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Gomory Hu Tree and Pendant Pairs of a Symmetric Submodular System

Saeid Hanifehnezhad and Ardeshir Dolati*

Department of Mathematics, Shahed University Tehran, Iran {s.hanifehnezhad,dolati}@shahed.ac.ir

Abstract. Let S = (V, f), be a symmetric submodular system. For two distinct elements s and l of V, let $\Gamma(s, l)$ denote the set of all subsets of V which separate s from l. By using every Gomory Hu tree of S we can obtain an element of $\Gamma(s, l)$ which has minimum value among all the elements of $\Gamma(s, l)$. This tree can be constructed iteratively by solving |V| - 1 minimum sl-separator problem. An ordered pair (s, l) is called a pendant pair of S if $\{l\}$ is a minimum sl-separator. Pendant pairs of a symmetric submodular system play a key role in finding a minimizer of this system. In this paper, we obtain a Gomory Hu tree of a contraction of S with respect to some subsets of V only by using contraction in Gomory Hu tree of S. Furthermore, we obtain some pendant pairs of Sand its contractions by using a Gomory Hu tree of S.

Keywords: Symmetric submodular system, Contraction of a system, Pendant pair, Maximum adjacency ordering, Gomory-Hu tree.

1 Introduction

Let V be a finite set. A set function $f: 2^V \mapsto \mathbb{R}$ is called a submodular function if and only if for all $X, Y \in 2^V$, we have

$$f(X) + f(Y) \ge f(X \cup Y) + f(X \cap Y). \tag{1}$$

Submodular functions play a key role in combinatorial optimization, see [3] for further discussion. Rank functions of matroids, cut capacity functions and entropy functions are some well known examples of submodular functions. For a given system S = (V, f), let $f : 2^V \to \mathbb{R}$ be a submodular function. The problem in which we want to find a subset $X \subseteq V$, for which $f(X) \leq f(Y)$ for all $Y \subseteq V$ is called submodular system minimization problem. Minimizing a submodular system is one of the most important problems in combinatorial optimization. Many problems in combinatorial optimization, such as finding minimum cut and minimum st-cut in graphs, or finding the largest common independent set in two matroids can be modeled as a submodular function minimization. Image segmentation [1, 8, 9], speech analysis [11, 12], wireless and power networks

^{*} Corresponding author. Email: dolati@shahed.ac.ir

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[20] are only a small part of applications of minimizing submodular functions. Grotchel, Lovasz and Scherijver have developed the first weakly and strongly polynomial time algorithm for minimizing submodular systems in [6] and [13], respectively. Each of them is designed based on the ellipsoid method. Then, nearly simultaneously Scherijver [18] and Iwata, Fleischer, and Fujishige [7] gave a combinatorial strongly polynomial time algorithms for this problem. Later, a faster algorithm for minimizing submodular system was proposed by Orlin [16]. To the best of our knowledge, the fastest algorithm to find a minimizer of a submodular system S = (V, f) is due to Lee et al [10]. Their algorithm runs in $O(|V|^3 log^2 |V|\tau + |V|^4 log^{O(1)} |V|)$ time, where τ is the time taken to evaluate the function.

Stoer and Wagner [19] and Frank [2] independently have presented an algorithm that finds a minimum cut of a graph G = (V, E) in $O(|E||V|+|V|^2 \log |V|)$ time. Their algorithms are based on Nagamochi and Ibaraki's algorithm [15] which finds a minimum cut of an undirected graph. Queyranne [17] developed a faster algorithm to find a minimizer in a special case of a submodular system. This algorithm was proposed to find a minimizer of a symmetric submodular system $\mathcal{S} = (V, f)$. It is a generalization of Stoer and Wagner's algorithm [19] and runs in $O(|V|^3)$ time. This algorithm, similar to Stoer and Wagner's algorithm uses pendant pairs to obtain a minimizer of a symmetric submodular system.

For a given weighted undirected graph G = (V, E), Gomory and Hu constructed a weighted tree, named as Gomory Hu tree [5]. By using a Gomory Hu tree of the graph G, one can solve the all pairs minimum *st*-cut problem with |V| - 1 calls to the maximum flow subroutine instead of the $\binom{|V|}{2}$ calls. Goemans and Ramakrishnan [4] illustrated that for every symmetric submodular system, there exists a Gomory-Hu tree. It is worth to mention that, there is neither an algorithm to construct a Gomory Hu tree of a symmetric submodular system by using pendant pairs nor any method to obtain pendant pairs of a symmetric submodular system by using a Gomory Hu tree of it.

