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# Gomory Hu Tree and Pendant Pairs of a Symmetric Submodular System 

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#### Abstract

Let $\mathcal{S}=(V, f)$, be a symmetric submodular system. For two distinct elements $s$ and $l$ of $V$, let $\Gamma(s, l)$ denote the set of all subsets of $V$ which separate $s$ from $l$. By using every Gomory Hu tree of $\mathcal{S}$ we can obtain an element of $\Gamma(s, l)$ which has minimum value among all the elements of $\Gamma(s, l)$. This tree can be constructed iteratively by solving $|V|-1$ minimum $s l$-separator problem. An ordered pair $(s, l)$ is called a pendant pair of $\mathcal{S}$ if $\{l\}$ is a minimum $s l$-separator. Pendant pairs of a symmetric submodular system play a key role in finding a minimizer of this system. In this paper, we obtain a Gomory Hu tree of a contraction of $\mathcal{S}$ with respect to some subsets of $V$ only by using contraction in Gomory Hu tree of $\mathcal{S}$. Furthermore, we obtain some pendant pairs of $\mathcal{S}$ and its contractions by using a Gomory Hu tree of $\mathcal{S}$.


Keywords: Symmetric submodular system, Contraction of a system, Pendant pair, Maximum adjacency ordering, Gomory-Hu tree.

## 1 Introduction

Let $V$ be a finite set. A set function $f: 2^{V} \mapsto \mathbb{R}$ is called a submodular function if and only if for all $X, Y \in 2^{V}$, we have

$$
\begin{equation*}
f(X)+f(Y) \geq f(X \cup Y)+f(X \cap Y) \tag{1}
\end{equation*}
$$

Submodular functions play a key role in combinatorial optimization, see [3] for further discussion. Rank functions of matroids, cut capacity functions and entropy functions are some well known examples of submodular functions. For a given system $\mathcal{S}=(V, f)$, let $f: 2^{V} \mapsto \mathbb{R}$ be a submodular function. The problem in which we want to find a subset $X \subseteq V$, for which $f(X) \leq f(Y)$ for all $Y \subseteq V$ is called submodular system minimization problem. Minimizing a submodular system is one of the most important problems in combinatorial optimization. Many problems in combinatorial optimization, such as finding minimum cut and minimum $s t$-cut in graphs, or finding the largest common independent set in two matroids can be modeled as a submodular function minimization. Image segmentation $[1,8,9]$, speech analysis $[11,12]$, wireless and power networks

[^0][20] are only a small part of applications of minimizing submodular functions. Grotchel, Lovasz and Scherijver have developed the first weakly and strongly polynomial time algorithm for minimizing submodular systems in [6] and [13], respectively. Each of them is designed based on the ellipsoid method. Then, nearly simultaneously Scherijver [18] and Iwata, Fleischer, and Fujishige [7] gave a combinatorial strongly polynomial time algorithms for this problem. Later, a faster algorithm for minimizing submodular system was proposed by Orlin [16]. To the best of our knowledge, the fastest algorithm to find a minimizer of a submodular system $\mathcal{S}=(V, f)$ is due to Lee et al [10]. Their algorithm runs in $O\left(|V|^{3} \log ^{2}|V| \tau+|V|^{4} \log ^{O(1)}|V|\right)$ time, where $\tau$ is the time taken to evaluate the function.

Stoer and Wagner [19] and Frank [2] independently have presented an algorithm that finds a minimum cut of a graph $G=(V, E)$ in $O\left(|E||V|+|V|^{2} \log |V|\right)$ time. Their algorithms are based on Nagamochi and Ibaraki's algorithm [15] which finds a minimum cut of an undirected graph. Queyranne [17] developed a faster algorithm to find a minimizer in a special case of a submodular system. This algorithm was proposed to find a minimizer of a symmetric submodular system $\mathcal{S}=(V, f)$. It is a generalization of Stoer and Wagner's algorithm [19] and runs in $O\left(|V|^{3}\right)$ time. This algorithm, similar to Stoer and Wagner's algorithm uses pendant pairs to obtain a minimizer of a symmetric submodular system.

