^2(C^2y^2 + C^2k^2\lambda + g^2). \quad \text{Hence,}$$

$$\begin{aligned} & e^{\beta t} \bar{Y}_t^2 + \mathbb{E}\left[\beta \int_t^T e^{\beta s} \bar{Y}_s^2 ds + \int_t^T e^{\beta s} (\bar{Z}_s^2 + \bar{K}_s^2 \lambda_s) ds | \mathcal{G}_t\right] \leq \mathbb{E}\left[e^{\beta T} \bar{Y}_T^2 | \mathcal{G}_t\right] + \\ & + \mathbb{E}\left[\left(2C + \frac{1}{\varepsilon^2}\right) \int_t^T e^{\beta s} \bar{Y}_s^2 ds + 3C^2\varepsilon^2 \int_t^T e^{\beta s} (\bar{Z}_s^2 + \bar{K}_s^2 \lambda_s) ds + 3\varepsilon^2 \int_t^T e^{\beta s} \bar{g}_s^2 ds | \mathcal{G}_t\right]. \end{aligned} \quad (11)$$

Let us make the change of variable  $\eta = 3\varepsilon^2$ . Then, for each  $\beta, \eta > 0$  chosen as in the proposition, this inequality leads to (6). By integrating (6), we obtain (7). Using (7) and inequality (11), we derive (8).

*Remark 5.* By classical results on the norms of semimartingales, one similarly shows that  $\|\bar{Y}\|_{\mathcal{G}^2} \leq K(\mathbb{E}[\bar{\xi}^2] + \|\bar{g}\|_{H^2})$ , where  $K$  is a positive constant only depending on  $T$  and  $C$ .

### 2.2.2 Existence and uniqueness result for BSDEs with default jump

By the representation property of  $\mathbb{G}$ -martingales (Lemma 1) and the a priori estimates given in Proposition 1, we derive the existence and the uniqueness of the solution associated with a *generalized  $\lambda$ -admissible driver*.

**Proposition 2.** *Let  $g$  be a  $\lambda$ -admissible driver, let  $\xi \in L^2(\mathcal{G}_T)$ , and let  $D$  be an (optional) process belonging to  $\mathcal{A}^2$ . There exists a unique solution  $(Y, Z, K)$  in  $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}_\lambda^2$  of BSDE (5).*

*Remark 6.* Suppose that  $D = 0$ . Suppose also that  $\xi$  is  $\mathcal{G}_{\vartheta \wedge T}$ -measurable and that  $g$  is replaced by  $g\mathbf{1}_{t \leq \vartheta}$  (which is a  $\lambda$ -admissible driver). Then, the solution  $(Y, Z, K)$  of the associated BSDE (4) is equal to the solution of the BSDE with random terminal time  $\vartheta \wedge T$ , driver  $g$  and terminal condition  $\xi$ , as considered in [6]. Note also that in the present paper, contrary to papers [6, 12], we do not suppose that the default intensity process  $\lambda$  is bounded (which is interesting since this is the case in some models with default).

*Proof.* Let us first consider the case when  $g(t)$  does not depend on  $(y, z, k)$ . Then the solution  $Y$  is given by  $Y_t = E[\xi + \int_t^T g(s)ds + D_T - D_t | \mathcal{G}_t]$ . The processes  $Z$  and  $K$  are obtained by applying the representation property of  $\mathbb{G}$ -martingales to the square integrable martingale  $E[\xi + \int_0^T g(s)ds + D_T | \mathcal{G}_t]$ . Hence, there thus exists a unique solution of BSDE (5) associated with terminal condition  $\xi \in L^2(\mathcal{F}_T)$  and *generalized driver*  $g(t)dt + dD_t$ . Let us now turn to the case with a general  $\lambda$ -admissible driver  $g(t, y, z, k)$ . Denote by  $\mathbb{H}_\beta^2$  the space  $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}_\lambda^2$  equipped with the norm  $\|Y, Z, K\|_\beta^2 := \|Y\|_\beta^2 + \|Z\|_\beta^2 + \|K\|_{\lambda, \beta}^2$ . We define a mapping  $\Phi$  from  $\mathbb{H}_\beta^2$  into itself as follows. Given  $(U, V, L) \in \mathbb{H}_\beta^2$ , let  $(Y, Z, K) = \Phi(U, V, L)$  be the solution of the BSDE associated with *generalized driver*  $g(s, U_s, V_s, L_s)ds + dD_s$  and terminal condition  $\xi$ . Let us prove that the mapping  $\Phi$  is a contraction from  $\mathbb{H}_\beta^2$  into  $\mathbb{H}_\beta^2$ . Let  $(U', V', L')$  be another element of  $\mathbb{H}_\beta^2$  and let  $(Y', Z', K') := \Phi(U', V', L')$ , that is, the solution of the BSDE associated with the *generalized driver*  $g(s, U'_s, V'_s, L'_s)ds + dD_s$  and terminal condition  $\xi$ .

Set  $\bar{U} = U - U'$ ,  $\bar{V} = V - V'$ ,  $\bar{L} = L - L'$ ,  $\bar{Y} = Y - Y'$ ,  $\bar{Z} = Z - Z'$ ,  $\bar{K} = K - K'$ .

Set  $\Delta g_t := g(t, U_t, V_t, L_t) - g(t, U'_t, V'_t, L'_t)$ . By Remark 4 applied to the driver processes  $g_1(t) := g(t, U_t, V_t, L_t)$ <sup>4</sup> and  $g_2(t) := g(t, U'_t, V'_t, L'_t)$ , we derive that for all  $\eta, \beta > 0$  such that  $\beta \geq \frac{3}{\eta}$ , we have

$$\|\bar{Y}\|_\beta^2 + \|\bar{Z}\|_\beta^2 + \|\bar{K}\|_{\lambda, \beta}^2 \leq \eta(T+1)\|\Delta g\|_\beta^2.$$

Since the driver  $g$  is  $\lambda$ -admissible with  $\lambda$ -constant  $C$ , we get

$$\|\bar{Y}\|_\beta^2 + \|\bar{Z}\|_\beta^2 + \|\bar{K}\|_{\lambda, \beta}^2 \leq \eta(T+1)3C^2(\|\bar{U}\|_\beta^2 + \|\bar{V}\|_\beta^2 + \|\bar{L}\|_{\lambda, \beta}^2),$$

for all  $\eta, \beta > 0$  with  $\beta \geq \frac{3}{\eta}$ . Choosing  $\eta = \frac{1}{(T+1)6C^2}$  and  $\beta \geq \frac{3}{\eta} = 18(T+1)C^2$ , we derive that  $\|(\bar{Y}, \bar{Z}, \bar{K})\|_\beta^2 \leq \frac{1}{2}\|(\bar{U}, \bar{V}, \bar{K})\|_\beta^2$ . Hence, for  $\beta \geq 18(T+1)C^2$ ,  $\Phi$  is a contraction from  $\mathbb{H}_\beta^2$  into  $\mathbb{H}_\beta^2$  and thus admits a unique fixed point  $(Y, Z, K)$  in the Banach space  $\mathbb{H}_\beta^2$ , which is the (unique) solution of BSDE (4).

### 2.3 $\lambda$ -linear BSDEs with default jump

We introduce the notion of  $\lambda$ -linear BSDEs in our framework with default jump.

**Definition 3 ( $\lambda$ -linear driver).** A driver  $g$  is called  $\lambda$ -linear if it is of the form:

$$g(t, y, z, k) = \delta_t y + \beta_t z + \gamma_t k \lambda_t + \varphi_t, \quad (12)$$

<sup>4</sup> Note that the driver processes  $g_1(t)$  admits  $C = 0$  as  $\lambda$ -constant.

where  $(\varphi_t) \in \mathbb{H}^2$ , and  $(\delta_t)$ ,  $(\beta_t)$  and  $(\gamma_t)$  are  $\mathbf{R}$ -valued predictable processes such that  $(\delta_t)$ ,  $(\beta_t)$  and  $(\gamma_t \sqrt{\lambda_t})$  are bounded. By extension,

$$(\delta_t y + \beta_t z + \gamma_t k \lambda_t) dt + dD_t,$$

where  $D \in \mathcal{A}^2$ , is called a *generalized  $\lambda$ -linear driver*.

*Remark 7.* Note that  $g$  given by (12) can be rewritten as

$$g(t, y, z, k) = \varphi_t + \delta_t y + \beta_t z + v_t k \sqrt{\lambda_t}, \quad (13)$$

where  $v_t := \gamma_t \sqrt{\lambda_t}$  is a bounded predictable process.<sup>5</sup> From this remark, it clearly follows that a  $\lambda$ -linear driver is  $\lambda$ -admissible.

We will now prove that the solution of a  $\lambda$ -linear BSDE (or more generally a *generalized  $\lambda$ -linear driver*) can be written as a conditional expectation via an exponential semimartingale. We first show a preliminary result on exponential semimartingales.

Let  $(\beta_s)$  and  $(\gamma_s)$  be two real-valued  $\mathbb{G}$ -predictable processes such that the stochastic integrals  $\int_0^\cdot \beta_s dW_s$  and  $\int_0^\cdot \gamma_s dM_s$  are well-defined. Let  $(\zeta_s)$  be the process satisfying the forward SDE:

$$d\zeta_s = \zeta_{s-} (\beta_s dW_s + \gamma_s dM_s); \quad \zeta_0 = 1. \quad (14)$$

*Remark 8.* Recall that the process  $(\zeta_s)$  satisfies the so-called Doléans-Dade formula, that is

$$\zeta_s = \exp\left\{\int_0^s \beta_r dW_r - \frac{1}{2} \int_0^s \beta_r^2 dr\right\} \exp\left\{-\int_0^s \gamma_r \lambda_r dr\right\} (1 + \gamma_\vartheta \mathbf{1}_{\{s \geq \vartheta\}}), \quad s \geq 0 \quad \text{a.s.}$$

Hence, if  $\gamma_\vartheta \geq -1$  (resp.  $> -1$ ) a.s., then  $\zeta_s \geq 0$  (resp.  $> 0$ ) for all  $s \geq 0$  a.s.

*Remark 9.* The inequality  $\gamma_\vartheta \geq -1$  a.s. is equivalent to the inequality  $\gamma_t \geq -1$ ,  $\lambda_t dt \otimes dP$ -a.s. Indeed, we have  $\mathbb{E}[\mathbf{1}_{\gamma_\vartheta < -1}] = \mathbb{E}[\int_0^{+\infty} \mathbf{1}_{\gamma_r < -1} dN_r] = \mathbb{E}[\int_0^{+\infty} \mathbf{1}_{\gamma_r < -1} \lambda_r dr]$ , because the process  $(\int_0^\cdot \lambda_r dr)$  is the  $\mathbb{G}$ -predictable compensator of the default jump process  $N$ .

