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# 1 How long does it take for all users in a social 2 network to choose their communities?

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## 12 — Abstract —

13 We consider a community formation problem in social networks, where the users are either  
14 friends or enemies. The users are partitioned into conflict-free groups (*i.e.*, independent sets  
15 in the *conflict graph*  $G^- = (V, E)$  that represents the enmities between users). The dynamics  
16 goes on as long as there exists any set of at most  $k$  users,  $k$  being any fixed parameter, that  
17 can change their current groups in the partition *simultaneously*, in such a way that they all  
18 strictly increase their utilities (number of friends *i.e.*, the cardinality of their respective groups  
19 minus one). Previously, the best-known upper-bounds on the maximum time of convergence were  
20  $\mathcal{O}(|V|\alpha(G^-))$  for  $k \leq 2$  and  $\mathcal{O}(|V|^3)$  for  $k = 3$ , with  $\alpha(G^-)$  being the independence number of  
21  $G^-$ . Our first contribution in this paper consists in reinterpreting the initial problem as the study  
22 of a dominance ordering over the vectors of integer partitions. With this approach, we obtain for  
23  $k \leq 2$  the tight upper-bound  $\mathcal{O}(|V| \min\{\alpha(G^-), \sqrt{|V|}\})$  and, when  $G^-$  is the empty graph, the  
24 exact value of order  $\frac{(2|V|)^{3/2}}{3}$ . The time of convergence, for any fixed  $k \geq 4$ , was conjectured to  
25 be polynomial [7, 14]. In this paper we disprove this. Specifically, we prove that for any  $k \geq 4$ ,  
26 the maximum time of convergence is an  $\Omega(|V|^{\Theta(\log |V|)})$ .

27 **2012 ACM Subject Classification** Networks, Theory of computation

28 **Keywords and phrases** communities, social networks, integer partitions, coloring games, graphs

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30 **Foreword:** The organizers of a wedding (party) have difficulties in arranging place  
31 settings for the guests as there are many incompatibilities among those who do not want  
32 to be at the same table as an "enemy" (ex girl (boy) friend, boss or employee, student or  
33 supervisor, etc. . . ). The organizers realize that they have no set of 5 pairwise friends and  
34 so allow people place themselves. Successively each person joins a table where she has no  
35 enemies or starts a new table. At any time a person can move from one table to another  
36 table (of course where she has no enemy) if in doing so she increases strictly the number of

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<sup>2</sup> Part of this work has been done during his post-doc at Columbia University in the City of New York.



37 friends she has at the new table. The process converges relatively fast (linear time). Some  
 38 time later the organizers of FUN having heard about this scenario decide to use the same  
 39 process to place the participants in different groups for the social activities of the afternoon.  
 40 Each participant registers first in her own group. The organizers decide to accelerate the  
 41 process by authorizing not just one person but any subset of 4 persons to change their mind  
 42 and leave the group in which they are registered to join another group or create a new group;  
 43 these persons move only if they desire to do so, that is, they increase strictly the number  
 44 of friends. Surprisingly the process takes a very long (exponential) time and night arrives  
 45 before groups are formed. As we will discover, the exponential time derives from the fact  
 46 that at FUN all the persons are friends and there are no enemies due to the use of moves  
 47 implying 4 persons. At this point the reader (and the organizers) might ask why we see such  
 48 a difference in behaviors and how long does it takes for users of a social network to form  
 49 groups. The answers to these questions and "all you wanted to know but were afraid to ask"  
 50 will be revealed in this paper.

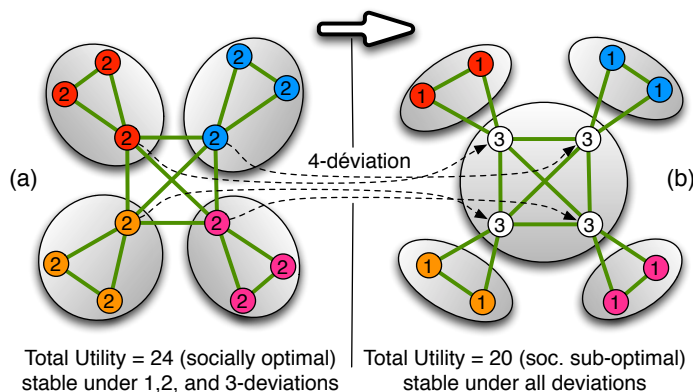
## 51    **1    Introduction**

52 Community formation is a fundamental problem in social network analysis. It has already  
 53 been modeled in several ways, each trying to capture key aspects of the problem. The model  
 54 studied in this paper has been proposed in [14] in order to reflect the impact of information  
 55 sharing on the community formation process. Although it is a simplified model, we show that  
 56 its understanding requires us to solve combinatorial problems that are surprisingly intricate.  
 57 More precisely, we consider the following dynamics of formation of groups (communities)  
 58 in social networks. Each group represents a set of users sharing about some information  
 59 topic. We assume for simplicity that each user shares about a given topic in only one group.  
 60 Therefore the groups will partition the set of users. We follow the approach of [14]. An  
 61 important feature is the emphasis on incompatibility between some pairs of users that we  
 62 will call enemies. Two enemies do not want to share information and so will necessarily  
 63 belong to different groups. In the general model one consider different degrees of friendship  
 64 or incompatibilities. Here we will restrict to the case where two users are either friends  
 65 or enemies – as noted in [14], even a little beyond this case, the problem quickly becomes  
 66 intractable. As example, if we add a neutral (indifference) relation, there are instances for  
 67 which there is no stability.

68     The social network is often modeled by the friendship graph  $G^+$  where the vertices are  
 69 the users and an edge represents a friendship relation. We will use this graph to present the  
 70 first notions and examples. However, for the rest of the article and the proofs we will use the  
 71 complementary graph, that we call the *conflict graph* and denote by  $G^-$ ; here the vertices  
 72 represent users and the edges represent the incompatibility relation. We assign each user a  
 73 *utility* which is the number of friends in the group to which she belongs. Equivalently, the  
 74 utility is the size of the group minus one, as in a group there is no pair of enemies; in [14]  
 75 this is modeled by putting the utility as  $-\infty$  when there is an enemy in the group.

76     In the example of Figure 1, the graph depicted is the friendship graph: the edges represent  
 77 the friendship relation, and if there is no edge it corresponds to a pair of enemies. Figure 1(a)  
 78 depicts a partition of 12 users composed of 4 non-empty groups each of size 3. The integers  
 79 on the vertices represent the utilities of the users which are all equal to 2. Figure 1(b)  
 80 depicts another partition consisting of 5 groups with one group of size 4 (where users have  
 81 utility 3) and 4 groups of size 2 (where users have utility 1).

82     In this study we are interested in the dynamics of formation of groups. Another important



■ **Figure 1** A friendship graph with 12 vertices (users). (a) 3-stable partition that is not 4-stable but it is optimal in terms of total utility. (b)  $k$ -stable partition for any  $k \geq 1$  that is not optimal in terms of total utility.

83 feature of [14], taken into account in the dynamics, is the notion of bounded cooperation  
 84 between users. More precisely, the dynamics is as follows: initially each user is alone in her  
 85 own group. In the simplest case, a move consists for a specific user to leave the group to  
 86 which she belongs to join another group but only if this action increases strictly her utility  
 87 (acting in a selfish manner); in particular, it implies that a user does not join a group where  
 88 she has an enemy. In the  $k$ -bounded mode of cooperation, a set of at most  $k$ -users can leave  
 89 their respective groups to join another group, again, only if each user increases strictly their  
 90 utility. If the group they join is empty it corresponds to creating a new group. We call such  
 91 a move a  $k$ -deviation. Note that this notion is slightly different from that of  $(k + 1)$ -defection  
 92 of [14]. We will say that a partition is  $k$ -stable if there does not exist a  $k$ -deviation for this  
 93 partition.

94 The partition of Figure 1(a) is  $k$ -stable when  $k \in \{1, 2, 3\}$ . Indeed each user has at least  
 95 one enemy in each non empty other group and so cannot join another group. Furthermore,  
 96 when  $k \leq 3$ , if  $k$  users join an empty group their utility will be at most 2 and so will not  
 97 strictly increase. However, this partition is not 4-stable because there is a 4-deviation: the  
 98 four central users can join an empty group and so they increase their utilities from 2 to 3.  
 99 The partition obtained after such a 4-deviation is depicted in Figure 1(b). This partition  
 100 is  $k$ -stable for any  $k \geq 1$ . Note that the utility of the other users is now 1 (instead of 2).  
 101 Thus, we deduce that this partition is not optimal in terms of total utility (the total utility  
 102 has decreased from 24 to 20); but it is now stable under all deviations. This illustrates the  
 103 fact that users act in a selfish manner as some increase their utility, but on the contrary the  
 104 global utility decreases. For more information on the suboptimality of  $k$ -stable partitions,  
 105 *i.e.*, bounds on the price of anarchy and the price of stability, the reader is referred to [14].

### 106 1.1 Related work.

107 This above dynamics has been also modeled in the literature with *coloring games*. A coloring  
 108 game is played on the conflict graph. Players must choose a color in order to construct a  
 109 proper coloring of the graph, and the individual goal of each agent is to maximize the number  
 110 of agents with the same color as she has. On a more theoretical side, coloring games have  
 111 been introduced in [18] as a game-theoretic setting for studying the chromatic number in  
 112 graphs. Specifically, the authors in [18] have shown that for every coloring game, there

exists a Nash equilibrium where the number of colors is exactly the chromatic number of the graph. Since then, these games have been used many times, attracting attention in the study of information sharing and propagation in graphs [4, 7, 14]. Coloring games are an important subclass of the more general Hedonic games, of which several variations have been studied in the literature in order to model coalition formation under selfish preferences of the agents [10, 12, 15, 5, 8, 16]. We stress that while every coloring game has a Nash equilibrium that can be computed in polynomial-time [18], deciding whether a given Hedonic game admits a Nash equilibrium is NP-complete [1].

If the set of edges of the conflict graph is empty (edgeless conflict graph), there exists a unique  $k$ -stable partition namely that consists of the group of all the users. In [14], it is proved that there always exists a  $k$ -stable partition for any conflict graph, but that it is NP-hard to compute one (this result was also proved independently in [7]). Indeed, if  $k$  is equal to the number of users, a largest group in such a partition must be a maximum independent set of the conflict graph. In contrast, it can be computed a  $k$ -stable partition in polynomial time for every fixed  $k \leq 3$ , by using simple *better-response dynamics* [18, 7, 14]. In such an algorithm one does a  $k$ -deviation until there does not exist any one. That corresponds to the dynamics of formation of groups that we study in this work for larger values of  $k$ .

## 1.2 Additional related work and our results.

In this paper we are interested in analyzing in this simple model the convergence of the dynamics with  $k$ -deviations, in particular in the worst case. It has been proved implicitly in [14] that the dynamics always converges within at most  $\mathcal{O}(2^n)$  steps. Let  $L(k, G^-)$  be the size of a longest sequence of  $k$ -deviations on a conflict graph  $G^-$ . We first observe that the maximum value, denoted  $L(k, n)$ , of  $L(k, G^-)$  over all the graphs with  $n$  vertices is attained on the edgeless conflict graph  $G^0$  of order  $n$ . Prior to this work, no lower bound on  $L(k, n)$  was known, and the analysis was limited to potential function that only applies when  $k \leq 3$  [7, 14] giving upper bounds of  $\mathcal{O}(n^2)$  in the case  $k = 1, 2$  and  $\mathcal{O}(n^3)$  in the case  $k = 3$ . In order to go further in our analysis, the key observation is that when the conflict graph is edgeless, the dynamics depends only of the size of the groups of the partitions generated. Following [3], let an integer partition of  $n \geq 1$ , be a non-increasing sequence of integers  $Q = (q_1, q_2, \dots, q_n)$  such that  $q_1 \geq q_2 \geq \dots \geq q_n \geq 0$  and  $\sum_{i=1}^n q_i = n$ . If we rank the groups by non increasing order of their size, there is a natural relation between partition in groups and integer partitions (the size  $q_i$  of the group  $X_i$  corresponding to the integers  $q_i$  of the partition of  $n$ ). Using this relation, we prove in Section 3 that the better response dynamics algorithm reaches a stable partition in  $p_n$  steps, where  $p_n = \Theta((e^\pi \sqrt{2n/3})/n)$  denotes the number of integer partitions. This is already far less than  $2^n$ , which was shown to be the best upper bound that one can obtain for  $k \geq 4$  when using an additive potential function [14].

Table 1 summarizes our contributions described below.

- For  $k = 1, 2$ , we refine the relation between partitions into groups and integer partitions as follows.
  - In the case  $k = 1$  (Section 4.1), we prove that there is a one to one mapping between sequences of 1-deviations in the edgeless conflict graph and chains in the dominance lattice of integer partitions. Then, we use the value of the longest chain in this dominance lattice obtained in [9] to determine exactly  $L(1, n)$ . More precisely, if  $n = \frac{m(m+1)}{2} + r$ , with  $0 \leq r \leq m$ ,  $L(1, n) = 2^{\binom{m+1}{3}} + mr$ . The latter implies in particular  $L(1, n)$  is of order  $\mathcal{O}(n^{\frac{3}{2}})$ , thereby improving the previous bound  $\mathcal{O}(n^2)$ .

$k$	Prior to our work	Our results	
1	$\mathcal{O}(n^2)$ [14]	exact analysis, which implies $L(1, n) \sim \frac{(2n)^{3/2}}{3}$	Theorem 6
2	$\mathcal{O}(n^2)$ [14]	exact analysis, which implies $L(2, n) \sim \frac{(2n)^{3/2}}{3}$	Theorem 9
1-2	$\mathcal{O}(n\alpha(G^-))$ [18]	$L(k, G^-) = \Omega(n\alpha(G^-))$ for some $G^-$ and $\alpha(G^-) = \mathcal{O}(\sqrt{n})$	Theorem 12
3	$\mathcal{O}(n^3)$ [7, 14]	$L(3, n) = \Omega(n^2)$	Theorem 13
$\geq 4$	$\mathcal{O}(2^n)$ [14]	$L(k, n) = \Omega(n^{\Theta(\ln(n))})$ , $L(k, n) = \mathcal{O}(\exp(\pi\sqrt{2n/3})/n)$	Theorem 14

■ **Table 1** Previous bounds and results we obtained on  $L(k, n)$  and  $L(k, G^-)$ .