For a given weighted undirected graph $G=(V, E)$, Gomory and Hu constructed a weighted tree, named as Gomory Hu tree [5]. By using a Gomory Hu tree of the graph $G$, one can solve the all pairs minimum st-cut problem with $|V|-1$ calls to the maximum flow subroutine instead of the $\binom{|V|}{2}$ calls. Goemans and Ramakrishnan [4] illustrated that for every symmetric submodular system, there exists a Gomory-Hu tree. It is worth to mention that, there is neither an algorithm to construct a Gomory Hu tree of a symmetric submodular system by using pendant pairs nor any method to obtain pendant pairs of a symmetric submodular system by using a Gomory Hu tree of it.

In this paper, we obtain a Gomory Hu tree of a contraction of a symmetric submodular system $\mathcal{S}=(V, f)$, under some subsets of $V$ only by using a Gomory Hu tree of $\mathcal{S}$. In other words, without solving any minimum st-separator problem of the contracted system, we obtain a Gomory Hu tree of it only by contracting the Gomory Hu tree of the original system. Furthermore, we obtain some pendant pairs of a symmetric submodular system $\mathcal{S}$ and its contractions by using a Gomory Hu tree of $\mathcal{S}$.

The outline of this paper is as follows. Section 2 provides preliminaries and basic definitions. In Section 3, we obtain some pendant pairs of a symmetric submodular system by using a Gomory Hu tree of the system. In Section 4, we construct a Gomory Hu tree of a contracted system by using a Gomory Hu tree of the original system.

## 2 Preliminaries

Let $V$ be a finite nonempty set. A function $f: 2^{V} \mapsto \mathbb{R}$ is called a set function on $V$. For every $X \subseteq V, x \in X$ and $y \in V \backslash X$, we use $X+y$ and $X-x$ instead of $X \cup\{y\}$ and $X \backslash\{x\}$, respectively. Also, for $x \in V$ we use $f(x)$ instead of $f(\{x\})$.

A pair $\mathcal{S}=(V, f)$, is called a system if $f$ is a set function on $V$. A system $\mathcal{S}=(V, f)$ is called a submodular system if for all $X, Y \subseteq V$, we have

$$
\begin{equation*}
f(X)+f(Y) \geq f(X \cup Y)+f(X \cap Y) \tag{2}
\end{equation*}
$$

Furthermore, it is called a symmetric system if for every $X \subseteq V$, we have

$$
\begin{equation*}
f(X)=f(V \backslash X) \tag{3}
\end{equation*}
$$

Consider a symmetric submodular system $\mathcal{S}=(V, f)$. Suppose that $A$ and $B$ are two disjoint subsets of $V$. A subset $X \subseteq V$ is called an $A B$-separator in $\mathcal{S}$, if $X \cap(A \cup B)=A$ or $X \cap(A \cup B)=B$. Let $\Gamma(A, B)$ denote the set of all $A B$-separators in $\mathcal{S}$. A subset $X \in \Gamma(A, B)$ is called a minimum $A B$-separator in $\mathcal{S}$ if $f(X)=\min _{Y \in \Gamma(A, B)} f(Y)$. If $A$ and $B$ are singletones $\{a\}$ and $\{b\}$, then we use $a b$-separator and $\Gamma(a, b)$ instead of $\{a\}\{b\}$-separator and $\Gamma(\{a\},\{b\})$, respectively.

Let $G=(V, E)$ be a weighted undirected graph with the weight function $w: E \mapsto \mathbb{R}^{+} \cup\{0\}$. Suppose that $X$ is a nonempty proper subset of $V$. The set of all edges connecting $X$ to $V \backslash X$, is called the cut associated with $X$ and is denoted by $\delta(X)$. The capacity of $\delta(X)$ is denoted by $C(X)$ and defined by

$$
\begin{equation*}
C(X)=\sum_{e \in \delta(X)} w(e) \tag{4}
\end{equation*}
$$

By setting $C(\emptyset)=C(V)=0,(V, C)$ is a symmetric submodular system [14]. For two distinct vertices $u$ and $v$ of $G$, every minimum $u v$-separator of $(V, C)$ is a minimum $u v-$ cut of $G$.

