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The Rank Pricing Problem: models and branch-and-cut algorithms

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Abstract

One of the main concerns in management and economic planning is to sell the right product to the right customer for the right price. Companies in retail and manufacturing employ pricing strategies to maximize their revenues. The Rank Pricing Problem aims at maximizing the revenue of a company by setting the prices of their products, taking into account the customers' preferences. In this paper, we present a bilevel and a non linear single level formulation of the problem, carry out two different linearisations of the models and develop some preprocessing techniques. We also study the polyhedral structure of the model, taking advantage of the fact that a subset of its constraints constitutes a special case of the Set Packing Problem, and characterize all the clique inequalities. Besides, we obtain two families of valid inequalities which, embedded in the formulation by implementing a branch-and-cut algorithm, allow us to tighten the upper bound given by the linear relaxation of the models. Finally, we show the efficiency of the formulations, the preprocessing and the branch-and-cut algorithm through an extensive computational study.

Keywords: Pricing Problems, Bilevel Programming, Rank Pricing Problem, Set Packing

1 Introduction

Over the last few decades, the broad development of information and communication technologies has led to extensive changes in society. The impact of the new technologies and globalization has originated different consumer trends and strong competition in the market. This has forced companies to adapt their marketing strategies in order to fit customer's preferences. In this context, many retail and manufacturing companies have opted for the use of pricing strategies designed to increase their revenue, also motivated by the rising amount of data available from surveys and web pages.

Pricing optimization problems aim at determining the prices of a series of products in order to maximize the revenue of a company. Setting a low price can lead to a loss of income if clients were willing to pay a higher price, but it can also make a product

available to a greater amount of customers; on the contrary, a higher price can generate greater revenue, but clients may not purchase it if it is too high. Therefore, it becomes obvious that a pricing problem is a bilevel program, in other words, has a hierarchical structure with a first optimization problem given by the company, which aims at maximizing its profit, and part of the constraints that force the solution to be optimal to another optimization problem, which is minimizing the customers' purchasing cost.

Bilevel Programming (BP) problems were introduced in 1973 by Bracken and McGill in [2] as mathematical programs with optimization problems in constraints, and their hierarchical structure appears in a wide variety of problems related to management, economic planning and engineering, among others. According to [9], price setting problems stated using bilevel programming fit multiple applications in sectors such as networks (an overview can be found in [15]), the trucking industry and in the context of air traffic management, to cite but a few. An equivalent problem in Game Theory is known as the Stackelberg game ([14]). In this two-player game, one of the players (known as the leader) plays first and decides his best strategy taking into account that the second player (or follower) will react to his movement optimally, knowing the leader's choice.

Customers whose purchasing decision is modelled by means of a budget and who intend to buy at most one product amongst the several they are interested in are called unit-demand customers. These customers will be able to purchase a product provided that its price is at most their budget. Unit-demand customers as stated here were introduced by Guruswami et al. [8], where they show that the single item pricing problem with unlimited supply is not only NP-complete, but APX-hard, and they focus on approximation algorithms. Regarding pricing problems with unlimited supply and unit-demand customers, Rusmevichientong et al. are the first to propose in [12] maximum and minimum utility objectives, as well as a rank-buying objective. In the first two cases, the customer purchases the item which maximizes or minimizes the difference between his budget and the price of the product, whereas in the rank-buying objective the client buys the product that ranks highest and he can afford. In this case, the authors represent a customer by its budget and an ordered list of recommended products. They show that these problems are NP-complete in the strong sense and introduce new constraints on the ordering of prices (price-ladder arises naturally, for example, when selling two different versions of a product) and a heuristic approximation algorithm together with performance bounds. Posterior works include MIP formulations for the maximum utility model with reservation prices given by Shioda et al. [13] along with approximation algorithms and complexity analysis for variations of these models: Aggarwal et al. [1] take up the work presented by Rusmevichientong et al. and present a polynomial time algorithm and an approximation algorithm for the rank-buying with price-ladder problem and related models; whereas Briest and Krysta [3] obtain a tight lower bound showing that the min-buying and the general rank-buying models are not approximable within $o(\log^\epsilon |C|)$ for some $\epsilon > 0$.

In this paper, we present a bilevel non linear formulation for the Rank Pricing Problem with unlimited supply and a single level formulation. We develop some preprocessing techniques and linearise the formulations by means of two types of auxiliary variables. We also take advantage of the fact that some of its constraints constitute a special case of the Set Packing Problem and other properties of the problem to derive new valid inequalities in order to strengthen the formulation. These inequalities are separated and included into the models through the development of a branch-and-cut algorithm. Finally, we present

the results of our computational analysis, in which we compare the formulations and show that our previous work reduces the computational effort when obtaining optimal solutions.

The remainder of the paper is organized as follows. Section 2 is devoted to a bilevel formulation for the Rank Pricing Problem (RPP); in Section 3, the RPP is formulated straightforwardly as a single level non linear model; Section 4 includes some preprocessing results, and Section 5 some families of clique facets are studied attending to the formulation; Sections 6 and 7 include two linearisations of the single level model and the development of other valid inequalities to strengthen its linear relaxation through the implementation of a branch-and-cut algorithm; Section 8 is devoted to testing the performance of the models by means of a computational study; and Section 9 constitutes a conclusion of the paper.

2 Bilevel formulation

Let $K = \{1, \dots, |K|\}$ denote the set of customers and $I = \{1, \dots, |I|\}$ the set of products. Each customer $k \in K$ is represented by a positive budget, a subset of products $S^k \subseteq I$ he is interested in and a preference for each product of this subset s_i^k , $i \in S^k$, where $s_i^k > s_j^k$ if customer k prefers product i over product j . We will also assume that for each customer all preferences are strict, so that he never likes two products the same. As budgets can be equal for different customers, let $B = \{b^1, \dots, b^M\}$, $M \leq |K|$, denote the set of different budgets, where $b^1 < b^2 < \dots < b^M$. To describe the budget of customer k , we define a function $\sigma : K \rightarrow \{1, \dots, M\}$ such that $\sigma(k) = \ell$ if the budget of customer k is b^ℓ . We will say that a customer k is *richer* than customers k_1, \dots, k_m if $\sigma(k) > \sigma(k_r)$ $\forall r \in \{1, \dots, m\}$, and the *richest* customers will be those whose budget is b^M . Without loss of generality, we will suppose that for any product $i \in I$ there exists $k \in K$ such that $i \in S^k$, and also $S^k \neq \emptyset \forall k \in K$. Since it will be useful in following sections, we will set $b^0 = 0$.

Thus, the Rank Pricing problem aims at establishing the prices of a set of products sold by a company so as to maximise its revenue, which will be the sum of the prices of all items sold, taking into account that customers have a list of preferences including a subset of the offered and they will purchase their most preferred one among the ones they can afford (if any).

Let us now introduce continuous variables p_i , $i \in I$ representing the leader variables, i.e., the prices of the products; and binary variables x_i^k , $k \in K$, $i \in S^k$, for every customer's purchase decision, that is, $x_i^k = 1$ if and only if customer k buys product i . The formulation of the bilevel problem is:

$$\text{(BNL}^p) \quad \max_{p,x} \quad \sum_{k \in K} \sum_{i \in S^k} p_i x_i^k \quad (1a)$$

$$\text{s.t.} \quad p_i \geq 0, \quad \forall i \in I, \quad (1b)$$

where $\forall k \in K$, x^k is an optimal solution of

$$\max_{x^k} \sum_{i \in S^k} s_i^k x_i^k \quad (1c)$$

$$\text{s.t.} \quad \sum_{i \in S^k} x_i^k \leq 1 \quad (1d)$$

$$\sum_{i \in S^k} p_i x_i^k \leq b^{\sigma(k)} \quad (1e)$$

$$x_i^k \in \{0, 1\}, \quad \forall i \in S^k, \quad (1f)$$

where constraint (1d) forces customer k to buy one product or none and (1e) establishes that customer k will only buy a product if he can afford it. (BNL P) is a bilevel problem with multiple independent followers. Note that the preference of a customer for each product, s_i^k , $k \in K$, $i \in S^k$, can be fixed in many different but equivalent ways giving rise to different formulations.

For the sake of completeness, we shall prove a property already observed by Rusmevichientong et al. in [12]:

Proposition 2.1. *An optimal solution of (BNL P) exists such that $p_i \in B \forall i \in I$.*

Proof. Let us first assume that there exists an optimal solution (\hat{p}, \hat{x}) of (BNL P) and a product i_0 such that $\hat{p}_{i_0} \neq b^\ell$, for $\ell \in \{1, \dots, M\}$. Our aim is to build another optimal solution (\tilde{p}, \tilde{x}) in which $\tilde{p}_{i_0} = b^\ell$ for some $\ell \in \{1, \dots, M\}$.

Let us first assume product i_0 is bought by one or more customers. In this case, it is clear that $\hat{p}_{i_0} \leq b^M$, so let us suppose $b^\ell < p_i < b^{\ell+1}$ for $\ell \in \{1, \dots, M-1\}$. If customer k is buying product i_0 , then $\sigma(k) \geq \ell + 1$, and therefore we can build a feasible solution (\tilde{p}, \tilde{v}) such that $\tilde{p}_i = \hat{p}_i \forall i \neq i_0$, $\tilde{p}_{i_0} = b^{\ell+1}$ and $\tilde{x}_i^k = \hat{x}_i^k \forall k \in K, i \in S^k$, whose objective value is $\sum_{k=1}^m \sum_{i \in S^k} \tilde{p}_i \tilde{x}_i^k > \sum_{k=1}^m \sum_{i \in S^k} \hat{p}_i \hat{x}_i^k$. Thus, this is a contradiction with the fact that (\hat{p}, \hat{x}) is optimal.

Let us now assume product i_0 is not sold in an optimal solution (\hat{p}, \hat{x}) and $b^\ell < \hat{p}_{i_0} < b^{\ell+1}$ for $\ell \in \{1, \dots, M-1\}$. Let \tilde{p} be another vector of prices such that $\tilde{p}_i = \hat{p}_i \forall i \neq i_0$, $\tilde{p}_{i_0} = b^M$. Customers $k : \sigma(k) \leq \ell$ can afford the same products than in the previous solution, so they will purchase the same items if we settle prices this way; and customers $k : \sigma(k) \geq \ell + 1$ could afford product i_0 in solution (\hat{p}, \hat{x}) , so they were already buying another product more preferable to them than i_0 . All in all, we can conclude that (\tilde{p}, \hat{x}) is also optimal and satisfies $\tilde{p}_{i_0} = b^M$.

Finally, let us suppose product i_0 is not sold in an optimal solution (\hat{p}, \hat{x}) and $\hat{p}_{i_0} > b^M$. Like in the former case, let \tilde{p} be another vector of prices such that $\tilde{p}_i = \hat{p}_i \forall i \neq i_0$, $\tilde{p}_{i_0} = b^M$. In this case, customers $k : \sigma(k) < M$ can afford the same products than in the previous solution, so they will purchase the same items, i.e. $\tilde{x}_i^k = \hat{x}_i^k \forall k : \sigma(k) < M, i \in S^k$; however, customers $k : \sigma(k) = M$ are now able to afford product i_0 , therefore $\tilde{x}_i^k = \hat{x}_i^k \forall k : \sigma(k) = M$ and $\sum_{i \in S^k} s_i^k \hat{x}_i^k > s_{i_0}^k, \forall i \in S^k$; and $\tilde{x}_{i_0}^k = 1, \tilde{x}_i^k = 0 \forall k : \sigma(k) = M$ and $\sum_{i \in S^k} s_i^k \hat{x}_i^k < s_{i_0}^k, \forall i \in S^k \setminus \{i_0\}$.

We have proved that, given an optimal solution (\hat{p}, \hat{v}) and a product $i_0 \in I$ we can

obtain another optimal solution (\tilde{p}, \tilde{v}) in which $\tilde{p}_{i_0} = b^\ell$ for $\ell \in \{1, \dots, M\}$. The result follows by induction on i . \square

From the proof of Proposition 2.1 we can conclude that, if we have an optimal solution (\hat{p}, \hat{x}) and a product i_0 which is not sold, we can obtain a different optimal solution (\tilde{p}, \hat{x}) for every price vector \tilde{p} such that $\tilde{p}_i = \hat{p}_i \forall i \neq i_0$, $\tilde{p}_{i_0} > \hat{p}_{i_0}$, i.e., setting a higher price for i_0 results in another optimal solution in which products are assigned to customers in the same way as previously.

Proposition 2.1 allows us to define new binary variables v_i^ℓ , $i \in I$, $\ell \in \{1, \dots, M\}$ representing the prices of products, that is, $v_i^\ell = 1$ if and only if product i has a price of b^ℓ . By definition of v -variables, the price of product i can be expressed as $p_i = \sum_{\ell=1}^M b^\ell v_i^\ell$. We can now reformulate the problem replacing p_i variables by v_i^ℓ variables, adding the following constraints in order to ensure products have at most one price:

$$\sum_{\ell=1}^M v_i^\ell \leq 1, \quad \forall i \in I$$

and replacing constraints (1e) of the lower level by:

$$x_i^k \leq \sum_{\ell=1}^{\sigma(k)} v_i^\ell, \quad \forall k \in K, i \in S^k. \quad (2)$$

Establishing strictly different values for the preferences s_i^k guarantees that the lower level problem has a unique optimal solution for each customer. Besides, the fact that the matrix corresponding to the feasible set of each lower level problem is totally unimodular enables us to relax the integrality constraints (1f), according to [16, Propositions 3.2 and 3.3]. The lower level problems can be further simplified taking into account that once the leader variables v_i^ℓ are known, a subset of variables is determined. If we consider the subset $I(k) = \{i \in S^k : \sum_{\ell=1}^{\sigma(k)} v_i^\ell = 1\}$, variables $\{x_i^k, k \in K, i \in S^k \setminus I(k)\}$ are automatically settled to 0. Hence, constraints (2) can be eliminated and the lower level problem can be formulated as:

$$\begin{aligned} \max_{x^k} \quad & \sum_{i \in I(k)} s_i^k x_i^k \\ \text{s.t.} \quad & \sum_{i \in I(k)} x_i^k \leq 1 \\ & x_i^k \geq 0, \quad i \in I(k). \end{aligned}$$

For each customer k , the dual problem of the lower level problem is:

$$\begin{aligned}
\min_{u^k} \quad & u^k \\
\text{s.t.} \quad & u^k \geq s_i^k, \quad i \in I(k) \\
& u^k \geq 0.
\end{aligned}$$

By strong duality of linear problems, we can obtain a single level formulation from the bilevel one by adding the constraints imposing optimality on both x_i^k and u^k , as explained by Dempe in [4, Chapter 3]. Thus, the resultant formulation after substitution of u^k is:

$$(\text{BNL}^v) \quad \max_{v,x} \quad \sum_{k \in K} \sum_{i \in S^k} \left(\sum_{\ell=1}^M b^\ell v_i^\ell \right) x_i^k \quad (3a)$$

$$\text{s.t.} \quad \sum_{i \in S^k} x_i^k \leq 1, \quad \forall k \in K \quad (3b)$$

$$\sum_{\ell=1}^M v_i^\ell \leq 1, \quad \forall i \in I \quad (3c)$$

$$\sum_{j \in S^k} s_j^k x_j^k \geq s_i^k \sum_{\ell=1}^{\sigma(k)} v_i^\ell, \quad \forall k \in K, i \in S^k \quad (3d)$$

$$x_i^k \leq \sum_{\ell=1}^{\sigma(k)} v_i^\ell, \quad \forall k \in K, i \in S^k \quad (3e)$$

$$v_i^\ell, x_i^k \in \{0, 1\}, \quad \forall k \in K, i \in S^k, \ell \in \{1, \dots, M\}. \quad (3f)$$

3 Single level formulations

In this section, we formulate the problem straightforwardly as a single level optimization problem. First of all, we introduce some definitions based on the ones introduced by Cánovas et al. in [5] for the plant location problem with order.