**Proposition 3.** *Let  $T > 0$ . Suppose that the random variable  $\int_0^T (\beta_r^2 + \gamma_r^2 \lambda_r) dr$  is bounded. Then, the process  $(\zeta_s)_{0 \leq s \leq T}$ , defined by (14), is a martingale and satisfies  $\mathbb{E}[\sup_{0 \leq s \leq T} \zeta_s^2] < +\infty$ .*

*Proof.* By definition, the process  $(\zeta_s)$  is a local martingale. Let  $T > 0$ . Let us show that  $\mathbb{E}[\sup_{0 \leq s \leq T} \zeta_s^2] < +\infty$ . By Itô's formula applied to  $\zeta_s^2$ , we get  $d\zeta_s^2 = 2\zeta_{s-} d\zeta_s + d[\zeta, \zeta]_s$ . We have

$$d[\zeta, \zeta]_s = \zeta_{s-}^2 \beta_s^2 ds + \zeta_{s-}^2 \gamma_s^2 dN_s.$$

Using (1), we thus derive that

$$d\zeta_s^2 = \zeta_{s-}^2 [2\beta_s dW_s + (2\gamma_s + \gamma_s^2) dM_s + (\beta_s^2 + \gamma_s^2 \lambda_s) ds].$$

It follows that  $\zeta^2$  is an exponential semimartingale which can be written:

$$\zeta_s^2 = \eta_s \exp\left\{\int_0^s (\beta_r^2 + \gamma_r^2 \lambda_r) dr\right\}, \quad (15)$$

where  $\eta$  is the exponential local martingale satisfying

$$d\eta_s = \eta_{s-} [2\beta_s dW_s + (2\gamma_s + \gamma_s^2) dM_s],$$

with  $\eta_0 = 1$ . By equality (15), the local martingale  $\eta$  is non negative. Hence, it is a supermartingale, which yields that  $\mathbb{E}[\eta_T] \leq 1$ . Now, by assumption,  $\int_0^T (\beta_r^2 + \gamma_r^2 \lambda_r) dr$  is bounded. By (15), it follows that

<sup>5</sup> Actually the formulation (13) is equivalent to (12).



$$\mathbb{E}[\zeta_T^2] \leq \mathbb{E}[\eta_T] K \leq K,$$

where  $K$  is a positive constant. By martingale inequalities, we derive that  $\mathbb{E}[\sup_{0 \leq s \leq T} \zeta_s^2] < +\infty$ . Hence, the process  $(\zeta_s)_{0 \leq s \leq T}$  is a martingale.

*Remark 10.* Note that, under the assumption from Proposition 3, one can prove by an induction argument (as in the proof of Proposition A.1 in [21]) that for all  $p \geq 2$ , we have  $\mathbb{E}[\sup_{0 \leq s \leq T} \zeta_s^p] < +\infty$ .

We now show a representation property of the solution of a *generalized  $\lambda$ -linear BSDE* when the finite variational process  $D$  is supposed to be predictable.

**Theorem 1 (Representation result for *generalized  $\lambda$ -linear BSDEs with  $D$  predictable*).** *Let  $(\delta_t)$ ,  $(\beta_t)$  and  $(\gamma_t)$  be  $\mathbf{R}$ -valued predictable processes such that  $(\delta_t)$ ,  $(\beta_t)$  and  $(\gamma_t \sqrt{\lambda_t})$  are bounded. Let  $\xi \in L^2(\mathcal{G}_T)$  and let  $D$  be a process belonging to  $\mathcal{A}_p^2$ , that is, a finite variational RCLL predictable process with  $D_0 = 0$  and square integrable total variation process. Let  $(Y, Z, K)$  be the solution in  $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}_\lambda^2$  of the BSDE associated with generalized  $\lambda$ -linear driver  $(\delta_t y + \beta_t z + \gamma_t k \lambda_t) dt + dD_t$  and terminal condition  $\xi$ , that is*

$$-dY_t = (\delta_t Y_t + \beta_t Z_t + \gamma_t K_t \lambda_t) dt + dD_t - Z_t dW_t - K_t dM_t; \quad Y_T = \xi. \quad (16)$$

For each  $t \in [0, T]$ , let  $(\Gamma_{t,s})_{s \geq t}$  (called the adjoint process) be the unique solution of the following forward SDE

$$d\Gamma_{t,s} = \Gamma_{t,s^-} [\delta_s ds + \beta_s dW_s + \gamma_s dM_s]; \quad \Gamma_{t,t} = 1. \quad (17)$$

The process  $(Y_t)$  satisfies

$$Y_t = \mathbb{E}[\Gamma_{t,T} \xi + \int_t^T \Gamma_{t,s^-} dD_s \mid \mathcal{G}_t], \quad 0 \leq t \leq T, \quad \text{a.s.} \quad (18)$$

*Remark 11.* From Remark 8, it follows that the process  $(\Gamma_{t,s})_{s \geq t}$ , defined by (17), satisfies

$$\Gamma_{t,s} = e^{\int_t^s \delta_r dr} \exp\left\{ \int_t^s \beta_r dW_r - \frac{1}{2} \int_t^s \beta_r^2 dr \right\} e^{-\int_t^s \gamma_r \lambda_r dr} (1 + \gamma_\vartheta \mathbf{1}_{\{s \geq \vartheta > t\}}) \quad s \geq t \quad \text{a.s.}$$

Hence, if  $\gamma_\vartheta \geq -1$  (resp.  $> -1$ ) a.s., we then have  $\Gamma_{t,s} \geq 0$  (resp.  $> 0$ ) for all  $s \geq t$  a.s.

Note also that the process  $(e^{\int_t^s \delta_r dr})_{t \leq s \leq T}$  is positive, and bounded since  $\delta$  is bounded. Using Proposition 3, since  $\beta$  and  $\gamma \sqrt{\lambda}$  are bounded, we derive that  $\mathbb{E}[\sup_{t \leq s \leq T} \Gamma_{t,s}^2] < +\infty$ .

*Proof.* Fix  $t \in [0, T]$ . Note first that since  $D$  is a finite variational RCLL process, here supposed to be *predictable*, and since the process  $\Gamma_{t,\cdot}$  admits only one jump at the totally inaccessible stopping time  $\vartheta$ , we get  $[\Gamma_{t,\cdot}, D] = 0$ . By applying the Itô product formula to  $Y_s \Gamma_{t,s}$ , we get

$$\begin{aligned} -d(Y_s \Gamma_{t,s}) &= -Y_s^- d\Gamma_{t,s} - \Gamma_{t,s^-} dY_s - d[Y, \Gamma]_s \\ &= -Y_s \Gamma_{t,s^-} \delta_s ds + \Gamma_{t,s^-} [\delta_s Y_s + \beta_s Z_s + \gamma_s K_s \lambda_s] ds + \Gamma_{t,s^-} dD_s \\ &\quad - \beta_s Z_s \Gamma_{t,s^-} ds - \Gamma_{t,s^-} \gamma_s K_s \lambda_s ds - \Gamma_{t,s^-} (Y_s \beta_s + Z_s) dW_s - \Gamma_{t,s^-} [K_s (1 + \gamma_s) + Y_s^- \gamma_s] dM_s. \end{aligned} \quad (19)$$

Setting

$$dm_s = -\Gamma_{t,s^-} (Y_s \beta_s + Z_s) dW_s - \Gamma_{t,s^-} [K_s (1 + \gamma_s) + Y_s^- \gamma_s] dM_s,$$

we get

$$-d(Y_s \Gamma_{t,s}) = \Gamma_{t,s^-} dD_s - dm_s. \quad (20)$$

By integrating between  $t$  and  $T$ , we obtain

$$Y_t = \xi \Gamma_{t,T} + \int_t^T \Gamma_{t,s^-} dD_s - (m_T - m_t) \quad \text{a.s.} \quad (21)$$

By Remark 11, we have  $(\Gamma_{t,s})_{t \leq s \leq T} \in \mathcal{S}^2$ . Moreover,  $Y \in \mathcal{S}^2$ ,  $Z \in \mathbb{H}^2$ ,  $K \in \mathbb{H}_\lambda^2$ , and  $\beta$  and  $\gamma\sqrt{\lambda}$  are bounded. It follows that the local martingale  $m = (m_s)_{t \leq s \leq T}$  is a martingale. Hence, by taking the conditional expectation in equality (21), we get equality (18).

When the finite variational process  $D$  is no longer supposed to be predictable (which is often the case in the literature on default risk <sup>6</sup>), the representation formula (18) does not generally hold. We now provide a representation property of the solution in that case, that is, when the finite variational process  $D$  is only supposed to be RCLL and adapted, which is new in the literature on BSDEs.

**Theorem 2 (Representation result for generalized  $\lambda$ -linear BSDEs with  $D$  optional).** *Suppose that the assumptions of Theorem 1 hold, except that  $D$  is supposed to belong to  $\mathcal{A}^2$  instead of  $\mathcal{A}_p^2$ . Let  $D' \in \mathcal{A}_p^2$  and  $\eta \in \mathbb{H}_\lambda^2$  be such that for all  $t \in [0, T]$ ,*<sup>7</sup>

$$D_t = D'_t + \int_0^t \eta_s dN_s \quad \text{a.s.} \quad (22)$$

Let  $(Y, Z, K)$  be the solution in  $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}_\lambda^2$  of the BSDE associated with generalized  $\lambda$ -linear driver  $(\delta_t y + \beta_t z + \gamma_t k \lambda_t) dt + dD_t$  and terminal condition  $\xi$ , that is BSDE (16).

Then, a.s. for all  $t \in [0, T]$ ,

$$Y_t = \mathbb{E}[\Gamma_{t,T} \xi + \int_t^T \Gamma_{t,s^-} (dD'_s + (1 + \gamma_s) \eta_s dN_s) | \mathcal{G}_t] = \mathbb{E}[\Gamma_{t,T} \xi + \int_t^T \Gamma_{t,s^-} dD'_s + \Gamma_{t,\vartheta} \eta_\vartheta \mathbf{1}_{\{t < \vartheta \leq T\}} | \mathcal{G}_t] \quad (23)$$

where  $(\Gamma_{t,s})_{s \in [t, T]}$  satisfies (17).