Let $T=(V, F)$ be a tree and $u v$ be an arbitrary edge of it. By $T-u v$, we mean the forest obtained from $T$ by removing $u v$. The set of vertices of two components of $T-u v$ which respectively contain $u$ and $v$, is denoted by $V_{u}(T-u v)$ and $V_{v}(T-u v)$. Also, for $u v \in F$, define $\mathcal{F}_{u}(T-u v)=\left\{X \mid u \in X \subseteq V_{u}(T-u v)\right\}$, $\mathcal{F}_{v}(T-u v)=\left\{X \mid v \in X \subseteq V_{v}(T-u v)\right\}$.

Suppose that $X$ is a nonempty subset of vertices of a given graph $G=(V, E)$. We denote by $G_{>X<}$ the graph obtained from $G$ by contracting all the vertices in $X$ into a single vertex.

Let $\mathcal{S}=(V, f)$ be a symmetric submodular system. Suppose that $T=(V, F)$ is a weighted tree with the weight function $w: E \mapsto \mathbb{R}^{+}$. If for all $u, v \in V$, the minimum weight of the edges on the path between $u$ and $v$ in $T$ is equal to minimum $u v$-separator in $\mathcal{S}$, then $T$ is called a flow equivalent tree of $\mathcal{S}$. Also, we say that $T$ has the cut property with respect to $\mathcal{S}$ if $w(e)=f\left(V_{u}(T-u v)\right)=$ $f\left(V_{v}(T-u v)\right)$ for every $e=u v \in F$. A flow equivalent tree of $\mathcal{S}$ is called a Gomory Hu tree of $\mathcal{S}$ if it has cut property with respect to $\mathcal{S}$.

Consider a system $\mathcal{S}=(V, f)$. A pair of elements $(x, y)$ of $V$ is called a pendant pair for $\mathcal{S}$, if $\{y\}$ is a minimum $x y$-separator in $\mathcal{S}$.

Let $\mathcal{S}=(V, f)$ be a symmetric submodular system. Suppose that $\rho=$ $\left(v_{1}, v_{2}, \cdots, v_{|V|}\right)$ is an ordring of the elements of $V$, where $v_{1}$ can be chosen arbitrarily. If for all $2 \leq i \leq j \leq|V|$, we have

$$
\begin{equation*}
f\left(V_{i-1}+v_{i}\right)-f\left(v_{i}\right) \leq f\left(V_{i-1}+v_{j}\right)-f\left(v_{j}\right), \tag{5}
\end{equation*}
$$

where $V_{i}=\left\{v_{1}, v_{2}, \cdots, v_{i}\right\}$, then $\rho$ is called a maximum adjacency ordering (MA-ordering) of $\mathcal{S}$.

For a symmetric submodular system $\mathcal{S}=(V, f)$, Queyranne [17] showed that the last two elements $\left(v_{|V|-1}, v_{|V|}\right)$ of an MA-ordering of $\mathcal{S}$, is a pendant pair of this system.

Let $\mathcal{S}=(V, f)$ be a system and $X$ be an arbitrary subset of $V$. By $\varphi(X)$, we mean a single element obtained by unifying all elements of $X$. The contraction of $\mathcal{S}$ with respect to a subset $A \subseteq V$ is denoted by $\mathcal{S}_{A}=\left(V_{A}, f_{A}\right)$ and defined by $V_{A}=(V \backslash A)+\varphi(A)$ and

$$
f_{A}(X)= \begin{cases}f(X) & \text { if } \varphi(A) \notin X  \tag{6}\\ f((X-\varphi(A)) \cup A) & \text { if } \varphi(A) \in X\end{cases}
$$

Suppose that $A$ and $B$ are two nonempty disjoint subsets of $V$. We denote by $\left(\mathcal{S}_{A}\right)_{B}$, the contraction of $\mathcal{S}_{A}$ with respect to $B$.

## 3 Obtaining Pendant Pairs from a Gomory Hu tree

Stoer and Wagner [19] obtained a pendant pair of a weighted undirected graph $G=(V, E)$ by using MA-ordering in $O(|E|+|V| \log |V|)$ time. By generalizing their algorithm to a symmetric submodular system $\mathcal{S}=(V, f)$, Queyranne [17] obtained a pendant pair of this system in $O\left(|V|^{2}\right)$ time.

In this section, by using the fact that there exists a Gomory Hu tree for a symmetric submodular system $\mathcal{S}=(V, f)$, we obtain some pendant pairs of it from a Gomory Hu tree of this system. Also, we show that a Gomory Hu tree of a symmetric submodular system can be constructed by pendant pairs. Firstly, we prove the following lemma.