Definition 3.1. Let $k \in K$ be a customer and $i, j \in S^k$ two products. It is said that i is k -better than j if customer k prefers product i over product j , and it is denoted $i >_k j$. The set of products k -better than i is denoted by $B(k, i) = \{j \in S^k : j >_k i\}$.

Definition 3.2. Let $k \in K$ be a customer and $i, j \in S^k$ two products. It is said that i is k -worse than j if customer k prefers product j over product i , and it is denoted $i <_k j$. The set of products k -worse than i is denoted by $\overline{B}(k, i) = \{j \in S^k : j <_k i\}$.

Since preferences are strict, for any given products $i, j \in S^k$, it follows $i >_k j$ or $i <_k j$. It is also worth noticing that a customer k buys product i if and only if $i \in S^k$, its price is below the customer budget and all the products more preferred than i have a price higher than his budget. In terms of the binary variables x_i^k, v_i^ℓ previously defined:

$$x_i^k = 1 \Leftrightarrow \sum_{\ell=1}^{\sigma(k)} v_i^\ell = 1 \text{ and } \sum_{\ell=1}^{\sigma(k)} v_j^\ell = 0 \forall j \in B(k, i).$$

Using this notation and decision variables x_i^k, v_i^ℓ , a single level non linear formulation is:

$$(SLNL) \quad \max_{v, x} \sum_{k \in K} \sum_{i \in S^k} \left(\sum_{\ell=1}^M b^\ell v_i^\ell \right) x_i^k \quad (4a)$$

$$\text{s.t.} \quad \sum_{i \in S^k} x_i^k \leq 1, \quad \forall k \in K \quad (4b)$$

$$\sum_{\ell=1}^M v_i^\ell \leq 1, \quad \forall i \in I \quad (4c)$$

$$x_i^k + \sum_{\ell=1}^{\sigma(k)} v_j^\ell \leq 1, \quad \forall k \in K, i \in S^k, j \in B(k, i) \quad (4d)$$

$$x_i^k + \sum_{\ell=\sigma(k)+1}^M v_i^\ell \leq 1, \quad \forall k \in K, i \in S^k \quad (4e)$$

$$v_i^\ell, x_i^k \in \{0, 1\}, \quad \forall k \in K, i \in S^k, \ell \in \{1, \dots, M\}, \quad (4f)$$

where constraints (3e) have been replaced by (4e) using constraints (4c). Constraints (4d), also called *preference constraints*, are given by the previous reasoning and can be strengthened by means of the following results:

Proposition 3.3. *The following constraints:*

$$\sum_{i \in \overline{B(k, j)}} x_i^k + \sum_{\ell=1}^{\sigma(k)} v_j^\ell \leq 1, \quad \forall k \in K, j \in S^k : \overline{B(k, j)} \neq \emptyset, \quad (5)$$

are valid for (SLNL) and dominate constraints (4d).

Proof. First of all, we shall prove the validity of (5). We have $\sum_{i \in \overline{B(k, j)}} x_i^k \leq \sum_{i \in I} x_i^k \leq 1$ using (4b) and $\sum_{\ell=1}^{\sigma(k)} v_j^\ell \leq \sum_{\ell=1}^M v_j^\ell \leq 1$ because of (4c). Furthermore, provided that product j is within k 's budget, i.e., if $\sum_{\ell=1}^{\sigma(k)} v_j^\ell = 1$, then customer k will not buy any product less preferable for him than j , so $\sum_{i \in \overline{B(k, j)}} x_i^k = 0$, so (5) are valid.

Furthermore, we have:

$$x_i^k + \sum_{\ell=1}^{\sigma(k)} v_j^\ell \leq \sum_{i \in \overline{B(k, j)}} x_i^k + \sum_{\ell=1}^{\sigma(k)} v_j^\ell \leq 1.$$

Therefore, we have proved that (5) are stronger than (4d). \square

4 Preprocessing

In this section, our aim is to fix x - and v -variables to zero in order to reduce the size of the RPP instances before solving it.

Let us begin by recursively defining a function $u : K \rightarrow I$ as follows:

1. If $\sigma(k) = M$, then $u(k) = i$ if and only if $i \in S^k$ and $B(k, i) = \emptyset$.
2. If $\sigma(k) \neq M$ and $\exists i \in S^k$ such that $\forall k' : \sigma(k') > \sigma(k)$, $u(k') \neq i$, then $u(k) = i$ if and only if $i \in S^k$, $\nexists k'$ with $\sigma(k') > \sigma(k)$ such that $u(k') = i$ and $\forall j \in B(k, i) \cap S^k$, $\exists k', \sigma(k') > \sigma(k)$, such that $u(k') = j$.
3. If $\sigma(k) \neq M$ and $\forall i \in S^k$, $\exists k'$ with $\sigma(k') > \sigma(k)$ and $u(k') = i$, then $u(k) = i$ if and only if $i \in S^k$ and $\overline{B(k, i)} = \emptyset$.

Function u assigns, to the richest customers, their most preferred product; and to the rest of the customers, their most preferred product among the ones which have not been previously assigned to any richer customer (or their least preferred one if all of them have already been assigned).

Based on the definition of u , we are going to establish a partition of the set of customers. Thus, let C_r , $r \in \{1, 2, 3\}$, be such that $k \in C_r \Leftrightarrow u(k)$ has been defined for k making use of item r of the definition of u . It is clear that $\cup_{r \in \{1, 2, 3\}} C_r = K$, but given this definition it is possible that both C_2 and C_3 are empty or C_3 is. If $C_2 = C_3 = \emptyset$, then $\sigma(k) = M \forall k \in K$, and the problem becomes trivial: it suffices to establish $v_i^M = 1 \forall i \in I$, every customer will purchase his most preferred item and the objective value will be the sum of every client's budget, i.e., $b^M |K|$. If $C_1 \neq \emptyset \neq C_2$ and $C_3 = \emptyset$, then we will see in Corollary 4.7 that an optimal solution can be found by inspection.

The following result shows the usefulness of this function when fixing x -variables to zero:

Proposition 4.1. *There exist optimal solutions (\tilde{v}, \tilde{x}) of (BNL^v) and (SLNL) such that $\tilde{x}_i^k = 0 \forall k \in K, \forall i \in \overline{B(k, u(k))}$.*

Proof. Suppose we have an optimal solution (\hat{v}, \hat{x}) which does not satisfy the statement conditions. By slightly modifying (\hat{v}, \hat{x}) , we aim at building another solution (\tilde{v}, \tilde{x}) , with the same objective value, which does satisfy them.

Let us proceed by induction on k . First consider k_0 such that $\sigma(k_0) = M$. Then, by Proposition 2.1 we know k_0 is able to afford every product he is interested in, and therefore in every optimal solution he will purchase his most preferred product. Therefore, $\hat{x}_i^{k_0} = 0$ must hold for all k_0 such that $\sigma(k_0) = M$ and $i \in \overline{B(k_0, u(k_0))}$.

Since (\hat{v}, \hat{x}) does not satisfy the statement conditions, there will exist $k_0 \in K$ such that $\sigma(k_0) = \ell_0 < M$ and $\forall k$ such that $\sigma(k) > \ell_0$ $\hat{x}_i^k = 0 \forall i \in \overline{B(k, u(k))}$ but $\hat{x}_{i_0}^{k_0} = 1$ for a product $i_0 \in \overline{B(k_0, u(k_0))}$. It is clear that $k_0 \in C_2$. The fact that k_0 buys product i_0 implies he can not afford product $u(k_0)$, i.e., $\sum_{\ell=1}^{\ell_0} \hat{v}_{u(k_0)}^\ell = 0$ and $\hat{x}_{u(k_0)}^{k_0} = 0$. We are going to show that $\hat{x}_{u(k_0)}^k = 0 \forall k$, that is to say, that product $u(k_0)$ has not been sold

in the considered optimal solution. On the one hand, it is clear that $\hat{x}_{u(k_0)}^k = 0$ for all k such that $\sigma(k) \leq \sigma(k_0)$ because these customers cannot afford it either. On the other hand, let us prove that for all k such that $\sigma(k) > \sigma(k_0)$, it holds $u(k_0) \in \overline{B(k, u(k))}$ or $u(k_0) \notin S^k$. First of all, we know $u(k_0) \neq u(k) \forall k : \sigma(k) > \sigma(k_0)$ because $k_0 \in C_2$. Besides, let us suppose $u(k_0) \in B(k_1, u(k_1))$ for $k_1 : \sigma(k_1) > \sigma(k_0)$. If $\sigma(k_1) = M$, we have $B(k_1, u(k_1)) = \emptyset$, hence $M > \sigma(k_1) > \sigma(k_0)$ and $k_1 \in C_2 \cup C_3$. But then, by definition of u , $u(k_0) \in B(k_1, u(k_1)) \Rightarrow$ there exists $k_2 : \sigma(k_2) > \sigma(k_1)$ and $u(k_2) = u(k_0)$, which is a contradiction with $k_0 \in C_2$. Therefore, we have proved that customers with budget greater than k_0 do not purchase product $u(k_0)$ because they buy others that prefer more, and customers k such that $\sigma(k) \leq \sigma(k_0)$ cannot afford product $u(k_0)$. Hence, $u(k_0)$ is not sold in this optimal solution.

Let us consider now a price vector \tilde{v} defined by $\tilde{v}_i^\ell = \hat{v}_i^\ell \forall \ell, \forall i \neq u(k_0)$ and $\tilde{v}_{u(k_0)}^{\ell_0} = 1$, $\tilde{v}_{u(k_0)}^\ell = 0 \forall \ell \neq \ell_0$. If prices are settled this way, customers k such that $\sigma(k) < \ell_0$ can afford the same products as before, so they purchase the same item; customers k with $\sigma(k) = \ell_0$ are now able to afford product $u(k_0)$, but if they purchase it (because they prefer it over the one they were buying in the previous solution) they spend their whole budget, therefore the revenue does not decrease; and customers k with $\sigma(k) > \ell_0$ were already buying a product more preferable than $u(k_0)$ in the previous solution, so they buy the same as previously. Thus, $\tilde{x}_i^k = \hat{x}_i^k \forall k : \sigma(k) \neq \ell_0, \forall i \in S^k$; $\tilde{x}_i^k = \hat{x}_i^k \forall k : \sigma(k) = \ell_0$ and $u_{k_0} \in \overline{B(k, j)}$ for $j : \hat{x}_j^k = 1, \forall i \in S^k$; and $\tilde{x}_{u(k_0)}^k = 1, \tilde{x}_i^k = 0 \forall k : \sigma(k) = \ell_0$ and $u(k_0) \in B(k, j)$ for $j : \hat{x}_j^k = 1$, and $\forall i \neq u(k_0)$.

Therefore, through \tilde{v} we have built a feasible solution (\tilde{v}, \tilde{x}) with the same objective value as the one given by solution (\hat{v}, \hat{x}) and such that $\tilde{x}_i^{k_0} = 0 \forall i \in \overline{B(k_0, u(k_0))}$. Proceeding by induction on k , we deduce that we can obtain an optimal solution satisfying the statement conditions. \square

Remark 4.2. Besides being useful when fixing variables to zero, the proof of Proposition 4.1 derives an optimal solution (\tilde{v}, \tilde{x}) from another solution (\hat{v}, \hat{x}) which satisfies $\sum_{i \in S^k} s_i^k \tilde{x}_i^k \geq \sum_{i \in S^k} s_i^k \hat{x}_i^k \forall k \in K$, that is, it allows us to obtain an optimal solution in which customers either buy the same product, or buy another product they prefer more. It is also remarkable that there may be more than one optimal solution satisfying Proposition 4.1.

Function u also allows us to conclude that some products will not be sold in any optimal solution that satisfies Proposition 4.1:

Corollary 4.3. Let (\tilde{v}, \tilde{x}) be an optimal solution of (BNL^v) or (SLNL) satisfying Proposition 4.1. Then for every product $i \in I$ such that $u^{-1}(i) = \emptyset$, it follows $\tilde{x}_i^k = 0$ for every client $k \in K$ with $i \in S^k$, i.e., product i is not sold.

Proof. Let us consider an optimal solution (\tilde{v}, \tilde{x}) which meets the requirements given by Proposition 4.1, and a customer k and a product i such that $\hat{x}_i^k = 1$. Then $u(k) = i$ or $i \in B(k, u(k))$, and in the last case by definition of u there exists a customer k' with $\sigma(k') > \sigma(k)$ and $u(k') = i$. \square

Remark 4.4. Corollary 4.3 allows us to eliminate $x_i^k \forall k$ and $v_i^\ell \forall \ell$ for all products i we know will not be sold, thus reducing the size of the problem. Furthermore, after this

procedure, and by definition of u , we will always obtain instances of the problem with $|I| \leq |K|$. However, there might still remain products which will not be sold in one or more optimal solutions.

The following result allows us to fix v -variables to zero, reducing the size of the problem:

Proposition 4.5. *There exists an optimal solution (\tilde{v}, \tilde{x}) of (BNL^v) or (SLNL) such that $\forall i, \ell : \nexists k$ with $\sigma(k) = \ell$ and $i \in S^k$, it follows $v_i^\ell = 0$.*

Proof. Let us suppose we have an optimal solution (\hat{v}, \hat{x}) which does not satisfy the statement requirements, that is, there exist i_0, ℓ_0 such that $\hat{v}_{i_0}^{\ell_0} = 1$ and $\nexists k$ with $\sigma(k) = \ell_0$ and $i_0 \in S^k$. We aim at building another solution (\tilde{v}, \tilde{x}) in which $\tilde{v}_{i_0}^{\ell_0} = 0$ and the objective value does not decrease.