- 159 ■ In Section 4.2, we prove that any 2-deviation can be “replaced” (in some precise way)  
160 either by one or two 1-deviations, and so,  $L(2, n) = L(1, n)$ .
- 161 ■ For  $k = 1, 2$  and a general conflict graph  $G^-$ , the value of  $L(k, G^-)$  depends on the  
162 independence number  $\alpha(G^-)$  (cardinality of a largest independent set) of the conflict  
163 graph. In [18] it was proved that the convergence of the dynamics is in  $\mathcal{O}(n\alpha(G^-))$ .  
164 In the case of edgeless conflict graph, we have seen that  $L(1, n) = \Omega(n^{3/2})$  and so  
165 the preceding upper-bound was not tight. So we inferred that the convergence of the  
166 dynamics was in  $\mathcal{O}(n\sqrt{\alpha(G^-)})$ . Yet in fact we prove in Section 4.3 that, for any  
167  $\alpha(G^-) = \mathcal{O}(\sqrt{n})$ , there exists a conflict graph  $G^-$  with  $n$  vertices and independence  
168 number  $\alpha(G^-)$  for which we need a sequence of at least  $\Omega(n\alpha(G^-))$  1-deviations to  
169 reach a stable partition. *For the wedding’s example of the foreword,  $\alpha(G^-) = 4$  and  
170 so the sequence is linearly bounded.*
- 171 ■ Finally, our main contribution is obtained for  $k \geq 3$ . Prior to our work, it was known  
172 that  $L(3, n) = \mathcal{O}(n^3)$ , which follows from another application of the potential function  
173 method [14]. But nothing proved that  $L(3, n) > L(2, n)$ , and in fact it was conjectured  
174 in [7] that both values are equal. In Section 5.2, we prove (Theorem 13) that  $L(3, n) =$   
175  $\Omega(n^2)$  and thus we show for the first time that deviations can delay convergence and that  
176 the gap between  $k = 2$  and  $k = 3$  obtained from potential function is indeed justified. It  
177 was also conjectured in [14] that  $L(k, n)$  was polynomial in  $n$  for  $k$  fixed. In Section 5.1 we  
178 disprove this conjecture and prove in Theorem 14 that  $L(4, n) = \Omega(n^{\Theta(\ln(n))})$ . This shows  
179 that 4-deviations are responsible for a sudden complexity increase, as no polynomial  
180 bounds exist for  $L(4, n)$ . *This explains why in the foreword it takes an exponential time  
181 for the organizers of FUN to schedule the groups.*

## 2 Notations

183 **Conflict graph.** We refer to [2] for standard graph terminology. For the remaining of the  
184 paper, we suppose that we are given a *conflict graph*  $G^- = (V, E)$  where  $V$  is the set of  
185 vertices (called users or players in the introduction) and edges represent the incompatibility  
186 relation (*i.e.*, an edge means that the two users are enemies). The number of vertices is  
187 denoted by  $n = |V|$ . The independence number of  $G^-$ , denoted  $\alpha(G^-)$ , is the maximum  
188 cardinality of an independent set in  $G^-$ . In particular, if  $\alpha(G^-) = n$  then the conflict graph  
189 is edgeless and we denote it by  $G^\emptyset = (V, E = \emptyset)$  and call it the empty graph.

190 **Partitions and utilities.** We consider any partition  $P = X_1, \dots, X_i, \dots, X_n$  of the  
191 vertices into  $n$  independent sets  $X_i$  called groups (colors in coloring games), with some of  
192 them being possibly empty. In particular, two enemies are not in the same group. We rank  
193 the groups by non increasing size, that is  $|X_i| \geq |X_{i+1}|$ . For any  $1 \leq i \leq n$  and for any

194  $v \in X_i$ , the *utility* of vertex  $v$  is the number of other vertices in the same group as it, that  
 195 is  $|X_i| - 1$ .

196 We use in our proofs two alternative representations of the partition  $P$ . The *partition*  
 197 *vector* associated to  $P$  is defined as  $\vec{\Lambda}(P) = (\lambda_n(P), \dots, \lambda_1(P))$ , where  $\lambda_i(P)$  is the number  
 198 of groups of size  $i$ . The *integer partition* associated to  $P$  is defined as  $Q = (q_1, q_2, \dots, q_n)$   
 199 such that  $q_1 \geq q_2 \geq \dots \geq q_n \geq 0$  and  $\sum_{i=1}^n q_i = n$ , where  $q_i = |X_i|$ .

200 In the example of Figure 1(a) we have a partition  $P$  of the 12 vertices into 4 groups  
 201 each of size 3 and so  $\lambda_3(P) = 4$  and  $\lambda_i(P) = 0$  for  $i \neq 3$ ; in other words  $\vec{\Lambda}(P) =$   
 202  $(0, 0, 0, 0, 0, 0, 0, 0, 0, 4, 0, 0)$ . The integer partition  $Q(P) = (3, 3, 3, 3, 0, 0, 0, 0, 0, 0, 0, 0)$ . In  
 203 the example of Figure 1(b) we have a partition  $P'$  of the 12 vertices into one group of  
 204 size 4 and 4 groups each of size 2 and so  $\lambda_4(P') = 1$ ,  $\lambda_2(P') = 4$  and  $\lambda_i(P') = 0$   
 205 for  $i \neq 2, 4$ ; in other words  $\vec{\Lambda}(P') = (0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 4, 0)$ . The integer partition  
 206  $Q(P') = (4, 2, 2, 2, 2, 0, 0, 0, 0, 0, 0, 0)$ .

207 **k-deviations and k-stability.** We can think of a  $k$ -deviation as a move of at most  $k$   
 208 vertices which leave the groups to which they belong in  $P$ , to join another group (or create  
 209 a new group) with the necessary condition that each vertex strictly increases its utility,  
 210 thereby leading to a new partition  $P'$ . A *k-stable partition* is simply a partition for which  
 211 there exists no  $k$ -deviation. We write  $L(k, G^-)$ , resp.  $L(k, n)$ , for the length of a longest  
 212 sequence of  $k$ -deviations to reach a stable partition in  $G^-$ , resp. in any conflict graph with  
 213  $n$  vertices. Recall that we start with the partition consisting of  $n$  groups of size 1, that is,  
 214  $\vec{\Lambda}(P) = (\dots, 0, 0, 0, n)$ .

215 We next define a natural vector representation for  $k$ -deviations. The *difference vector*  $\vec{\varphi}$   
 216 associated to a  $k$ -deviation  $\varphi$  from  $P$  to  $P'$  is equal to  $\vec{\varphi} = \vec{\Lambda}(P') - \vec{\Lambda}(P)$ . In concluding  
 217 this section, we define the difference vectors for some of the  $k$ -deviations used in our proofs:

- 218 ■  $\vec{\alpha}[p, q]$ , the 1-deviation where a vertex leaves a group of size  $q + 1$  for a group of size  
 219  $p - 1$  (valid when  $p \geq q + 2$ ). In that case  $\alpha_p = 1, \alpha_{p-1} = -1, \alpha_{q+1} = -1, \alpha_q = -1$ , and  
 220  $\alpha_i = 0$  for any  $i \notin \{q, q + 1, p - 1, p\}$  (we omit for ease of reading the brackets  $[p, q]$ ).
- 221 ■  $\vec{\gamma}[p]$ , the 3-deviation where one vertex in each of 3 groups of size  $p - 1$  moves to a group  
 222 of size  $p - 3$  to form a new group of size  $p$  (valid if there are at least 3 groups of size  
 223  $p - 1$  and one of size  $p - 3$ ). In that case  $\gamma_p = 1, \gamma_{p-1} = -3, \gamma_{p-2} = 3, \gamma_{p-3} = -1$ , and  
 224  $\gamma_i = 0$  for any  $i \notin \{p - 3, p - 2, p - 1, p\}$ .
- 225 ■  $\vec{\delta}[p]$ , the 4-deviation where one vertex in each of 4 groups of size  $p - 1$  moves to a group  
 226 of size  $p - 4$  to form a new group of size  $p$  (valid if there are at least 4 groups of size  
 227  $p - 1$  and one of size  $p - 4$ ). In that case  $\delta_p = 1, \delta_{p-1} = -4, \delta_{p-2} = 4, \delta_{p-4} = -1$ , and  
 228  $\delta_i = 0$  for any  $i \notin \{p - 4, p - 2, p - 1, p\}$ . As an example, the move from the partition of  
 229 Figure 1(a) to the partition of Figure 1(b), is a 4-deviation with difference vector  $\vec{\delta}[4]$ .

### 230 3 Preliminary results

231 In [14], the authors prove that there always exists a  $k$ -stable partition, but that it is NP-hard  
 232 to compute one (this result was also proved independently in [7]). In contrast, it can be  
 233 computed a  $k$ -stable partition in polynomial time for every fixed  $k \leq 3$ , by using simple  
 234 *better-response dynamics* [18, 7, 14]. The latter results question the role of the value of  $k$  in  
 235 the complexity of computing stable partitions.

236 Formally, a better-response dynamics proceeds as follows. We start from the trivial  
 237 partition  $P_1$  consisting of  $n$  groups with one vertex in each of them. In particular, the  
 238 partition vector  $\vec{\Lambda}(P_1)$  is such that  $\lambda_1(P_1) = n$  and, for all other  $j \neq 1$ ,  $\lambda_j(P_1) = 0$ .

239 Provided there exists a  $k$ -deviation with respect to the current partition  $P_i$ , we pick any  
 240 one of these  $k$ -deviations  $\varphi$  and in so doing we obtain a new partition  $P_{i+1}$ . If there is no  
 241  $k$ -deviation, the partition  $P_i$  is  $k$ -stable. An algorithmic presentation is given in Algorithm 1.

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**Dynamics of the system (Algorithm 1)**

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**Input:** a positive integer  $k \geq 1$ , and a conflict graph  $G^-$ .

**Output:** a  $k$ -stable partition for  $G^-$ .

- 1: Let  $P_1$  be the partition composed of  $n$  singletons groups.
  - 2: Set  $i = 1$ .
  - 3: **while** there exists a  $k$ -deviation for  $P_i$  **do**
  - 4:   Set  $i = i + 1$ .
  - 5:   Choose one  $k$ -deviation and compute the partition  $P_i$  after this  $k$ -deviation.
  - 6: Return the partition  $P_i$ .
- 

242 We now prove in Proposition 1 that better-response dynamics can be used for computing  
 243 a  $k$ -stable partition for every fixed  $k \geq 1$  (but not necessarily in polynomial time). It  
 244 shows that for every fixed  $k \geq 1$ , the problem of computing a  $k$ -stable partition is in the  
 245 complexity class PLS (Polynomial Local Search), that is conjectured to lie strictly between  
 246 P and NP [13]. Recall that the problem becomes NP-hard when  $k$  is part of the input.

247 ► **Proposition 1.** For any  $k \geq 1$ , for any conflict graph  $G^-$ , Algorithm 1 converges to a  
 248  $k$ -stable partition.

249 **Proof.** Let  $P_i, P_{i+1}$  be two partitions for  $G^-$  such that  $P_{i+1}$  is obtained from  $P_i$  after some  
 250  $k$ -deviation  $\varphi$ . Let  $S$  be the set of vertices which move ( $|S| \leq k$ ) and let  $j$  be the size of  
 251 the group they join ( $j = 0$  if they create a new group). Then, the new group obtained has  
 252 size  $p = j + |S|$ . Note that all the vertices of  $S$  have increased their utilities and so, they  
 253 belonged in  $P_i$  to groups of size  $< p$ . Therefore, the coordinates of the difference vector  
 254  $\vec{\varphi}$  satisfy  $\varphi_p = 1$  and  $\varphi_j = 0$  for  $j > p$ , and so  $\vec{\Lambda}(P_i) <_L \vec{\Lambda}(P_{i+1})$  where  $<_L$  is the  
 255 lexicographical ordering. Finally, as the number of possible partition vectors is finite, we  
 256 obtain the convergence of Algorithm 1. ◀

257 An instrumental observation for our next proofs is the following:

258 ► **Observation 1.**  $L(k, n)$  is always attained on the empty conflict graph  $G^\emptyset$  of order  $n$ .

259 Indeed, any sequence of  $k$ -deviations on a conflict graph  $G^-$  is also a sequence in the empty  
 260 conflict graph with the same vertices. Note that the converse is not true as it can happen  
 261 that some moves allowed in the empty conflict graph are not allowed in  $G^-$  as they bring  
 262 two enemies in the same group.

263 Recall that we can associate to any partition  $P = X_1, \dots, X_i, \dots, X_n$  of the vertices the  
 264 integer partition  $Q = (q_1, q_2, \dots, q_n)$  such that  $q_1 \geq q_2 \geq \dots \geq q_n \geq 0$  and  $\sum_{i=1}^n q_i = n$  by  
 265 letting  $q_i = |X_i|$ . The converse is not true in general; as example it suffices to consider a  
 266 partition with  $q_1 > \alpha(G^-)$ . However the converse is true when the conflict graph is empty;  
 267 indeed it suffices to associate to an integer partition any partition of the vertices obtained  
 268 by putting in the group  $X_i$  a set of  $q_i$  vertices .

269 We can now use the value of the number  $p_n$  of the number of integer partitions (see [11])  
 270 to obtain the following proposition which follows from Proposition 1.

271 ► **Proposition 2.** Algorithm 1 reaches a stable partition in at most  $p_n = \Theta((e^\pi \sqrt{\frac{2n}{3}})/n)$  steps.



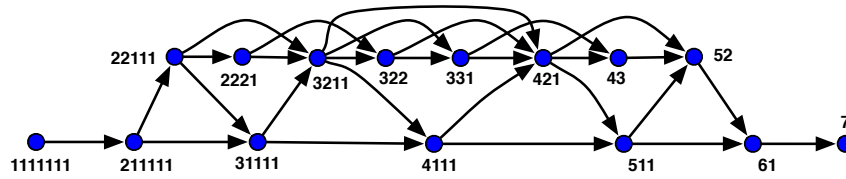


Figure 2 The lattice of integer partitions for  $n = 7$ .

Note that this is already far less than  $2^n$ , which was shown to be the best upper bound that one can obtain for  $k \geq 4$  when using an additive potential function [14].

#### 4 Analysis for $k \leq 2$

In [14], the authors proved that for  $k \leq 2$ , Algorithm 1 converges to a stable partition in at most a quadratic time (by using a potential function). Indeed when performing a 1-deviation  $\vec{\alpha}[p, q]$ , a vertex moves from a group of size  $q + 1$  to a group of size  $p - 1$  (with  $p \geq q + 2$ ); the utility of this vertex increases by  $p - q - 1$ , the utility of the  $q$  other vertices of the group of size  $q + 1$  decreases by 1, while the utility of the vertices of the group of size  $p - 1$  increases by 1 and so the global utility increases by  $2p - 2q - 2 \geq 2$  as  $p \geq q + 2$ . Furthermore, in a  $k$ -stable partition, the utility of a vertex is at most  $n - 1$  and the global utility is at most  $n(n - 1)/2$  and so  $L(k, n) = O(n^2)$ .

In the next subsections we improve this result as we completely solve this case and give the exact (non-asymptotic) value of  $L(k, n)$  when  $k \leq 2$ . The gist of the proof is to use a partial ordering that was introduced in [3], and is sometimes called the dominance ordering.

##### 4.1 Exact analysis for $k = 1$ and empty conflict graph

In [3] the author has defined an ordering over the integer partitions, sometimes called the dominance ordering which creates a lattice of integer partitions. This ordering is a direct application of the theory of majorization to integer partitions [17].

**Definition 3.** (dominance ordering) Given two integer partitions of  $n \geq 1$ ,  $Q = (q_1, q_2, \dots, q_n)$  and  $Q' = (q'_1, q'_2, \dots, q'_n)$ , we say that  $Q'$  dominates  $Q$  if  $\sum_{j=1}^i q'_j \geq \sum_{j=1}^i q_j$ , for all  $1 \leq i \leq n$ .

The example of Figure 2 shows the dominance lattice for  $n = 7$ . We did not write in the figure the integers equal to 0. Now we will show that there is a one to one mapping between chains in the dominance lattice and sequences of 1-deviations in the empty conflict graph.

The two next lemmas show that there is a one to one mapping between chains in the dominance lattice and sequences of 1-deviations in the empty conflict graph.

**Lemma 4.** Let  $P$  be a partition of the vertices and  $P'$  be the partition obtained after a 1-deviation  $\varphi$ . Then, the integer partition  $Q' = Q(P')$  dominates  $Q = Q(P)$ .