Lemma 1. Let $T=(V, F)$ be a flow equivalent tree of a symmetric submodular system $\mathcal{S}=(V, f)$ with the weight function $w: E \mapsto \mathbb{R}^{+}$. If $e=u v$ is an arbitrary edge of $T$, then for every $A \in \mathcal{F}_{u}(T-u v)$ and $B \in \mathcal{F}_{v}(T-u v)$, we have $w(e) \leq \min \{f(X) \mid X \in \Gamma(A, B)\}$.

Proof. Since $T$ is a flow equivalent tree of $\mathcal{S}$, then the value of a minimum $u v$-separator in $\mathcal{S}$ is equal to $w(e)$. In other words $w(e)=\min \{f(X) \mid X \in$ $\Gamma(u, v)\}$. Since $\mathcal{F}_{u}(T-u v)$ and $\mathcal{F}_{v}(T-u v)$ are two subsets of $\Gamma(u, v)$, then $w(e) \leq \min \{f(X) \mid X \in \Gamma(A, B)\}$.

Theorem 1. Let $T=(V, F)$ be a Gomory Hu tree of a symmetric submodular system $\mathcal{S}=(V, f)$ with the weight function $w: E \mapsto \mathbb{R}^{+}$. If $e=u v$ is an arbitrary edge of $T$, then for every $A \in \mathcal{F}_{u}(T-u v)$ and $B \in \mathcal{F}_{v}(T-u v), V_{v}(T-u v)$ is a minimum $A B$-separator in $\mathcal{S}$.

Proof. Since $T$ has the cut property, then $w(e)=f\left(V_{v}(T-u v)\right.$. Now, according to Lemma 1 we have $\mathrm{f}\left(\mathrm{V}_{v}(T-u v)\right) \leq \min \{f(X) \mid X \in \Gamma(A, B)\}$. On the other hand, $V_{v}(T-u v)$ is one of the elements of $\Gamma(A, B)$ then $\mathrm{f}\left(\mathrm{V}_{v}(T-u v)\right)=$ $\min \{f(X) \mid X \in \Gamma(A, B)\}$, and the proof is completed.

The following theorem is immediate from Theorem 1.
Theorem 2. Let $T=(V, F)$ be a Gomory Hu tree of a symmetric submodular system $\mathcal{S}=(V, f)$. If $e=u v$ is an arbitrary edge of $T$, then $\left(\varphi(A), \varphi\left(V_{v}(T-u v)\right)\right.$ for every $A \in \mathcal{F}_{u}(T-u v)$ is a pendant pair of $\left(\mathcal{S}_{A}\right)_{V_{v}(T-u v)}$.

Theorem 2 shows that every Gomory Hu tree of a symmetric submodular system can be obtained by using pendant pairs. We know that every Gomory Hu tree of a symmetric submodular system $\mathcal{S}$ is a flow equivalent tree having the cut propety. We show by an example that Theorem 2 is not necessarily true for every flow equivalent tree of $\mathcal{S}$. Let $V=\{1,2,3,4\}$. Consider the symmetric submodular system $\mathcal{S}=(V, f)$ presented in Table 1 .

Table 1. A symmetric submodular system $\mathcal{S}=(V, f)$.

| $A$ | $\{1\}$ | $\{2\}$ | $\{3\}$ | $\{4\}$ | $\{1,2\}$ | $\{1,3\}$ | $\{1,4\}$ | $V$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $f(A)=f(V \backslash A)$ | 4 | 6 | 3 | 5 | 4 | 5 | 9 | 0 |

It can be shown that the tree $T=(V, f)$ depicted in Figure 1 is a flow equivalent tree of $\mathcal{S}$. Suppose that $w: E \mapsto \mathbb{R}^{+}$is the weight function of $T$. By considering the edge $e=24$ of $T$, we have $V_{4}(T-24)=\{4\}$. Now, choose the element $\{2\}$ from $\mathcal{F}_{2}(T-24)$. According to Figure $1,\{4\}$ is a minimum 24 -cut and $w(24)=4$. Since $T$ is a flow equivalent tree of $\mathcal{S}$, then the value of minimum 24 -separator in $\mathcal{S}$ is equal to 4 . However, in the given system we have $f(\{4\})=5$. Therefore, $(2,4)$ cannot be a pendant pair of $\mathcal{S}$.