*Proof.* Since  $D$  satisfies (22), we get  $d[\Gamma_{t,\cdot}, D]_s = \Gamma_{t,s^-} \gamma_s \eta_s dN_s$ . The computations are then similar to those of the proof of Theorem 1, with  $\Gamma_{t,s^-} dD_s$  replaced by  $\Gamma_{t,s^-} (dD_s + \gamma_s \eta_s dN_s)$  in equations (19), (20) and (21). We thus derive that  $Y_t = \mathbb{E}[\Gamma_{t,T} \xi + \int_t^T \Gamma_{t,s^-} (dD_s + \gamma_s \eta_s dN_s) | \mathcal{G}_t]$  a.s. From this together with (22), the first equality of (23) follows. Now, we have a.s.

$$\mathbb{E}[\int_t^T \Gamma_{t,s^-} (1 + \gamma_s) \eta_s dN_s | \mathcal{G}_t] = \mathbb{E}[\Gamma_{t,\vartheta^-} (1 + \gamma_\vartheta) \eta_\vartheta \mathbf{1}_{\{t < \vartheta \leq T\}} | \mathcal{G}_t] = \mathbb{E}[\Gamma_{t,\vartheta} \eta_\vartheta \mathbf{1}_{\{t < \vartheta \leq T\}} | \mathcal{G}_t],$$

where the second equality is due to the fact that  $\Gamma_{t,\vartheta^-} (1 + \gamma_\vartheta) = \Gamma_{t,\vartheta}$  a.s. (cf. Remark 11). This yields the second equality of (23).

*Remark 12.* By adapting the arguments of the above proof, this result can be generalized to the case of a BSDE driven by a Brownian motion and a Poisson random measure <sup>8</sup>, which provides a new result in the theory of BSDEs in this framework.

## 2.4 Comparison theorems for BSDEs with default jump

We now provide a comparison theorem and a strict comparison theorem for BSDEs with *generalized  $\lambda$ -admissible drivers* associated with finite variational RCLL adapted processes.

**Theorem 3 (Comparison theorems).** *Let  $\xi_1$  and  $\xi_2 \in L^2(\mathcal{G}_T)$ . Let  $g_1$  and  $g_2$  be two  $\lambda$ -admissible drivers. Let  $D^1$  and  $D^2$  be two (optional) processes in  $\mathcal{A}^2$ .*

*For  $i = 1, 2$ , let  $(Y^i, Z^i, K^i)$  be the solution in  $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}_\lambda^2$  of the following BSDE*

$$-dY_t^i = g_i(t, Y_t^i, Z_t^i, K_t^i) dt + dD_t^i - Z_t^i dW_t - K_t^i dM_t; \quad Y_T^i = \xi_i.$$

<sup>6</sup> In the case of a contingent claim or a contract subjected to default,  $\Delta D_\vartheta$  represents the cashflow generated by the claim at the default time  $\vartheta$  (see Section 3). It is sometimes called "rebate" (cf. [16, 4]).

<sup>7</sup> see Remark 2.

<sup>8</sup> since in this case, the jumps times of the Poisson random measure are totally inaccessible.

(i) (Comparison theorem). Assume that there exists a predictable process  $(\gamma_t)$  with

$$(\gamma_t \sqrt{\lambda_t}) \text{ bounded and } \gamma_t \geq -1, \quad dP \otimes dt - a.e. \quad (24)$$

such that

$$g_1(t, Y_t^2, Z_t^2, K_t^1) - g_1(t, Y_t^2, Z_t^2, K_t^2) \geq \gamma_t (K_t^1 - K_t^2) \lambda_t, \quad t \in [0, T], \quad dP \otimes dt - a.e. \quad (25)$$

Suppose that  $\xi_1 \geq \xi_2$  a.s., that the process  $\bar{D} := D^1 - D^2$  is non decreasing, and that

$$g_1(t, Y_t^2, Z_t^2, K_t^1) \geq g_2(t, Y_t^2, Z_t^2, K_t^2), \quad t \in [0, T], \quad dP \otimes dt - a.e. \quad (26)$$

We then have  $Y_t^1 \geq Y_t^2$  for all  $t \in [0, T]$  a.s.

(ii) (Strict Comparison Theorem). Suppose moreover that  $\gamma_\vartheta > -1$  a.s.

If  $Y_{t_0}^1 = Y_{t_0}^2$  a.s. for some  $t_0 \in [0, T]$ , then  $\xi^1 = \xi^2$  a.s., and the inequality (26) is an equality on  $[t_0, T]$ . Moreover,  $\bar{D} = D^1 - D^2$  is constant on  $[t_0, T]$  and  $Y^1 = Y^2$  on  $[t_0, T]$ .

*Remark 13.* We stress that the above comparison theorems hold even in the case when the generalized drivers are associated with non-predictable finite variational processes, which thus may admit a jump at the default time  $\vartheta$ . This is important for the applications to nonlinear pricing of contingents claims. Indeed, in a market with default, contingent claims often generate a cashflow at the default time (see Section 3.3 for details).

As seen in the proof below, the treatment of the case of non-predictable finite variational processes requires some additional arguments, compared to the case of predictable ones.

*Proof.* Setting  $\bar{Y}_s = Y_s^1 - Y_s^2$ ;  $\bar{Z}_s = Z_s^1 - Z_s^2$ ;  $\bar{K}_s = K_s^1 - K_s^2$ , we have

$$-d\bar{Y}_s = h_s ds + d\bar{D}_s - \bar{Z}_s dW_s - \bar{K}_s dM_s; \quad \bar{Y}_T = \xi_1 - \xi_2,$$

where  $h_s := g_1(s, Y_{s^-}^1, Z_s^1, K_s^1) - g_2(s, Y_{s^-}^2, Z_s^2, K_s^2)$ .

Set  $\delta_s := \frac{g_1(s, Y_{s^-}^1, Z_s^1, K_s^1) - g_1(s, Y_{s^-}^2, Z_s^1, K_s^1)}{\bar{Y}_{s^-}}$  if  $\bar{Y}_{s^-} \neq 0$ , and 0 otherwise.

Set  $\beta_s := \frac{g_1(s, Y_{s^-}^2, Z_s^1, K_s^1) - g_1(s, Y_{s^-}^2, Z_s^2, K_s^1)}{\bar{Z}_s}$  if  $\bar{Z}_s \neq 0$ , and 0 otherwise.

By definition, the processes  $\delta$  and  $\beta$  are *predictable*. Moreover, since  $g_1$  satisfies condition (3), the processes  $\delta$  and  $\beta$  are bounded. Now, we have

$$h_s = \delta_s \bar{Y}_{s^-} + \beta_s \bar{Z}_s + g_1(s, Y_{s^-}^2, Z_s^2, K_s^1) - g_1(s, Y_{s^-}^2, Z_s^2, K_s^2) + \varphi_s,$$

where  $\varphi_s := g_1(s, Y_{s^-}^2, Z_s^2, K_s^2) - g_2(s, Y_{s^-}^2, Z_s^2, K_s^2)$ .<sup>9</sup>

Using the assumption (25) and the equality  $\bar{Y}_{s^-} = \bar{Y}_s dP \otimes ds$ -a.e. (cf. Remark 3), we get

$$h_s \geq \delta_s \bar{Y}_s + \beta_s \bar{Z}_s + \gamma_s \bar{K}_s \lambda_s + \varphi_s \quad dP \otimes ds - a.e. \quad (27)$$

Fix  $t \in [0, T]$ . Let  $\Gamma_{t,\cdot}$  be the process defined by (17). Since  $\delta$ ,  $\beta$  and  $\gamma\sqrt{\lambda}$  are bounded, it follows from Remark 11 that  $\Gamma_{t,\cdot} \in \mathcal{S}^2$ . Also, since  $\gamma_s \geq -1$ , we have  $\Gamma_{t,\cdot} \geq 0$  a.s. Let us first consider the simpler case when the processes  $D^1$  and  $D^2$  are predictable. By Itô's formula and similar computations to those of the proof of Theorem 1, we derive that

$$-d(\bar{Y}_s \Gamma_{t,s}) = \Gamma_{t,s} (h_s - \delta_s \bar{Y}_s - \beta_s \bar{Z}_s - \gamma_s \bar{K}_s \lambda_s) ds + \Gamma_{t,s} d\bar{D}_s - dm_s,$$

where  $m$  is a martingale (because  $\Gamma_{t,\cdot} \in \mathcal{S}^2$ ,  $\bar{Y} \in \mathcal{S}^2$ ,  $\bar{Z} \in \mathbf{H}^2$ ,  $\bar{K} \in \mathbf{H}_\lambda^2$  and  $\beta$ ,  $\gamma\sqrt{\lambda}$  are bounded). Using inequality (27) together with the non negativity of  $\Gamma$ , we thus get  $-d(\bar{Y}_s \Gamma_{t,s}) \geq \Gamma_{t,s} \varphi_s ds + \Gamma_{t,s} d\bar{D}_s - dm_s$ . By integrating between  $t$  and  $T$  and by taking the conditional expectation, we obtain

<sup>9</sup> Note that, by Remark 3, we have  $\varphi_s = g_1(s, Y_s^2, Z_s^2, K_s^2) - g_2(s, Y_s^2, Z_s^2, K_s^2) dP \otimes ds$ -a.e.

$$\bar{Y}_t \geq \mathbb{E}[\Gamma_{t,T}(\xi_1 - \xi_2) + \int_t^T \Gamma_{t,s^-}(\varphi_s ds + d\bar{D}_s) | \mathcal{G}_t], \quad 0 \leq t \leq T, \quad \text{a.s.} \quad (28)$$

By assumption (26),  $\varphi_s \geq 0$   $dP \otimes ds$ -a.e. Moreover,  $\xi_1 - \xi_2 \geq 0$  and  $\bar{D}$  is non decreasing, which, together with the non negativity of  $\Gamma_{t,\cdot}$ , implies that  $\bar{Y}_t = Y_t^1 - Y_t^2 \geq 0$  a.s. Since this inequality holds for all  $t \in [0, T]$ , the assertion (i) follows.