First of all, let us suppose product i_0 is sold in solution (\hat{v}, \hat{x}) , i.e., there exists at least a customer k with $\sigma(k) > \ell_0$ and $\hat{x}_{i_0}^k = 1$. Let $\ell_1 = \min\{\sigma(k) : k \in K, \sigma(k) > \ell_0 \text{ and } \hat{x}_{i_0}^k = 1\}$. Let \tilde{v} be such that $\tilde{v}_i^\ell = \hat{v}_i^\ell \forall \ell, \forall i \neq i_0$ and $\tilde{v}_{i_0}^{\ell_1} = 1, \tilde{v}_{i_0}^\ell = 0 \forall \ell \neq \ell_1$. If we set prices this way, it is clear that every customer who was previously purchasing product i_0 will continue to purchase it, and will pay b^{ℓ_1} for it instead of b^{ℓ_0} . Moreover, the rest of the customers will also purchase the same products than in solution (\hat{v}, \hat{x}) . Therefore, we have obtained another solution (\tilde{v}, \tilde{x}) with $\tilde{x}_i^k = \hat{x}_i^k \forall k \in K, i \in S^k$, and objective value strictly greater than the one given by (\hat{v}, \hat{x}) , which is a contradiction.

On the other hand, let us assume product i_0 is not sold in solution (\hat{v}, \hat{x}) . Let $\ell_1 = \sigma(k_1)$ for a client $k_1 : i_0 \in S^{k_1}$ and $\sigma(k) \leq \sigma(k_1) \forall k : i_0 \in S^k$. Let $\tilde{v}_i^\ell = \hat{v}_i^\ell \forall \ell, i \neq i_0$ and $\tilde{v}_{i_0}^{\ell_1} = 1, \tilde{v}_{i_0}^\ell = 0 \forall \ell \neq \ell_1$. Given this price vector, it is clear that $\tilde{x}_i^k = \hat{x}_i^k \forall k : \sigma(k) \neq \sigma(k_1), i \in S^k$. As for customers $k : \sigma(k) = \sigma(k_1)$, they will purchase their most preferred product between the one they were purchasing in solution (\hat{v}, \hat{x}) and i_0 , but if they prefer i_0 they will spend their whole budget. Thus, the objective value of solution (\tilde{v}, \tilde{x}) is equal to the one given by solution (\hat{v}, \hat{x}) and it satisfies $\tilde{v}_{i_0}^{\ell_0} = 0$. We obtain the result by induction on i .

□

Remark 4.6. *Although optimal solutions satisfying Proposition 4.1 do not necessarily satisfy Proposition 4.5, there exist optimal solutions satisfying both propositions. More specifically, by applying the procedure indicated in the proof of Proposition 4.5 to a solution (\hat{v}, \hat{x}) satisfying Proposition 4.1, we can obtain a solution (\tilde{v}, \tilde{x}) satisfying both. Furthermore, we can assume that if a variable x_i^k can be fixed to zero in an optimal solution (\hat{v}, \hat{x}) according to Proposition 4.1, then i no longer belongs to the list of preferences of customer k , i.e., $i \notin S^k$, thus fixing more v -variables to zero when applying Proposition 4.5.*

By recursively building function u and using the previous results, x_i^k -variables with $i \in \overline{B(k, u(k))}$ can be removed from all formulations based on v - and x -variables. This will imply that $x_{u(k)}^k = 1$ for all richest customers k and their preferred products $u(k)$ such that $B(k, u(k)) = \emptyset$. Variables in the conditions of Proposition 4.5 can also be removed. In some cases, as shown in the following result, an optimal solution to the problem can be directly obtained from the preprocessing phase:

Corollary 4.7. *If for all customers $k \in K$ with $\sigma(k) < M$ an $i \in S^k$ exists such that $\forall k' : \sigma(k') > \sigma(k)$, $u(k') \neq i$, that is, if $C_3 = \emptyset$, an optimal solution can be derived by inspection.*

Proof. Let (\tilde{v}, \tilde{x}) be defined as follows: for all $k \in K$, $\tilde{x}_{u(k)}^k = 1$, $\tilde{x}_i^k = 0 \forall i \neq u(k)$ and $\tilde{v}_{u(k)}^{\sigma(k)} = 1$, $\tilde{v}_{u(k)}^\ell = 0 \forall \ell \neq \sigma(k)$; for all $i : u^{-1}(i) = \emptyset$, $\tilde{v}_i^M = 1$, $\tilde{v}_i^\ell = 0 \forall \ell \neq M$. We are going to show that solution (\tilde{v}, \tilde{x}) is optimal.

First of all, we know by hypothesis that $u(k) = u(k') \Rightarrow \sigma(k) = \sigma(k')$, and therefore \tilde{v} is well defined. Moreover, \tilde{x} is also well defined because for all $k \in K$, $i \in B(k, u(k))$ there exists $k' : \sigma(k') > \sigma(k)$ with $u(k') = i$, and thus $\tilde{v}_i^{\sigma(k')} = 1$ and k cannot afford product i . Finally, since in this solution all customers k are purchasing a product for their whole budget $\sigma(k)$, then the objective value is equal to the sum of the budgets of every customer (which is an upper bound), and therefore (\tilde{v}, \tilde{x}) is optimal. \square

Corollary 4.8. *If $|K| \leq |I|$ and $S^k = I \forall k \in K$, then an optimal solution can be derived by inspection.*

Proof. It suffices to notice that Corollary 4.7 can be applied. \square

5 Polyhedral analysis of the set packing subproblem

In this section, we make use of the fact that a subset of the constraints of the model (SLNL) constitutes a special case of the *Set Packing Problem* (SPP). An SPP is a problem in the form of

$$\max\{ct : At \leq \mathbf{1}_w, t \in \{0, 1\}^u\},$$

where $c \in \mathbb{R}^u$, $A \in \{0, 1\}^{w \times u}$ and $\mathbf{1}_w$ is a w -vector of ones. In the model (SLNL), constraints (4b), (4c) and (4e) are set packing constraints and, as stated before, constraints (5) dominate (4d) and are also set packing constraints.

The polyhedral structure of the SPP has been widely studied in the literature. The interested reader is referred to [11], where the basis of this section are presented, and to [6], where further results are presented and the main papers on the topic are referenced. In the following paragraphs we briefly expose the notation and results necessary to tackle the section. For details, the reader may consult [10].

Associated with each instance of a set packing problem, let the intersection graph be $G = (V, E)$, where each node in the set V is associated to a variable of the problem and $(v_i, v_j) \in E$ if and only if $a_{ki} + a_{kj} = 2$ in some row of A . The neighborhood of a node v is the set of nodes adjacent to v . A non empty subset of pairwise non adjacent nodes in G is known as a packing, and the problem of obtaining an optimal solution of SPP is equivalent to that of obtaining a packing of maximum cardinality on its intersection graph. A complete graph is that in which all the nodes are pairwise adjacent, and a clique in G is a maximal complete subgraph. The incidence vector of a subset $V' \subset V$ is

a binary vector $(t_1, \dots, t_{|V|})$ where $t_j = 1$ if and only if the j^{th} node of V belongs to V' , for $j \in \{1, \dots, |V|\}$.

Let $P(G)$ be the convex hull of the incidence vectors of all the packings of the intersection graph G , i.e., the convex hull of all the feasible solutions of SPP. A valid inequality for $P(G)$ is a facet of $P(G)$ if and only if it is satisfied as an equality by $|V|$ linearly independent points of $P(G)$. Since facets are not dominated by any other valid inequality, one way of confirming that our formulation (or part of it) is tight consists in proving that its constraints are facet-defining. And, as stated in [11], an inequality in the form $\sum_{v \in V'} x_v \leq 1$, where $V' \subset V$, is a facet for $P(G)$ if and only if the subgraph induced by V' is a clique in G .

In order to apply the SPP properties to our problem, we begin by identifying the intersection graph GSLNL associated to our formulation (SLNL). To represent the graph, we will have a matrix of nodes associated to variables x_i^k with $k \in K$ and $i \in S^k$, in which each row will correspond to the preferences of a customer k and each column to the preferences associated with product i (see Figures 1 and 3). If customer k is not interested in product i , i.e. $i \notin S^k$, we will represent it with '-'. Similarly, we will have another matrix given by nodes associated to variables v_i^ℓ such that rows will represent the different prices that products can have and every column will represent one of these products (see Figures 2 and 4). The large amount of edges of this graph makes drawing it impractical, so we will follow a different approach in order to describe the intersection graph based on the following proposition:

Proposition 5.1. *Given the intersection graph GSLNL associated to (SLNL):*

- (1) *Two nodes $x_i^k, x_j^k, i \neq j$, are adjacent $\forall i, j \in S^k$.*
- (2) *Two nodes $x_i^k, x_i^{k'}, k \neq k'$, are never adjacent.*
- (3) *Two nodes $x_i^k, x_j^{k'}, k \neq k', i \neq j$, are adjacent if and only if $\sigma(k) \geq \sigma(k')$ and $j \in B(k, i)$ (or, equivalently, $i \in \overline{B(k, j)}$).*
- (4) *Two nodes x_i^k, v_i^ℓ , are adjacent if and only if $\ell > \sigma(k)$.*
- (5) *Two nodes $x_i^k, v_j^\ell, i \neq j$ are adjacent if and only if $\ell \leq \sigma(k)$ and $j \in B(k, i)$.*
- (6) *Two nodes $v_i^\ell, v_i^{\ell'}, \ell \neq \ell'$, are adjacent $\forall \ell, \ell'$.*
- (7) *Two nodes $v_i^\ell, v_j^{\ell'}, i \neq j$, are never adjacent.*

Proof.

- (1) A customer k purchases at most one product.
- (2) The fact that a customer k purchases a product i does not imply that another customer can not afford it (that depends on i 's price), and therefore does not allow us to determine whether another customer is going to buy it or not.

	i = 1	i = 2	i = 3	i = 4	i = 5	Budget
k = 1	$\triangle 2$	4	-	5	3	20
k = 2	4	$\triangle 1$	$\circledast 2$	$\circ 5$	3	14
k = 3	-	$\square 2$	$\square 5$	$\square 4$	$\bullet 3$	14
k = 4	-	-	$\circ 5$	$\circ 3$	4	8
k = 5	5	4	-	$\circ 3$	-	5

Figure 1: Preference matrix of an instance of (SLNL) and nodes x_i^k adjacent to x_5^3 (x_5^3 in grey).

	i = 1	i = 2	i = 3	i = 4	i = 5	b^ℓ
$\ell = 4$	\circ	\circ	\circ	\circ	\bullet	20
$\ell = 3$	\circ	\circ	\bullet	\bullet	\circ	14
$\ell = 2$	\circ	\circ	\bullet	\bullet	\circ	8
$\ell = 1$	\circ	\circ	\bullet	\bullet	\circ	5

Figure 2: Matrix (v_i^ℓ) of an instance of (SLNL) and nodes v_i^ℓ adjacent to x_5^3 .

- (3) Let us suppose $x_i^k = 1$, i.e., customer k purchases product i . That implies k is not able to afford any product $j \in B(k, i)$, and therefore no customer k' with $\sigma(k') \leq \sigma(k)$ will be able to afford it either, hence $x_j^{k'} = 0$. However, the fact that k purchases product i does not allow us to infer which products will not be purchased by other customers k' richer than k or which customers will not purchase a product $j \in \overline{B(k, i)} \cup \{i\}$.
- (4) If $x_i^k = 1$, k can afford product i , so there must exist $\ell_0 \leq \sigma(k)$ such that $v_i^{\ell_0} = 1$. Since product i can have one price at most, it follows $v_i^\ell = 0 \forall \ell > \sigma(k)$.
- (5) Let us suppose $x_i^k = 1$, i.e., customer k purchases product i . That implies k is not able to afford any product $j \in B(k, i)$, i.e., $v_j^\ell = 0 \forall j \in B(k, i), \forall \ell \leq \sigma(k)$. However, it does not provide any insight into the prices of products $j \in \overline{B(k, i)}$.
- (6) A product i can have at most one price.
- (7) Knowing the price of a product does not provide any insight into the price of the rest.

□

In what follows, and for the sake of clarity, we show an example of the neighborhoods corresponding with given nodes x_i^k and v_i^ℓ in the intersection graph we have just introduced.

	i = 1	i = 2	i = 3	i = 4	i = 5	Budget
k = 1	2	4	-	5	3	20
k = 2	4	1	2	5	3	14
k = 3	-	2	5	4	3	14
k = 4	-	-	5	3	4	8
k = 5	5	4	-	3	-	5

Figure 3: Preference matrix of an instance of (SLNL) and nodes x_i^k adjacent to v_4^3 .

	i = 1	i = 2	i = 3	i = 4	i = 5	b^ℓ
$\ell = 4$	○	○	○	●	○	20
$\ell = 3$	○	○	○	◐	○	14
$\ell = 2$	○	○	○	●	○	8
$\ell = 1$	○	○	○	●	○	5

Figure 4: Matrix (v_i^ℓ) of an instance of (SLNL) and nodes v_i^ℓ adjacent to v_4^3 (v_4^3 in grey).

Example 5.2. Figure 1 shows an instance of the RPP with 5 customers with budgets 20, 14, 14, 8 and 5, respectively, and 5 products. If i is the most preferred product for customer k , we have set $s_i^k = |I| = 5$; if j is the second most preferred product for k , $s_j^k = 4$, *et cetera*. Similarly, making use of the previously described notation, we have $M = 4$, and $b^1 = 5$, $b^2 = 8$, $b^3 = 14$ and $b^4 = 20$. We have not applied any preprocessing to the instance.

Let us suppose $x_5^3 = 1$, i.e., customer 3 purchases product 5 ($s_5^3 = 3$ and it is grey in Figure 1). Our aim is to determine all nodes of GSLNL adjacent to x_5^3 by making use of Proposition 5.1. Due to **(1)**, we know x_5^3 is adjacent to x_2^3 , x_3^3 and x_4^3 (marked with a square in Figure 1). By **(3)** applied to $k = 1$, $k' = 3$, $j = 5$, we obtain that there exists a link between nodes x_5^3 and $x_i^1 \forall i \in \overline{B(1,5)} = \{1\}$. **(3)** applied to $k = 2$, $k' = 3$ and $j = 5$ indicates x_5^3 is adjacent to x_2^2 and x_3^2 (all of these nodes are marked in Figure 1 through a triangle).

Since $\sigma(2) = \sigma(3)$, we can also apply **(3)** to $k = 3$, $i = 5$, $k' = 2$, obtaining edges between nodes x_5^3 and x_3^2 , x_4^2 . Similarly, applying **(3)** to $k = 3$, $i = 5$ and $k' = 4, 5$ leads to edges between x_5^3 and the nodes x_3^4 , x_4^4 and x_4^5 (all of which are circled in Figure 1).

Next, for the same instance of the problem we are going to determine which nodes v_i^ℓ are linked by edges with the same node x_5^3 , by making use of Figure 2. By Proposition 5.1**(4)** we obtain that x_5^3 and v_5^ℓ are adjacent for $\ell > \sigma(3) = 3$, i.e., for $\ell = 4$ (in black in Figure 2). For **(5)**, there exist edges between x_5^3 and v_j^ℓ for $j \in B(3,5) = \{3,4\}$ and $\ell \leq 3$ (coloured in dark grey in Figure 2).