Fig. 1. Flow Equivalent tree of $\mathcal{S}$.

Theorem 3. Let $T=(V, F)$ be a flow equivalent tree of a symmetric submodular system $\mathcal{S}=(V, f)$ with the weight function $w: F \mapsto \mathbb{R}^{+}$. If for every edge $e=u v$
of $T$, there exists a set $A \in \mathcal{F}_{u}(T-u v)$ such that $\left(\varphi(A), \varphi\left(V_{v}(T-u v)\right)\right.$ is a pendant pair of $\left(\mathcal{S}_{A}\right)_{V_{v}(T-u v)}$, then $T$ is a Gomory Hu tree.

Proof. Since for every $e=u v$ of $T$, there exists a subset $A$ in $\mathcal{F}_{u}(T-u v)$, that $\left(\varphi(A), \varphi\left(V_{v}(T-u v)\right)\right.$ is a pendant pair of $\left(\mathcal{S}_{A}\right)_{V_{v}(T-u v)}$ then $w(e)=f\left(V_{v}(T-\right.$ $u v)$ ). Thus, $T$ has the cut property. Therefore, $T$ is a Gomory Hu tree of $\mathcal{S}$.

In the rest of this section, we will prove some properties of pendant pairs of a system.

Theorem 4. Let $(s, l)$ be a pendant pair of a system $\mathcal{S}=(V, f)$. For every $A \subseteq V \backslash\{s, l\},(s, l)$ is a pendant pair of $\mathcal{S}_{A}$.

Proof. Since $(s, l)$ is a pendant pair of $\mathcal{S}$, then $f(l)=\min \{f(X) \mid X \in \Gamma(s, l)\}$. From (6) we have $\mathrm{f}(\mathrm{l})=\mathrm{f}_{A}(l)=\min \left\{f_{A}(X) \mid X \in \Gamma(s, l)\right\}$.

Thus, $(s, l)$ is a pendant pair of $\mathcal{S}_{A}$. The proof is completed.
Note that, the converse of Theorem 4 is not generally true. Consider the given system in Table 1. Table 2 contains MA-orderings of $\mathcal{S}$ and $\mathcal{S}_{\{1,3\}}$ and also pendant pairs, obtained from these MA-orderings.

Table 2. MA-orderings of $\mathcal{S}$ and $\mathcal{S}_{\{1,3\}}$.

| system | MA-ordering | Pendant Pair |
| :---: | :---: | :---: |
| $\mathcal{S}$ | $4,2,1,3$ | $(1,3)$ |
| $S_{\{1,3\}}$ | $13,2,4$ | $(2,4)$ |

It can be observed that $(2,4)$ is a pendant pair of $S_{\{1,3\}}$; however, it is not a pendant pair of $\mathcal{S}$.

Proposition 1. If $(s, l)$ is a pendant pair of a system $\mathcal{S}=(V, f)$, then $(l, s)$ is a pendant pair of $\mathcal{S}$ iff $f(l)=f(s)$.

Proof. Let $(l, s)$ be a pendant pair of $\mathcal{S}$. Thus, $\quad f(l)=\min \{f(X) \mid X \in \Gamma(s, l)\}$. Since $(s, l)$ is also a pendant pair of $\mathcal{S}$, then $f(s)=f(l)$. Now, suppose that $f(s)=f(l)$. Since $(s, l)$ is a pendant pair of $\mathcal{S}$, then $(l, s)$ is also a pendant pair of $\mathcal{S}$.

## 4 Gomoru Hu Tree of the Contraction of a System

Let $T=(V, F)$ be a tree. A subset $X \subseteq V$ is called a $T$-connected subset of $V$, if the graph induced by $X$ in $T$ is a subtree. The following theorem shows that by having a flow equivalent tree $T$ of a symmetric submodular system $\mathcal{S}=(V, f)$, we can easily obtain a flow equivalent tree of $\mathcal{S}_{X}$ for every T-connected subset of $V$.