Suppose moreover that  $Y_{t_0}^1 = Y_{t_0}^2$  a.s. and that  $\gamma > -1$ . Since  $\gamma_\vartheta > -1$  a.s., we have  $\Gamma_{t,s} > 0$  a.s. for all  $s \geq t$ . From this, together with (28) applied with  $t = t_0$ , we get  $\xi_1 = \xi_2$  a.s. and  $\varphi_t = 0$ ,  $t \in [t_0, T]$   $dP \otimes dt$ -a.e. On the other hand, set  $\bar{D}_t := \int_{t_0}^t \Gamma_{t_0,s^-} d\bar{D}_s$ , for each  $t \in [t_0, T]$ . By assumption,  $\bar{D}_T \geq 0$  a.s. By (28), we thus get  $\mathbb{E}[\bar{D}_T | \mathcal{G}_{t_0}] = 0$  a.s. Hence  $\bar{D}_T = 0$  a.s. Now, since  $\Gamma_{t_0,s} > 0$ , for all  $s \geq t_0$  a.s., we can write  $\bar{D}_T - \bar{D}_{t_0} = \int_{t_0}^T \Gamma_{t_0,s}^{-1} d\bar{D}_s$ . We thus get  $\bar{D}_T = \bar{D}_{t_0}$  a.s. The proof of (ii) is thus complete.

Let us now consider the case when the processes  $D^1$  and  $D^2$  are not predictable. By Remark 2, for  $i = 1, 2$ , there exist  $D^i \in \mathcal{A}_p^2$  and  $\eta^i \in \mathcal{H}_\lambda^2$  such that

$$D_t^i = D_t^i + \int_0^t \eta_s^i dN_s \quad \text{a.s.}$$

Since  $\bar{D} := D^1 - D^2$  is non decreasing, we derive that the process  $\bar{D}' := D'^1 - D'^2$  is non decreasing and that  $\eta_\vartheta^1 \geq \eta_\vartheta^2$  a.s. on  $\{\vartheta \leq T\}$ . By Itô's formula and similar computations to those of the proof of Theorems 1 and 2, we get

$$-d(\bar{Y}_s \Gamma_{t,s}) = \Gamma_{t,s}(h_s - \delta_s \bar{Y}_s - \beta_s \bar{Z}_s - \gamma_s \bar{K}_s \lambda_s) ds + \Gamma_{t,s^-} [d\bar{D}_s + (\eta_s^1 - \eta_s^2) \gamma_s dN_s] - dm_s,$$

where  $m$  is a martingale. Using inequality (27) and the equality  $\bar{D}_t = \bar{D}_t^1 + \int_0^t (\eta_s^1 - \eta_s^2) dN_s$  a.s., we thus derive that

$$\bar{Y}_t \geq \mathbb{E}[\Gamma_{t,T}(\xi_1 - \xi_2) + \int_t^T \Gamma_{t,s^-}(\varphi_s ds + d\bar{D}_s + (\eta_s^1 - \eta_s^2)(1 + \gamma_s) dN_s) | \mathcal{G}_t], \quad 0 \leq t \leq T, \quad \text{a.s.} \quad (29)$$

Since  $\eta_\vartheta^1 \geq \eta_\vartheta^2$  a.s. on  $\{\vartheta \leq T\}$  and  $\gamma_\vartheta \geq -1$  a.s., we have  $(\eta_\vartheta^1 - \eta_\vartheta^2)(1 + \gamma_\vartheta) \geq 0$  a.s. on  $\{\vartheta \leq T\}$ . Hence, using the other assumptions made in (i), we derive that  $\bar{Y}_t = Y_t^1 - Y_t^2 \geq 0$  a.s. Since this inequality holds for all  $t \in [0, T]$ , the assertion (i) follows.

Suppose moreover that  $Y_{t_0}^1 = Y_{t_0}^2$  a.s. and that  $\gamma_\vartheta > -1$  a.s. By the inequality (29) applied with  $t = t_0$ , we derive that  $\xi^1 = \xi^2$  a.s.,  $\varphi_s = 0$   $dP \otimes ds$ -a.e. on  $[t_0, T]$ ,  $\eta_\vartheta^1 = \eta_\vartheta^2$  a.s. on  $\{t_0 < \vartheta \leq T\}$ . Moreover,  $\bar{D}' = D'^1 - D'^2$  is constant on the time interval  $[t_0, T]$ . Hence,  $\bar{D}$  is constant on  $[t_0, T]$ . The proof is thus complete.

*Remark 14.* By adapting the arguments of the above proof, this result can be generalized to the case of BSDEs driven by a Brownian motion and a Poisson random measure (since the jumps times associated with the Poisson random measure are totally inaccessible). This extends the comparison theorems given in the literature on BSDEs with jumps (see [21, Theorems 4.2 and 4.4]) to the case of generalized drivers of the form  $g(t, y, z, k)dt + dD_t$ , where  $D$  is a finite variational RCLL adapted process (not necessarily predictable).

When the assumptions of the comparison theorem (resp. strict comparison theorem) are violated, the conclusion does not necessarily hold, as shown by the following example.

*Example 1.* Suppose that the process  $\lambda$  is bounded. Let  $g$  be a  $\lambda$ -linear driver (see (12)) of the form

$$g(\omega, t, y, z, k) = \delta_t(\omega)y + \beta_t(\omega)z + \gamma k \lambda_t(\omega), \quad (30)$$

where  $\gamma$  is here a real constant. At terminal time  $T$ , the associated adjoint process  $\Gamma_0$  satisfies (see (17) and Remark 11):

$$\Gamma_{0,T} = H_T \exp\left\{-\int_0^T \gamma \lambda_r dr\right\} (1 + \gamma \mathbf{1}_{\{T \geq \vartheta\}}). \quad (31)$$

where  $(H_t)$  satisfies  $dH_t = H_t(\delta_t dt + \beta_t dW_t)$  with  $H_0 = 1$ .

Let  $Y$  be the solution of the BSDE associated with driver  $g$  and terminal condition

$$\xi := \mathbf{1}_{\{T \geq \vartheta\}}.$$

The representation property of  $\lambda$ -linear BSDEs with default jump (see (18)) gives

$$Y_0 = \mathbb{E}[I_{0,T} \xi] = \mathbb{E}[I_{0,T} \mathbf{1}_{\{T \geq \vartheta\}}].$$

Hence, by (31), we get

$$Y_0 = \mathbb{E}[I_{0,T} \mathbf{1}_{\{T \geq \vartheta\}}] = \mathbb{E}[H_T e^{-\gamma \int_0^T \lambda_s ds} (1 + \gamma \mathbf{1}_{\{T \geq \vartheta\}}) \mathbf{1}_{\{T \geq \vartheta\}}] = (1 + \gamma) \mathbb{E}[H_T e^{-\gamma \int_0^T \lambda_s ds} \mathbf{1}_{\{T \geq \vartheta\}}]. \quad (32)$$

Equation (32) shows that under the additional assumption  $P(T \geq \vartheta) > 0$ , when  $\gamma < -1$ , we have  $Y_0 < 0$  although  $\xi \geq 0$  a.s.

This example also gives a counter-example for the strict comparison theorem by taking  $\gamma = -1$ . Indeed, in this case, the relation (32) yields that  $Y_0 = 0$ . Under the additional assumption  $P(T \geq \vartheta) > 0$ , we have  $P(\xi > 0) > 0$ , even though  $Y_0 = 0$ .

### 3 Nonlinear pricing in a financial market with default

#### 3.1 Financial market with defaultable risky asset

We consider a complete financial market with default as in [5], which consists of one risk-free asset, with price process  $S^0$  satisfying  $dS_t^0 = S_t^0 r_t dt$  with  $S_0^0 = 1$ , and two risky assets with price processes  $S^1, S^2$  evolving according to the equations:

$$\begin{aligned} dS_t^1 &= S_t^1 [\mu_t^1 dt + \sigma_t^1 dW_t] \quad \text{with } S_0^1 > 0; \\ dS_t^2 &= S_t^2 [\mu_t^2 dt + \sigma_t^2 dW_t - dM_t] \quad \text{with } S_0^2 > 0, \end{aligned} \quad (33)$$

where the process  $(M_t)$  is given by (1).

The processes  $\sigma^1, \sigma^2, r, \mu^1, \mu^2$  are predictable (that is  $\mathcal{P}$ -measurable). We set  $\sigma = (\sigma^1, \sigma^2)'$ , where  $\prime$  denotes transposition.

We suppose that  $\sigma^1, \sigma^2 > 0$ , and  $r, \mu^1, \mu^2, \sigma^1, \sigma^2, (\sigma^1)^{-1}, (\sigma^2)^{-1}$  are bounded. Note that the intensity process  $(\lambda_t)$  is not necessarily bounded, which is useful in market models with default where the intensity process is modeled by the solution (which is not necessarily bounded) of a forward stochastic differential equation.

*Remark 15.* By Remark 11, we have

$$S_t^2 = e^{\int_0^t \mu_r^2 dr} \exp\left\{\int_0^t \sigma_r^2 dW_r - \frac{1}{2} \int_0^t (\sigma_r^2)^2 dr\right\} e^{\int_0^t \lambda_r dr} (1 - \mathbf{1}_{\{t \geq \vartheta\}}), \quad t \geq 0 \quad \text{a.s.}$$

The second risky asset is thus defaultable with total default: we have  $S_t^2 = 0, t \geq \vartheta$  a.s.

We consider an investor who, at time 0, invests an initial amount  $x \in \mathbf{R}$  in the three assets. For  $i = 1, 2$ , we denote by  $\varphi_t^i$  the amount invested in the  $i^{\text{th}}$  risky asset. After time  $\vartheta$ , the investor does not invest in the defaultable asset since its price is equal to 0. We thus have  $\varphi_t^2 = 0$  on  $t > \vartheta$ . A process  $\varphi = (\varphi^1, \varphi^2)'$  belonging to  $\mathbb{H}^2 \times \mathbb{H}_\lambda^2$  is called a *risky assets strategy*. Let  $C$  be a finite variational optional process belonging to  $\mathcal{A}^2$ , representing the *cumulative cash amount withdrawn* from the portfolio.

The value at time  $t$  of the portfolio (or *wealth*) associated with  $x, \varphi$  and  $C$  is denoted by  $V_t^{x, \varphi, C}$ . The amount invested in the risk-free asset at time  $t$  is then given by  $V_t^{x, \varphi, C} - (\varphi_t^1 + \varphi_t^2)$ .