**Proof.** In the 1-deviation  $\varphi$  a vertex  $v$  moves from a group  $X_k$  to a group  $X_j$  with  $q_j = |X_j| \geq q_k = |X_k|$ . W.l.o.g. we can suppose that the groups (ranked in non increasing order of size) are ranked in a such a way that  $X_j$  is the first group with size  $|X_j|$  and  $X_k$  the last group with size  $|X_k|$ . Thus, the integer partition  $Q(P)$  associated to  $P$  satisfies  $q_1 \geq q_2 \geq \dots \geq q_{j-1} > q_j \geq q_{j+1} \geq \dots \geq q_k > q_{k+1} \geq \dots \geq q_n$ . After the move the groups of  $P'$  are the same as those of  $P$  except we have replaced  $X_j$  with the group  $X_j \cup v$  and  $X_k$  with  $X_k - v$ . Therefore the integer partition  $Q'$  associated to  $P'$  has the same elements as  $Q$

306 except  $q'_j = q_j + 1$  and  $q'_k = q_k - 1$  and so  $Q'$  dominates  $Q$ . Note that this lemma holds for  
 307 any conflict graph. ◀

308 In the case  $n = 7$ , consider the partition  $P$  with one group of size 3, one of size 2 and  
 309 two of size 1. The integer partition associated to  $P$  is  $Q = (3, 2, 1, 1, 0, 0, 0)$ . Let  $\varphi$  be the  
 310 1-deviation where a vertex in the group of size 1 moves to the group of size 3. We obtain the  
 311 partition  $P'$  with one group of size 4, one of size 2 and one of size 1. The integer partition  
 312 associated to  $P'$  is  $Q' = (4, 2, 1, 0, 0, 0, 0)$  which dominates  $Q$ .

313 ▶ **Lemma 5.** *Let  $G^0$  be the empty conflict graph and let  $Q, Q'$  be two integer partitions*  
 314 *of  $n = |V|$  such that  $Q'$  dominates  $Q$ . For any partition  $P$  associated to  $Q$ , there exists*  
 315 *another partition  $P'$  associated to  $Q'$  such that  $P'$  is obtained from  $P$  by doing a sequence*  
 316 *of 1-deviations.*

317 **Proof.** As proved in [3], we have that if  $Q'$  dominates  $Q$  then there is a finite sequence of  
 318 integer partitions  $Q^0, \dots, Q^r, \dots, Q^s$ , with  $Q = Q^0$  and  $Q' = Q^s$  such that for each  $0 \leq r < s$ ,  
 319  $Q^{r+1}$  dominates  $Q^r$  and differs from it only in two elements  $j_r$  and  $k_r$  with  $q_{j_r}^{r+1} = q_{j_r}^r + 1$   
 320 and  $q_{k_r}^{r+1} = q_{k_r}^r - 1$ .

321 The proof is now by induction on  $r$ , starting from any partition  $P^0 = P$  associated to  $Q$ .  
 322 For  $r > 0$ , we consider the partition  $P^r$  associated to  $Q^r$ . Recall that  $Q^r$  and  $Q^{r+1}$  differ  
 323 only in the two groups  $X_{j_r}$  and  $X_{k_r}$ . As  $q_{j_r}^{r+1} = q_{j_r}^r + 1$  and  $q_{k_r}^{r+1} = q_{k_r}^r - 1$ ,  $P^{r+1}$  can be  
 324 obtained from  $P^r$  by moving a vertex from  $X_{k_r}$  to  $X_{j_r}$ . This move is valid as the conflict  
 325 graph is empty. (Note that the lemma is not valid for a general conflict graph.) ◀

326 As an example, consider the two integer partitions  $Q = (2, 2, 2, 1, 0, 0, 0)$  and  $Q' =$   
 327  $(5, 1, 1, 0, 0, 0, 0)$  where  $Q'$  dominates  $Q$ . The sequence of integer partitions is  $Q_0 = Q$ ,  
 328  $Q_1 = (3, 2, 1, 1, 0, 0, 0)$ ,  $Q_2 = (4, 1, 1, 1, 0, 0, 0)$ ,  $Q_3 = (5, 1, 1, 0, 0, 0, 0)$ . Partition  $P^1$  is  
 329 obtained from  $P^0$  by moving a vertex of a group of size 2 to another group of size 2. Then,  
 330  $P^2$  is obtained by moving a vertex of the group of size 2 to the group of size 3 and  $P'$  is  
 331 obtained from  $P^2$  by moving a vertex of one group of size 1 to that of size 4.

332 In summary we conclude that a sequence of 1-deviations with an empty conflict graph  
 333 corresponds to a chain of integer partitions, and vice versa. Therefore, by Observation 1,  
 334 the length of the longest sequence of 1-deviations with an empty conflict graph is the same  
 335 as the length of the longest chain in the dominance lattice of integer partitions. Since it has  
 336 been proven in [9] that for  $n = \frac{m(m+1)}{2} + r$ , the longest chain in the Dominance Lattice has  
 337 length  $2\binom{m+1}{3} + mr$ , we obtain the *exact* value for  $L(1, n)$ .

338 ▶ **Theorem 6.** *Let  $m$  and  $r$  be the unique non negative integers such that  $n = \frac{m(m+1)}{2} + r$ ,*  
 339 *and  $0 \leq r \leq m$ . Then,  $L(1, n) = 2\binom{m+1}{3} + mr$ .*

340 We note that the proof in [9] is not straightforward. One can think that the longest  
 341 chain is obtained by taking among the possible 1-deviations the one which leads to the  
 342 smallest partition in the lexicographic order. Unfortunately this is not true. Indeed let  
 343  $n = 9$ . After 6 steps we get the integer partition  $(3, 3, 2, 1, 0, 0, 0, 0, 0)$ . Then, by choosing  
 344 the 1-deviation that gives the smallest partition (in the lexicographic order), we get the  
 345 partition  $(3, 3, 3, 0, 0, 0, 0, 0, 0)$  and then  $(4, 3, 2, 0, 0, 0, 0, 0, 0)$ . But there is a longer chain  
 346 of length 3 from  $(3, 3, 2, 1, 0, 0, 0, 0, 0)$  to  $(4, 3, 2, 0, 0, 0, 0, 0, 0)$  namely  $(4, 2, 2, 1, 0, 0, 0, 0, 0)$ ,  
 347  $(4, 3, 1, 1, 0, 0, 0, 0, 0)$ ,  $(4, 3, 2, 0, 0, 0, 0, 0, 0)$ . However the proof in [9] implies that the follow-  
 348 ing simple construction works for any  $n$ .

349 ▶ **Proposition 7.** A longest sequence of 1-deviations in the empty conflict graph is obtained  
 350 by choosing, at a given step, among all the possible 1-deviations, any one of which leads to  
 351 the smallest increase of the global utility.

352 **Proof.** We need to introduce the terminology of [9] (this will not be used elsewhere in the  
 353 paper). Note that they consider their chains starting from the end, and so, we need to  
 354 reverse the steps in their construction in order to make them correspond with 1-deviations.

- 355 ■ A *V*-step is corresponding to a user leaving her group of size  $p$  for another group of size  
 356  $p$ , thereby increasing her utility from  $p - 1$  to  $p$ ; in other words, the deviation vector of  
 357 such 1-deviation is  $\vec{\alpha}[p + 1, p - 1]$  for some  $p$ .
- 358 ■ An *H*-step is corresponding to a user leaving her group  $X_i$  of size  $p$  for another group  
 359  $X_j$  of size  $p \leq |X_j| \leq p + 1$ , but *only* if there is no other group of size  $p$  than  $X_i$  and  
 360 (possibly)  $X_j$ ; in particular, if groups are ordered by decreasing size, this means that  
 361  $j = i - 1$ . The deviation vector of an *H*-step is either  $\vec{\alpha}[p + 1, p - 1]$  (and then,  $X_i, X_j$   
 362 are the only two groups of size  $p$ ) or  $\vec{\alpha}[p + 2, p - 1]$  (and then  $X_i$  is the unique group of  
 363 size  $p$ ). Furthermore, note that an *H*-step can also be a *V*-step.

364 The relationship between *V*-steps, *H*-steps and our construction is as follows. At every  
 365 1-deviation, the global utility has to increase by at least two, and this is attained if and only  
 366 if the deviation vector is  $\vec{\alpha}[p + 1, p - 1]$  for some  $p$ ; equivalently, this move is corresponding to  
 367 a *V*-step. Furthermore, if no such a move is possible, then the global utility has to increase by  
 368 at least four, and this is attained if and only if the deviation vector is  $\vec{\alpha}[p + 2, p - 1]$  for some  
 369  $p$ ; in such a case, there cannot exist any other group of size  $p$  than the one involved in the 1-  
 370 deviation (otherwise, a *V*-step would have been possible), hence this move is corresponding  
 371 to a *H*-step. As proved, *e.g.*, by Brylawski [3], starting from any integer partition with at  
 372 least two summands (*i.e.*,  $q_n \neq 1$ ), it is always possible to perform one of the two types  
 373 of move defined above (these two types of move actually correspond to the two cases when  
 374 an integer partition can cover another one). Therefore, our strategy leads to a sequence of  
 375 1-deviations where all the moves correspond to either a *V*-step or (only if the first type of  
 376 move is not possible) to an *H*-step. By a commutativity argument (Lemma 3 in [9]) it can  
 377 be proved that as soon as no move  $\vec{\alpha}[p + 1, p - 1]$  (corresponding to *V*-steps) is possible  
 378 for any  $p$ , every ulterior move of this type will correspond to both a *V*-step and an *H*-step  
 379 simultaneously (*i.e.*, the deviation vector will be  $\vec{\alpha}[p + 1, p - 1]$  for some  $p$ , and there will  
 380 be no other groups of size  $p$  than the two groups involved in the 1-deviation). Therefore,  
 381 the sequences we obtain are corresponding to a particular case of the so-called *HV*-chains  
 382 in [9]. Finally, the main result in [9] is that every *HV*-chain is of maximum length. ◀

383 In what follows, we will reuse part of the construction in [9] for proving Theorem 10.

## 384 4.2 Analysis for $k = 2$

385 Interestingly we will prove that any 2-deviation can be replaced either by one or two 1-  
 386 deviations and so we will prove in Theorem 9 that  $L(2, n) = L(1, n)$ .

387 ► **Claim 8.** If the conflict graph  $G^-$  is empty, then any 2-deviation can be replaced either  
 388 by one or two 1-deviations

389 *Proof.* Consider a 2-deviation which is not a 1-deviation. In that case case two vertices  $u_i$   
 390 and  $u_j$  leave their respective group  $X_i$  and  $X_j$  (which can be the same) to join a group  
 391  $X_k$ . Let  $|X_i| \geq |X_j|$ ; in order for the utility of the vertices to increase, we should have  
 392  $|X_k| \geq |X_i| - 1$  ( $\geq |X_j| - 1$ ).

- 393 ■ Case 1:  $|X_k| \geq |X_j|$ . In that case the 2-deviation can be replaced by a sequence of two  
 394 1-deviations where firstly a vertex  $u_j$  leaves  $X_j$  to join  $X_k$  and then a vertex  $u_i$  leaves  
 395  $X_i$  to join the group  $X_k \cup u_j$  whose size is now at least that of  $X_i$ .

396 ■ Case 2:  $|X_k| = |X_i| - 1 = |X_j| - 1 = p - 2$  and  $X_i = X_j$ . In that case the effect of the  
 397 2-deviation is to replace the group  $X_i$  of size  $p - 1$  with a group of size  $p - 3$  and to replace  
 398 the group  $X_k$  of size  $p - 2$  with a group of size  $p$ . Said otherwise, the difference vector  
 399  $\vec{\varphi}$  associated to the 2-deviation has as non null coordinates  $\varphi_p = 1, \varphi_{p-1} = -1, \varphi_{p-2} =$   
 400  $-1, \varphi_{p-3} = 1$ . We obtain the same effect by doing the 1-deviation  $\vec{\alpha}[p - 1, p - 2]$  where  
 401 a vertex leaves  $X_k$  to join  $X_i$ .

402 ■ Case 3:  $|X_k| = |X_i| - 1 = |X_j| - 1 = p - 2$  and  $X_i \neq X_j$ . In that case the effect of the  
 403 2-deviation is to replace the 2 groups  $X_i$  and  $X_j$  of size  $p - 1$  with two groups of size  
 404  $p - 2$  and to replace the group  $X_k$  of size  $p - 2$  with a group of size  $p$ . Said otherwise,  
 405 the difference vector  $\vec{\varphi}$  associated to the 2-deviation has as non null coordinates  $\varphi_p =$   
 406  $1, \varphi_{p-1} = -2, \varphi_{p-2} = 1$ . We obtain the same effect by doing the 1-deviation  $\vec{\alpha}[p - 1, p - 1]$   
 407 where a vertex leaves  $X_j$  to join  $X_i$ .

408 Note that the fact  $G^-$  is empty is needed for the proof. Indeed, in the case 2, it might happen  
 409 that all the vertices of  $X_k$  have some enemy in  $X_i$  and so the 1-deviation we describe is not  
 410 valid. Similarly, in case 3, it might happen that all the vertices of  $X_i$  have some enemy in  
 411  $X_j$  and so the 1-deviation we describe is not valid.  $\diamond$

412 ► **Theorem 9.**  $L(2, n) = L(1, n)$ .

413 **Proof.** Clearly,  $L(2, n) \geq L(1, n)$  as any 1-deviation is also a 2-deviation. By Observation 1,  
 414 the value of  $L(2, n)$  is obtained when the conflict graph  $G^-$  is empty. In that case, Claim 8  
 415 implies that  $L(2, n) \leq L(1, n)$ .  $\blacktriangleleft$

### 416 4.3 Analysis for $k \leq 2$ and a general conflict graph

417 Using the potential function introduced at the beginning of this section, Panagopoulou and  
 418 Spirakis ([18]) proved that for every conflict graph  $G^-$  with independence number  $\alpha(G^-)$ ,  
 419 the convergence of the dynamics is in  $\mathcal{O}(n\alpha(G^-))$ . Indeed as we have seen each 1-deviation  
 420 increases the global utility by at least 2. But the global utility of a stable partition is at most  
 421  $n(\alpha(G^-) - 1)$  as the groups have maximum size  $\alpha(G^-)$ . If the conflict graph is empty we  
 422 have seen that  $L(1, n) = \Theta(n^{3/2})$  that is in that case  $\mathcal{O}(n\sqrt{\alpha(G^-)})$ . This leads one of us ([6],  
 423 page 131) to conjecture that in the case of 1-deviations the worst time of convergence of the  
 424 dynamics is  $\mathcal{O}(n\sqrt{\alpha(G^-)})$ . We disprove the conjecture by proving the following theorem:

425 ► **Theorem 10.** For  $n = \binom{m+1}{2}$ , there exists a conflict graph  $G^-$  with  $\alpha(G^-) = m = \Theta(\sqrt{n})$   
 426 and a sequence of  $\binom{m+1}{3}$  valid 1-deviations, that is a sequence of  $\Omega(n^{\frac{3}{2}}) = \Omega(n\alpha(G^-))$   
 427 1-deviations.

428 **Proof.** We will use part of the construction of Greene and Kleitman ([9]). Namely they  
 429 prove that, if  $n = \binom{m+1}{2}$ , there is a sequence of  $\binom{m+1}{3}$  1-deviations transforming the partition  
 430  $P_1$  consisting of  $n$  groups each of size 1 (the coordinates of  $\vec{\lambda}(P_1)$  satisfy  $\lambda_1 = n$ ) into the  
 431 partition  $P_m$  consisting of  $m$  groups, one of each possible size  $i$  for  $1 \leq i \leq m$  (the coordinates  
 432 of  $\vec{\lambda}(P_m)$  satisfy  $\lambda_i = 1$  for  $1 \leq i \leq m$ ). Furthermore they prove that the moves used are  
 433  $V$ -steps (see the proof of proposition 7 that is  $\vec{\alpha}[p + 1, p - 1]$  for some  $p$  (one vertex leaves  
 434 a group of size  $p$  to join a group of the same size  $p$ ). One can note that in such a move the  
 435 utility increases only by 2 and as the total utility of  $P_m$  is  $\sum_{i=1}^m i(i-1) = (m+1)m(m-1)/3$   
 436 the number of moves is  $(m+1)m(m-1)/6$ .