Now let us suppose $v_4^3 = 1$ (i.e., product 4 has a price of $b^3 = 14$), so as to determine

which nodes x_i^k are adjacent to it. By Proposition 5.1(4), v_4^3 is adjacent to x_4^k for $k \in \{4, 5\}$ (marked with a square in Figure 3). Since $\sigma(1) = 4 \geq 3$, for (5) v_4^3 and x_i^1 are adjacent for $i \in \overline{B(1, 4)} = \{1, 2, 5\}$. Similarly, we can obtain edges between v_4^3 and x_i^2 for $i \in \overline{B(2, 4)} = \{1, 2, 3, 5\}$ and x_i^3 for $i \in \overline{B(3, 4)} = \{2, 5\}$ (all of them circled in Figure 3).

Finally, nodes v_i^ℓ adjacent to v_4^3 (in grey) can be seen in Figure 4. Thus, by (6) v_4^3 is adjacent to v_4^ℓ for $\ell \neq 3$ (nodes in black), and (7) establishes that v_4^3 is not adjacent to any other node. \square

5.1 Characterization of all the cliques in the intersection graph

We first include a lemma that will be useful when characterizing all the cliques:

Lemma 5.3. *Any clique in GSLNL which contains nodes $v_i^{\ell_1}, v_i^{\ell_2}$ with $\ell_1 < \ell_2$, contains $v_i^\ell \forall \ell$ such that $\ell_1 < \ell < \ell_2$.*

Proof. Let (V', E') be a clique in GSLNL and suppose $v_i^{\ell_1}, v_i^{\ell_2} \in V'$, for $\ell_1 < \ell_2$.

Let us suppose that there exists $k \in K$ with $x_i^k \in V'$. Then, x_i^k is adjacent to $v_i^{\ell_1}$, and thus for Prop. 5.1(4) it follows $\sigma(k) < \ell_1$. Therefore, for every $\ell > \ell_1 > \sigma(k)$, the same result implies x_i^k is adjacent to v_i^ℓ .

Now let us suppose that there exist $k \in K$ and $j \in S^k$, $j \neq i$, with $x_j^k \in V'$. By hypothesis we have x_j^k adjacent to $v_i^{\ell_2}$, which for Proposition 5.1(5) implies $i \in B(k, j)$ and $\sigma(k) \geq \ell_2$. Thus, for every $\ell < \ell_2 \leq \sigma(k)$, it follows from the same result that x_j^k is adjacent to v_i^ℓ .

Finally, we know from Proposition 5.1(6) and (7) that v_j^ℓ adjacent to $v_i^{\ell_1} \Leftrightarrow j = i$, hence v_i^ℓ is adjacent to $v_i^{\ell'} \forall \ell \neq \ell'$ and $v_j^\ell \notin V'$ for $j \neq i$.

All in all, we have proven that for ℓ such that $\ell_1 < \ell < \ell_2$, any variable x_j^k or $v_i^{\ell'} \in V'$ is adjacent to v_i^ℓ . Thus, the statement follows. \square

Before proving the main results of this section, we need to introduce now some sets that generalize $B(k, i)$.

Definition 5.4. Let k be a customer and $P \subseteq S^k$ a subset of products k is interested in. Then we define $B(k, P)$ as the set $\{i \in S^k : \overline{i >_k j} \forall j \in P\}$ of products that are preferred by k to all the products in P . Similarly $\overline{B(k, P)} := \{i \in S^k : i <_k j \forall j \in P\}$. In the special case when $P = \emptyset$ we define $B(k, \emptyset) := I$ and $\overline{B(k, \emptyset)} := I$.

Now we can state the two main results in this section. Note that, in order to keep a consistent notation, a set $\{k_2, \dots, k_n\}$ is defined in Theorem 5.5 that will be extended to $\{k_1, \dots, k_n\}$ in Theorem 5.7.

Theorem 5.5. *Given a set of customers $\{k_2, \dots, k_n\}$, $n \geq 2$, with $\sigma(k_2) \leq \dots \leq \sigma(k_n)$, and non empty pairwise disjoint sets of products $P^{k_q} \subseteq S^{k_q}$, $q = 2, \dots, n$, such that*

$$P^{k_q} \subseteq \left(\bigcap_{\substack{r=2: \\ \sigma(k_r) < \sigma(k_q)}}^{q-1} \overline{B(k_q, P^{k_r})} \right) \cap \left(\bigcap_{\substack{r=2: \\ \sigma(k_r) = \sigma(k_q)}}^{q-1} \left(\overline{B(k_q, P^{k_r})} \cup B(k_r, P^{k_r}) \right) \right) \quad \forall q \in \{3, \dots, n\},$$

the following inequalities are valid for (SLNL):

$$\sum_{q=2}^n \sum_{j \in P^{k_q}} x_j^{k_q} \leq 1. \quad (6)$$

Valid inequalities (6) are facets for (SLNL) if and only if $\#(k_0, i_0) \in K \times S^{k_0}$ satisfying

1. $i_0 \in B(k_q, P^{k_q}) \forall q \in \{2, \dots, n\} : \sigma(k_q) \geq \sigma(k_0)$,
2. $i_0 \in \overline{B(k_0, P^{k_q})} \forall q \in \{2, \dots, n\} : \sigma(k_q) \leq \sigma(k_0)$,

and $|\bigcup_{\substack{q=2: \\ \sigma(k_q)=\sigma(k_2)}}^n P^{k_q}| \geq 2$. Furthermore, all the clique facets for (SLNL) containing only x -variables are in family (6).

Proof. Let $\text{GSLNL} = (V_G, E_G)$ be the intersection graph of (SLNL) and let $Q = (V', E')$ be a clique of GSLNL containing only x -variables.

Let k_2 be a customer with minimum budget in the clique and a subset of products $P^{k_2} \subseteq S^{k_2}$ such that $x_j^{k_2} \in V' \forall j \in P^{k_2}$ (taking into account that, by Proposition 5.1(1), $x_i^{k_2}$ is adjacent to $x_j^{k_2} \forall i \neq j$).

Provided that there exist customers $k_q, \forall q \in \{3, \dots, n\}$ such that $\sigma(k_2) \leq \sigma(k_3) \leq \dots \leq \sigma(k_n)$ and sets of products $P^{k_q} \subseteq S^{k_q}, P^{k_q} \neq \emptyset \forall q \in \{3, \dots, n\}$, such that $x_j^{k_q} \in V' \forall j \in P^{k_q}$, then by Proposition 5.1(2) P^{k_2}, \dots, P^{k_n} are pairwise disjoint, and verify the following conditions:

•

$$P^{k_q} \subseteq \bigcap_{\substack{r=2 \\ \sigma(k_r) < \sigma(k_j)}}^{q-1} \overline{B(k_q, P^{k_r})}, \quad \forall q \in \{3, \dots, n\}.$$

Otherwise, there exist k_r with $\sigma(k_r) < \sigma(k_q)$ and products $i \in P^{k_r}, j \in P^{k_q}$ such that $x_i^{k_r}, x_j^{k_q} \in V'$ but $j \notin \overline{B(k_q, i)}$, and by Proposition 5.1(3) this implies $x_i^{k_r}, x_j^{k_q}$ are not neighbors in the intersection graph. Therefore, V' does not induce a complete graph.

•

$$P^{k_q} \subseteq \bigcap_{\substack{r=2 \\ \sigma(k_r) = \sigma(k_j)}}^{q-1} \left(\overline{B(k_q, P^{k_r})} \cup B(k_r, P^{k_r}) \right), \quad \forall q \in \{3, \dots, n\}.$$

Otherwise, there exist k_r with $\sigma(k_r) = \sigma(k_q)$ and products $i \in P^{k_r}, j \in P^{k_q}$ such that $x_i^{k_r}, x_j^{k_q} \in V'$ but Proposition 5.1(3) does not hold for $k = k_r, k' = k_q$ or for $k = k_q, k' = k_r$, and hence V' does not induce a complete graph.

Therefore, the above conditions guarantee that the nodes corresponding with the x -variables in an inequality in the form of (6) induce a complete graph, so the family of inequalities (6) is valid.

In addition, if there exist $(k_0, i_0) \in K \times S^{k_0}$ meeting the conditions of the statement, then $x_{i_0}^{k_0}$ is adjacent in the intersection graph to every other node in V' by Proposition 5.1(3) and conditions 1 and/or 2, and therefore the complete subgraph is not maximal.

On the other hand, if $|\bigcup_{\substack{q=2: \\ \sigma(k_q)=\sigma(k_2)}}^n P^{k_q}| \geq 2$ holds, no v -variable can be adjacent in the intersection graph to all nodes in V' . Otherwise, $P^{k_2} = \{i\}$ and either $n = 2$ or $\sigma(k_2) < \sigma(k_3)$, and hence variable $v_i^{\sigma(k_2)+1}$ would be adjacent to every node in V' and the complete subgraph would not be maximal. \square

Example 5.6. By making use of the instance given in Figure 1, we are going to give examples of clique facets containing only x -variables and valid inequalities that are not cliques. Consider first the set of x -variables given by $\{x_1^2, x_2^2, x_3^2, x_5^2\}$. Given that all these variables share the superindex, their corresponding nodes are neighbors in the intersection graph and then $x_1^2 + x_2^2 + x_3^2 + x_5^2 \leq 1$ is a valid inequality for (SLNL). This inequality corresponds with $n = 2$, $k_2 = 2$ ($\sigma(k_2) = 3$) and $P^2 = \{1, 2, 3, 5\}$ in Theorem 5.5. Nevertheless, it does not define a facet for (SLNL) since a pair $(k_0, i_0) = (4, 4)$ exists (with $\sigma(4) = 2 \leq 3$) in the case 1 of the theorem, with $4 \in B(2, \{1, 2, 3, 5\}) = \{4\}$. Then, variable x_4^4 could be added to the inequality to produce the tighter inequality $x_1^2 + x_2^2 + x_3^2 + x_5^2 + x_4^4 \leq 1$. This inequality corresponds now to $n = 3$, $k_2 = 4$, $k_3 = 2$, $P^4 = \{4\}$, $P^2 = \{1, 2, 3, 5\}$ and it is not a facet for (SLNL) either, since $|\bigcup_{\substack{q=2: \\ \sigma(k_q)=\sigma(k_2)}}^n P^{k_q}| = |\bigcup_{\substack{q=2: \\ \sigma(k_q)=2}}^3 P^{k_q}| = |P^{k_2}| = 1$ and, in this case, variable v_4^3 could be added to the inequality to produce the tighter one given by $x_1^2 + x_2^2 + x_3^2 + x_5^2 + x_4^4 + v_4^3 \leq 1$.

Consider, instead, $(k_0, i_0) = (3, 4)$, i.e., $n = 3$, $k_2 = 3$, $k_3 = 2$, $P^3 = \{4\}$, $P^2 = \{1, 2, 3, 5\}$ where $|\bigcup_{\substack{q=2: \\ \sigma(k_q)=\sigma(k_2)}}^n P^{k_q}| = |\bigcup_{\substack{q=2: \\ \sigma(k_q)=3}}^3 P^{k_q}| = |P^3 \cup P^2| = 5$. No v -variables can be added to the inequality, and conditions 1 and 2 in the theorem are also satisfied, thus producing the facet $x_1^2 + x_2^2 + x_3^2 + x_5^2 + x_4^3 \leq 1$. \square

Table 1 shows every clique facet that can be obtained applying Theorem 5.5 to the instance of the RPP given in Figure 1, previously preprocessed setting $x_4^1 = 1$, $x_i^1 = 0 \forall i \neq 4$. Each row of the table represents a clique, and a variable $x_i^{k'}$ belongs to that clique if and only if the number i appears in column $k = k'$ in that row, for $k' = \{2, \dots, 5\}$, $i \in S^{k'}$ (variables related to customer 1 do not appear in any clique because they have been previously settled to either zero or one). For instance, the first row of Table 1 refers to clique $\sum_{i=1}^5 x_i^2 \leq 1$, whereas the fourth one makes reference to clique $x_2^2 + x_3^2 + x_5^2 + x_1^5 + x_4^5 \leq 1$.

	x			
	$k = 2$	$k = 3$	$k = 4$	$k = 5$
i	12345			
	1235	4		
	234	5		
	235			14
	23	45		
	23		45	
	2	345		
	2	5	34	
	2		345	
	345	2		
	34	25		
	35	24		
	3	245		
	3	2	45	
		2345		
		25	34	
		2	345	
				124

Table 1: Facets obtained applying Theorem 5.5 to the instance of the RPP given in Figure 1

	x				v			
	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
$L = \{\ell_1, \dots, \ell_p\}$	235			1		1	1	
		245	3				3	
	2	45	3				3	
	1235			4		4	4	
	3	25		4		4	4	
	23	5		4		4	4	
	35	2		4		4	4	
	1235		4				4	
	3	25	4				4	
	23	5	4				4	
	35	2	4				4	
	23		5				5	
3	2	5				5		
$L = \{1, \dots, \ell_p\}$	235				1	1	1	
				24	1			
	235			4	1			
				4	2	2	2	
	2	45			3	3	3	
		245			3	3	3	
	2		45		3	3		
		2	45		3	3		
	2	5	4		3	3		
		25	4		3	3		
	3	25			4	4	4	
	35	2			4	4	4	
	23	5			4	4	4	
	1235				4	4	4	
	23				5	5	5	
3	2			5	5	5		
23		4		5	5			
3	2	4		5	5			
$L = \{\ell_1, \dots, M\}$	1			1		1	1	1
	2							2
		2						2
	3			2		2	2	2
		3						3
	4		3				3	3
	4		4				4	
		4				4	4	
5			4		4	4	4	
	5			5			5	
		5				5	5	
$L = \{1, \dots, M\}$					1	1	1	1
					2	2	2	2
					3	3	3	3
					4	4	4	4
					5	5	5	5

Table 2: Facets obtained applying Theorem 5.7 to the instance of the RPP given in Figure 1

Theorem 5.7. Given a nonempty set $L = \{\ell_1, \dots, \ell_p\} \subseteq \{1, \dots, M\}$, a product $i \in I$ and

- if $\ell_1 > 1$, a customer k_1 such that $\sigma(k_1) = \ell_1 - 1$, $i \in S^{k_1}$, and a set $P^{k_1} = \{i\}$; otherwise, $P^{k_1} = \emptyset$;
- if $\ell_p < M$, customers k_2, \dots, k_n , $n \geq 2$, such that $\ell_p = \sigma(k_2) \leq \dots \leq \sigma(k_n)$ ($n = 1$ otherwise) and non empty pairwise disjoint sets of products $P^{k_q} \subseteq S^{k_q} \setminus \{i\}$, $q = 2, \dots, n$ such that $P^{k_2} \subseteq \overline{B(k_2, i)}$ and

$$P^{k_q} \subseteq \left(\bigcap_{\substack{r=1: \\ \sigma(k_r) < \sigma(k_q)}}^{q-1} \overline{B(k_q, P^{k_r})} \right) \cap \left(\bigcap_{\substack{r=1: \\ \sigma(k_r) = \sigma(k_q)}}^{q-1} \left(\overline{B(k_q, P^{k_r})} \cup B(k_r, P^{k_r}) \right) \right)$$

$$\forall q \in \{3, \dots, n\},$$

the following inequalities are valid for (SLNL):

$$\sum_{\ell \in L} v_i^\ell + \sum_{q=1}^n \sum_{j \in P^{k_q}} x_j^{k_q} \leq 1. \quad (7)$$

Valid inequalities (7) are facets for (SLNL) if and only if $\nexists (k_0, i_0) \in K \times S^{k_0} \setminus \{i\}$: $\sigma(k_0) \geq \ell_p$ satisfying

1. $i_0 \in B(k_q, P^{k_q}) \forall q \in \{1, \dots, n\} : \sigma(k_q) \geq \sigma(k_0)$,
2. $i_0 \in \overline{B(k_0, P^{k_q})} \forall q \in \{1, \dots, n\} : \sigma(k_q) \leq \sigma(k_0)$.