### 3.2 Pricing of European options with dividends in a perfect (linear) market with default

In this section, we place ourselves in a perfect (linear) market model with default. In this case, by the self financing condition, the wealth process  $V^{x, \varphi, C}$  (simply denoted by  $V$ ) follows the dynamics:

$$\begin{aligned} dV_t &= (r_t V_t + \varphi_t^1 (\mu_t^1 - r_t) + \varphi_t^2 (\mu_t^2 - r_t)) dt - dC_t + (\varphi_t^1 \sigma_t^1 + \varphi_t^2 \sigma_t^2) dW_t - \varphi_t^2 dM_t \\ &= (r_t V_t + \varphi_t^1 \sigma_t \theta_t^1 - \varphi_t^2 \theta_t^2 \lambda_t) dt - dC_t + \varphi_t^1 \sigma_t dW_t - \varphi_t^2 dM_t, \end{aligned} \quad (34)$$

where  $\varphi_t^1 \sigma_t = \varphi_t^1 \sigma_t^1 + \varphi_t^2 \sigma_t^2$ , and

$$\theta_t^1 := \frac{\mu_t^1 - r_t}{\sigma_t^1}; \quad \theta_t^2 := -\frac{\mu_t^2 - \sigma_t^2 \theta_t^1 - r_t}{\lambda_t} \mathbf{1}_{\{\lambda_t \neq 0\}}.$$

Suppose that the processes  $\theta^1$  and  $\theta^2 \sqrt{\lambda}$  are bounded.

Let  $T > 0$ . Let  $\xi$  be a  $\mathcal{G}_T$ -measurable random variable belonging to  $L^2$ , and let  $D$  be a finite variational optional process belonging to  $\mathcal{A}_T^2$ . We consider a European option with maturity  $T$ , which generates a *terminal payoff*  $\xi$ , and intermediate cashflows called *dividends*, which are not necessarily positive (cf. for example [7, 8]). For each  $t \in [0, T]$ ,  $D_t$  represents the cumulative intermediate cashflows paid by the option between time 0 and time  $t$ . The process  $D = (D_t)$  is called the *cumulative dividend process*. Note that  $D$  is not necessarily non decreasing.

The aim is to price this contingent claim. Let us consider an agent who wants to sell the option at time 0. With the amount the seller receives at time 0 from the buyer, he/she wants to be able to construct a portfolio which allows him/her to pay to the buyer the amount  $\xi$  at time  $T$ , as well as the intermediate dividends.

Now, setting

$$Z_t := \varphi_t^1 \sigma_t; \quad K_t := -\varphi_t^2, \quad (35)$$

by (34), we derive that the process  $(V, Z, K)$  satisfies the following dynamics:

$$-dV_t = -(r_t V_t + \theta_t^1 Z_t + \theta_t^2 K_t \lambda_t) dt + dD_t - Z_t dW_t - K_t dM_t.$$

We set for each  $(\omega, t, y, z, k)$ ,

$$g(\omega, t, y, z, k) := -r_t(\omega)y - \theta_t^1(\omega)z - \theta_t^2(\omega)k \lambda_t(\omega). \quad (36)$$

Since by assumption, the coefficients  $r, \theta^1, \theta^2 \sqrt{\lambda}$  are predictable and bounded, it follows that  $g$  is a  $\lambda$ -linear driver (see Definition 3). By Proposition 2, there exists a unique solution  $(X, Z, K) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}_\lambda^2$  of the BSDE associated with terminal time  $T$ , *generalized  $\lambda$ -linear driver*  $g(t, y, z, k) dt + dD_t$  (with  $g$  defined by (36)) and terminal condition  $\xi$ .

Let us show that the process  $(X, Z, K)$  provides a replicating portfolio. Let  $\varphi$  be the risky-assets strategy such that (35) holds. Note that this defines a change of variables  $\Phi$  as follows:

$\Phi : \mathbb{H}^2 \times \mathbb{H}_\lambda^2 \rightarrow \mathbb{H}^2 \times \mathbb{H}_\lambda^2; (Z, K) \mapsto \Phi(Z, K) := \varphi$ , where  $\varphi = (\varphi^1, \varphi^2)$  is given by (35), which is equivalent to

$$\varphi_t^2 = -K_t; \quad \varphi_t^1 = \frac{Z_t - \varphi_t^2 \sigma_t^2}{\sigma_t^1} = \frac{Z_t + \sigma_t^2 K_t}{\sigma_t^1}. \quad (37)$$

The process  $D$  corresponds here to the cumulative cash withdrawn by the seller from his/her hedging portfolio. The process  $X$  thus coincides with  $V^{X_0, \varphi, D}$ , the value of the portfolio associated with initial wealth  $x = X_0$ , risky-assets strategy  $\varphi$  and cumulative cash withdrawal  $D$ . We deduce that this portfolio is a replicating portfolio for the seller since, by investing the initial amount  $X_0$  in the reference assets along the strategy  $\varphi$ , the seller can pay the terminal payoff  $\xi$  to the buyer at time  $T$ , as well as the intermediate dividends (since the cash withdrawals perfectly replicate the dividends of the option). We derive that  $X_0$  is the initial price of the option, called *hedging price*, denoted by  $X_{0,T}(\xi, D)$ ,

and that  $\varphi$  is the *hedging risky-assets strategy*. Similarly, for each time  $t \in [0, T]$ ,  $X_t$  is the *hedging price* at time  $t$  of the option, and is denoted by  $X_{t,T}(\xi, D)$ .

Suppose that the *cumulative dividend process*  $D$  is predictable. Since the driver  $g$  given by (36) is  $\lambda$ -linear, the representation property of the solution of a *generalized  $\lambda$ -linear BSDE* (see Theorem 1) yields

$$X_{t,T}(\xi, D) = \mathbb{E}[e^{-\int_t^T r_s ds} \zeta_{t,T} \xi + \int_t^T e^{-\int_t^s r_u du} \zeta_{t,s^-} dD_s | \mathcal{G}_t] \quad \text{a.s.}, \quad (38)$$

where  $\zeta$  satisfies

$$d\zeta_{t,s} = \zeta_{t,s^-} [-\theta_s^1 dW_s - \theta_s^2 dM_s]; \quad \zeta_{t,t} = 1.$$

Suppose now that  $\theta_t^2 < 1$   $dP \otimes dt$ -a.e. By Proposition 3 and Remark 8, the process  $\zeta_{0,\cdot}$  is a square integrable positive martingale. Let  $Q$  be the probability measure which admits  $\zeta_{0,T}$  as density with respect to  $P$  on  $\mathcal{G}_T$ .<sup>10</sup> By the equality (38), we have

$$X_{t,T}(\xi, D) = \mathbb{E}_Q[e^{-\int_t^T r_s ds} \xi + \int_t^T e^{-\int_t^s r_u du} dD_s | \mathcal{G}_t] \quad \text{a.s.} \quad (39)$$

When the *cumulative dividend process*  $D$  is not predictable and thus admits a jump at time  $\vartheta$ , the representation formulas (38) and (39) for the no-arbitrage price of the contingent claim do not generally hold. In this case, by Remark 2, there exist a (unique) process  $D' \in \mathcal{A}_P^2$  and a (unique) process  $\eta \in \mathbb{H}_\lambda^2$  such that for all  $t \in [0, T]$ ,

$$D_t = D'_t + \int_0^t \eta_s dN_s \quad \text{a.s.} \quad (40)$$

The random variable  $\eta_\vartheta$  (sometimes called "rebate" in the literature) represents the cash flow generated by the contingent claim at the default time  $\vartheta$  (see e.g. [7, 4, 8, 16] for examples of such contingent claims). By Theorem 2, we get

$$X_{t,T}(\xi, D) = \mathbb{E}[e^{-\int_t^T r_s ds} \zeta_{t,T} \xi + \int_t^T e^{-\int_t^s r_u du} \zeta_{t,s^-} dD'_s + e^{-\int_t^\vartheta r_s ds} \zeta_{t,\vartheta} \eta_\vartheta \mathbf{1}_{\{t < \vartheta \leq T\}} | \mathcal{G}_t] \quad \text{a.s.},$$

or equivalently

$$X_{t,T}(\xi, D) = \mathbb{E}_Q[e^{-\int_t^T r_s ds} \xi + \int_t^T e^{-\int_t^s r_u du} dD'_s + e^{-\int_t^\vartheta r_s ds} \eta_\vartheta \mathbf{1}_{\{t < \vartheta \leq T\}} | \mathcal{G}_t] \quad \text{a.s.}$$

We thus recover the risk-neutral pricing formula of [4, 16], which we have established here by working under the primitive probability measure, using BSDE techniques.

We note that the pricing system (for a fixed maturity  $T$ ):  $(\xi, D) \mapsto X_{\cdot,T}(\xi, D)$  is linear.

### 3.3 Nonlinear pricing of European options with dividends in an imperfect market with default

From now on, we assume that there are imperfections in the market which are taken into account via the *nonlinearity* of the dynamics of the wealth. More precisely, we suppose that the *wealth process*  $V_t^{x, \varphi, C}$  (or simply  $V_t$ ) associated with an initial wealth  $x$ , a risky-assets strategy  $\varphi = (\varphi^1, \varphi^2)$  in  $\mathbb{H}^2 \times \mathbb{H}_\lambda^2$  and a cumulative withdrawal process  $C \in \mathcal{A}^2$  satisfies the following dynamics:

<sup>10</sup> Note that the discounted price process  $(e^{-\int_0^t r_s ds} S_t^1)_{0 \leq t \leq T}$  (resp.  $(e^{-\int_0^t r_s ds} S_t^2)_{0 \leq t \leq T}$ ) is a martingale (resp. local martingale) under  $Q$ . Suppose now that  $E[e^{q \int_0^T \lambda_r dr}] < +\infty$  for some  $q > 2$ . Using Remark 15, we show that  $e^{-\int_0^T r_s ds} S_T^2 \in L_Q^2$ , which, by martingale inequalities, implies that  $(e^{-\int_0^t r_s ds} S_t^2)_{0 \leq t \leq T}$  is a martingale under  $Q$ . In other terms,  $Q$  is a *martingale probability measure*. By classical arguments,  $Q$  can be shown to be the unique *martingale probability measure*.

$$-dV_t = g(t, V_t, \varphi_t' \sigma_t, -\varphi_t^2) dt - \varphi_t' \sigma_t dW_t + dC_t + \varphi_t^2 dM_t; V_0 = x, \quad (41)$$

where  $g$  is a nonlinear  $\lambda$ -admissible driver (see Definition 1). Equivalently, setting  $Z_t = \varphi_t' \sigma_t$  and  $K_t = -\varphi_t^2$ , we have

$$-dV_t = g(t, V_t, Z_t, K_t) dt - Z_t dW_t + dC_t - K_t dM_t; V_0 = x. \quad (42)$$

Let us consider a European option with maturity  $T$ , terminal payoff  $\xi \in L^2(\mathcal{G}_T)$ , and dividend process  $D \in \mathcal{A}^2$  (with a possible jump at the default time  $\vartheta$ ) in this market model. Let  $(X_{\cdot, T}^g(\xi, D), Z_{\cdot, T}^g(\xi, D), K_{\cdot, T}^g(\xi, D))$ , simply denoted by  $(X, Z, K)$ , be the solution of BSDE associated with terminal time  $T$ , *generalized driver*  $g(t, y, z, k) dt + dD_t$  and terminal condition  $\xi$ , that is satisfying

$$-dX_t = g(t, X_t, Z_t, K_t) dt + dD_t - Z_t dW_t - K_t dM_t; \quad X_T = \xi.$$

The process  $X = X_{\cdot, T}^g(\xi, D)$  is equal to the wealth process associated with initial value  $x = X_0$ , strategy  $\varphi = \Phi(Z, K)$  (see (37)) and cumulative amount  $D$  of cash withdrawals, that is  $X = V^{X_0, \varphi, D}$ . Its initial value  $X_0 = X_{0, T}^g(\xi, D)$  is thus a sensible price (at time 0) of the option for the seller since this amount allows him/her to construct a risky-assets strategy  $\varphi$ , called *hedging* strategy, such that the value of the associated portfolio is equal to  $\xi$  at time  $T$ , and such that the cash withdrawals perfectly replicate the dividends of the option. We call  $X_0 = X_{0, T}^g(\xi, D)$  the *hedging price* at time  $t$  of the option. Similarly, for each  $t \in [0, T]$ ,  $X_t = X_{t, T}^g(\xi, D)$  is the *hedging price* at time  $t$  of the option.