437 The conflict graph of the counterexample will consist of  $m$  complete graphs  $K^j, 1 \leq j \leq$   
 438  $m$  where  $K^j$  has exactly  $j$  vertices. An independent set is therefore formed by taking at  
 439 most one vertex in each  $K^j$  and  $\alpha(G^-) = m$ . We will denote the elements of  $K^j$  by  $\{x_i^j\}$

440 with  $1 \leq i \leq j \leq m$ . The group of  $P_m$  of size  $i$  will be  $X_i = \bigcup x_i^j$  with  $m + 1 - i \leq j \leq m$ .  
 441 So these groups are independent sets.

442 Recall that  $n = m(m + 1)/2$ . For each  $p, 1 \leq p \leq m$  let us denote by  $P_p$  the partition  
 443 consisting of 1 group of each size  $i$  for  $1 \leq i \leq p$  and  $n - p(p + 1)/2$  groups of size 1 (said  
 444 otherwise the coordinates of  $\vec{\lambda}(P_p)$  satisfy  $\lambda_i = 1$  for  $2 \leq i \leq p$  and  $\lambda_1 = 1 + n - p(p + 1)/2$ ).  
 445 We will now describe the sequence  $\vec{\sigma}[p - 1]$  of  $p(p - 1)/2$  1-deviations which transform the  
 446 partition  $P_{p-1}$  into  $P_p$ . One way to do the Greene-Keitman sequence is obtained by doing  
 447 successively the sequences  $\sum_{p=2}^m \vec{\sigma}[p - 1]$ . More precisely we will prove by induction the  
 448 following fact:

449  
 450 ► **Claim 11.** There exists a sequence  $\vec{\sigma}[p - 1]$  of  $p(p - 1)/2$  valid 1-deviations which transform  
 451 the partition  $P_{p-1}$  into  $P_p$  such that after this sequence the group  $X_i[p]$  of size  $i, 1 \leq i \leq p$   
 452 contains exactly the vertices  $X_i[p] = \bigcup x_{i+m-p}^j$  with  $m + 1 - i \leq j \leq m$ .

453 *Proof.* (the reader can follow the construction on the example after)

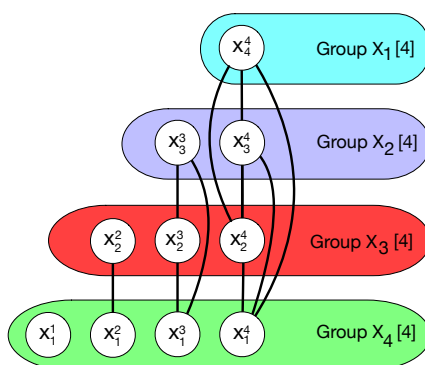
454 We suppose we have built the sequence till  $p - 1$  and that, for  $1 \leq i \leq p - 1, X_i[p - 1] =$   
 455  $\bigcup x_{i+m-p+1}^j$  with  $m + 1 - i \leq j \leq m$ . In a first phase we consider the subpartition of  $n - p + 1$   
 456 elements obtained by removing the group  $X_{p-1}[p - 1]$ . Namely, this above subpartition  
 457 consists of the groups  $X_i[p - 1]$  for  $1 \leq i \leq p - 2$  and groups of size 1. In particular, the  
 458 subpartition is isomorphic to  $P_{p-2}$  with  $p - 1$  singleton groups removed. Our construction  
 459 ensure that these  $p - 1$  singleton groups that are missing are not used for  $\vec{\sigma}[p - 2]$ . So,  
 460 we can do the transformation  $\vec{\sigma}[p - 2]$  consisting of  $(p - 1)(p - 2)/2$  valid moves on the  
 461 partition of  $n - p + 1$  elements not contained in  $X_{p-1}[p - 1]$ . It gives rise to the groups  
 462  $X_i[p] = X_{i-1}[p - 1] + x_{i+m-p}^{m+1-i}$ . Note that at this stage we have two groups of size  $p - 1$ ,  
 463 namely the original one  $X_{p-1}[p - 1]$  and the new one constructed  $X_{p-1}[p]$ . The second phase  
 464 consists in doing  $p - 1$  successive 1-deviations with the vertex  $x_m^{m+1-p}$ . More precisely we  
 465 move this vertex to the group  $X_1[p]$  created in the first phase, then from this group to  $X_2[p]$   
 466 and so on till  $X_{p-2}[p]$  and finally from  $X_{p-2}[p]$  to the original  $X_{p-1}[p - 1]$ . The moves  
 467 are valid as we move a vertex from  $K^{m+1-p}$  and the groups did not contain any vertex of  
 468 this complete graph. Groups created in the first phase are eventually left unchanged as  
 469  $x_m^{m+1-p}$  joins this group and then leaves it. Finally we have constructed a new group  
 470  $X_p[p] = X_{p-1}[p - 1] \cup x_m^{m+1-p}$ . The groups are exactly those described in the claim. ◊

471 To end the proof of Theorem 10, it suffices to note that the groups  $X_i$  form an independen-  
 472 dent set and that after  $\sum_{p=2}^m \vec{\sigma}[p - 1]$  we have obtained the desired groups of  $P_m$  which  
 473 gives the counterexample. ◀

474 **Example for  $m = 4$ .** (See Figure 3.)

- 475 ■ After  $\vec{\sigma}[1]$ , we have the 2 groups  $X_2[2] = x_4^4 \cup x_4^3$  and  $X_1[2] = x_3^4$ .
- 476 ■ First phase of  $\vec{\sigma}[2]$ : we do the move of  $\vec{\sigma}[1]$  on the vertices not in  $X_2[2]$  and create the  
 477 groups  $X_2[3] = x_3^4 \cup x_3^3$  and  $X_1[3] = x_2^4$ .
- 478 ■ Second phase of  $\vec{\sigma}[2]$ : now we move  $x_4^2$  to  $X_1[3]$  and then from  $X_1[3]$  to the original  
 479  $X_2[2] = x_4^4 \cup x_4^3$ , thereby creating the group  $X_3[3] = x_4^4 \cup x_4^3 \cup x_4^2$ .
- 480 ■ First phase of  $\vec{\sigma}[3]$ : we do the 3 moves of  $\vec{\sigma}[2]$  on the vertices not in  $X_3[3]$  and create  
 481 the groups  $X_3[4] = x_3^4 \cup x_3^3 \cup x_3^2, X_2[4] = x_2^4 \cup x_2^3, X_1[4] = x_1^4$ .
- 482 ■ Second phase of  $\vec{\sigma}[3]$ : now we move  $x_4^1$  to  $X_1[4]$ , then from  $X_1[4]$  to  $X_2[4]$  and finally  
 483 from  $X_2[4]$  to the original  $X_3[3] = x_4^4 \cup x_4^3 \cup x_4^2$ , thereby creating the group  $X_4[4] =$   
 484  $x_4^4 \cup x_4^3 \cup x_4^2 \cup x_4^1$ .

485 We can prove a theorem analogous to Theorem 10 for any independence number  $\alpha(G^-)$ .



■ **Figure 3** Illustration for Example 4.

486 ▶ **Theorem 12.** For any  $\alpha = \mathcal{O}(\sqrt{n})$ , there exists a conflict graph  $G^-$  with  $n$  vertices and  
 487 independence number  $\alpha(G^-) = \alpha$ , and a sequence of at least  $\Omega(n\alpha)$  1-deviations to reach a  
 488 stable partition.

489 **Proof.** Let  $G_0^-$  be the graph of Theorem 10 for  $m = \alpha$ .  $G_0^-$  has  $n_0 = \mathcal{O}(\alpha^2)$  vertices,  
 490 independence number  $\alpha$ , and furthermore there exists a sequence of  $\Theta(\alpha^3)$  valid 1-deviations  
 491 for  $G_0^-$ . Let  $G^-$  be the graph obtained by taking the complete join of  $k = n/n_0$  copies of  
 492  $G_0^-$  (i.e., we add all possible edges between every two copies of  $G_0^-$ ). By construction,  $G^-$   
 493 has order  $n = kn_0 = \mathcal{O}(n\alpha^2)$  and the same independence number  $\alpha$  as  $G_0^-$ . Furthermore,  
 494 there exists a sequence of  $k\Theta(\alpha^3) = \Omega(n\alpha)$  valid 1-deviations for  $G^-$ . ◀

495 Note that in any 2-deviation the global utility increases by at least 2 and so the number  
 496 of 2 deviations when the conflict graph has independence number  $\alpha(G^-)$  is also at most  
 497  $\mathcal{O}(n\alpha(G^-))$ . This bound is attained by using only 1-deviations as proved in Theorem 12,  
 498 which is also valid for  $k = 2$ .

## 5 Lower bounds for $k > 2$

500 The classical dominance ordering does not suffice to describe all  $k$ -deviations as soon as  
 501  $k \geq 3$ . As noted before, there is only one  $k$ -stable partition  $P_{max}$  in the empty conflict  
 502 graph  $G^\emptyset$ , namely the one consisting of one group of size  $n$ , with integer partition  $Q_{max} =$   
 503  $(n, 0, \dots, 0)$  and partition vector  $(1, 0, \dots, 0)$ . Let  $d(Q)$  be the length of a longest sequence in  
 504 the dominance lattice from the integer partition  $Q$  to the integer partition  $Q_{max}$ . For  $k = 4$   
 505 let  $P$  be the partition consisting of 4 groups of size 4 and one group of size 1 with integer  
 506 partition  $Q = (4, 4, 4, 4, 1)$ . Apply the 4-deviation where one vertex of each group of size 4  
 507 joins the group of size 1; it leads to the partition  $P'$  with integer partition  $Q' = (5, 3, 3, 3, 3)$ .  
 508  $Q$  is covered in the dominance lattice by the integer partition  $(5, 4, 4, 3, 1)$  while  $Q'$  is at  
 509 distance 3 from it via  $(5, 4, 3, 3, 2)$  and  $(5, 4, 4, 2, 2)$  and so  $d(Q') = d(Q) + 2$ .

510 Prior to our work, it was known  $L(3, n) = \mathcal{O}(n^3)$  ([14]). But nothing proved that  
 511  $L(3, n) > L(2, n)$ , and in fact it was conjectured in [7] that both values are equal. Theorem 13  
 512 proves for the first time that deviations can delay convergence and that the gap between  
 513  $k = 2$  and  $k = 3$  obtained from potential function is indeed justified. It was also conjectured  
 514 in [14] that  $L(k, n)$  was polynomial in  $n$  for  $k$  fixed. We disprove this conjecture and prove  
 515 in Theorem 14 a much more significant result: 4-deviations are responsible for a sudden  
 516 complexity increase, as we prove that no polynomial bounds exist for  $L(4, n)$ .

517 ► **Theorem 13.**  $L(3, n) = \Omega(n^2)$ .

518 ► **Theorem 14.**  $L(4, n) = \Omega(n^{\Theta(\ln(n))})$ .

519 The main idea of the proofs consists in doing repeated shifted sequences (called cascades)  
 520 of deviations similar to the ones given in the example above. The proof of Theorems 13  
 521 can be found in the appendix (Section 5.2). In the next section, we give the the proof of  
 522 Theorem 14 for  $k = 4$ . We use sequences (cascades) of 4-deviations, called  $\delta[p]$ , and various  
 523 additional tricks such that the repetition of the process by using cascades of cascades. Our  
 524 motivation for using  $\delta[p]$  as a basic building block for our construction is that it is the only  
 525 type of 4-deviation which decreases the global utility.

### 526 5.1 Case $k = 4$ . Proof of Theorem 14

527 **Definition of  $\delta[p]$ :** Consider a partition  $P$  containing at least 4 groups of size  $p - 1$  and 1  
 528 group of size  $p - 4$ . In the 4-deviation  $\delta[p]$  one vertex in each of the 4 groups of size  $p - 1$   
 529 moves to the group of size  $p - 4$  to form a new group of size  $p$ . The example given at the  
 530 beginning of this section corresponds to the case  $p = 5$ . The coordinates of the associated  
 531 difference vector (where we omit the bracket  $[p]$  for ease of reading) are:

532

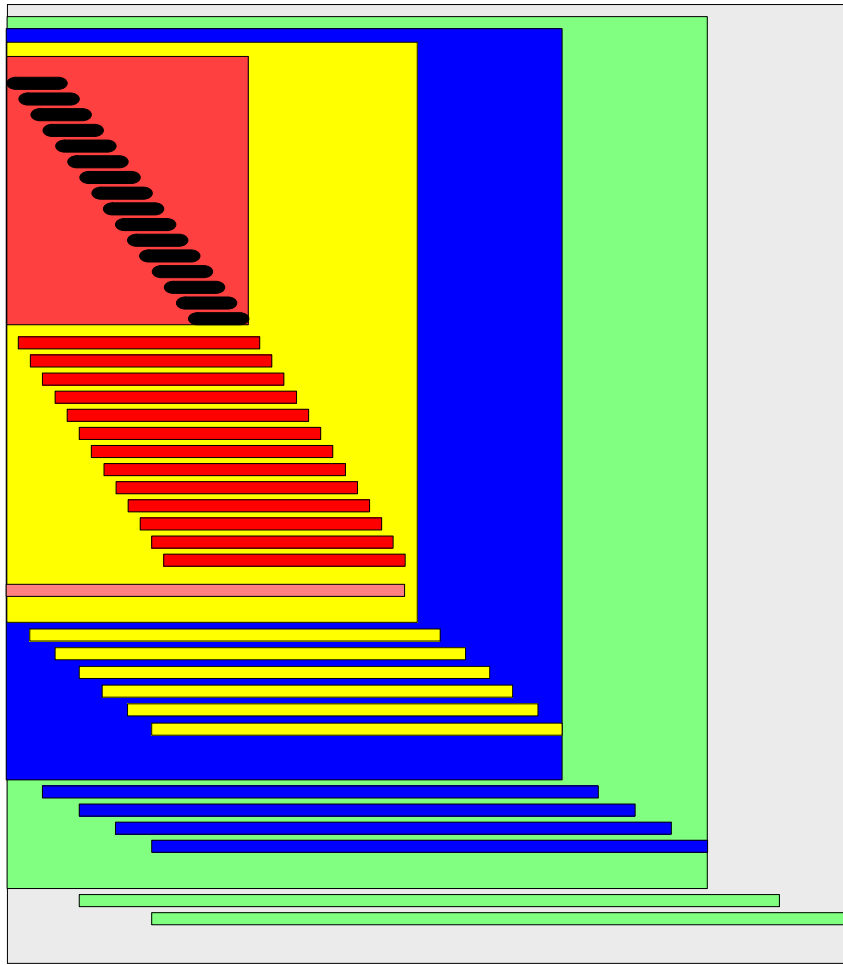
...	$\delta_p$	$\delta_{p-1}$	$\delta_{p-2}$	$\delta_{p-3}$	$\delta_{p-4}$	...
...0	1	-4	4	0	-1	0...

533 Figure 4 gives a visual description of these cascades. Here we start with a sequence of  
 534  $t$  4-deviations  $\delta[p]$  represented by black rectangles ( $t = 16$  in the figure). The cascade so  
 535 obtained, called  $\vec{\delta}^1[p, t]$ , is represented in red. Then we do  $(t - 2)$  such cascades represented  
 536 by red rectangles getting the cascade  $\vec{\delta}^2[p, t - 2]$  represented in yellow which contains  
 537  $224 (= 16 \cdot 14)$  4- deviations. We apply some 1-deviations to get a deviation called  $\vec{\tau}^2[p]$   
 538 with the so-called Nice Property enabling us to do recursive constructions. We do a cascade  
 539 of these  $\vec{\tau}^2[p]$  (shifted by 2) represented by yellow rectangles getting the blue cascade called  
 540  $\vec{\tau}^3[p]$ . We do a cascade of these  $\vec{\tau}^3[p]$  (shifted by 3) represented by blue rectangles getting  
 541 the green cascade called  $\vec{\tau}^4[p]$  and we finally do a cascade of these  $\vec{\tau}^4[p]$  (shifted by 5)  
 542 represented by green rectangles getting the grey cascade called  $\vec{\tau}^5[p]$ . The reader has to  
 543 realize that, in this example,  $\vec{\tau}^5[p]$  contains 3 cascades  $\vec{\tau}^4[p]$  each containing 5 cascades  
 544  $\vec{\tau}^3[p]$  each consisting of 7 cascades  $\vec{\tau}^2[p]$ . Altogether the cascade  $\vec{\tau}^5[p]$  of this example  
 545 contains 23520 4-deviations  $\delta[p]$ .