Furthermore, all the clique facets for (SLNL) containing v -variables are in family (7).

Proof. Let $\text{GSLNL} = (V_G, E_G)$ be the intersection graph of (SLNL) and let $Q = (V', E')$ be a clique of GSLNL containing v -variables. Taking into account Proposition 5.1(7), all v -variables in the same clique must share the subindex, and by Lemma 5.3, all v -variables in the same clique must have consecutive superindices. We represent with $L = \{\ell_1, \dots, \ell_p\}$ this set of consecutive superindices and with i the common subindex. We thus distinguish several cases depending on L :

1. $L = \{1, \dots, M\}$.

Then by Proposition 5.1(5) we know that a node x_j^k in the neighborhood of v_i^1, \dots, v_i^M must satisfy $\sigma(k) = M$ and $j \in \overline{B(k, i)}$. However, we have already proved in Section 4 that the richest customers always purchase their most preferred product, and therefore we have removed all these x -nodes from the intersection graph, i.e., $P^{k_2} = \dots = P^{k_n} = \emptyset$.

Since Proposition 5.1(4) does not either provide any node adjacent to $v_i^\ell \forall \ell$, we obtain $P^{k_1} = \emptyset$ and thus the set of nodes $\{v_i^\ell : \ell \in \{1, \dots, M\}\}$ induces a maximal complete subgraph in GSLNL .

2. $L = \{\ell_1, \dots, M\}$ for some $\ell_1 > 1$.

As $v_i^\ell \notin V' \forall \ell \in \{1, \dots, \ell_1 - 1\}$, a node adjacent to v_i^ℓ for $\ell \geq \ell_1$ but not to $v_i^{\ell_1-1}$ must belong to the clique. Applying Lemma 5.3 and Proposition 5.1, we know this node corresponds with an x -variable, i.e., there exists a node $x_j^k \in V'$ for some customer k and product j . As in the previous case, Proposition 5.1(5) does not provide any node adjacent to v_i^M , thus $P^{k_2} = \dots = P^{k_n} = \emptyset$. Therefore, node x_j^k must be adjacent to v_i^ℓ for $\ell \geq \ell_1$ by Proposition 5.1(4), so $j = i$ and $k = k_1$ for a customer k_1 : $\sigma(k_1) < \ell_1$ and $P^{k_1} = \{i\}$. Since $x_i^{k_1}$ is not adjacent to $v_i^{\ell_1-1}$, also by Proposition 5.1(4) $\sigma(k_1) \geq \ell_1 - 1$, and hence $\sigma(k_1) = \ell_1 - 1$.

If we suppose there exists another node $x_j^k \in V'$, then x_j^k must be adjacent to $v_i^\ell \forall \ell \geq \ell_1$ by Proposition 5.1(4), and therefore $j = i$. However, x_i^k and $x_i^{k_1}$ are not adjacent for any customer $k \neq k_1$ by (2), so the set $\{v_i^\ell : \ell \geq \ell_1\} \cup \{x_i^{k_1}\}$ induces a clique in GSLNL.

3. $L = \{1, \dots, \ell_p\}$ for some $\ell_p < M$.

Since $v_i^\ell \notin V' \forall \ell > \ell_p$, applying Lemma 5.3 and Proposition 5.1 there must exist a node $x_{i_0}^k \in V'$ such that $x_{i_0}^k$ is adjacent to $v_i^{\ell_p}$ but not to $v_i^{\ell_p+1}$. Proposition 5.1(4) does not provide any node adjacent to v_i^1 , hence $P^{k_1} = \emptyset$ and $x_{i_0}^k$ has to be adjacent to v_i^ℓ , $\ell \leq \ell_p$, by Proposition 5.1(5). Hence, there exists a customer k_2 : $\sigma(k_2) \geq \ell_p$ and a subset of products $P^{k_2} \subseteq \overline{B(k_2, i)}$ such that $i_0 \in P^{k_2}$ and $x_j^{k_2} \in V' \forall j \in P^{k_2}$ (taking into account that, by Proposition 5.1(1), $x_j^{k_2}$ is adjacent to $x_{j'}^{k_2} \forall j \neq j'$). Since $x_{i_0}^{k_2}$ is not adjacent to $v_i^{\ell_p+1}$, it follows $\sigma(k_2) = \ell_p$.

Provided that there exist customers $k_q, \forall q \in \{3, \dots, n\}$ such that $\sigma(k_2) \leq \sigma(k_3) \leq \dots \leq \sigma(k_n)$ and sets of products $P^{k_q} \subseteq S^{k_q}$, $P^{k_q} \neq \emptyset \forall q \in \{3, \dots, n\}$, such that $x_j^{k_q} \in V' \forall j \in P^{k_q}$, then by Proposition 5.1(2) P^{k_1}, \dots, P^{k_n} are pairwise disjoint. Moreover, $P^{k_q} \subseteq S^{k_q} \setminus \{i\} \forall q \in \{3, \dots, n\}$; otherwise, $x_i^{k_q} \in V'$ for some k_q : $\sigma(k_q) \geq \ell_p$ and is not adjacent to $v_i^{\ell_p}$ (Proposition 5.1(4)), thus V' does not induce a complete graph.

Applying arguments analogous to those of Theorem 5.5, the rest of the conditions stated above must hold.

4. $L = \{\ell_1, \dots, \ell_p\}$ for some $\ell_1 > 1, \ell_p < M$.

Applying arguments analogous to those of the previous items, we can conclude that there exist customers $k_1 \in K$: $\overline{\sigma(k_1)} = \ell_1 - 1$, $i \in S^{k_1}$ such that $P^{k_1} = \{i\}$ and $k_2 \in K$: $\sigma(k_2) = \ell_p$ with $P^{k_2} \subseteq \overline{B(k_2, i)}$, $P^{k_2} \neq \emptyset$. The rest of the conditions also hold applying a reasoning analogous to that of Theorem 5.5.

□

Example 5.8. By making use of the instance given in Figure 1, we are going to give examples of each of the different types of clique facets that can be encountered. We consider $i = 4$ and different possibilities for set L .

Consider first $L = \{1, 2, 3, 4\}$. Due to condition 1 in Theorem 5.7, $P^{k_1} = \emptyset$, and by condition 2, $n = 1$. Then a facet in the shape of $v_4^1 + v_4^2 + v_4^3 + v_4^4 \leq 1$ is obtained.

If we remove 1 from L , $L = \{2, 3, 4\}$, now $\ell_1 = 2$ and condition k_1 : $\sigma(k_1) = 1$ and $4 \in S^{k_1}$ applies, giving $k_1 = 5$. Then $P^5 = \{4\}$ and the new facet is $v_4^2 + v_4^3 + v_4^4 + x_4^5 \leq 1$. Removing also 2 from L , $L = \{3, 4\}$, facet $v_4^3 + v_4^4 + x_4^4 \leq 1$ is produced.

If, instead, we remove 4 from L , $L = \{1, 2, 3\}$, by condition 2 a new customer k_2 such that $\sigma(k_2) = \ell_p = 3$ is required. Customer k_2 can be chosen in the set $\{2, 3\}$. If we choose $n = 2$ and $k_2 = 2$, condition 2.a, $P^2 \subseteq \overline{B(2, 4)} = \{1, 2, 3, 5\}$, applies. Taking $P^2 = \{1, 2, 3, 5\}$ the new facet is $v_4^1 + v_4^2 + v_4^3 + x_1^2 + x_2^2 + x_3^2 + x_5^2 \leq 1$, since no pair (k_0, i_0) in the conditions of the theorem exists and condition 2.b is also satisfied. If, instead, we choose $n = 3$, $k_2 = 2$ and $k_3 = 3$, a possibility that satisfies all the requirements is to take $P^2 = \{3, 5\}$, $P^3 = \{2\}$ to produce the facet $v_4^1 + v_4^2 + v_4^3 + x_3^2 + x_5^2 + x_2^3 \leq 1$. However, if we also remove 3 from L , i.e., $L = \{1, 2\}$, 4 is the only customer whose budget meets the requirement $\sigma(k_2) = \ell_p = 2$, but $\overline{B(4, 4)} = \emptyset$, and thus condition 2.a in Theorem 5.7 can not be verified for any customer. Therefore, no clique facet in this instance contains only v -variables v_4^1 and v_4^2 .

If we take now $L = \{2, 3\}$, again with $k_1 = 5$, $n = 3$, $k_2 = 2$ and $k_3 = 3$, we can obtain facets such as $v_4^2 + v_4^3 + x_4^5 + x_3^2 + x_5^2 + x_2^3 \leq 1$.

□

Now that we have established the different shapes that clique facets can adopt, we are able to determine whether constraints (4b), (4c), (5) and (4e) always define clique facets. Thus, we can conclude that constraints (4c) and (4e) always define clique facets by applying cases 1 and 2 of the proof of Theorem 5.7, respectively. By Theorem 5.5, and given that $B(k, S^k) = \emptyset \forall k$, we know a valid inequality from the family (4b) will be a clique if and only if $|S^k| \geq 2$ and $\exists (k_0, i_0) \in K \times S^{k_0}$ satisfying $\sigma(k_0) \geq \sigma(k)$ and $i_0 \in \overline{B(k_0, S^k)}$. For example, the valid inequality $\sum_{i \in S^4} x_i^4 \leq 1$ is not a clique facet in the instance of the RPP given by Figure 1, since there exists $(3, 2) \in K \times S^3$ satisfying $\sigma(3) \geq \sigma(2)$ and $2 \in \overline{B(3, S^4)}$, and therefore x_2^3 is adjacent to $x_i^4 \forall i \in S^4$. As for constraints (5), they do not necessarily define clique facets either: in the same instance, $\sum_{\ell=1}^3 v_4^\ell + \sum_{i \in \overline{B(3, 4)}} x_i^3 \leq 1$ is valid but is not a clique facet, because x_2^3 could be added to the left hand side and the constraint would still be valid, since the pair $(2, 3) \in K \times S^2 \setminus \{4\}$ satisfies $\sigma(2) \geq \ell_3$ and meets conditions 1 and 2 of Theorem 5.7.

Table 2 includes every facet that can be obtained applying Theorem 5.7 to the instance given by Figure 1, previously preprocessed setting $x_4^1 = 1$, $x_i^1 = 0 \forall i \neq 4$.

6 Linearising formulations

The model formulation is non linear because of the objective function (4a). In order to linearize it, one approach consists in introducing variables z^k , $k \in K$, representing the profit obtained from customer k . Thus, the objective (4a) can be replaced by:

$$\max_{v, x, z} \sum_{k \in K} z^k$$

and the following constraints need to be added to the formulation:

$$z^k \leq \sum_{\ell=1}^M b^\ell v_i^\ell + b^{\sigma(k)} (1 - x_i^k), \quad \forall k \in K, i \in S^k \quad (8a)$$

$$z^k \leq b^{\sigma(k)} \sum_{i \in S^k} x_i^k, \quad \forall k \in K, \quad (8b)$$

where constraints (8a) ensure that if customer k buys product k , $z^k = \sum_{\ell=1}^M b^\ell v_i^\ell$ and (8b) guarantee $z^k \leq 0$ if customer k does not make any purchase. Constraints (8a) can be strengthened taking into account that customer k buys at most one item, and he only buys a product below his budget, obtaining:

$$z^k \leq \sum_{\ell=1}^{\sigma(k)} b^\ell v_i^\ell + b^{\sigma(k)} \sum_{j \in S^k: j \neq i} x_j^k, \quad \forall k \in K, i \in S^k. \quad (9)$$

$\sum_{j \in S^k: j \neq i} x_j^k$ is less than $(1 - x_i^k)$ and the first summation in (9) has less terms than in (8a), making constraints (9) stronger than (8a).

Therefore, we can reformulate problem (SLNL) obtaining a linear model as follows:

$$(SLL_1) \quad \max_{v, x, z} \quad \sum_{k \in K} z^k \quad (10a)$$

$$\text{s.t.} \quad \sum_{i \in S^k} x_i^k \leq 1, \quad \forall k \in K \quad (10b)$$

$$\sum_{\ell=1}^M v_i^\ell \leq 1, \quad \forall i \in I \quad (10c)$$

$$\sum_{j \in B(k, i)} x_j^k + \sum_{\ell=1}^{\sigma(k)} v_i^\ell \leq 1, \quad \forall k \in K, i \in S^k \quad (10d)$$

$$x_i^k + \sum_{\ell=\sigma(k)+1}^M v_i^\ell \leq 1, \quad \forall k \in K, i \in S^k \quad (10e)$$

$$z^k \leq \sum_{\ell=1}^{\sigma(k)} b^\ell v_i^\ell + b^{\sigma(k)} \sum_{j \in S^k: j \neq i} x_j^k, \quad \forall k \in K, i \in S^k \quad (10f)$$

$$z^k \leq b^{\sigma(k)} \sum_{i \in S^k} x_i^k, \quad \forall k \in K \quad (10g)$$

$$v_i^\ell, x_i^k \in \{0, 1\}, z^k \geq 0 \quad \forall k \in K, i \in S^k, \ell \in \{1, \dots, M\}. \quad (10h)$$

Another alternative to deal with the nonlinearity of the objective function (4a) is to introduce variables z_i^k , $k \in K$, $i \in S^k$, representing the profit obtained from customer k when he buys product i . With these variables, the objective is:

$$\max_{v,x,z} \sum_{k \in K} \sum_{i \in S^k} z_i^k$$

and the following constraints ought to be added to the model:

$$\begin{aligned} z_i^k &\leq \sum_{\ell=1}^{\sigma(k)} b^\ell v_i^\ell, \quad \forall k \in K, i \in S^k \\ z_i^k &\leq b^{\sigma(k)} x_i^k, \quad \forall k \in K, i \in S^k. \end{aligned}$$

Thus, the resulting model is:

$$(SLL_2) \quad \max_{v,x,z} \sum_{k \in K} \sum_{i \in S^k} z_i^k \quad (11a)$$

$$\text{s.t.} \quad \sum_{i \in S^k} x_i^k \leq 1, \quad \forall k \in K \quad (11b)$$

$$\sum_{\ell=1}^M v_i^\ell \leq 1, \quad \forall i \in I \quad (11c)$$

$$\sum_{j \in B(k,i)} x_j^k + \sum_{\ell=1}^{\sigma(k)} v_i^\ell \leq 1, \quad \forall k \in K, i \in S^k \quad (11d)$$

$$x_i^k + \sum_{\ell=\sigma(k)+1}^M v_i^\ell \leq 1, \quad \forall k \in K, i \in S^k \quad (11e)$$

$$z_i^k \leq \sum_{\ell=1}^{\sigma(k)} b^\ell v_i^\ell, \quad \forall k \in K, i \in S^k \quad (11f)$$

$$z_i^k \leq b^{\sigma(k)} x_i^k, \quad \forall k \in K, i \in S^k \quad (11g)$$

$$v_i^\ell, x_i^k \in \{0, 1\}, z_i^k \geq 0 \quad \forall k \in K, i \in S^k, \ell \in \{1, \dots, M\}. \quad (11h)$$

7 Strengthening of the linear formulations

In the formulations (SLL₁) and (SLL₂), the values of the z -variables associated to an assignment of prices to products (v -variables) and products to clients (x -variables) are obtained, respectively, by means of constraints (10f)-(10g) and (11f)-(11g). Although these constraints suffice to obtain the desired values of the z -variables, they suffer from a great weakness when using the corresponding linear relaxations. Given the shape of the objective function, such weakness is directly transmitted to the upper bounds in the branch-and-bound method. Furthermore, in (10f) (resp. (11f)), a bound for z is obtained exclusively from the v -variables, and in (10g) (resp. (11g)), from the x -variables. These two issues invite to the development of stronger constraints which, by means of the

reduction of the values of the z -variables obtained in the LP solution of the pertinent formulation, produce a smaller value of the LP objective function, i.e., tighter upper bounds to be used in the branch-and-cut solution procedure.