Thus, for each maturity  $S \in [0, T]$  and for each pair *payoff-dividend*  $(\xi, D) \in L^2(\mathcal{G}_S) \times \mathcal{A}_S^2$ , the process  $X_{\cdot, S}^g(\xi, D)$  is called the *hedging price* process of the option with maturity  $S$  and *payoff-dividend*  $(\xi, D)$ . This leads to a *pricing* system

$$\mathbf{X}^g : (S, \xi, D) \mapsto X_{\cdot, S}^g(\xi, D), \quad (43)$$

which is generally *nonlinear* with respect to  $(\xi, D)$ .

We now give some properties of this *nonlinear pricing* system  $\mathbf{X}^g$  which generalize those given in [15] to the case with a default jump and dividends.

- **Consistency.** By the flow property for BSDEs, the pricing system  $\mathbf{X}^g$  is *consistent*. More precisely, let  $S' \in [0, T]$ ,  $\xi \in L^2(\mathcal{G}_{S'})$ ,  $D \in \mathcal{A}_{S'}^2$ , and let  $S \in [0, S']$ . Then, the *hedging price* of the option associated with payoff  $\xi$ , cumulative dividend process  $D$  and maturity  $S'$  coincides with the *hedging price* of the option associated with maturity  $S$ , payoff  $X_{S, S'}^g(\xi, D)$  and dividend process  $(D_t)_{t \leq S}$  (still denoted by  $D$ ), that is

$$X_{\cdot, S'}^g(\xi, D) = X_{\cdot, S}^g(X_{S, S'}^g(\xi, D), D).$$

- When  $g(t, 0, 0, 0) = 0$ <sup>11</sup>, then the price of the European option with null payoff and no dividends is equal to 0, that is, for all  $S \in [0, T]$ ,  $X_{\cdot, S}^g(0, 0) = 0$ .

Due the presence of the default, the *nonlinear pricing* system  $\mathbf{X}^g$  is not necessarily monotone with respect to the payoff and the dividend. We introduce the following assumption.

**Assumption 4** *Assume that there exists a map*

$$\gamma : \Omega \times [0, T] \times \mathbf{R}^4 \rightarrow \mathbf{R}; (\omega, t, y, z, k_1, k_2) \mapsto \gamma_t^{y, z, k_1, k_2}(\omega)$$

$\mathcal{P} \otimes \mathcal{B}(\mathbf{R}^4)$ -measurable, satisfying  $dP \otimes dt$ -a.e., for each  $(y, z, k_1, k_2) \in \mathbf{R}^4$ ,

$$|\gamma_t^{y, z, k_1, k_2} \sqrt{\lambda_t}| \leq C \quad \text{and} \quad \gamma_t^{y, z, k_1, k_2} \geq -1, \quad (44)$$

and

<sup>11</sup> Note that when the market is perfect,  $g$  is given by (36) and thus satisfies  $g(t, 0, 0, 0) = 0$ .



$$g(t, y, z, k_1) - g(t, y, z, k_2) \geq \gamma_t^{y, z, k_1, k_2} (k_1 - k_2) \lambda_t \quad (45)$$

(where  $C$  is a positive constant).

*Remark 16.* Suppose  $(\lambda_t)$  bounded (as in [12]). Then the first inequality in (44) holds if, for example,  $\gamma$  is bounded.

Recall that  $\lambda$  vanishes after  $\vartheta$  and  $g(t, \cdot)$  does not depend on  $k$  on  $\{t > \vartheta\}$ . Hence, inequality (45) is always satisfied on  $\{t > \vartheta\}$ . Note that Assumption 4 holds when  $g(t, \cdot)$  is non decreasing with respect to  $k$ , or when  $g$  is  $\mathcal{C}^1$  in  $k$  with  $\partial_k g(t, \cdot) \geq -\lambda_t$ .

Before giving some additional properties of the nonlinear pricing system under Assumption 4, we introduce the following partial order relation, defined for each fixed time  $S \in [0, T]$ , on the set of pairs "payoff-dividends" by: for each  $(\xi^1, D^1), (\xi^2, D^2) \in L^2(\mathcal{G}_S) \times \mathcal{A}_S^2$

$$(\xi^1, D^1) \succ (\xi^2, D^2) \quad \text{if} \quad \xi^1 \geq \xi^2 \text{ a.s. and } D^1 - D^2 \text{ is non decreasing.}$$

Loosely speaking, the non decreasing property of  $D^1 - D^2$  corresponds to the fact that the dividends paid by the option associated with  $(\xi^1, D^1)$  are greater than or equal to those paid by the option associated with  $(\xi^2, D^2)$ .

Using the comparison theorem for BSDEs with generalized drivers (Theorem 3 (i)), we derive the following properties:

- **Monotonicity.** Under Assumption 4, the nonlinear pricing system  $\mathbf{X}^g$  is non decreasing with respect to the payoff and the dividend. More precisely, for all maturity  $S \in [0, T]$ , for all payoffs  $\xi_1, \xi_2 \in L^2(\mathcal{G}_S)$ , and cumulative dividend processes  $D^1, D^2 \in \mathcal{A}_S^2$ , the following property holds: If  $(\xi^1, D^1) \succ (\xi^2, D^2)$ , then we have  $X_{t,S}^g(\xi_1, D^1) \geq X_{t,S}^g(\xi_2, D^2)$ ,  $t \in [0, S]$  a.s.<sup>12</sup>
- **Convexity.** Under Assumption 4, if  $g$  is convex with respect to  $(y, z, k)$ , then the nonlinear pricing system  $\mathbf{X}^g$  is convex with respect to  $(\xi, D)$ , that is, for any  $\alpha \in [0, 1]$ ,  $S \in [0, T]$ ,  $\xi_1, \xi_2 \in L^2(\mathcal{G}_S)$ ,  $D^1, D^2 \in \mathcal{A}_S^2$ , for all  $t \in [0, S]$ , we have

$$X_{t,S}^g(\alpha \xi_1 + (1 - \alpha) \xi_2, \alpha D^1 + (1 - \alpha) D^2) \leq \alpha X_{t,S}^g(\xi_1, D^1) + (1 - \alpha) X_{t,S}^g(\xi_2, D^2) \quad \text{a.s.}$$

- **Nonnegativity.** Under Assumption 4, when  $g(t, 0, 0, 0) \geq 0$ , the nonlinear pricing system  $\mathbf{X}^g$  is nonnegative, that is, for each  $S \in [0, T]$ , for all non negative  $\xi \in L^2(\mathcal{G}_S)$  and all non decreasing processes  $D \in \mathcal{A}_S^2$ , we have  $X_{t,S}^g(\xi, D) \geq 0$  for all  $t \in [0, S]$  a.s.

By the strict comparison theorem (see Theorem 3 (ii)), we have the following additional property.

- **No arbitrage.** Under Assumption 4 with  $\gamma_\vartheta^{y, z, k_1, k_2} > -1$ , the nonlinear pricing system  $\mathbf{X}^g$  satisfies the *no arbitrage* property: for all maturity  $S \in [0, T]$ , for all payoffs  $\xi^1, \xi^2 \in L^2(\mathcal{G}_S)$ , and cumulative dividend processes  $D^1, D^2 \in \mathcal{A}_S^2$ , for each  $t_0 \in [0, S]$ , the following holds: If  $(\xi^1, D^1) \succ (\xi^2, D^2)$  and if the prices of the two options are equal at time  $t_0$ , that is,  $X_{t_0,S}^g(\xi_1, D^1) = X_{t_0,S}^g(\xi_2, D^2)$  a.s., then,  $\xi_1 = \xi_2$  a.s. and  $(D_t^1 - D_t^2)_{t_0 \leq t \leq S}$  is a.s. constant.<sup>13</sup>

*Remark 17.* In the perfect market model with default, the driver is given by (36). When  $\theta_t^2 \leq 1$ , then Assumption 4 is satisfied with  $\gamma_t^{y, z, k_1, k_2} = -\theta_t^2$ , which ensures in particular the **monotonicity** property of the pricing system. Note that when  $(\theta_t^2)$  is a constant  $\theta > 1$  and  $P(T \geq \vartheta) > 0$ , the pricing system is no longer monotone (see Example 1 with  $\delta_t = -r_t$ ,  $\beta_t = -\theta_t^1$  and  $\gamma = -\theta$ ). Moreover, when  $\theta_t^2 < 1$ , then the above **no arbitrage** property holds. This is no longer the case when, for example,  $\theta_t^2 = 1$  and  $P(T \geq \vartheta) > 0$  (see Example 1 with  $\delta_t = -r_t$ ,  $\beta_t = -\theta_t^1$  and  $\gamma = -1$ ).

<sup>12</sup> This property follows from Theorem 3 (i) applied to  $g^1 = g^2 = g$  and  $\xi^1, \xi^2, D^1, D^2$ . Indeed by Assumption 4, Assumption (25) holds with  $\gamma_t$  replaced by the predictable process  $\gamma_t^{y^2, z^2, k^1, k^2}$ .

<sup>13</sup> In other words, the intermediate dividends paid between  $t_0$  and  $S$  are equal a.s.