546 **The cascade  $\vec{\delta}^1[p, t]$ :** we first do a cascade consisting of a sequence of  $t$  shifted 4-  
 547 deviations  $\delta[p], \delta[p - 1], \dots, \delta[p - t + 1]$ , for some parameter  $t$  which will be chosen later to  
 548 give the maximum number of 4-deviations.

549 The reader can follow the construction in Table 2 with  $t = 7$ . The coordinates  $\vec{\delta}^1[p, t]$ ,  
 550 are given in Claim 15 and Table 10. We note that there are lot of cancellations and only 8  
 551 non zero coordinates. Indeed consider the groups of size  $p - i$  for  $4 \leq i \leq t - 1$ ; we have  
 552 deleted such a group when doing the 4-deviation  $\vec{\delta}[p + 4 - i]$ , then created 4 such groups  
 553 with  $\vec{\delta}[p + 2 - i]$ , then deleted 4 such groups with  $\vec{\delta}[p + 1 - i]$ , and finally created one with  
 554  $\vec{\delta}[p - i]$ . The reader can follow these cancellations in Table 2 for  $i = 4, 5, 6$ . The variation  
 555 of the number of groups of a given size  $p - i$  (which correspond to the coordinate  $\delta_{p-i}^1$ ) is  
 556 obtained by summing the coefficients appearing in the corresponding column and so is 0 for  
 557  $p - 4, p - 5, p - 6$ .

558 ► **Claim 15.** For  $3 \leq t \leq p - 3$ , the coordinates of the cascade  $\vec{\delta}^1[p, t] = \sum_{i=0}^{t-1} \vec{\delta}[p - i]$   
 559 satisfy:  $\delta_p^1 = 1$ ,  $\delta_{p-1}^1 = -3$ ,  $\delta_{p-2}^1 = 1$ ,  $\delta_{p-3}^1 = 1$ ,  $\delta_{p-t}^1 = -1$ ,  $\delta_{p-t-1}^1 = 3$ ,  $\delta_{p-t-2}^1 = -1$ ,  
 560  $\delta_{p-t-3}^1 = -1$ , and  $\delta_j^1 = 0$  for all the others  $j$  (see Table 10).



■ **Figure 4** Cascades of cascades.

561 *Proof.* We have  $\delta_j^1 = \sum_{i=0}^{t-1} \delta_j[p-i]$ . For a given  $j$ ,  $\delta_j[p-i] = 0$  except for the following values  
 562 of  $i$  such that  $0 \leq i \leq t-1$ :  $i = p-j$  where  $\delta_j[j] = 1$ ;  $i = p-j-1$  where  $\delta_j[j+1] = -4$ ;  
 563  $i = p-j-2$  where  $\delta_j[j+2] = 4$ ;  $i = p-j-4$  where  $\delta_j[j+4] = -1$  (in the table it  
 564 corresponds to the non zero values in a column, whose number is at most 4). Therefore, for  
 565  $j > p$ :  $\delta_j^1 = 0$ ;  $\delta_p^1 = 1$ ;  $\delta_{p-1}^1 = -4 + 1 = -3$ ;  $\delta_{p-2}^1 = 4 - 4 + 1 = 1$ ;  $\delta_{p-3}^1 = 0 + 4 - 4 + 1 = 1$ ;  
 566 for  $p-4 \geq j \geq p-t+1$ ,  $\delta_{p-j}^1 = -1 + 0 + 4 - 4 + 1 = 0$ ;  $\delta_{p-t}^1 = -1 + 0 + 4 - 4 = -1$ ;  
 567  $\delta_{p-t-1}^1 = -1 + 0 + 4 = 3$ ;  $\delta_{p-t-2}^1 = -1 + 0 = -1$ ;  $\delta_{p-t-3}^1 = -1$  and, for  $j \leq p-t-4$ ,  $\delta_j^1 = 0$ .  
 568  $\diamond$

569 **Validity of the cascades.** We have to see when the cascades are valid, that is, to  
 570 determine how many groups we need at the beginning. For the cascade  $\vec{\delta}^1[p, t]$  we note  
 571 that the coordinates of any subsequence of the cascade, *i.e.*, the coordinates of some  $\vec{\delta}^1[p, r]$ ,  
 572 are all at least  $-1$  except  $\delta_{p-1}^1$ : which is  $-4$  when  $r = 1$  and then  $-3$ . Therefore such a  
 573 cascade is valid as soon as we have at least 4 groups of size  $p-1$  and one group of each  
 574 other size  $p-i$  ( $2 \leq i \leq t+3$ ). To deal in general with the validity of cascades let us now  
 575 introduce the notion of *h-balanced sequence*.

576 ► **Definition 16.** Let  $h$  be a positive integer and let  $\vec{\Phi} = \sum_{j=1}^s \vec{\varphi}^j$  be a cascade consisting  
 577 of  $s$   $k$ -deviations. We call this cascade *h-balanced* if, for any  $1 \leq i \leq s$ , the sum of the  $i$



**XX:16** How long does it take for all users in a social network to choose their communities?

	...0	p	p-1	p-2	p-3	p-4	p-5	p-6	p-7	p-8	p-9	p-10	0...
$\delta[p]$	...0	1	-4	4	0	-1	0	0	0	0	0	0	0...
$+\delta[p-1]$	...0	0	1	-4	4	0	-1	0	0	0	0	0	0...
$+\delta[p-2]$	...0	0	0	1	-4	4	0	-1	0	0	0	0	0...
$+\delta[p-3]$	...0	0	0	0	1	-4	4	0	-1	0	0	0	0...
$+\delta[p-4]$	...0	0	0	0	0	1	-4	4	0	-1	0	0	0...
$+\delta[p-5]$	...0	0	0	0	0	0	1	-4	4	0	-1	0	0...
$+\delta[p-6]$	...0	0	0	0	0	0	0	1	-4	4	0	-1	0...
$= \vec{\delta}^1[p, 7]$	...0	1	-3	1	1	0	0	0	-1	3	-1	-1	0...

■ **Table 2** Computation of  $\delta^1[p, 7]$ .

...	$\delta_p^1$	$\delta_{p-1}^1$	$\delta_{p-2}^1$	$\delta_{p-3}^1$	...	$\delta_{p-t}^1$	$\delta_{p-t-1}^1$	$\delta_{p-t-2}^1$	$\delta_{p-t-3}^1$	...
...0	1	-3	1	1	0...0	-1	3	-1	-1	0...

■ **Table 3** Difference vector  $\delta^1[p, t]$ .

578 first vectors, namely  $\sum_{j=1}^i \vec{\varphi}^j$ , has all its coordinates greater than or equal to  $-h$ .

579 For example, the cascade  $\vec{\delta}^1[p, t]$  described before is 4-balanced. The interest of this  
 580 notion lies in the following fact: Let  $p_{\max}$  be the largest index  $j$  that satisfies  $\vec{\Phi}_j \neq 0$ . Then,  
 581 if we start from a partition with at least  $h$  groups of each size  $j$ , for  $1 \leq j \leq p_{\max}$ , an  
 582  $h$ -balanced sequence is valid.

583 Note that a sequence is itself composed of sub-sequences and the following lemma will  
 584 be useful to bound the value  $h$  of a sequence.

585 ► **Lemma 17.** *Let  $\vec{\Phi}^1$  be an  $h_1$ -balanced sequence and  $\vec{\Phi}^2$  be an  $h_2$ -balanced sequence. Then,*  
 586  $\vec{\Phi}^1 + \vec{\Phi}^2$  *is a  $(\max\{h_1, h_2 - \min_i \Phi_i^1\})$ -balanced sequence.*

587 *Proof.* As  $\vec{\Phi}^1$  is  $h_1$ -balanced, the coordinates of any subsequence of  $\vec{\Phi}^1$  are greater than or  
 588 equal to  $-h_1$ . Consider a subsequence  $\vec{\Phi}^1 + \vec{\Phi}^3$  where  $\vec{\Phi}^3$  is a subsequence of  $\vec{\Phi}^2$ . The  $j$ -th  
 589 coordinate is  $\Phi_j^1 + \Phi_j^3$ ; by definition  $\Phi_j^3 \geq -h_2$  and so  $\Phi_j^1 + \Phi_j^3 \geq \Phi_j^1 - h_2 \geq \min_i \Phi_i^1 - h_2$ . ◊

590 **The cascade  $\vec{\delta}^2[p, t-2]$ :** We do now the following sequence of  $t-2$  cascades  $\vec{\delta}^2[p, t-2]$   
 591  $= \sum_{i=0}^{t-3} \vec{\delta}^1[p-i, t]$ . Altogether we have a sequence of  $t(t-2)$  4-deviations. There are  
 592 a lot of cancellations and in fact, as shown in Claim 18,  $\vec{\delta}^2[p, t-2]$  has only 10 non zero  
 593 coordinates. Table 4 describes an example of computation of  $\vec{\delta}^2[p, t-2]$  with  $t = 7$ .

	...	p	p-1			p-5			p-9			p-13	p-14	...
$\delta^1[p, 7]$	...0	1	-3	1	1	0	0	0	-1	3	-1	-1	0	...
$+\delta^1[p-1, 7]$	...0	0	1	-3	1	1	0	0	0	-1	3	-1	-1	0
$+\delta^1[p-2, 7]$	...0	0	0	1	-3	1	1	0	0	0	-1	3	-1	-1
$+\delta^1[p-3, 7]$	...0	0	0	0	1	-3	1	1	0	0	-1	3	-1	-1
$+\delta^1[p-4, 7]$	...0	0	0	0	0	1	-3	1	1	0	0	-1	3	-1
$= \vec{\delta}^2[p, 5]$	...0	1	-2	-1	0	0	-1	2	0	2	1	0	0	1

■ **Table 4** Computation of  $\delta^2[p, 5]$ .

594 ► **Claim 18.** For  $3 \leq t \leq \frac{p}{2}$ , the coordinates of the cascade  $\vec{\delta}^2[p, t-2] = \sum_{i=0}^{t-3} \vec{\delta}^1[p-i, t]$   
 595 satisfy:  $\delta_p^2 = 1$ ,  $\delta_{p-1}^2 = -2$ ,  $\delta_{p-2}^2 = -1$ ,  $\delta_{p-t+2}^2 = -1$ ,  $\delta_{p-t+1}^2 = 2$ ,  $\delta_{p-t-1}^2 = 2$ ,  $\delta_{p-t-2}^2 = 1$ ,  
 596  $\delta_{p-2t+2}^2 = 1$ ,  $\delta_{p-2t+1}^2 = -2$ ,  $\delta_{p-2t}^2 = -1$ , and  $\delta_j^2 = 0$  for all the others  $j$  (see Table 5).  
 597 Furthermore this cascade is 4-balanced.

...	$\delta_p^2$	$\delta_{p-1}^2$	$\delta_{p-2}^2$	...	$\delta_{p-t+2}^2$	$\delta_{p-t+1}^2$	$\delta_{p-t}^2$	$\delta_{p-t-1}^2$	$\delta_{p-t-2}^2$	...	$\delta_{p-2t+2}^2$	$\delta_{p-2t+1}^2$	$\delta_{p-2t}^2$	...
0	1	-2	-1	0	-1	2	0	2	1	0	1	-2	-1	0

■ **Table 5** Difference vector  $\delta^2[p, t-2]$ .

598 *Proof.* We have  $\delta_j^2 = \sum_{i=0}^{t-3} \delta_j^1[p-i, t]$ . Using the values of  $\delta_j^1[p-i, t]$  given in Claim 23,  
 599 we get that: for  $j > p$ ,  $\delta_j^2 = 0$ ;  $\delta_p^2 = 1$ ;  $\delta_{p-1}^2 = -3 + 1 = -2$ ;  $\delta_{p-2}^2 = 1 - 3 + 1 = -1$ ; for  
 600  $p-3 \geq j \geq p-t+3$ ,  $\delta_j^2 = 1 + 1 - 3 + 1 = 0$ ;  $\delta_{p-t+2}^2 = 1 + 1 - 3 = -1$ ;  $\delta_{p-t+1}^2 = 1 + 1 = 2$ ;  
 601  $\delta_{p-t}^2 = -1 + 1 = 0$ ;  $\delta_{p-t-1}^2 = 3 - 1 = 2$ ;  $\delta_{p-t-2}^2 = -1 + 3 - 1 = 1$ ; for  $p-t-3 \geq j \geq p-2t+3$ ,  
 602  $\delta_j^2 = -1 - 1 + 3 - 1 = 0$ ;  $\delta_{p-2t+2}^2 = -1 - 1 + 3 = 1$ ;  $\delta_{p-2t+1}^2 = -1 - 1 = -2$ ,  $\delta_{p-2t}^2 = -1$ ,  
 603 and for  $j < p-2t$ ,  $\delta_j^2 = 0$ .

604 Using Lemma 17 we get that  $\vec{\delta}^2[p, t-2]$  is 7-balanced; but a careful analysis shows  
 605 that this sequence is in fact 4-balanced. Indeed we will prove by induction that  $\vec{\delta}^2[p, r] =$   
 606  $\sum_{i=0}^{r-1} \vec{\delta}^1[p-i, t]$  is 4-balanced for any  $r \leq t-3$ . That is true for  $r=1$ , as  $\vec{\delta}^1[p, t]$  is 4-  
 607 balanced. Suppose that it is true for  $r$ . We have  $\vec{\delta}^2[p, r+1] = \vec{\delta}^2[p, r] + \vec{\delta}^1[p-r-1, t]$ . All  
 608 the coordinates of  $\vec{\delta}^2[p, r]$  are by the computation above at least  $-3$ , and the coordinates  
 609 of  $\vec{\delta}^1[p-r-1, t]$  are greater than  $-1$  except for  $j = p-r-2$  where  $\delta_{p-r-2}^1[p-r-1] = -4$   
 610 ; but  $\delta_{p-r-2}^2[p, r] = 1$  (case  $r=1$ ) or  $2$  (case  $r > 1$ ) and so all the coordinates of  $\vec{\delta}^2[p, r+1]$   
 611 are at least  $-4$ .  $\diamond$

612 At this stage we could continue and do a cascade of  $\vec{\delta}^2[p, t-2]$  but there is no more  
 613 the phenomenon of cancellation. In fact we will use the following ‘‘symmetrization’’ trick.  
 614 We will transform the cascade  $\vec{\delta}^2[p, t-2]$  into a sequence  $\vec{\zeta}^2[p]$  by doing some sequence  
 615 of 1-deviations whose coordinates are given in Claim 19 The sequence obtained has only 8  
 616 non zero coefficients (4 with values 1 and 4 with values  $-1$ ) arranged in a very symmetric  
 617 nice way (that we will call *Nice Property*). Furthermore we will be able to iterate a cascade  
 618 process on it many times keeping the property.

619 For  $p \geq q+2$ , we will denote by  $\vec{\alpha}[p, q]$  the 1-deviation, where a vertex leaves a group of  
 620 size  $q+1$  for a group of size  $p-1$  (valid as  $p \geq q+2$ ). Let  $\vec{\alpha}^1[p, q, r] = \sum_{i=0}^{r-1} \vec{\alpha}[p-i, q+i]$   
 621 denote a cascade of  $r$  such 1-deviations (we need  $p-r+1 \geq q+r+2$  in order it is valid).  
 622 The coordinates of  $\vec{\alpha}^1[p, q, r]$  are given in the following Claim 19.