In what follows, two families of valid inequalities for (SLL₁) and (SLL₂) are presented. As will be shown in the computational study, they produce the desired improvement in the bounds, and they have the particularity of relating the z -variables with both the x - and the v -variables at a time.

Proposition 7.1. *The following inequalities are valid for (SLL₁):*

$$z^k \leq \sum_{i \in S^k} \left(b^{r_i^k} x_i^k + \sum_{\ell=r_i^k+1}^{\sigma(k)} (b^\ell - b^{r_i^k}) v_i^\ell + \sum_{\ell \in Q_i^k} (b^\ell - b^{r_i^k}) (x_i^k + v_i^\ell - 1) \right), \quad (12)$$

$\forall k \in K$, integers $r_i^k \in \{0, \dots, \sigma(k)\} \forall i \in S^k$ and subsets $Q_i^k \subseteq \{1, \dots, r_i^k - 1\} \forall i \in S^k$.

Proof. We aim at proving that constraints (12) are valid for (SLL₁). Let us assume $x_{i_0}^k = 1$ for some $i_0 \in S^k$, and prove that the sum of the addends corresponding to product i_0 in the right hand side of the constraint is greater than or equal to its price. Thus, such sum is

$$b^{r_{i_0}^k} + \sum_{\ell=r_{i_0}^k+1}^{\sigma(k)} (b^\ell - b^{r_{i_0}^k}) v_{i_0}^\ell + \sum_{\ell \in Q_{i_0}^k} (b^\ell - b^{r_{i_0}^k}) v_{i_0}^\ell, \quad (13)$$

and we know that $v_{i_0}^{\ell_0} = 1$ for some $\ell_0 \leq \sigma(k)$. If $\ell_0 > r_{i_0}^k$, then $v_{i_0}^\ell = 0 \forall \ell \in Q_{i_0}^k$ and we get $b^{r_{i_0}^k} + (b^{\ell_0} - b^{r_{i_0}^k}) = b^{\ell_0}$, which is exactly the price of i_0 . On the other hand, if $\ell_0 \leq r_{i_0}^k$ we have $v_{i_0}^\ell = 0 \forall \ell : r_{i_0}^k < \ell \leq \sigma(k)$, and therefore (13) becomes

$$b^{r_{i_0}^k} + \sum_{\ell \in Q_{i_0}^k} (b^\ell - b^{r_{i_0}^k}) v_{i_0}^\ell.$$

If $\ell_0 \notin Q_{i_0}^k$, we obtain $b^{r_{i_0}^k}$, which is greater than or equal to b^{ℓ_0} because $r_{i_0}^k \geq \ell_0$; otherwise, if $\ell_0 \in Q_{i_0}^k$, then the term becomes $b^{r_{i_0}^k} + (b^{\ell_0} - b^{r_{i_0}^k}) = b^{\ell_0}$.

Now, let us suppose $x_{i_0}^k = 0$ for $i_0 \in S^k$. Then the addends corresponding to product i_0 become

$$\sum_{\ell=r_{i_0}^k+1}^{\sigma(k)} (b^\ell - b^{r_{i_0}^k}) v_{i_0}^\ell + \sum_{\ell \in Q_{i_0}^k} (b^\ell - b^{r_{i_0}^k}) (v_{i_0}^\ell - 1).$$

Since $(b^\ell - b^{r_{i_0}^k}) > 0$ for $\ell : r_{i_0}^k < \ell \leq \sigma(k)$ and $(b^\ell - b^{r_{i_0}^k}) < 0$ for $\ell \in Q_{i_0}^k$, then the sum is greater than or equal to zero.

Therefore, if $x_i^k = 0 \forall i \in S^k$, z^k will be upperly bounded by a sum of non-negative values. Otherwise, since, at any feasible solution, at most one x -variable can take value 1 for a fixed customer k , say $x_{i_0}^k$, the upper bound will be obtained as the sum of the term corresponding to product i_0 , which has been proved to be greater than or equal to the price assigned to i_0 , plus some non-negative addends. \square

Remark 7.2. *The family of inequalities (12) contains all of the previous upper bound constraints on z^k of (SLL₁). Constraints (10f) are obtained by, given a customer $k \in K$ and a product $i \in S^k$, setting $r^k(i) = 0$, $r^k(j) = \sigma(k) \forall j \in S^k \setminus \{i\}$ and $Q_j^k = \emptyset \forall j \in S^k$ in (12); constraints (10g), by, given a customer $k \in K$, setting $r^k(i) = \sigma(k)$ and $Q_i^k = \emptyset \forall i \in S^k$.*

Proposition 7.3. *The inequalities of the following family are valid for (SLL₂):*

$$z_i^k \leq b^{r_i^k} x_i^k + \sum_{\ell=r_i^k+1}^{\sigma(k)} (b^\ell - b^{r_i^k}) v_i^\ell + \sum_{\ell \in Q_i^k} (b^\ell - b^{r_i^k}) (x_i^k + v_i^\ell - 1), \quad (14)$$

$\forall k \in K, i \in S^k$, any integer $r_i^k \in \{0, \dots, \sigma(k)\}$ and any subset $Q_i^k \subseteq \{1, \dots, r_i^k - 1\}$.

Proof. First assume that $x_i^k = 1$. This implies $v_i^{\ell_0} = 1$ for some $\ell_0 \leq \sigma(k)$. If $\ell_0 \leq r_i^k$, then $v_i^\ell = 0 \forall \ell : r_i^k < \ell \leq \sigma(k)$ and (14) becomes $z_i^k \leq b^{r_i^k} + \sum_{\ell \in Q_i^k} (b^\ell - b^{r_i^k}) v_i^\ell$. If $\ell_0 \in Q_i^k$, then the right hand side of the constraint is $b^{r_i^k} + (b^{\ell_0} - b^{r_i^k}) = b^{\ell_0}$, which is valid as it is the exact price of product i ; otherwise, the right hand side of the constraint is $b^{r_i^k}$, valid since $b^{r_i^k} \geq b^{\ell_0}$. If $\ell_0 > r_i^k$, then $v_i^\ell = 0 \forall \ell \in Q_i^k$ and the inequality we obtain is $z_i^k \leq b^{r_i^k} + (b^{\ell_0} - b^{r_i^k})$, also valid.

On the other hand, if we assume $x_i^k = 0$, then the inequality holds trivially because its right hand side is non negative and $z_i^k = 0$. \square

Remark 7.4. *The family of inequalities (14) contains all of the previous upper bound constraints on z_i^k of (SLL₂): constraints (11f) are obtained by making $r_i^k = 0$ and $Q_i^k = \emptyset \forall k \in K, i \in S^k$, whereas constraints (11g) appear as a result of setting $r_i^k = \sigma(k)$, $Q_i^k = \emptyset \forall k, i \in S^k$.*

The number of inequalities of Propositions 7.1 and 7.3 exponentially as the number of customers and products grows. However, these inequalities can be efficiently separated and added dynamically to formulations (SLL₁) and (SLL₂), respectively, in a branch-and-cut mode. Thus, regarding the family of valid inequalities (12), and given a fractional optimal solution of the linear relaxation of (SLL₁), $(\bar{v}_i^\ell, \bar{x}_i^k, \bar{z}^k)$, our aim is to find, for each $k \in K$, integers r_i^k and subsets $Q_i^k \forall i \in S^k$ such that the upper bound given by the right hand side of the resultant constraint of the family is as tight as possible. As the sum given by the right hand side of (12) can be decomposed by products and given that \bar{z} is fixed, our problem reduces to

$$\min_{\substack{r \in \{0, \dots, \sigma(k)\}, \\ Q \subseteq \{1, \dots, r-1\}}} b^r \bar{x}_i^k + \sum_{\ell=r+1}^{\sigma(k)} (b^\ell - b^r) \bar{v}_i^\ell + \sum_{\ell \in Q} (b^\ell - b^r) (\bar{x}_i^k + \bar{v}_i^\ell - 1), \quad (15)$$

where $(k, i) \in K \times S^k$ is fixed, and we have denoted r_i^k as r and Q_i^k as Q so as to simplify notation. It is worth noticing that this pair (r, Q) also minimizes the right hand side of the corresponding constraint of family (14) when given an optimal fractional solution of the linear relaxation of (SLL₂), $(\bar{v}_i^\ell, \bar{x}_i^k, \bar{z}_i^k)$, and fixed $(k, i) \in K \times S^k$. Thus, finding a pair (r, Q) that minimizes (15) for a given customer k and product i not only allows us

to develop an efficient separation algorithm for the set of valid inequalities (12), but also for the set (14).

The fact that $(b^\ell - b^r) \leq 0 \forall \ell \leq r$ implies that, for a given r , $Q^r := \{\ell \in \{1, \dots, r-1\} : \bar{x}_i^k + \bar{v}_i^\ell > 1\}$ minimizes (15). Therefore, if $W(r)$ is the value of the sum (15) when $Q = Q^r$, our problem consists in minimizing $W(r)$ for $r \in \{0, \dots, \sigma(k)\}$.

To do so, we shall study the variation of $W(r)$ as r increases. Given that $Q^{r+1} = Q^r \cup \{r\}$ if $\bar{x}_i^k + \bar{v}_i^r > 1$, $Q^{r+1} = Q^r$ otherwise, for $r < \sigma(k)$ we get:

$$\begin{aligned}
W(r+1) - W(r) &= \left(b^{r+1} \bar{x}_i^k + \sum_{\ell=r+2}^{\sigma(k)} (b^\ell - b^{r+1}) \bar{v}_i^\ell + \sum_{\ell \in Q^{r+1}} (b^\ell - b^{r+1}) (\bar{x}_i^k + \bar{v}_i^\ell - 1) \right) \\
&\quad - \left(b^r \bar{x}_i^k + \sum_{\ell=r+1}^{\sigma(k)} (b^\ell - b^r) \bar{v}_i^\ell + \sum_{\ell \in Q^r} (b^\ell - b^r) (\bar{x}_i^k + \bar{v}_i^\ell - 1) \right) \\
&= (b^{r+1} - b^r) \bar{x}_i^k + \sum_{\ell=r+2}^{\sigma(k)} (b^r - b^{r+1}) \bar{v}_i^\ell - (b^{r+1} - b^r) \bar{v}_i^{r+1} \\
&\quad + \sum_{\ell \in Q^{r+1}} (b^r - b^{r+1}) (\bar{x}_i^k + \bar{v}_i^\ell - 1) \\
&= (b^{r+1} - b^r) \left(\bar{x}_i^k - \sum_{\ell=r+1}^{\sigma(k)} \bar{v}_i^\ell + \sum_{\ell \in Q^{r+1}} (1 - \bar{x}_i^k - \bar{v}_i^\ell) \right). \tag{16}
\end{aligned}$$

First of all, we are going to prove that, when r increases from 0 to $\sigma(k)$, $W(r)$ first decreases and then increases. We can achieve that by proving that $W(r) - W(r-1) \geq 0 \Rightarrow W(r+1) - W(r) \geq 0$. Since $b^{r+1} - b^r > 0 \forall r < \sigma(k)$, it follows from (16) that $W(r+1) - W(r) \geq 0 \Leftrightarrow \bar{x}_i^k - \sum_{\ell=r+1}^{\sigma(k)} \bar{v}_i^\ell + \sum_{\ell \in Q^{r+1}} (1 - \bar{x}_i^k - \bar{v}_i^\ell) \geq 0 \forall r < \sigma(k)$, and therefore demonstrating the above is equivalent to proving $\bar{x}_i^k - \sum_{\ell=r+1}^{\sigma(k)} \bar{v}_i^\ell + \sum_{\ell \in Q^{r+1}} (1 - \bar{x}_i^k - \bar{v}_i^\ell) - \left(\bar{x}_i^k - \sum_{\ell=r}^{\sigma(k)} \bar{v}_i^\ell + \sum_{\ell \in Q^r} (1 - \bar{x}_i^k - \bar{v}_i^\ell) \right) \geq 0$. But we have

$$\begin{aligned}
&\bar{x}_i^k - \sum_{\ell=r+1}^{\sigma(k)} \bar{v}_i^\ell + \sum_{\ell \in Q^{r+1}} (1 - \bar{x}_i^k - \bar{v}_i^\ell) - \left(\bar{x}_i^k - \sum_{\ell=r}^{\sigma(k)} \bar{v}_i^\ell + \sum_{\ell \in Q^r} (1 - \bar{x}_i^k - \bar{v}_i^\ell) \right) \\
&= \bar{v}_i^r + \min \{0, 1 - \bar{x}_i^k - \bar{v}_i^r\} = \min \{\bar{v}_i^r, 1 - \bar{x}_i^k\} \geq 0.
\end{aligned}$$

Hence, $W(r)$ reaches its minimum value for the smallest r such that $W(r) - W(r-1) \leq 0$ and $W(r+1) - W(r) > 0$.

Furthermore, noticing in (16) that $\sum_{\ell \in Q^{r+1}} (1 - \bar{x}_i^k - \bar{v}_i^\ell) \leq 0 \forall r$ allows us to deduce that $W(r) - W(r-1) \leq 0$ provided that $\bar{x}_i^k - \sum_{\ell=r}^{\sigma(k)} \bar{v}_i^\ell \leq 0$, i.e., if r is such that $\bar{x}_i^k \leq \sum_{\ell=r}^{\sigma(k)} \bar{v}_i^\ell$. This fact saves us having to compute the whole sum (16) in order to know if $W(r) - W(r-1) \leq 0$ whenever $\bar{x}_i^k \leq \sum_{\ell=r}^{\sigma(k)} \bar{v}_i^\ell$.