### 3.4 The $(g, D)$ -conditional expectation $\mathcal{E}^{g,D}$ and $\mathcal{E}^{g,D}$ -martingales

Let  $g$  be a  $\lambda$ -admissible driver and let  $D$  be an optional singular process belonging to  $\mathcal{A}_T^2$ . We define the  $(g, D)$ -conditional expectation for each  $S \in [0, T]$  and each  $\xi \in L^2(\mathcal{G}_S)$  by

$$\mathcal{E}_{t,S}^{g,D}(\xi) := X_{t,S}^g(\xi, D), \quad 0 \leq t \leq S.$$

In other terms,  $\mathcal{E}_{t,S}^{g,D}(\xi)$  is defined as the first coordinate of the solution of the BSDE associated with terminal time  $S$ , *generalized driver*  $g(t, y, z, k)dt + dD_t$  and terminal condition  $\xi$ .

In the case where  $D = 0$ , it reduces to the  $g$ -conditional expectation  $\mathcal{E}^g$  (in the case of default).

Note that  $\mathcal{E}_{t,S}^{g,D}(\xi)$  can be defined on the whole interval  $[0, T]$  by setting  $\mathcal{E}_{t,S}^{g,D}(\xi) := \mathcal{E}_{t,T}^{g^S, D^S}(\xi)$  for  $t \geq S$ , where  $g^S(t, \cdot) := g(t, \cdot)\mathbf{1}_{t \leq S}$  and  $D_t^S := D_{t \wedge S}$ .

We also define  $\mathcal{E}_{t,\tau}^{g,D}(\xi)$  for each stopping time  $\tau \in \mathcal{T}$  and each  $\xi \in L^2(\mathcal{G}_\tau)$  as the solution of the BSDE associated with terminal time  $T$ , driver  $g^\tau(t, \cdot) := g(t, \cdot)\mathbf{1}_{t \leq \tau}$  and singular process  $D_t^\tau := D_{t \wedge \tau}$ .

We now give some properties of the  $(g, D)$ -conditional expectation which generalize those given in [20] to the case of a default jump and generalized driver.

The  $(g, D)$ -conditional expectation  $\mathcal{E}^{g,D}$  is **consistent**. More precisely, let  $\tau'$  be a stopping time in  $\mathcal{T}_0$ ,  $\xi \in L^2(\mathcal{G}_{\tau'})$ , and let  $\tau$  be a stopping time smaller or equal to  $\tau'$ .

We then have  $\mathcal{E}_{t,\tau'}^{g,D}(\xi) = \mathcal{E}_{t,\tau}^{g,D}(\mathcal{E}_{\tau,\tau'}^{g,D}(\xi))$  for all  $t \in [0, T]$  a.s.

The  $(g, D)$ -conditional expectation  $\mathcal{E}^{g,D}$  satisfies the following property: for all  $\tau \in \mathcal{T}_0$ ,  $\xi \in L^2(\mathcal{G}_\tau)$ , and for all  $t \in [0, T]$  and  $A \in \mathcal{F}_t$ , we have:

$$\mathcal{E}_{t,\tau}^{g^A, D^A}(\mathbf{1}_A \xi) = \mathbf{1}_A \mathcal{E}_{t,\tau}^{g,D}(\xi) \text{ a.s., where } g^A(s, \cdot) = g(s, \cdot)\mathbf{1}_{A \cap [s, T]}(s) \text{ and } D_s^A := (D_s - D_t)\mathbf{1}_{s \geq t}.^{14}$$

Using the comparison theorem for BSDEs with default and generalized drivers (Theorem 3 (i)), we derive that, under Assumption 4, the  $(g, D)$ -conditional expectation  $\mathcal{E}^{g,D}$  is **monotone** with respect to  $\xi$ . If moreover  $g$  is convex with respect to  $(y, z, k)$ , then  $\mathcal{E}^{g,D}$  is convex with respect to  $\xi$ .

From the strict comparison theorem (see Theorem 3 (ii)), we derive that, under Assumption 4 with  $\gamma_\vartheta^{y,z,k_1,k_2} > -1$ ,  $\mathcal{E}^{g,D}$  satisfies the **no arbitrage** property. More precisely, for all  $S \in [0, T]$ ,  $\xi^1, \xi^2 \in L^2(\mathcal{G}_S)$ , and for all  $t_0 \in [0, S]$  and  $A \in \mathcal{G}_{t_0}$ , we have:

If  $\xi^1 \geq \xi^2$  a.s. and  $\mathcal{E}_{t_0,S}^{g,D}(\xi^1) = \mathcal{E}_{t_0,S}^{g,D}(\xi^2)$  a.s. on  $A$ , then  $\xi^1 = \xi^2$  a.s. on  $A$ .

The **no arbitrage** property also ensures that when  $\gamma_\vartheta^{y,z,k_1,k_2} > -1$ , the  $(g, D)$ -conditional expectation  $\mathcal{E}^{g,D}$  is **strictly monotone**.<sup>15</sup>

We now introduce the definition of an  $\mathcal{E}^{g,D}$ -martingale which generalizes the classical notion of  $\mathcal{E}^g$ -martingale.

**Definition 4.** Let  $Y \in \mathcal{S}^2$ . The process  $Y$  is said to be a  $\mathcal{E}^{g,D}$ -martingale if  $\mathcal{E}_{\sigma,\tau}^{g,D}(Y_\tau) = Y_\sigma$  a.s. on  $\sigma \leq \tau$ , for all  $\sigma, \tau \in \mathcal{T}_0$ .

**Proposition 4.** For all  $S \in [0, T]$ , payoff  $\xi \in L^2(\mathcal{G}_S)$  and dividend process  $D \in \mathcal{A}_S^2$ , the associated hedging price process  $\mathcal{E}_{\cdot,S}^{g,D}(\xi)$  is an  $\mathcal{E}^{g,D}$ -martingale.

Moreover, for all  $x \in \mathbb{R}$ , risky-assets strategy  $\varphi \in \mathbb{H}^2 \times \mathbb{H}_\lambda^2$  and cash withdrawal process  $D \in \mathcal{A}^2$ , the associated wealth process  $V^{x,\varphi,D}$  is an  $\mathcal{E}^{g,D}$ -martingale.

*Proof.* The first assertion follows from the consistency property of  $\mathcal{E}^{g,D}$ . The second one is obtained by noting that  $V^{x,\varphi,D}$  is the solution of the BSDE with *generalized driver*  $g(t, \cdot)dt + dD_t$ , terminal time  $T$  and terminal condition  $V_T^{x,\varphi,D}$ .

*Remark 18.* The above result is used in [12, Section 5.4] to study the nonlinear pricing of game options with intermediate dividends in an imperfect financial market with default.

<sup>14</sup> From this property, we derive the following **Zero-one law**: if  $g(\cdot, 0, 0, 0) = 0$ , then  $\mathcal{E}_{t,\tau}^{g^A, D^A}(\mathbf{1}_A \xi) = \mathbf{1}_A \mathcal{E}_{t,\tau}^{g,D}(\xi)$  a.s.

<sup>15</sup> In the case without default, it is well-known that, up to a minus sign, the  $g$ -conditional expectation  $\mathcal{E}^g$  can be seen as a dynamic risk measure (see e.g. [20, ?, 21]). In our framework, we can define a dynamic risk measure  $\rho^g$  by setting  $\rho^g := -\mathcal{E}^g (= -\mathcal{E}^{g,0})$ . This dynamic risk-measure thus satisfies similar properties to those satisfied by  $\mathcal{E}^g$ .

Some examples of market models with default and imperfections or constraints, leading to a nonlinear pricing are given in [13, 14, 8, 7, 19]. We now provide another example.

### 3.5 Example: Large seller who affects the default probability

We consider a European option with maturity  $T$ , terminal payoff  $\xi \in L^2(\mathcal{G}_T)$ , and dividend process  $D \in \mathcal{A}_T^2$ . We suppose that the seller of this option is a *large trader*. More precisely, her hedging strategy (as well as its associated cost) may affect the prices of the risky assets and the default probability. She takes into account these feedback effects in her market model in order to price the option. To the best of our knowledge, the possible impact on the default probability has not been considered in the literature before.

In order to simplify the presentation, we consider the case when the seller's strategy affects only the default intensity. We also suppose in this example that the default intensity is bounded.

We are given a family of probability measures parametrized by  $V$  and  $\varphi$ . More precisely, for all  $V \in \mathcal{S}^2$ ,  $\varphi \in \mathbb{H}^2 \times \mathbb{H}_\lambda^2$ , let  $Q^{V,\varphi}$  be the probability measure equivalent to  $P$ , which admits  $L^{V,\varphi}$  as density with respect to  $P$ , where  $L^{V,\varphi}$  is the solution of the following SDE:

$$dL_t^{V,\varphi} = L_t \gamma(t, V_t, \varphi_t) dM_t; \quad L_0^{V,\varphi} = 1.$$

Here,  $\gamma: (\omega, t, y, \varphi_1, \varphi_2) \mapsto \gamma(\omega, t, y, \varphi_1, \varphi_2)$  is a  $\mathcal{P} \otimes \mathcal{B}(\mathbf{R}^3)$ -measurable function defined on  $\Omega \times \mathbf{R}^+ \times \mathbf{R}^3$ , bounded, and such that the map  $y \mapsto \gamma(\omega, t, y, \varphi_1, \varphi_2)/\varphi_2$  is uniformly Lipschitz. We suppose that  $\gamma(t, \cdot) > -1$ . Note that by Proposition 3 and Remark 8, the process  $L^{V,\varphi}$  is positive and belongs to  $\mathcal{S}^2$ .

By Girsanov's theorem, the process  $W$  is a  $Q^{V,\varphi}$ -Brownian motion and the process  $M^{V,\varphi}$  defined as

$$M_t^{V,\varphi} := N_t - \int_0^t \lambda_s (1 + \gamma(s, V_s, \varphi_s)) ds = M_t - \int_0^t \lambda_s \gamma(s, V_s, \varphi_s) ds \quad (46)$$

is a  $Q^{V,\varphi}$ -martingale. Hence, under  $Q^{V,\varphi}$ , the  $\mathbb{G}$ -default intensity process is equal to  $\lambda_t (1 + \gamma(t, V_t, \varphi_t))$ . The process  $\gamma(t, V_t, \varphi_t)$  represents the *impact of the seller's strategy on the default intensity* in the case when  $\varphi$  is the seller's risky-assets strategy and  $V$  is the value of her portfolio.