Theorem 5. Let $T=(V, F)$ be a flow equivalent tree of a symmetric submodular system $\mathcal{S}=(V, f)$. If $X$ is a $T$-connected subset of $V$, then $T_{>X}<$ is a flow equivalent tree of $\mathcal{S}_{X}=\left(V_{X}, f_{X}\right)$.

Proof. Let $u$ and $v$ be two distinct elements of $V_{X}$. Consider the path $P_{u v}$ connecting $u$ and $v$ in $T$. Suppose that $T^{\prime}=\left(V^{\prime}, F^{\prime}\right)$ is the induced subtree by $X$ in $T$. Let $E^{\prime}$ be the set of edges in $P_{u v}$ with the minimum weight. If $E^{\prime} \nsubseteq F^{\prime}$, then there is nothing to prove. Now, suppose that $E^{\prime}$ is a subset of $F^{\prime}$. Assume that $m_{1}=\min \left\{f_{X}(A) \mid A \in \Gamma(u, \varphi(X))\right\}, m_{2}=\min \left\{f_{X}(A) \mid A \in \Gamma(\varphi(X), v)\right\}$ and $m^{*}=\min \left\{f_{X}(A) \mid A \in \Gamma(u, v)\right\}$. Obviously, the values of the minimum $u \varphi(X)$-cut, the minimum $\varphi(X) v$-cut and the minimum $u v$-cut in tree $T_{>X<}$ are equal to $m_{1}, m_{2}$ and $\min \left\{m_{1}, m_{2}\right\}$, respectively. Thus, to show that $T_{>X}<$ is a flow equivalent tree of $\mathcal{S}_{X}$ it suffices to prove that $m^{*}=\min \left\{m_{1}, m_{2}\right\}$. Let $T^{\prime}$ be a flow equivalent tree of $\mathcal{S}_{X}$ and $P_{u v}^{\prime}$ be a path connecting $u$ and $v$ in $T^{\prime}$. Now, we have two cases: case (i), $\varphi(X)$ is appeared in $P_{u v}^{\prime}$. Therefore, the value of $u v$-separator in $\mathcal{S}_{X}$ is equal to $\min \left\{m_{1}, m_{2}\right\}$ which is equal to the value of minimum $u v$-cut in $T_{>X<}$. Case (ii), if $\varphi(X)$ is not appeared in $P_{u v}^{\prime}$, then we can easily conclude that the value of minimum $u v-$ cut in $T^{\prime}$ and $T_{>X}<$ are equal. The proof is completed.

Theorem 6. Let $T=(V, F)$ be a Gomory Hu tree of a symmetric submodular system $\mathcal{S}=(V, f)$. If $X$ is a $T$-connected subset of $V$, then $T_{>X}<$ is a Gomory Hu tree of $\mathcal{S}_{X}=\left(V_{X}, f_{X}\right)$.

Proof. According to Theorem $5, T_{>X}<$ is a flow equivalent of $\mathcal{S}_{X}$. Furthermore, from (6), $T_{>X}<$ has the cut property. Then, $T_{>X}<$ is a Gomory Hu tree of $\mathcal{S}_{X}$.

Then, by having a Gomory Hu tree of a symmetric submodular system $\mathcal{S}=$ $(V, f)$, we can find a Gomory Hu tree of the contracted system, with respect to a connected set $X$, without finding any minimum $s t$-separators in $\mathcal{S}_{X}$. Also, we can deduce that $T_{>V_{v}(T-u v)<}$ is a Gomory Hu tree of $\mathcal{S}_{V_{v}(T-u v)}$.

Corollary 1. Let $T=(V, f)$ be a Gomory Hu tree of a symmetric submodular system $\mathcal{S}=(V, f)$ and uv be an arbitrary edge of $T$. For every $A \in \mathcal{F}_{u}(T-u v)$, for which $A$ is a connecting set in $T$, the tree $T_{>A \cup V_{v}(T-u v)<}$ is a Gomory $H u$ tree of $\left(\mathcal{S}_{A}\right)_{V_{v}(T-u v)}$.

## 5 Conclusion

In this paper, we obtained some pendant pairs of a symmetric submodular system by using its Gomory Hu tree. Furthermore, for a contraction of $\mathcal{S}$ with respect to a connected set, we constructed a Gomory Hu tree only by contracting the connected set in Gomory Hu tree of $\mathcal{S}$.

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