623 ► **Claim 19.** For  $p-r \geq q+r+1$ ,  $\vec{\alpha}^1[p, q, r] = \sum_{i=0}^{r-1} \vec{\alpha}[p-i, q+i]$  has only 4 non zero  
 624 coordinates namely  $\alpha_p^1 = 1$ ,  $\alpha_{p-r}^1 = -1$ ,  $\alpha_{q+r}^1 = -1$ , and  $\alpha_q^1 = 1$ .

625 ► **Claim 20.** For  $3 \leq t \leq \frac{p+1}{2}$ , the coordinates of the sequence  $\vec{\zeta}^2[p] = \vec{\delta}^2[p, t-2] + \vec{\alpha}^1[p-1, p-2t-1, t-2] + \vec{\alpha}^1[p-1, p-2t, 2] + \vec{\alpha}^1[p-t+2, p-2t+1, 1] + \vec{\alpha}^1[p-t, p-t-3, 1]$   
 626 satisfy:  $\zeta_p^2 = 1$ ,  $\zeta_{p-2}^2 = -1$ ,  $\zeta_{p-3}^2 = -1$ ,  $\zeta_{p-t}^2 = 1$ ,  $\zeta_{p-t-1}^2 = 1$ ,  $\zeta_{p-2t+2}^2 = -1$ ,  $\zeta_{p-2t+1}^2 = -1$ ,  
 627  $\zeta_{p-2t-1}^2 = 1$  (see Table 6). Furthermore this cascade is still 4-balanced.  
 628

629 *Proof.* By Claim 19, we have the following coordinates:

- 630 ■ for  $\vec{\alpha}^1[p-1, p-2t-1, t-2]$ ,  $\alpha_{p-1}^1 = 1$ ,  $\alpha_{p-t+1}^1 = -1$ ,  $\alpha_{p-t-3}^1 = -1$ ,  $\alpha_{p-2t-1}^1 = 1$ ;
- 631 ■ for  $\vec{\alpha}^1[p-1, p-2t, 2]$ ,  $\alpha_{p-1}^1 = 1$ ,  $\alpha_{p-3}^1 = -1$ ,  $\alpha_{p-2t+2}^1 = -1$ ,  $\alpha_{p-2t}^1 = 1$ ;
- 632 ■ for  $\vec{\alpha}^1[p-t+2, p-2t+1, 1]$ ,  $\alpha_{p-t+2}^1 = 1$ ,  $\alpha_{p-t+1}^1 = -1$ ,  $\alpha_{p-2t+2}^1 = -1$ ,  $\alpha_{p-2t+1}^1 = 1$ ;
- 633 ■ for  $\vec{\alpha}^1[p-t, p-t-3, 1]$ ,  $\alpha_{p-t}^1 = 1$ ,  $\alpha_{p-t-1}^1 = -1$ ,  $\alpha_{p-t-2}^1 = -1$ ,  $\alpha_{p-t-3}^1 = 1$ .

**XX:18** How long does it take for all users in a social network to choose their communities?

...	$\zeta_p^2$	$\zeta_{p-1}^2$	$\zeta_{p-2}^2$	$\zeta_{p-3}^2$	...	$\zeta_{p-t}^2$	$\zeta_{p-t-1}^2$	...	$\zeta_{p-2t+2}^2$	$\zeta_{p-2t+1}^2$	$\zeta_{p-2t}^2$	$\zeta_{p-2t-1}^2$	...
...0	1	0	-1	-1	0...0	1	1	0...0	-1	-1	0	1	0...

■ **Table 6** Difference vector  $\zeta^2[p]$ .

634 Therefore, using these values and the values of the coordinates of  $\delta_j^2$  given in claim 18, we  
 635 get  $\zeta_p^2 = 1$ ,  $\zeta_{p-1}^2 = -2 + 1 + 1 = 0$ ,  $\zeta_{p-2}^2 = -1$ ,  $\zeta_{p-3}^2 = 0 - 1 = -1$ ,  $\zeta_{p-t+2}^2 = -1 + 1 = 0$ ,  
 636  $\zeta_{p-t+1}^2 = 2 - 1 - 1 = 0$ ,  $\zeta_{p-t}^2 = 0 + 1 = 1$ ,  $\zeta_{p-t-1}^2 = 2 - 1 = 1$ ,  $\zeta_{p-t-2}^2 = 1 - 1 = 0$ ,  
 637  $\zeta_{p-t-3}^2 = 0 - 1 + 1 = 0$ ,  $\zeta_{p-2t+2}^2 = 1 - 1 - 1 = -1$ ,  $\zeta_{p-2t+1}^2 = -2 + 1 = -1$ ,  $\zeta_{p-2t}^2 = -1 + 1 = 0$   
 638  $\zeta_{p-2t-1}^2 = 0 + 1$ .

639 To prove that  $\vec{\zeta}^2[p]$  is 4-balanced, apply Lemma 17 with  $\vec{\Phi}^1 = \vec{\delta}^2[p, t-2]$  and  $\vec{\Phi}^2 =$   
 640  $\vec{\alpha}^1[p-1, p-2t-1, t-2] + \vec{\alpha}^1[p-1, p-2t, 1] + \vec{\alpha}^1[p-t+2, p-2t+1, 1] + \vec{\alpha}^1[p-t, p-t-3, 1]$ .  
 641 We have that  $h_1 = 4$  and furthermore all the coefficients of  $\vec{\Phi}^1$  are greater than  $-2$  and  $\vec{\Phi}^2$   
 642 is 2-balanced. Hence,  $\vec{\zeta}^2[p]$  is  $\max(4, 2+2) = 4$ -balanced.  $\diamond$

643 Table 7 shows an example with  $t = 7$ .

	...	p	p-1	...	p-7	p-8	...	p-15	...									
$\delta^2[p, 5]$	0	1	-2	-1	0	0	-1	2	0	2	1	0	0	1	-2	-1	0	
$+\alpha[p-1, p-15, 5]$	0	0	1	0	0	0	0	-1	0	0	0	-1	0	0	0	0	1	0
$+\alpha[p-1, p-14, 2]$	0	0	1	0	-1	0	0	0	0	0	0	-1	0	-1	0	1	0	0
$+\alpha[p-5, p-13, 1]$	0	0	0	0	0	0	1	-1	0	0	0	0	0	-1	1	0	0	0
$+\alpha[p-7, p-10, 1]$	0	0	0	0	0	0	0	0	1	-1	-1	1	0	0	0	0	0	0
$= \vec{\zeta}^2[p]$	0	1	0	-1	-1	0	0	0	1	1	0	0	0	-1	-1	0	1	0

■ **Table 7** Computation of  $\zeta^2[p]$  with  $t = 7$ .

644 ► **Definition 21. Nice Property:** Let  $k \geq 2$  be a positive integer. We will say the  
 645 sequence  $\vec{\zeta}^k[p]$  has the *Nice Property*, if there exist 3 integers  $a(k)$ ,  $b(k)$ , and  $s(k)$  satisfying  
 646  $1 < a(k) < b(k) < 2a(k)$  and  $b(k) < s(k) - 1 < p/2$  and such that all coordinates of  $\vec{\zeta}^k$  are  
 647 null except for:

- 648 ■  $\zeta_p^k = \zeta_{p+1-2s(k)}^k = 1$ ,
- 649 ■  $\zeta_{p-a(k)}^k = \zeta_{p-b(k)}^k = \zeta_{p+1-2s(k)+b(k)}^k = \zeta_{p+1-2s(k)+a(k)}^k = -1$ , and
- 650 ■  $\zeta_{p+1-s(k)}^k = \zeta_{p-s(k)}^k = 1$ .

651 We note the symmetry of the coordinates, as for any  $j$ ,  $\zeta_{p-j}^k = \zeta_{p+1-2s(k)+j}^k$ . As an  
 652 example, the sequence  $\vec{\zeta}^2[p]$  satisfies the Nice Property with  $a(2) = 2$ ,  $b(2) = 3$  and  $s(2) =$   
 653  $t + 1$  and is 4-balanced. Now we will show how starting with a sequence  $\vec{\zeta}^k[p]$  satisfying  
 654 the Nice Property we can construct a sequence  $\vec{\zeta}^{k+1}[p]$  having still the Nice Property.

655 ► **Claim 22. Main construction:** Let  $\vec{\zeta}^k[p]$  be a sequence satisfying the Nice Property  
 656 with parameters  $a(k), b(k), s(k)$ . Then, we can construct a sequence  $\vec{\zeta}^{k+1}[p]$  satisfying the  
 657 following properties:

- 658 ■  $\vec{\zeta}^{k+1}[p]$  satisfies the Nice Property with parameters
  - 659 •  $a(k+1) = b(k)$ ,
  - 660 •  $b(k+1) = b(k) + a(k)$ ,

- 661 •  $s(k+1) = s(k) + a(k)r(k)/2$ , where  $r(k)$  is the greatest even integer such that
- 662  $r(k)a(k) + b(k) < s(k) - 1$ ;
- 663 ■ if  $\vec{\zeta}^k[p]$  is  $h(k)$ -balanced, then  $\vec{\zeta}^{k+1}[p]$  is  $(h(k) + 1)$ -balanced;
- 664 ■  $\vec{\zeta}^{k+1}[p]$  contains  $r(k) + 1$  sequences  $\vec{\zeta}^k[p]$ .

665 *Proof.* We will first do a cascade of  $\vec{\zeta}^k[p]$ , but we will take values of the parameters differing  
 666 by a multiple of  $a(k)$  in order for some of the coordinates to cancel. More precisely let us  
 667 define  $\vec{\Psi}^r = \sum_{j=0}^r \vec{\zeta}^k[p - ja(k)]$ . Using the values of Definition 21, we get the following  
 668 values for the non zero coordinates:

- 669 (1)  $\psi_p^r = 1$ ;  $\psi_{p-ja(k)}^r = -1 + 1 = 0$  for  $0 < j \leq r$  (cancellation phenomenon);  $\psi_{p-(r+1)a(k)}^r =$   
 670  $-1$ ;
- 671 (2)  $\psi_{p+1-2s(k)+a(k)}^r = -1$ ;  $\psi_{p+1-2s(k)-(j-1)a(k)}^r = 1 - 1 = 0$  for  $0 < j \leq r$  (cancellation);  
 672  $\psi_{p+1-2s(k)-ra(k)}^r = 1$ ;
- 673 (3) for  $0 \leq j \leq r$ ,  $\psi_{p-b(k)-ja(k)}^r = -1$ ;
- 674 (4) for  $0 \leq j \leq r$ ,  $\psi_{p+1-2s(k)+b(k)-ja(k)}^r = -1$ ;
- 675 (5) for  $0 \leq j \leq r$ ,  $\psi_{p+1-s(k)-ja(k)}^r = \psi_{p-s(k)-ja(k)}^r = 1$ .

676 All the indices of the coordinates are different as  $a(k) < b(k) < 2a(k)$  and as soon as we  
 677 choose  $r$  even and nonzero such that  $p - b(k) - ra(k) > p + 1 - s(k)$  (that is equivalent  
 678 to  $ra(k) + b(k) < s(k) - 1$ ). Let us denote  $a(k+1) = b(k)$ ,  $b(k+1) = b(k) + a(k)$  and  
 679  $s(k+1) = s(k) + a(k)r/2$ . Then  $\vec{\Psi}^r$  has already part of the Nice Property for  $k+1$ . Indeed  
 680 we have:

- 681 ■  $\psi_p^r = 1$  by (1) and  $\psi_{p+1-2s(k+1)}^r = 1$  by (2) with  $j = r$  (as  $2s(k+1) = 2s(k) + ra(k)$ );
- 682 ■  $\psi_{p-a(k+1)}^r = \psi_{p-b(k)}^r = -1$ ,  $\psi_{p-b(k+1)}^r = \psi_{p-b(k)-a(k)}^r = -1$  by (3) with  $j = 0, 1$ ;
- 683 ■  $\psi_{p+1-2s(k+1)+b(k+1)}^r = -1$ ,  $\psi_{p+1-2s(k+1)+a(k+1)}^r = -1$  by (4) with  $j = r-1, r$ ;
- 684 ■  $\psi_{p+1-s(k+1)}^r = \psi_{p-s(k+1)}^r = 1$  by (5) with  $j = r/2$ .

685 The remaining non zero coordinates are in number  $4r$ : firstly there are  $r$  values  $-1$ ,  
 686 namely  $\psi_{p-b(k)-ja(k)}^r = -1$ , for  $2 \leq j \leq r$ , and  $\psi_{p-(r+1)a(k)}^r = -1$ ; then there are  $2r$   
 687 values 1, namely  $\psi_{p+1-s(k+1)}^r = \psi_{p-s(k+1)}^r = 1$ , for  $j \neq r/2$  and finally  $r$  values  $-1$ , namely  
 688  $\psi_{p+1-2s(k)+a(k)}^r = -1$  and  $\psi_{p+1-2s(k)+b(k)-ja(k)}^r = -1$ , for  $0 \leq j \leq r-2$ . These values are  
 689 disposed in a very symmetric way and can be written: for the values  $-1$ , in the form  $\psi_{p-x_m}^r$   
 690 and  $\psi_{p+1-2s(k+1)+x_m}^r$ ; and for the values 1, in the form  $\psi_{p-y_m}^r$  and  $\psi_{p+1-2s(k+1)+y_m}^r$  with  
 691  $x_m < y_m$  ( $0 \leq m \leq r-1$ ). Furthermore, these  $r$  quadruples of values can be canceled by  
 692 adding to  $\vec{\Psi}^r$  the  $r$  sequences  $\vec{\alpha}^1[p - x_m, p + 1 - 2s(k+1) + x_m, y_m - x_m]$ .

693 We claim that the sequence so obtained, with partition vector  $\vec{\Psi}^r + \sum_{m=0}^{r-1} \vec{\alpha}^1[p - x_m, p +$   
 694  $1 - 2s(k+1) + x_m, y_m - x_m]$ , satisfies the Nice Property with parameters  $a(k+1), b(k+1)$   
 695 and  $s(k+1)$ . Indeed  $a(k+1) = b(k) < b(k) + a(k) = b(k+1)$ ,  $b(k+1) = b(k) + a(k) <$   
 696  $b(k) + b(k) = 2a(k+1)$  and  $b(k+1) = b(k) + a(k) < s(k) - 1 + a(k) \leq s(k+1) - 1$  as  
 697  $r \geq 2$ . We also have to ensure in the computations that  $p$  is chosen so that  $p \geq 2s(k) - 1$ .  
 698 In order to get the maximum number of deviations we will consider this sequence for the  
 699 largest possible even integer  $r$  satisfying  $ra(k) + b(k) < s(k) - 1$ , denoted  $r(k)$  and we will  
 700 denote the sequence for this  $r(k)$  by  $\vec{\zeta}^{k+1}[p]$ .

701 We now prove that  $\vec{\zeta}^{k+1}[p]$  is  $h(k) + 1$ -balanced. We first prove by induction that  $\vec{\Psi}^r$   
 702 is  $(h(k) + 1)$ -balanced. That is true for  $r = 0$  as  $\vec{\zeta}^k[p]$  is  $h(k)$ -balanced. Then suppose it is  
 703 true for some  $r$ ; we apply Lemma 17 with  $\vec{\Phi}^1 = \vec{\Psi}^r$  and  $\vec{\Phi}^2 = \vec{\zeta}^k[p - (r+1)a(k)]$ . We have  
 704 that  $h_1 = h(k) + 1$  by induction hypothesis and furthermore all the coefficients of  $\vec{\Phi}^1$  are  
 705 greater than  $-1$ ; furthermore  $\vec{\Phi}^2$  is  $h(k)$ -balanced and so  $\vec{\Psi}^{r+1}$  is  $(\max(h(k) + 1, h(k) + 1) =$

706  $h(k) + 1$ )-balanced. Then when we add an  $\vec{\alpha}^1[p - x_m, p + 1 - 2s(k + 1) + x_m, y_m - x_m]$   
 707 which is 1-balanced we still get an  $(\max(h(k) + 1, 1 + 1) = h(k) + 1$ -balanced sequence.