Now that we have found a separation for valid inequalities (12), the next step consists in defining a procedure to incorporate these inequalities into formulation (SLL₁) dynamically in a branch-and-cut framework where the starting subproblem of every child node is the final formulation of the parent node with the corresponding branching x - or v -variable fixed to either zero or one. A scheme of this procedure is depicted in Algorithm 1, and these inequalities are only added into the formulation provided that the node depth in the branching tree is less than or equal to 4. The termination criteria is that the optimal solution of the linear relaxation of that node does not improve in the last iteration. Both the algorithm and the branch-and-cut procedure used to include dynamically inequalities (14) into model (SLL₂) are analogous to these ones.

Algorithm 1 Separation of inequalities (12)

Let $(\bar{x}_i^k, \bar{v}_i^\ell, \bar{z}^k)$ be an optimal fractional solution of the linear relaxation of (SLL₁).

For every customer $k \in K$ do

Step 1. For every product $i \in S^k$ do

Step 1.1. Set $r_i^k = 0$.

Step 1.2. If $r_i^k < \sigma(k)$ and $\sum_{\ell=1}^k v_i^\ell \leq x_i^k$, update $r_i^k := r_i^k + 1$ and go to Step 1.2.

Otherwise, go to Step 1.3.

Step 1.3. If $r_i^k < \sigma(k)$ and $W(r+1) - W(r) \leq 0$, $r_i^k := r_i^k + 1$ and go to Step 1.3.

Otherwise, go to Step 2.

Step 2. Set $Q_i^k := \{\ell \in \{1, \dots, r_i^k - 1\} : \bar{x}_i^k + \bar{v}_i^\ell > 1\} \forall i \in S^k$.

Step 3. Incorporate constraint

$$\bar{z}^k \leq \sum_{i \in S^k} \left(b^{r_i^k} \bar{x}_i^k + \sum_{\ell=r_i^k+1}^{\sigma(k)} (b^\ell - b^{r_i^k}) \bar{v}_i^\ell + \sum_{\ell \in Q_i^k} (b^\ell - b^{r_i^k}) (\bar{x}_i^k + \bar{v}_i^\ell - 1) \right)$$

to the formulation if and only if it is violated.

8 Computational results

A computational experiment has been carried out in order to compare the different models and check the performance of the valid inequalities and the preprocessing in terms of time and duality gap. The commercial IP solver used through all the testing was XPress-IVE version 1.24.12, on a Dell PowerEdge T110 II Server (Intel Xeon E3-1270, 3.40GHz) with 16 GB of RAM.

I	S ^k	(BL ₁)				(BL ₂)				(SLL ₁)				(SLL ₂)			
		Nodes	InGap	t(s)	Sol	Nodes	InGap	t(s)	Sol	Nodes	InGap	t(s)	Sol	Nodes	InGap	t(s)	Sol
5	1	1	0.0	0.0	10	1	0.0	0.0	10	1	0.0	0.0	10	1	0.0	0.0	10
5	2	486	25.2	0.3	10	566	24.5	0.3	10	198	7.0	0.2	10	60	3.6	0.1	10
5	3	2543	27.0	0.8	10	2493	26.8	0.7	10	1207	13.2	0.5	10	165	4.7	0.2	10
5	4	6681	26.2	1.5	10	5824	26.0	1.2	10	7012	15.3	1.2	10	282	6.3	0.4	10
5	5	9242	25.5	2.3	10	11903	25.3	2.1	10	33559	16.3	4.1	10	461	6.2	0.5	10
15	2	1413	14.2	0.3	10	1617	14.2	0.3	10	117	3.1	0.1	10	81	1.6	0.1	10
15	4	35402	8.8	4.0	10	47288	8.8	4.9	10	2091	4.7	0.6	10	153	1.6	0.2	10
15	8	1316143	7.5	313.2	7	1597499	7.6	331.0	6	133072	5.6	18.7	10	737	2.6	0.7	10
15	12	1587558	7.6	575.1	1	1803529	7.9	599.4	0	469085	5.6	77.0	10	1981	3.0	1.6	10
15	15	1232120	6.2	543.0	3	1396684	7.7	599.5	0	474677	4.8	112.2	10	3372	2.9	2.4	10
25	3	6614	5.1	0.9	10	5825	5.1	0.9	10	229	2.2	0.2	10	60	0.6	0.1	10
25	7	128768	1.7	16.0	10	1099694	1.7	138.6	9	1224	1.2	0.8	10	147	0.5	0.3	10
25	13	1684052	0.9	393.0	5	2159784	0.9	525.7	2	3763	0.6	2.3	10	352	0.4	0.8	10
25	19	1494673	2.3	599.5	0	1342037	3.2	599.4	0	21761	0.8	13.9	10	753	0.6	1.7	10
25	25	888882	2.0	599.4	0	893893	4.2	599.5	0	27361	0.6	22.1	10	641	0.4	1.8	10
Average		559638	10.7	203.3		691242	10.9	226.9		78357	5.4	16.9		616	2.3	0.7	

Table 3: Comparison of formulations (BL) and (SLNL) linearised both using z^k - and z_i^k -variables (10 instances averaged per line)

I	S ^k	(SLL ₁) + branch-and-cut						(SLL ₂) + branch-and-cut					
		Cuts	Nodes	InGap	CGap	t(s)	Sol	Cuts	Nodes	InGap	CGap	t(s)	Sol
5	1	11.2	1.0	0.0	0.0	0.0	10	11.1	1.0	0.0	0.0	0.0	10
5	2	12.6	1.1	7.0	0.0	0.0	10	11.8	1.3	3.6	0.0	0.0	10
5	3	15.4	2.7	13.2	0.2	0.1	10	13.2	2.1	4.7	0.2	0.1	10
5	4	18.1	3.2	15.3	0.5	0.3	10	14.6	4.9	6.3	0.5	0.2	10
5	5	20.7	2.5	16.3	0.2	0.5	10	16.1	3.3	6.2	0.2	0.2	10
15	2	22.9	1.0	3.1	0.0	0.0	10	16.4	1.0	1.6	0.0	0.0	10
15	4	27.6	1.1	4.7	0.0	0.1	10	19.1	1.0	1.6	0.0	0.1	10
15	8	31.1	2.3	5.6	0.1	1.0	10	21.5	1.6	2.6	0.1	0.3	10
15	12	35.0	5.0	5.6	0.3	7.0	10	23.9	5.8	3.0	0.3	1.2	10
15	15	37.0	4.4	4.8	0.2	14.4	10	25.7	3.8	2.9	0.2	1.5	10
25	3	40.8	1.0	2.2	0.0	0.1	10	27.7	1.0	0.6	0.0	0.0	10
25	7	45.6	1.6	1.2	0.0	0.7	10	33.0	1.3	0.5	0.0	0.2	10
25	13	48.8	1.0	0.6	0.0	3.2	10	36.9	1.1	0.4	0.0	0.6	10
25	19	52.7	1.5	0.8	0.0	21.8	10	39.8	1.1	0.6	0.0	1.4	10
25	25	57.0	1.5	0.6	0.0	47.2	10	43.7	1.1	0.4	0.0	2.5	10
Average		31.8	2.1	5.4	0.1	6.4		23.6	2.1	2.3	0.1	0.6	

Table 4: Comparison of models (SLL₁) and (SLL₂) including the branch-and-cut algorithms described in Section 7 (10 instances averaged per line)

$ I $	$ S^k $	%X	%V	(BL₁)			(BL₂)			(SLL₁)			(SLL₁) + branch-and-cut (SLL₂)			(SLL₂) + branch-and-cut							
				Nodes	InGap	t(s)	Nodes	InGap	t(s)	Nodes	InGap	t(s)	Cuts	Nodes	InGap	t(s)	Cuts	Nodes	InGap	t(s)			
5	1	0	76	1.0	0.0	0.0	1.0	0.0	0.0	1.0	0.0	0.0	1.0	0.0	0.0	4.7	1.0	0.0	0.0				
5	2	6	55	0.8	17.9	0.2	0.7	17.5	0.1	1.4	4.5	0.5	28.3	1.0	0.0	0.1	0.7	3.6	0.1	27.3	1.0	0.0	0.0
5	3	8	41	11.3	20.8	1.0	4.8	20.6	0.5	119.0	8.8	1.3	69.9	1.0	0.2	0.2	7.6	4.7	0.7	61.5	1.0	0.2	0.1
5	4	11	25	42.8	20.3	1.6	18.0	20.0	1.3	779.4	11.9	2.2	109.7	2.0	0.5	0.7	45.5	6.3	1.4	92.4	1.1	0.5	0.6
5	5	13	12	62.2	19.4	1.8	31.2	19.2	1.8	3181.6	12.5	3.4	148.4	2.0	0.2	0.9	71.9	6.2	1.9	138.8	1.3	0.2	0.4
15	2	17	87	0.8	3.7	0.0	0.6	3.7	0.0	0.6	1.8	0.1	14.7	1.0	0.0	0.0	0.7	1.6	0.0	15.7	1.0	0.0	0.0
15	4	28	77	0.7	4.7	0.2	0.7	4.7	0.1	6.6	1.8	0.3	31.5	1.0	0.0	0.1	0.8	1.6	0.2	24.5	1.0	0.0	0.0
15	8	37	62	28.8	5.7	2.2	2.4	5.7	1.4	109.9	3.4	2.2	82.8	1.0	0.1	0.7	1.8	2.6	1.3	57.9	1.0	0.1	0.3
15	12	42	49	147.4	5.9	3.6	56.4	5.9	2.4	2438.8	4.3	6.5	185.2	1.1	0.3	3.2	39.7	3.0	2.4	119.9	3.2	0.3	1.3
15	15	45	43	197.4	5.1	7.5	39.0	5.1	5.0	2818.1	4.0	11.0	228.8	1.5	0.2	5.2	87.6	2.9	4.5	151.2	1.0	0.2	1.1
25	3	39	91	0.5	1.0	0.0	0.7	0.9	0.0	0.2	0.6	0.0	10.8	1.0	0.0	0.0	0.6	0.6	0.0	9.4	1.0	0.0	0.0
25	7	59	86	0.4	0.8	0.1	0.8	0.8	0.1	0.6	0.5	0.1	18.9	0.9	0.0	0.1	0.5	0.5	0.1	14.0	1.0	0.0	0.1
25	13	70	81	0.4	0.5	0.1	0.8	0.5	0.2	0.7	0.4	0.2	25.6	1.0	0.0	0.1	0.7	0.4	0.2	18.8	1.0	0.0	0.1
25	19	73	77	7.6	0.8	0.7	0.6	0.8	0.7	4.4	0.7	0.9	45.7	1.0	0.0	0.3	0.9	0.6	0.8	32.2	1.0	0.0	0.1
25	25	77	74	4.8	0.6	0.5	0.8	0.6	0.6	8.5	0.5	0.8	46.3	1.0	0.0	0.4	0.7	0.4	0.7	32.5	1.0	0.0	0.2
Average		35	62	33.8	7.2	1.3	10.6	7.1	1.0	631.4	3.7	2.0	70.1	1.2	0.1	0.8	17.4	2.3	1.0	53.4	1.2	0.1	0.3

Table 5: Comparison of models (BL₁) and (BL₂) with models (SLL₁) and (SLL₂) including our branch-and-cut method, applying the preprocessing described in Section 4 to all the instances (10 instances averaged per line)

To begin with, we performed a preliminary computational study to test models (BNL^v) and (SLNL). So as to be able to compare the duality gaps and times of resolution provided by these models, we disabled automatic cuts and switched Xpress presolve settings off. In this experiment, the instances include 30 customers whose budgets have been generated independent and uniformly at random, sets of products of sizes 5, 15 and 25, and lists of preferences of sizes the 10, 25, 50, 75 and 100 percent of the number of products. The items included in the lists of preferences and their order have also been selected independent and uniformly at random, and the number of products in the list ($|S^k|$) is the same for every customer in all instances. As observed in Section 2, the values of s_i^k can be chosen in different but equivalent ways. For the computational study, we have fixed $s_i^k = |I| - n + 1$ if i is the n -th most preferred product for customer k , $\forall k \in K, i \in S^k$, in an analogous way as how they appear in Figures 1-4 in Section 5.

Table 3 shows the performance of models (BNL^v) and (SLNL) linearised by means of z^k - (obtaining, respectively, models (BL₁) and (SLL₁)) and z_i^k -variables (resulting in models (BL₂) and (SLL₂)), as described in Section 6. Each row of the table shows the average for 10 instances, and the only preprocessing applied to the instances consisted in setting $x_i^k = 1$ for every richest customer k and for every product $i \in S^k$ such that $B(k, i) = \emptyset$. The table shows the number of products ($|I|$), the number of products in which every customer is interested ($|S^k|$), the number of nodes of the branching tree (Nodes), the duality gap between the integer and the linear optimal values (InGap), the time in seconds needed to optimally solve the instance taking into account that the limit was settled to 600 seconds (t(s)), and the number of instances solved within that time period for the models for which not every instance was solved (Sol).

The average gap using both formulations (BL₁) and (BL₂) is of approximately 11%. This gap is halved when using model (SLL₁), and divided by four when using model (SLL₂), which corresponds with a reduction in the resolution times. Thus, it is remarkable how, whilst models (BL₁) and (BL₂) prove unable to solve to optimality most of the instances with 15 and 25 customers and dense matrices of preferences, models (SLL₁) and (SLL₂) solve all of these instances and the times of resolution decrease in two orders of magnitude (instances of 5 products are too easy to expect a great improvement). The average number of nodes is also drastically reduced (in one order of magnitude) when using the linearisations of formulation (SLNL) instead of the ones coming from model (BNL^v). Finally, we can see that the model (SLL₂) provides the best results in terms of duality gap and resolution times, solving all of the instances in an average time of less than a second, with an average gap of less than 3%.

In order to check the valid inequalities proposed for formulations (SLL₁) and (SLL₂) in Section 7, we implemented a branch-and-cut algorithm following the separation procedure explained in that section. Thus, in every node of the branching tree, a fractional solution $(\bar{v}, \bar{x}, \bar{z})$ was obtained after solving the linear relaxation of the corresponding subproblem, and, provided that the depth of this node in the tree was 4 or less, we checked for valid inequalities (12) or (14), respectively, and re-optimized the subproblem until no more valid inequalities were violated or the linear relaxation bound was no further improved. Table 4 shows the average number of cuts in the branching tree (Cuts), the average number of nodes of the branching tree (Nodes), the average duality gap (Gap) between the integer and the linear optimal values (extracted from Table 3), the average duality gap after the cuts in the root node (GapCuts) and the time in seconds (t(s)) spent to optimally solve

the instances.

With respect to the same models without the branch-and-cut algorithm, it is remarkable that the average duality gaps are reduced in one order of magnitude after adding the cuts in the root node, and the number of nodes in the branching tree is drastically reduced as well. The average resolution times are also improved for all instances except for the ones with 25 products and complete list of preferences.

With the aim of testing the performance of the preprocessing proposed in Section 4, we ran the same instances with the six previous models after fixing x - and v -variables to zero by applying Propositions 4.1 and 4.5, respectively. Table 5 shows the results obtained. Regarding the (SLL₁) and (SLL₂) models with the branch-and-cut procedure, we have not included in the table the average number of nodes of the branching tree in these two models is always below 3. Furthermore, the average duality gaps are reduced to zero in all cases after the cuts in the root node.