The large seller considers the following pricing model. For a fixed pair "wealth/risky-assets strategy"  $(V, \varphi) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}_\lambda^2$ , the dynamics of the risky-assets under the probability  $Q^{V,\varphi}$  are given by

$$\begin{aligned} dS_t^1 &= S_t^1 [\mu_t^1 dt + \sigma_t^1 dW_t]; \\ dS_t^2 &= S_t^2 [\mu_t^2 dt + \sigma_t^2 dW_t - dM_t^{V,\varphi}]. \end{aligned}$$

The value process  $(V_t)$  of the portfolio associated with an initial wealth  $x$ , a risky-assets strategy  $\varphi$ , and with a cumulative withdrawal process, that the seller chooses to be equal to the dividend process  $D$  of the option, must satisfy the following dynamics:

$$dV_t = (r_t V_t + \varphi_t' \sigma_t \theta_t^1 - \varphi_t^2 \theta_t^2 \lambda_t) dt - dD_t + \varphi_t' \sigma_t dW_t - \varphi_t^2 dM_t^{V,\varphi}. \quad (47)$$

Note that the dynamics of the wealth (47) can be written

$$dV_t = (r_t V_t + \varphi_t' \sigma_t \theta_t^1 - \varphi_t^2 \theta_t^2 \lambda_t + \gamma(t, V_t, \varphi_t) \lambda_t \varphi_t^2) dt - dD_t + \varphi_t' \sigma_t dW_t - \varphi_t^2 dM_t. \quad (48)$$

Let us suppose that the large seller has an initial wealth equal to  $x$  and follows a risky-assets strategy  $\varphi$ . By the assumptions made on  $\gamma$ , there exists a unique process  $V^{x,\varphi}$  satisfying (48) with initial condition  $V_0^{x,\varphi} = x$ . This model is thus well posed.

Moreover, it can be seen as a particular case of the general model described in Section 3.3. Indeed, setting  $Z_t = \varphi_t' \sigma_t$  and  $K_t = -\varphi_t^2$ , the dynamics (48) can be written

$$-dV_t = g(t, V_t, Z_t, K_t)dt + dD_t - Z_t dW_t - K_t dM_t, \quad (49)$$

where

$$g(t, y, z, k) = -r_t y - \theta_t^1 z - \theta_t^2 \lambda_t k + \gamma(t, y, (\sigma_t^1)^{-1}(z + \sigma_t^2 k), -k) \lambda_t k.$$

Assuming that there exists a positive constant  $C$  such that inequality (3) holds,  $g$  is  $\lambda$ -admissible. We are thus led to the model from Section 3.3 associated with this nonlinear driver  $g$ . Thus, by choosing this pricing model, the seller prices the option at time  $t$ , where  $t \in [0, T]$ , at the price  $X_{t,T}^g(\xi, D)$ . In other terms, the seller's price process<sup>16</sup> will be equal to  $X$ , where  $(X, Z, K)$  is the solution of the BSDE:

$$-dX_t = g(t, X_t, Z_t, K_t)dt + dD_t - Z_t dW_t - K_t dM_t; \quad X_T = \xi.$$

Moreover, her hedging risky-assets strategy  $\varphi$  will be such that  $Z_t = \varphi_t' \sigma_t$  and  $K_t = -\varphi_t^2$ , that is, equal to  $\Phi(Z, K)$ , where  $\Phi$  is given by (37).

This model can be easily generalized to the case when the coefficients  $\mu^1, \sigma^1, \mu^2, \sigma^2$  also depend on the hedging cost  $V$  (equal to the seller's price  $X$  of the option) and on the hedging strategy  $\varphi^2$ .<sup>17</sup>

## 4 Concluding remarks

In this paper, we have established properties of BSDEs with default jump and *generalized driver* which involves a finite variational process  $D$ . We treat the case when  $D$  is not necessarily predictable and may admit a jump at the default time. This allows us to study nonlinear pricing of European options generating intermediate dividends (with in particular a cashflow at the default time) in complete imperfect markets with default. Due to the default jump, we need an appropriate assumption on the driver  $g$  to ensure that the associated nonlinear pricing system  $\mathbf{X}^g : (T, \xi, D) \mapsto \mathcal{E}_{\cdot, T}^{g, D}(\xi)$  is monotonous, and a stronger condition to ensure that it satisfies the so-called **no-arbitrage** property. Some complements concerning the nonlinear pricing of European options are given in [13] (cf. Section 4 and Section 5.1). The nonlinear pricing of *American* options (resp. *game* options) in *complete* imperfect markets with default are addressed in [13] (resp. [12]). The case of American options in *incomplete* imperfect financial markets with default is studied in [14].

## Appendix

### BSDEs with default jump in $L^p$ , for $p \geq 2$

For  $p \geq 2$ , let  $\mathcal{S}^p$  be the set of  $\mathbb{G}$ -adapted RCLL processes  $\varphi$  such that  $\mathbb{E}[\sup_{0 \leq t \leq T} |\varphi_t|^p] < +\infty$ ,

$\mathbb{H}^p$  the set of  $\mathbb{G}$ -predictable processes such that  $\|Z\|_p^p := \mathbb{E}\left[\left(\int_0^T |Z_t|^2 dt\right)^{p/2}\right] < \infty$ ,

$\mathbb{H}_\lambda^p$  the set of  $\mathbb{G}$ -predictable processes such that  $\|U\|_{p, \lambda}^p := \mathbb{E}\left[\left(\int_0^T |U_t|^2 \lambda_t dt\right)^{p/2}\right] < \infty$ .

**Proposition 5.** *Let  $p \geq 2$  and  $T > 0$ . Let  $g$  be a  $\lambda$ -admissible driver such that  $g(t, 0, 0, 0) \in \mathbb{H}^p$ . Let  $\xi \in L^p(\mathcal{G}_T)$ . There exists a unique solution  $(Y, Z, K)$  in  $\mathcal{S}^p \times \mathbb{H}^p \times \mathbb{H}_\lambda^p$  of the BSDE with default (4).*

*Remark 19.* The above result still holds in the case when there is a  $\mathbb{G}$ -martingale representation theorem with respect to  $W$  and  $M$ , even if  $\mathbb{G}$  is not generated by  $W$  and  $M$ .

*Proof.* The proof relies on the same arguments as in the proof of Proposition A.2 in [21] together with the arguments used in the proof of Proposition 2.

<sup>16</sup> Note that the seller's price is not necessarily equal to the market price of the option.

<sup>17</sup> The coefficients may also depend on  $\varphi = (\varphi^1, \varphi^2)$ , but in this case, we have to assume that the map  $\Psi : (\omega, t, y, \varphi) \mapsto (z, k)$  with  $z = \varphi' \sigma_t(\omega, t, y, \varphi)$  and  $k = -\varphi^2$  is one to one with respect to  $\varphi$ , and such that its inverse  $\Psi_\varphi^{-1}$  is  $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^3)$ -measurable.

### BSDEs with default jump and change of probability measure

Let  $(\beta_s)$  and  $(\gamma_s)$  be two real-valued  $\mathbb{G}$ -predictable processes such that  $(\beta_s)$  and  $(\gamma_s \sqrt{\lambda_s})$  are bounded. Let  $(\zeta_s)$  be the process satisfying the forward SDE:

$$d\zeta_s = \zeta_{s-} (\beta_s dW_s + \gamma_s dM_s),$$

with  $\zeta_0 = 1$ . By Remark 10, we have  $\mathbb{E}[\sup_{0 \leq s \leq T} \zeta_s^p] < +\infty$  for all  $p \geq 2$ . We suppose that  $\gamma_\vartheta > -1$  a.s., which, by Remark 8, implies that  $\zeta_s > 0$  for all  $s \geq 0$  a.s. Let  $Q$  be the probability measure equivalent to  $P$  which admits  $\zeta_T$  as density with respect to  $P$  on  $\mathcal{G}_T$ .

By Girsanov's theorem (see [16] Chapter 9.4 Corollary 4.5), the process  $W_t^\beta := W_t - \int_0^t \beta_s ds$  is a  $Q$ -Brownian motion and the process  $M^\gamma$  defined as

$$M_t^\gamma := M_t - \int_0^t \lambda_s \gamma_s ds = N_t - \int_0^t \lambda_s (1 + \gamma_s) ds \quad (50)$$

is a  $Q$ -martingale. We now state a representation theorem for  $(Q, \mathbb{G})$ -local martingales with respect to  $W^\beta$  and  $M^\gamma$ .

**Proposition 6.** *Let  $m = (m_t)_{0 \leq t \leq T}$  be a  $(Q, \mathbb{G})$ -local martingale. There exists a unique pair of predictable processes  $(z_t, k_t)$  such that*

$$m_t = m_0 + \int_0^t z_s dW_s^\beta + \int_0^t k_s dM_s^\gamma \quad 0 \leq s \leq T \quad \text{a.s.} \quad (51)$$

*Proof.* Since  $m$  is a  $Q$ -local martingale, the process  $\bar{m}_t := \zeta_t m_t$  is a  $P$ -local martingale. By the martingale representation theorem (Lemma 1), there exists a unique pair of predictable processes  $(Z, K)$  such that

$$\bar{m}_t = \bar{m}_0 + \int_0^t Z_s dW_s + \int_0^t K_s dM_s \quad 0 \leq t \leq T \quad \text{a.s.}$$

Then, by applying Itô's formula to  $m_t = \bar{m}_t (\zeta_t)^{-1}$  and by classical computations, one can derive the existence of  $(z, k)$  satisfying (51).

From this result together with Proposition 5 and Remark 19, we derive the following corollary.

**Corollary 1.** *Let  $p \geq 2$  and let  $T > 0$ . Let  $g$  be a  $\lambda$ -admissible driver such that  $g(t, 0, 0, 0) \in \mathbb{H}_Q^p$ . Let  $\xi \in L_Q^p(\mathcal{G}_T)$ . There exists a unique solution  $(Y, Z, K)$  in  $\mathcal{S}_Q^p \times \mathbb{H}_Q^p \times \mathbb{H}_{Q, \lambda}^p$  of the BSDE with default:*

$$-dY_t = g(t, Y_t, Z_t, K_t) dt - Z_t W_t^\beta - K_t dM_t^\gamma; \quad Y_T = \xi.$$

Here the spaces  $\mathcal{S}_Q^p$ ,  $\mathbb{H}_Q^p$ , and  $\mathbb{H}_{Q, \lambda}^p$  are defined as  $\mathcal{S}^p$ ,  $\mathbb{H}^p$ , and  $\mathbb{H}_\lambda^p$ , by replacing the probability  $P$  by  $Q$ .

*Remark 20.* Note that the results given in the Appendix are used in [12] (Section 4.3) to study the nonlinear pricing problem of game options in an imperfect market with default and model uncertainty.

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