708 Finally, by construction, we get that  $\vec{\zeta}^{k+1}[p]$  contains  $r(k) + 1$  sequences  $\vec{\zeta}^k[p]$ .  $\diamond$

709 **End of the proof of Theorem 14.**

710 At this stage we have built a sequence  $\vec{\zeta}^2[p]$  which satisfies the Nice Property with  
 711  $a(2) = 2, b(2) = 3$  and  $s(2) = t + 1$  and is  $h(2) = 4$ -balanced. Furthermore, it contains  $t(t - 2)$   
 712 4-deviations. See Claim 20. Then, for some well-chosen  $K$  (to be defined later) we can apply  
 713  $K - 2$  times the main construction (Claim 22) to construct a sequence  $\vec{\zeta}^K[p]$  which satisfies  
 714 the Nice Property with parameters  $a(K), b(K)$  and  $s(K)$  and is  $h(K)$ -balanced.

We have  $a(k) = b(k - 1), b(k) = b(k - 1) + a(k - 1) = b(k - 1) + b(k - 2)$  and we recognize the Fibonacci recurrence relation. The  $k^{\text{th}}$  Fibonacci number  $F(k)$  is denoted as follows:

$$F(k) = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^k - \left( \frac{1 - \sqrt{5}}{2} \right)^k \right).$$

715 Then, as  $a(2) = 2 = F(3)$  and  $b(2) = 3 = F(4)$ , we get  $a(K) = F(K + 1)$  and  $b(K) =$   
 716  $F(K + 2)$ . In fact in what follows we will use only that  $a(K) \leq 2^{K-1}$  and  $b(K) \leq 2^K$ .  
 717 We have  $s(k + 1) = s(k) + a(k)r(k)/2$ ; but  $a(k)r(k) < s(k - 1) - b(k) < s(k - 1)$  and so  
 718  $s(k + 1) < (3/2) \times s(k)$  and  $s(K) < s(2)(3/2)^{K-2} = (t + 1)(3/2)^{K-2}$ .

719 Recall that we should have  $p \geq 2s(K) - 1$  so we choose  $p = 2s(K)$ . Furthermore by  
 720 induction we have that  $h(K) = K + 2$ . So we need to start with a partition containing at  
 721 least  $K + 2$  groups of each size  $i, 1 \leq i \leq p$ . It is easy to obtain such a starting partition  
 722 from the initial partition — which consists of  $n$  groups of size 1 — by doing a sequence of  
 723 1-deviation of size  $(K - 2)p(p + 1)/2$ ; indeed we can create a group of any size  $i$  with  $(i - 1)$   
 724 1-deviations. Therefore, we will take  $n = (K - 2)p(p + 1)/2 \leq (K - 2)s(K)(2s(K) + 1)$ .  
 725 Using the inequality  $s(K) < (t + 1)(3/2)^{K-2}$  we get that

$$726 \quad n = \mathcal{O}(t^2 K (3/2)^{2K}). \tag{1}$$

727 On the other hand we have to lower bound the number of deviations. By construction  
 728  $\vec{\zeta}^{k+1}[p]$  contains  $r(k) + 1$  sequences  $\vec{\zeta}^k[p]$  and so, contains  $t(t - 2) \prod_{k=2}^{K-1} (r(k) + 1)$  4-  
 729 deviations, as  $\vec{\zeta}^2[p]$  contains  $t(t - 2)$  4-deviations. Recall that  $r(k)$  is the greatest even  
 730 integer  $r$  such that  $ra(k) + b(k) < s(k) - 1$  and so  $r(k) \geq \lfloor \frac{s(k)-1-b(k)}{a(k)} \rfloor - 1$ . Using the  
 731 fact that  $b(k) + 1 \leq 2a(k)$  and  $s(k) > s(2) - 1 = t$ , and  $a(k) \leq a(K) < 2^{K-1}$  we get  
 732  $r(k) \geq \frac{t}{2^{K-1}} - 3$ . Then  $\prod_{k=2}^{K-1} (r(k) + 1) \geq (\frac{t}{2^{K-1}} - 2)^{K-2}$  and the number  $D$  of deviations  
 733 satisfies:

$$734 \quad D = \Omega(t^2 (\frac{t}{2^{K-1}} - 2)^{K-2}). \tag{2}$$

735 We have now to choose  $K$  as a function of  $t$ . In order for the number of deviations as given  
 736 by Equation 2 to increase we need that  $2^{K-1}$  is small compared to  $t$  that is  $K \ll \log_2(t)$ .  
 737 However in view of Equation 1 we want to choose  $K$  the largest possible. Therefore, a good  
 738 choice is  $K = 1/2(\log_2(t))$ .

739 In that case, we get by Equation 1 that  $n = \mathcal{O}(t^2 \log_2(t) (3/2)^{\log_2(t)})$ , or equivalently  
 740  $\log_2(n) = \mathcal{O}(2 \log_2(t) + \log_2(\log_2(t) + \log_2(t)(\log_2(3) - \log_2(2))))$ . Using  $\log_2(3) - \log_2(2) >$   
 741  $0.585$  and the fact that for  $t$  large enough  $\log_2(\log_2(t)) < 0.014 \log_2(t)$  we get  $\log_2(n) =$   
 742  $\mathcal{O}(2.6 \log_2(t))$ , that is  $n = \mathcal{O}(t^{2.6})$ . On the other hand we get by Equation 2:  $D =$   
 743  $\Omega((t^{1/2})^{1/2 \log_2(t)}) = \Omega(t^{1/4 \log_2(t)})$  and so  $D = \Omega(n^{c \log_2(n)})$  with  $c = \frac{1}{4 \times (2.6)^2} \simeq 1/27$ , thereby  
 744 proving Theorem 14.

745 ——— **References** ———

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782 **5.2 Appendix: Proof for the case  $k = 3$  (Theorem 13)**

783 The proof uses the same idea that for  $k = 4$  but is simpler as we can only do a limited  
 784 iteration of cascades, which use a lot of 3-deviations  $\gamma[p]$  (the only 3-deviation which does  
 785 not increase the global utility).

786 **Definition of  $\gamma[p]$ :** Consider a partition  $P$  containing at least three groups of size  $p - 1$   
 787 and a group of size  $p - 3$ . In the 3-deviation  $\gamma[p]$  one vertex in each of the 3 groups of size  
 788  $p - 1$  moves to the group of size  $p - 3$  to form a new group of size  $p$ . The example given at  
 789 the beginning of section 5 corresponds to the case  $p = 4$ . So, after this 3-deviation we get a  
 790 new partition  $P'$  with one more group of size  $p$ , 3 less groups of size  $p - 1$ , 3 more groups  
 791 of size  $p - 2$  and one less group of size  $p - 3$ . This is expressed by the coordinates of the  
 792 associated difference vector (where we omit the bracket  $[p]$  for ease of reading). Note that  
 793 such a deviation is valid only if there are 3 groups of size  $p - 1$  and one group of size  $p - 3$ .

794 **Difference vector  $\vec{\gamma}[p]$ :** The difference vector  $\vec{\gamma}[p]$  has the following coordinates:  
 795  $\gamma_p = 1, \gamma_{p-1} = -3, \gamma_{p-2} = 3, \gamma_{p-3} = -1$  and  $\gamma_j = 0$  for all other values of  $j$ . See Table 8.

...	$\gamma_p$	$\gamma_{p-1}$	$\gamma_{p-2}$	$\gamma_{p-3}$	...
...0	1	-3	3	1	0...

■ **Table 8** Difference vector of  $\gamma[p]$ .

796 **The cascade  $\vec{\gamma}^1[p, t]$ :** Like for  $k = 4$ , we do a cascade consisting of of  $t$  3-deviations  
 797  $\gamma[p], \gamma[p - 1], \gamma[p - 2], \dots, \gamma[p - t + 1]$ . We will denote this cascade by its difference vector  
 798  $\vec{\gamma}^1[p, t] = \sum_{i=0}^{t-1} \vec{\gamma}[p - i]$ . We will determine the coordinates of  $\vec{\gamma}^1[p, t]$  in Claim 23, but  
 799 first let us see how it works on a basic example.

p	13	12	11	10	9	8	7	6	5	4	3	2	1
$\vec{\gamma}[13]$	1	-3	3	-1									
$\vec{\gamma}[12]$		1	-3	3	-1								
$\vec{\gamma}[11]$			1	-3	3	-1							
$\vec{\gamma}[10]$				1	-3	3	-1						
$\vec{\gamma}[9]$					1	-3	3	-1					
$\vec{\gamma}[8]$						1	-3	3	-1				
$\gamma^1[13, 6]$	1	-2	1	0	0	0	-1	2	-1	0	0	0	0

■ **Table 9** Computation of  $\gamma^1[13, 6]$ .

800 **Basic example:** We consider the partition  $P^0$  consisting of  $g \geq 3$  groups of each size  $i$   
 801 for  $1 \leq i \leq 12$ . The partition vector associated  $\vec{\Lambda}(P^0)$  satisfies  $\lambda_i^0 = g$  for  $1 \leq i \leq 12$  and  
 802  $\lambda_j^0 = 0$  for  $j \geq 13$ . In the example of Table 9, we choose  $t = 6$  and so we do successively  
 803  $\vec{\gamma}[13 - i]$ . The non zero coordinates of  $\vec{\gamma}[13 - i]$  are indicated in the corresponding line.  
 804 The  $j$ th coordinate of  $\vec{\gamma}^1[13, r]$  is obtained by summing the numbers of the first  $r$  lines and  
 805 column  $j$ . The coordinate indicates the number of groups of size  $j$  created (if positive)  
 806 or deleted (if negative). If we look at the cascade obtained after 4 3-deviations we note  
 807 a remarkable phenomenon of cancellation as the coordinate  $\gamma_{10}^1$  of this cascade equals 0.  
 808 That comes from the fact that we have successively deleted a group of size 10 with  $\vec{\gamma}[13]$ ,  
 809 then created 3 new groups with  $\vec{\gamma}[12]$ , then deleted 3 groups with  $\vec{\gamma}[11]$  and finally created  
 810 a new group with  $\vec{\gamma}[10]$ . This cancellation stays for all the other deviations. Similarly,  
 811 the coordinate  $\gamma_9^1$  of the cascade  $\vec{\gamma}^1[13, 5]$  equals 0 and so on. In fact as indicated in the

812 Claim 23 the vector  $\vec{\gamma}^1[p, t]$  has only 6 non zero coordinates.

813 ► **Claim 23.** For  $3 \leq t \leq p - 3$ , the coordinates of the cascade  $\vec{\gamma}^1[p, t] = \sum_{i=0}^{t-1} \vec{\gamma}^1[p - i]$   
 814 satisfy:  $\gamma_p^1 = 1, \gamma_{p-1}^1 = -2, \gamma_{p-2}^1 = 1, \gamma_{p-t}^1 = -1, \gamma_{p-t-1}^1 = 2, \gamma_{p-t-2}^1 = -1$  and  $\gamma_j^1 = 0$  for  
 815 all the others  $j$ . Furthermore, this cascade is 3-balanced.

...	$\gamma_p^1$	$\gamma_{p-1}^1$	$\gamma_{p-2}^1$	...	$\gamma_{p-t}^1$	$\gamma_{p-t-1}^1$	$\gamma_{p-t-2}^1$	...
...0	1	-2	1	0...0	-1	2	-1	0...

■ **Table 10** Difference vector  $\gamma^1[p, t]$ .

816 *Proof.* We have  $\gamma_j^1 = \sum_{i=0}^{t-1} \gamma_j^1[p - i]$ . For a given  $j$ ,  $\gamma_j^1[p - i] = 0$  except for the following  
 817 values of  $i$  such that  $0 \leq i \leq t - 1$ :  $i = p - j$  where  $\gamma_j^1[j] = 1$ ;  $i = p - j - 1$  where  
 818  $\gamma_j^1[j + 1] = -3$ ;  $i = p - j - 2$  where  $\gamma_j^1[j + 2] = 3$ ;  $i = p - j - 3$  where  $\gamma_j^1[j + 3] = -1$  (in  
 819 the table it corresponds to the consecutive non zero values in a column which are at most  
 820 4). Therefore, for  $j > p$ :  $\gamma_j^1 = 0$ ;  $\gamma_p^1 = 1$ ;  $\gamma_{p-1}^1 = -3 + 1 = -2$ ;  $\gamma_{p-2}^1 = 3 - 3 + 1 = 1$ ; for  
 821  $p - 3 \geq j \geq p - t + 1$ ,  $\gamma_{p-j}^1 = -1 + 3 - 3 + 1 = 0$ ;  $\gamma_{p-t}^1 = -1 + 3 - 3 = -1$ ;  $\gamma_{p-t-1}^1 = -1 + 3 = 2$ ;  
 822  $\gamma_{p-t-2}^1 = -1$  and, for  $j < p - t - 2$ ,  $\gamma_j^1 = 0$ . Finally we note that the coordinates of any  
 823 subsequence of the cascade, *i.e.* the coordinates of  $\vec{\gamma}^1[p, r]$ , are all at least  $-1$  except  $\gamma_{p-1}^1$ :  
 824 which is  $-3$  when  $r = 1$  and then  $-2$ , thereby proving that the cascade is 3-balanced. ◊

825 In our example we can do the cascade till  $t = 10$  and in general we can do it till  
 826  $t = p - 3$ . However, it is better to choose a  $t$  smaller (we will see after that a good value  
 827 is such that  $p - 3t > 0$ ) and repeat the previous cascade but shifted, and so, to do a  
 828 cascade of cascades. More precisely we do now the following sequence of  $t - 1$  cascades  
 829  $\vec{\gamma}^2[p, t - 1] = \sum_{i=0}^{t-2} \vec{\gamma}^1[p - i, t]$ . Altogether we have a sequence of  $t(t - 1)$  3-deviations.

p	13	12	11	10	9	8	7	6	5	4	3	2	1
$\gamma^1[13, 4]$	1	-2	1	0	-1	2	-1						
$\gamma^1[12, 4]$		1	-2	1	0	-1	2	-1					
$\gamma^1[11, 4]$			1	-2	1	0	-1	2	-1				
$\gamma^2[13, 3]$	1	-1	0	-1	0	1	0	1	-1				
$\gamma^2[12, 3]$		1	-1	0	-1	0	1	0	1	-1			
$\gamma^3[13, 2]$	1	0	-1	-1	-1	1	1	1	0	-1			
$\alpha^1[11, 4, 3]$	0	0	1	0	0	-1	-1	0	0	1			
$\zeta^3[13, 2]$	1	0	0	-1	-1	0	0	1	0	0			
$\zeta^3[12, 2]$		1	0	0	-1	-1	0	0	1	0	0		
$\zeta^3[11, 2]$			1	0	0	-1	-1	0	0	1	0	0	
$\zeta^3[10, 2]$				1	0	0	-1	-1	0	0	1	0	0
$\zeta^4[13, 4]$	1	1	1	0	-2	-2	-2	0	1	1	1	0	0

■ **Table 11** Example of cascade of cascades.

830 In the example (see Table 11), we choose  $t = 4$  and so after having done  $\vec{\gamma}^1[13, 4]$  we do  
 831  $\vec{\gamma}^1[12, 4]$  and  $\vec{\gamma}^1[11, 4]$ . We can see again a phenomenon of cancellation as the coordinate  
 832  $\gamma_{11}^2$  of this cascade of cascade equals 0. That comes from the fact that we have successively  
 833 created a group of size 11 with  $\vec{\gamma}^1[13, 4]$ , then deleted 2 groups with  $\vec{\gamma}^1[12, 4]$  and finally  
 834 created a new group with  $\vec{\gamma}^1[11, 4]$ . Similarly the coordinate  $\gamma_7^2$  of this cascade of cascade  
 835 equals 0. This cancellation stays for all the other deviations. In the general case there are a  
 836 lot of cancellations and in fact, as shown in the next Claim 24,  $\vec{\gamma}^2[p, t - 1]$  has only 6 non  
 837 zero coordinates.