In order to create the instances of our main computational study, we have designed a model based on the Characteristics Model proposed by Fernandes et al. in [7]. This model has an economic interpretation, and focuses on the idea that each product has a profile of characteristics, and each customer is interested in several of them. In this way, a product will be more preferred by a client than another provided that more of its characteristics, or the most important ones, are among the ones he desires.

Let C be the set of characteristics, o the number of options for any characteristic and p the number of options in which a customer is interested for any characteristic. The characteristics of every product i are represented by means of a vector of options $E^i = (e_c^i)$, $c \in C$, whose entries are in the set $\{1, 2, \dots, o\}$, chosen independent and uniformly at random. The set of characteristics in which a customer k is interested is represented by a matrix $A_{|C| \times o}^k = (a_{cv}^k)$ where, for every row, p positions are set independent and uniformly at random to 1 (the ones in which k is interested) and $o - p$ positions to 0. The relevance of each characteristic c is determined by its weight $w(c)$, so that $w(c) > w(c')$ if characteristic c is considered (for every customer) more important than c' . In this way, the score each customer k gives to a product i is defined as the sum of the weights of the characteristics of i in which k is interested, i.e., $\text{score}_i^k := \sum_{c \in C} w(c) a_{ce_i}^k$. The preferences of a customer are based on the score he has given to each product, since customer k will prefer product i over product j if and only if $\text{score}_i^k > \text{score}_j^k$. Each customer is interested in $s \in \{1, \dots, |I|\}$ products; therefore, if $s < |I|$, the list of preferences of every customer will only include the s products with the greatest scores for each of them. Finally, the budgets of the customers are integers randomly selected.

The instances for the computational experiment have been created fixing the number of options of each characteristic as $o = 8$; the number of options preferred by each customer as $p = 7$; the number of characteristics as $|C| = 50|I|$; and the minimum and maximum values of a budget, respectively, as 1 and $2|K|$. With the aim of testing the performance of the models (SLL₁) and (SLL₂) with the branch-and-cut procedure, with and without preprocessing, using instances of different sizes and densities, we have created instances of 50, 100 and 150 customers and $0.1|K|$, $0.5|K|$ and $|K|$ products. And once the customers (including their budgets and scores for each product) and products are randomly created following the previously described procedure, we have created four different instances by modifying s , that is, including lists of preferences of 1 product, the 20 percent, half of

the products and all of them. The lists of preferences of every customer have equal sizes $s = |S^k|$ in all the instances.

Table 6 (resp. 7) shows the results obtained when running the previously described set of instances in Xpress with model (SLL₁) (resp. model (SLL₂)) with the branch-and-cut algorithm, with and without the preprocessing techniques described in Section 4. Each row of the tables show the average results for 10 instances. Both tables include the number of customers of the instance ($|K|$), the number of products ($|I|$), the number of products in which every customer is interested ($|S^k|$), the average number of nodes of the branching tree (Nodes), the average duality gap between the best integer solution found and the optimal solution of the linear relaxation (InGap), the average duality gap between the best solution found and the optimal fractional solution after the cuts in the root node (CGap), the average time in seconds needed to optimally solve the instances taking into account that the limit was settled to 1200 seconds (t(s)), and the number of instances solved within that time period (Sol). In the models which include preprocessing, the tables also show the average percentage of x - and v -variables fixed to zero during the preprocessing ((%X) and (%V), respectively). If for some instances of a row a model has not obtained an optimal solution with positive optimal value before the time limit, it is included in the corresponding table the average gap (InGap) and the average gap after the cuts in the root node (CGap) with the number of instances used to calculate such averages between parenthesis; if this occurs for all the instances of a row, the tables show a hyphen in columns (InGap) and (CGap) of such row.

Regarding the instances, it is noticeable from the results of Tables 6 and 7 that the difficulty to solve them increases when the number of products in which every customer is interested grows. Furthermore, if we compare both models without focusing on the preprocessing, it is clear that model (SLL₂) performs better than model (SLL₁). Thus, even though the number of cuts in some instances is inferior with model (SLL₁), the average number of nodes explored in the branching tree along with the average duality gaps (InGap) and (CGap) decrease for model (SLL₂), and as a result a greater number of instances are solved with the second model.

When comparing the results given by both models with and without the preprocessing techniques applied to the instances, we can observe that the preprocessing produces a clear improvement in the resolution of the instances in which the number of products is greater compared to the number of customers, i.e., those who allow to fix a greater number of x -variables. This fact becomes more obvious for the instances with $|K| = |I|$, which both models with the preprocessing solve within a few seconds. It is also remarkable that, even though we know from the results in Section 4 that an optimal solution can be derived by inspection for instances with $|K| = |I| = |S^k|$, we have decided to include them in the experiment to be able to compare the results.

9 Conclusion

We have studied the Rank Pricing Problem, proposing two new formulations and linearisation techniques. We have also carried out a characterisation of all the families of clique inequalities of the constraints of the model which constitute a special case of the SPP, and

K	I	S ^k	(SLL ₁) + branch-and-cut						(SLL ₁) + branch-and-cut + preprocessing							
			Cuts	Nodes	InGap	CGap	t(s)	Sol	%X	%V	Cuts	Nodes	InGap	CGap	t(s)	Sol
50	5	1	68	1	8.4	0.0	0.1	10	0	76	65	1	7.4	0.0	0.0	10
50	5	2	132	2	44.7	0.5	0.4	10	4	56	124	2	38.8	0.5	0.3	10
50	5	3	227	5	39.9	0.8	1.9	10	5	37	216	4	35.8	0.8	1.6	10
50	5	5	516	9	33.5	0.9	8.1	10	7	6	474	13	32.1	0.9	6.9	10
50	25	1	77	1	3.7	0.0	0.1	10	0	95	66	1	2.1	0.0	0.1	10
50	25	5	318	1	10.5	3.1	1.8	10	33	84	186	1	7.6	0.0	0.2	10
50	25	13	1338	35	7.8	7.8	29.9	10	41	66	541	16	7.6	0.3	10.7	10
50	25	25	2331	64839	9.8	9.8	812.3	5	45	45	1286	99	6.2	0.3	91.3	10
50	50	1	85	1	2.3	0.0	0.1	10	0	97	75	1	1.2	0.0	0.1	10
50	50	10	823	5033	1.3	1.3	22.2	10	73	93	137	1	1.2	0.0	0.1	10
50	50	25	358	252303	2.2	2.2	1184.0	1	87	92	140	1	1.0	0.0	0.1	10
50	50	50	282	143773	3.1	3.1	1199.4	0	93	92	144	1	1.0	0.0	0.2	10
100	10	1	134	1	7.3	0.0	0.1	10	0	87	130	1	5.9	0.0	0.1	10
100	10	2	270	1	44.1	0.0	0.6	10	4	76	243	1	36.5	0.0	0.4	10
100	10	5	1130	735	34.9	2.2	40.6	10	7	47	999	815	33.0	2.2	43.7	10
100	10	10	3348	2944	28.3	8.6	1130.7	1	8	8	3099	2969	27.5	3.1	1180.7	1
100	50	1	158	1	3.1	0.0	0.2	10	0	98	133	1	1.4	0.0	0.2	10
100	50	10	1797	8	7.5	7.5	61.2	10	38	85	874	5	7.1	0.1	14.5	10
100	50	25	3821	3790	52.1	52.1	1199.4	0	44	68	2790	532	5.9	0.6	834.8	6
100	50	50	1157	13096	48.6	48.6	1200.4	0	47	45	4841	0	-	-	1209.0	0
100	100	1	172	1	1.8	0.0	0.3	10	0	99	149	1	0.8	0.0	0.2	10
100	100	20	1507	8133	11.4	11.4	1199.6	0	84	96	297	1	0.6	0.0	0.4	10
100	100	50	969	3365	26.4	26.4	1200.0	0	92	95	314	1	0.5	0.0	0.8	10
100	100	100	1139	2075	28.6	28.6	1200.3	0	96	95	315	1	0.5	0.0	0.8	10
150	15	1	206	1	7.7	0.0	0.2	10	0	92	197	1	6.2	0.0	0.1	10
150	15	3	730	62	39.3	0.7	10.7	10	5	77	651	77	34.5	0.7	9.0	10
150	15	8	3320	2443	30.7	4.6	1199.8	0	8	45	2949	1900	29.4	3.9	1200.3	0
150	15	15	1821	8562	57.1	57.1	1201.7	0	9	8	7746	0	60.6 ₍₄₎	35.3 ₍₄₎	1207.3	0
150	75	1	237	1	2.7	0.0	0.4	10	0	98	203	1	1.2	0.0	0.3	10
150	75	15	5831	14	77.0	77.0	1022.6	4	41	86	2146	713	6.2	0.3	493.5	9
150	75	38	2094	807	91.6	91.6	1199.3	0	46	69	5447	0	-	-	1201.8	0
150	75	75	1316	331	70.9	70.9	1200.2	0	48	47	3542	0	-	-	1240.6	0
150	150	1	259	1	1.7	0.0	0.5	10	0	99	228	1	0.6	0.0	0.6	10
150	150	30	1380	1325	17.7	17.7	1199.4	0	88	97	445	1	0.4	0.0	1.4	10
150	150	75	1600	139	57.7	57.7	1201.3	0	94	97	462	1	0.3	0.0	2.0	10
150	150	150	357	2	133.6 ₍₃₎	133.6 ₍₃₎	1285.6	0	97	96	464	1	0.3	0.0	2.9	10

Table 6: Results obtained for model (SLL₁) with the branch-and-cut algorithm, without and with the preprocessing described in Section 4, for instances of 50, 100 and 150 customers (10 instances averaged per line)

K	I	S ^k	(SLL ₂) + branch-and-cut							(SLL ₂) + branch-and-cut + preprocessing						
			Cuts	Nodes	InGap	CGap	t(s)	Sol	%X	%V	Cuts	Nodes	InGap	CGap	t(s)	Sol
50	5	1	69	1	1.6	0.0	0.0	10	0	76	72	1	1.6	0.0	0.0	10
50	5	2	168	1	7.4	0.5	0.3	10	4	56	170	2	7.3	0.5	0.3	10
50	5	3	293	4	10.6	0.8	1.4	10	5	37	273	3	10.4	0.8	1.5	10
50	5	5	576	6	15.6	0.9	4.1	10	7	6	524	8	15.1	0.9	3.5	10
50	25	1	61	1	1.3	0.0	0.1	10	0	95	67	1	1.3	0.0	0.1	10
50	25	5	519	1	7.7	0.0	0.4	10	33	84	295	1	5.9	0.0	0.2	10
50	25	13	1777	13	17.0	0.3	6.0	10	41	66	767	8	11.2	0.3	2.2	10
50	25	25	4213	19	29.5	0.3	18.0	10	45	45	1651	12	17.6	0.3	7.5	10
50	50	1	57	1	1.1	0.0	0.1	10	0	97	76	1	1.1	0.0	0.1	10
50	50	10	1377	1	9.9	0.0	1.1	10	73	93	226	1	2.7	0.0	0.1	10
50	50	25	4623	1	24.3	0.0	6.7	10	87	92	268	1	3.1	0.0	0.1	10
50	50	50	11186	1	48.5	0.0	29.2	10	93	92	282	1	3.2	0.0	0.1	10
100	10	1	144	1	0.8	0.0	0.1	10	0	87	147	1	0.8	0.0	0.1	10
100	10	2	338	1	4.9	0.0	0.3	10	4	76	333	1	4.9	0.0	0.2	10
100	10	5	1292	173	12.7	2.2	16.6	10	7	47	1191	513	12.5	2.2	22.5	10
100	10	10	3122	2308	17.9	2.8	112.9	10	8	8	2754	4378	17.4	2.8	164.8	10
100	50	1	123	1	0.6	0.0	0.2	10	0	98	135	1	0.6	0.0	0.2	10
100	50	10	2528	4	8.1	0.1	7.8	10	38	85	1232	3	6.1	0.1	3.4	10
100	50	25	8345	247	16.2	0.5	125.9	10	44	68	3442	901	10.4	0.5	113.8	10
100	50	50	20934	719	29.3	0.7	900.5	6	47	45	8237	1956	17.2	0.8	815.3	5
100	100	1	118	1	0.6	0.0	0.2	10	0	99	149	1	0.6	0.0	0.2	10
100	100	20	6849	1	9.9	0.0	13.2	10	84	96	543	1	1.6	0.0	0.3	10
100	100	50	23424	1	24.5	0.0	123.3	10	92	95	638	1	2.0	0.0	0.5	10
100	100	100	58245	1	49.0	0.0	803.2	10	96	95	676	1	2.1	0.0	0.6	10
150	15	1	218	1	0.5	0.0	0.2	10	0	92	220	1	0.5	0.0	0.1	10
150	15	3	942	32	7.6	0.7	7.6	10	5	77	870	28	7.5	0.7	6.2	10
150	15	8	3476	9046	14.4	3.0	810.4	5	8	45	3004	20033	14.1	2.9	938.9	6
150	15	15	7548	5571	20.1	5.0	1199.7	0	9	8	6619	2463	19.5	5.0	1199.6	0
150	75	1	183	1	0.4	0.0	0.3	10	0	98	207	1	0.4	0.0	0.3	10
150	75	15	6569	118	8.0	0.3	69.8	10	41	86	2915	120	5.9	0.3	29.3	10
150	75	38	22443	418	16.4	1.1	1135.0	1	46	69	8638	1602	10.2	0.8	1108.3	1
150	75	75	55813	0	139.3	87.8	1201.5	0	48	47	20458	299	22.4	6.1	1201.8	0
150	150	1	174	1	0.4	0.0	0.4	10	0	99	230	1	0.4	0.0	0.5	10
150	150	30	17432	1	10.1	0.0	66.7	10	88	97	876	1	1.2	0.0	1.0	10
150	150	75	60096	1	25.3	0.0	998.2	9	94	97	987	1	1.4	0.0	1.2	10
150	150	150	122181	0	-	-	1258.3	0	97	96	1006	1	1.4	0.0	1.3	10

Table 7: Results obtained for model (SLL₂) with the branch-and-cut algorithm, without and with the preprocessing described in Section 4, for instances of 50, 100 and 150 customers (10 instances averaged per line)

as a result we have been able to determine which of them are part of our model. In order to improve the bound provided by the linear relaxation of the models, we have derived some families of valid inequalities by making an in-depth analysis of its structure. Some preprocessing techniques have also been applied so as to reduce the size of the instances before solving them. The different models and results have been tested through the development of a branch-and-cut algorithm and by means of a computational experiment which shows how these techniques allow the resolution of instances of greater size with reduced running times and duality gaps.

A challenging future line of research consists in generalizing the Rank Pricing Problem by considering limited supply. Adding capacity constraints on product supply would require the development of new formulations, since a decision must be taken about how to allocate the available units of each product to the customers. In particular, the development of models which provide an envy-free solution, i.e., a solution in which no customer can afford any other product more preferable for him than his own, or not, would lead to different solution strategies.

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