838 ► **Claim 24.** For  $3 \leq t \leq \frac{p-1}{2}$ , the coordinates of the cascade  $\vec{\gamma}^2[p, t-1] = \sum_{i=0}^{t-2} \vec{\gamma}^1[p-i, t]$   
 839 satisfy:  $\gamma_p^2 = 1$ ,  $\gamma_{p-1}^2 = -1$ ,  $\gamma_{p-t+1}^2 = -1$ ,  $\gamma_{p-t-1}^2 = 1$ ,  $\gamma_{p-2t+1}^2 = 1$ ,  $\gamma_{p-2t}^2 = -1$  and  $\gamma_j^2 = 0$   
 840 for all the others  $j$ . Furthermore this cascade is 3-balanced.

...	$\gamma_p^2$	$\gamma_{p-1}^2$	...	$\gamma_{p-t+1}^2$	...	$\gamma_{p-t-1}^2$	...	$\gamma_{p-2t+1}^2$	$\gamma_{p-2t}^2$	...
0	1	-1	0	-1	0	1	0	1	-1	0

■ **Table 12** Difference vector  $\delta^2[p, t-2]$ .

841 *Proof.* We have  $\gamma_j^2 = \sum_{i=0}^{t-2} \gamma_j^1[p-i, t]$ . Using the values of  $\gamma_j^1[p-i, t]$  given in Claim 23,  
 842 we get that: for  $j > p$ ,  $\gamma_j^2 = 0$ ;  $\gamma_p^2 = 1$ ;  $\gamma_{p-1}^2 = 1 - 2 = -1$ ; for  $p-2 \geq j \geq p-t+2$ ,  
 843  $\gamma_j^2 = 1 - 2 + 1 = 0$ ;  $\gamma_{p-t+1}^2 = 1 - 2 = -1$ ;  $\gamma_{p-t}^2 = -1 + 0 + 1 = 0$ ;  $\gamma_{p-t-1}^2 = 2 - 1 = 1$ ; for  
 844  $p-t-2 \geq j \geq p-2t+2$ ,  $\gamma_j^2 = -1 + 2 - 1 = 0$ ;  $\gamma_{p-2t+1}^2 = -1 + 2 = 1$ ;  $\gamma_{p-2t}^2 = -1$  and for  
 845  $j < p-2t$ ,  $\gamma_j^2 = 0$ .

846 Here again we can see that after any number  $r$  of 3-deviations the coordinates of the  
 847 sequence are always at least greater than or equal to  $-3$ . Indeed the  $-3$  appears only after  
 848 the first 3-deviation for groups of size  $p-1$ ; otherwise, when a  $-3$  appears it is after a  $+3$   
 849 and so all the other coordinates are in fact at least  $-2$ . Thus the cascade is 3-balanced. ◊

850 We can now repeat the cascade of cascades  $\vec{\gamma}^2[p, t-1]$  but shifted. More precisely we do  
 851 now the following sequence of  $t-2$  cascades  $\vec{\gamma}^3[p, t-2] = \sum_{i=0}^{t-3} \vec{\gamma}^2[p-i, t-1]$ . Altogether  
 852 we have a sequence of  $t(t-1)(t-2)$  3-deviations. In the example (see Table 11), after  
 853 having done  $\vec{\gamma}^2[13, 3]$  we do  $\vec{\gamma}^2[12, 3]$ . We can see again a phenomenon of cancellation as  
 854 the coordinate  $\gamma_{12}^3$  of this cascade of cascades equals 0. That comes from the fact that we  
 855 have successively deleted a group of size 12 after  $\vec{\gamma}^2[13, 3]$ , then created a new group after  
 856  $\vec{\gamma}^2[12, 3]$ . Similarly the coordinate  $\gamma_5^3$  of this cascade of cascade equals 0. Here we have also  
 857 deleted one group of size 5, then created one such group. This cancellation stays for all the  
 858 other deviations. In the general case there are a lot of cancellations and in fact as shown in  
 859 the next Claim 25,  $\vec{\gamma}^2[p, t-1]$  has only 8 non zero coordinates.

860  
 861 ► **Claim 25.** For  $3 \leq t \leq \frac{p+2}{3}$ , the coordinates of the cascade  $\vec{\gamma}^3[p, t-2] = \sum_{i=0}^{t-3} \vec{\gamma}^2[p-i, t-1]$   
 862  $i, t-1]$  satisfy:  $\gamma_p^3 = 1$ ,  $\gamma_{p-t+2}^3 = \gamma_{p-t+1}^3 = \gamma_{p-t}^3 = -1$ ,  $\gamma_{p-2t+3}^3 = \gamma_{p-2t+2}^3 = \gamma_{p-2t+1}^3 = 1$ ,  
 863  $\gamma_{p-3t+3}^3 = -1$ , and  $\gamma_j^3 = 0$  for all the others  $j$ . Furthermore this cascade is 4-balanced

...	$\gamma_p^3$	...	$\gamma_{p-t+2}^3$	$\gamma_{p-t+1}^3$	$\gamma_{p-t}^3$	...	$\gamma_{p-2t+3}^3$	$\gamma_{p-2t+2}^3$	$\gamma_{p-2t+1}^3$	...	$\gamma_{p-3t+3}^3$	...
0	1	0	-1	-1	-1	0	1	1	1	0	-1	0

■ **Table 13** Difference vector  $\gamma^3[p, t-2]$ .

864 *Proof.* We have  $\gamma_j^3 = \sum_{i=0}^{t-3} \gamma_j^2[p-i, t-1]$ . Using the values of  $\gamma_j^2[p-i, t-1]$  given in  
 865 Claim 24, we get that: for  $j > p$ ,  $\gamma_j^3 = 0$ ;  $\gamma_p^3 = 1$ ; for  $p-1 \geq j \geq p-t+3$ ,  $\gamma_j^3 = -1 + 1 = 0$ ;  
 866  $\gamma_{p-t+2}^3 = 0 - 1 = -1$ ;  $\gamma_{p-t+1}^3 = -1 + 0 = -1$ ;  $\gamma_{p-t}^3 = 0 - 1 = -1$ ; for  $p-t-1 \geq j \geq p-2t+4$ ,  
 867  $\gamma_j^3 = 1 + 0 - 1 = 0$ ;  $\gamma_{p-2t+3}^3 = 1 + 0 = 1$ ;  $\gamma_{p-2t+2}^3 = 0 + 1 = 1$ ;  $\gamma_{p-2t+1}^3 = 1 + 0 = 1$ ; for  
 868  $p-2t \geq j \geq p-3t+4$ ,  $\gamma_j^3 = -1 + 1 = 0$ ;  $\gamma_{p-3t+3}^3 = 0 - 1 = -1$ ; and for  $j < p-3t+3$ ,  
 869  $\gamma_j^3 = 0$ .

870 Here again a careful but tedious analysis of all the subsequence of deviations indicate  
 871 that all their coordinates are at least  $-3$ . However using Lemma 17 we can easily prove  
 872 that it is 4-balanced. In fact, we will prove by induction that  $\sum_{i=0}^r \vec{\gamma}^2[p-i, t-1]$  is 4-  
 873 balanced for any  $r \leq t-3$ . That is true for  $r = 0$ , as  $\vec{\gamma}^2[p, t-1]$  is 3-balanced. Suppose

874 that  $\sum_{i=0}^r \vec{\gamma}^2[p-i, t-1]$  is 4-balanced for some  $r \leq t-3$ . We apply Lemma 17 with  
 875  $\vec{\Phi}^1 = \sum_{i=0}^r \vec{\gamma}^2[p-i, t-1]$  and  $\vec{\Phi}^2 = \vec{\gamma}^2[p-r-1, t-1]$ . We have by induction hypothesis  
 876 that  $h_1 = 4$  and furthermore all the coefficients of  $\vec{\Phi}^1$  are greater than  $-1$  by Claim 24  
 877 when  $r = 0$  or Claim 25 when  $r > 0$  and so  $\min_i \Phi_i^1 = -1$ . Finally  $\vec{\Phi}^2$  is 3-balanced and  
 878 so  $\sum_{i=0}^{r+1} \vec{\gamma}^2[p-i, t-1]$  is also  $\max(4, 3+1) = 4$ -balanced and by induction  $\vec{\gamma}^3[p, t-2]$  is  
 879 4-balanced.  $\diamond$

880 We can now again repeat  $t$  times the cascades  $\vec{\gamma}^3[p, t-2]$  creating a sequence of cascades  
 881 that we call  $\vec{\gamma}^4[p, t] = \sum_{i=0}^{t-1} \vec{\gamma}^3[p-i, t-2]$ . This is enough to prove Theorem 13. But  
 882 we will use the trick used for  $k = 4$  which consists in doing after  $\vec{\gamma}^3[p, t-2]$  a sequence  
 883 of 1-deviations which will reduce to 4 the number of non zero coordinates. This slightly  
 884 improves the ratio between the number of 3-deviations and the number of vertices. So, after  
 885 having done  $\vec{\gamma}^3[p, t-2]$ , we apply the cascade  $\vec{\alpha}^1[p-t+2, p-3t+3, t-1]$  (see Claim 19).  
 886 Let  $\vec{\zeta}^3[p, t-2]$  be the sequence we obtain. As shown in Claim 26, it has only 4 non zero  
 887 coordinates.

888 **► Claim 26.** For  $3 \leq t \leq p/2$  the coordinates of  $\vec{\zeta}^3[p, t-2] = \vec{\gamma}^3[p, t-2] + \vec{\alpha}^1[p-t+2, p-$   
 889  $3t+3, t-1]$  satisfy:  $\zeta_p^3 = 1, \zeta_{p-t+1}^3 = \zeta_{p-t}^3 = -1, \zeta_{p-2t+1}^3 = 1$  and  $\zeta_j^3 = 0$  for all the others  
 890  $j$ . Furthermore it is 4-balanced.

891 *Proof.* Compared to  $\vec{\gamma}^3[p, t-2]$  only 4 coordinates have been changed and we get  $\zeta_{p-t+2}^3 =$   
 892  $\gamma_{p-t+2}^3 + \alpha^1(p-t+2) = -1+1 = 0, \zeta_{p-2t+3}^3 = \zeta_{p-2t+2}^3 = 1-1 = 0$  and  $\zeta_{p-3t+3}^3 = -1+1 = 0$ .  
 893 All the other coordinates remain the same. To prove that  $\vec{\zeta}^3[p, t-2]$  is 4-balanced, we apply  
 894 Lemma 17 with  $\vec{\Phi}^1 = \vec{\gamma}^3[p, t-2]$  and  $\vec{\Phi}^2 = \vec{\alpha}^1[p-t+2, p-3t+3, t-1]$ . We have proved  
 895 that  $\vec{\Phi}^1$  is 4-balanced and by Claim 25  $\min_i \Phi_i^1 = -1$ . Furthermore,  $\vec{\Phi}^2$  is 1-balanced and  
 896 so  $\vec{\zeta}^3[p, t-2]$  is  $\max(4, 1+1) = 4$ -balanced.  $\diamond$

897 Now we can do  $t$  cascades of  $\vec{\zeta}^3$  to obtain the cascade  $\vec{\zeta}^4[p, t] = \sum_{i=0}^{t-1} \vec{\zeta}^3[p-i, t-2]$ .

898 **► Claim 27.** For  $3 \leq t \leq p+2/3$  the coordinates of  $\vec{\zeta}^4[p, t] = \sum_{i=0}^{t-1} \vec{\zeta}^3[p-i, t-2]$  satisfy:  
 899 For  $p \geq j \geq p-t+2, \zeta_j^4 = 1$ ; for  $p-t \geq j \geq p-2t+2, \zeta_j^4 = -2$ ; for  $p-2t \geq j \geq p-3t+2,$   
 900  $\zeta_j^4 = 1$ ; and  $\zeta_j^4 = 0$  for all the others  $j$ . Furthermore it is 6-balanced.

901 *Proof.* This follows easily from the values of the coordinates of  $\vec{\zeta}^3[p-i, t-2]$ . Note that  
 902  $\zeta_{p-t+1}^4 = -1+1 = 0$  and  $\zeta_{p-2t+1}^4 = 1-1 = 0$ .

903 We will prove by induction that  $\sum_{i=0}^r \vec{\zeta}^3[p-i, t-2]$  is 6-balanced for any  $r \leq t-1$ .  
 904 That is true for  $r = 0$  as  $\vec{\zeta}^3[p, t-2]$  is 4-balanced. Suppose that  $\sum_{i=0}^r \vec{\zeta}^3[p-i, t-2]$   
 905 is 6-balanced for some  $r \leq t-2$ . We apply Lemma 17 with  $\vec{\Phi}^1 = \sum_{i=0}^r \vec{\zeta}^3[p-i, t-2]$   
 906 and  $\vec{\Phi}^2 = \vec{\zeta}^3[p-r-1, t-2]$ . We have  $h_1 = 6$  by induction hypothesis and furthermore  
 907 all the coefficients of  $\vec{\Phi}^1$  are greater than  $-2$  by Claim 26 when  $r = 0$  or Claim 27 when  
 908  $r > 0$  and so  $\min_i \Phi_i^1 = -2$ . Finally  $\vec{\Phi}^2$  is 4-balanced and so  $\sum_{i=0}^{r+1} \vec{\zeta}^3[p-i, t-2]$  is also  
 909  $\max(6, 4+2) = 6$ -balanced and by induction  $\vec{\zeta}^4[p, t]$  is 6-balanced.  $\diamond$

910 **End of the proof of Theorem 13:** The cascade  $\vec{\zeta}^4[p, t]$  consists of  $t$  cascades  $\vec{\zeta}^3[p-$   
 911  $i, t-2]$  each of them consisting of a cascade  $\vec{\gamma}^3[p-i, t-2]$  and  $t-1$  1-deviations.  $\vec{\gamma}^3[p-$   
 912  $i, t-2]$  itself consists of  $t-2$  cascades  $\vec{\gamma}^2[p-i, t-1]$  each of them consisting of  $t-1$  cascades  
 913  $\vec{\gamma}^1[p-i, t-1]$  each of them consisting of  $t$  3-deviations  $\vec{\gamma}[p-i]$ . So the cascade  $\vec{\zeta}^4[p, t]$   
 914 contains  $t^2(t-1)(t-2) = \theta(t^4)$  3-deviations (plus  $t(t-1)$  1-deviations which is negligible).

915 By Claim 27, it is 6-balanced and so if we choose as starting partition one with 6 groups  
 916 of each size  $i, 1 \leq i \leq p-1$  the cascade is valid. It is easy to obtain such a starting partition  
 917 from the initial partition which consists of  $n$  groups of size 1. Indeed we can create a group

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918 of any size  $i$ ; for that we choose a specific group of size 1 and successively move with  $(i - 1)$   
919 1-deviations one element of  $i - 1$  other groups of size 1 to form a group of size  $i$ . We do it  
920 6 times for each size  $i, 1 \leq i \leq p - 1$ . Of course that is possible only if  $n \geq 6p(p - 1)/2$ .

921 Finally, note that in order that the coordinates of all cascades have a meaning we must  
922 choose  $p$  and  $t$  such that  $p - 3t \geq 0$ . Let us choose  $p = 3t$ ; then the number of vertices is  
923  $n = 6 \sum_{i=0}^{p-1} i = 6p(p - 1)/2 = 9t(3t - 1) = \theta(t^2)$ .

924 In summary we have built a cascade which contains  $\theta(t^4) = \theta(n^2)$  3-deviations and so  
925  $L(3, n) = \Omega(n^2)$ .