In this paper, we obtain a Gomory Hu tree of a contraction of a symmetric submodular system S = (V, f), under some subsets of V only by using a Gomory Hu tree of S. In other words, without solving any minimum st-separator problem of the contracted system, we obtain a Gomory Hu tree of it only by contracting the Gomory Hu tree of the original system. Furthermore, we obtain some pendant pairs of a symmetric submodular system S and its contractions by using a Gomory Hu tree of S.

The outline of this paper is as follows. Section 2 provides preliminaries and basic definitions. In Section 3, we obtain some pendant pairs of a symmetric submodular system by using a Gomory Hu tree of the system. In Section 4, we construct a Gomory Hu tree of a contracted system by using a Gomory Hu tree of the original system.

2 Preliminaries

Let V be a finite nonempty set. A function $f: 2^V \mapsto \mathbb{R}$ is called a set function on V. For every $X \subseteq V$, $x \in X$ and $y \in V \setminus X$, we use X + y and X - x instead of $X \cup \{y\}$ and $X \setminus \{x\}$, respectively. Also, for $x \in V$ we use f(x) instead of $f(\{x\})$.

A pair S = (V, f), is called a system if f is a set function on V. A system S = (V, f) is called a submodular system if for all $X, Y \subseteq V$, we have

$$f(X) + f(Y) \ge f(X \cup Y) + f(X \cap Y).$$

$$\tag{2}$$

Furthermore, it is called a symmetric system if for every $X \subseteq V$, we have

$$f(X) = f(V \setminus X). \tag{3}$$

Consider a symmetric submodular system S = (V, f). Suppose that A and B are two disjoint subsets of V. A subset $X \subseteq V$ is called an AB-separator in S, if $X \cap (A \cup B) = A$ or $X \cap (A \cup B) = B$. Let $\Gamma(A, B)$ denote the set of all AB-separators in S. A subset $X \in \Gamma(A, B)$ is called a minimum AB-separator in S if $f(X) = \min_{Y \in \Gamma(A, B)} f(Y)$. If A and B are singletones $\{a\}$ and $\{b\}$, then we use ab-separator and $\Gamma(a, b)$ instead of $\{a\}\{b\}$ -separator and $\Gamma(\{a\}, \{b\})$, respectively.

Let G = (V, E) be a weighted undirected graph with the weight function $w : E \mapsto \mathbb{R}^+ \cup \{0\}$. Suppose that X is a nonempty proper subset of V. The set of all edges connecting X to $V \setminus X$, is called the cut associated with X and is denoted by $\delta(X)$. The capacity of $\delta(X)$ is denoted by C(X) and defined by

$$C(X) = \sum_{e \in \delta(X)} w(e).$$
(4)

By setting $C(\emptyset) = C(V) = 0$, (V, C) is a symmetric submodular system [14]. For two distinct vertices u and v of G, every minimum uv-separator of (V, C) is a minimum uv-cut of G.

Let T = (V, F) be a tree and uv be an arbitrary edge of it. By T-uv, we mean the forest obtained from T by removing uv. The set of vertices of two components of T - uv which respectively contain u and v, is denoted by $V_u(T - uv)$ and $V_v(T - uv)$. Also, for $uv \in F$, define $\mathcal{F}_u(T - uv) = \{X | u \in X \subseteq V_u(T - uv)\},$ $\mathcal{F}_v(T - uv) = \{X | v \in X \subseteq V_v(T - uv)\}.$

Suppose that X is a nonempty subset of vertices of a given graph G = (V, E). We denote by $G_{>X<}$ the graph obtained from G by contracting all the vertices in X into a single vertex.

Let S = (V, f) be a symmetric submodular system. Suppose that T = (V, F)is a weighted tree with the weight function $w : E \mapsto \mathbb{R}^+$. If for all $u, v \in V$, the minimum weight of the edges on the path between u and v in T is equal to minimum uv-separator in S, then T is called a flow equivalent tree of S. Also, we say that T has the cut property with respect to S if $w(e) = f(V_u(T - uv)) =$ $f(V_v(T - uv))$ for every $e = uv \in F$. A flow equivalent tree of S is called a Gomory Hu tree of S if it has cut property with respect to S.

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Consider a system S = (V, f). A pair of elements (x, y) of V is called a pendant pair for S, if $\{y\}$ is a minimum xy-separator in S.

Let S = (V, f) be a symmetric submodular system. Suppose that $\rho = (v_1, v_2, \dots, v_{|V|})$ is an ordring of the elements of V, where v_1 can be chosen arbitrarily. If for all $2 \le i \le j \le |V|$, we have

$$f(V_{i-1} + v_i) - f(v_i) \le f(V_{i-1} + v_j) - f(v_j), \tag{5}$$

where $V_i = \{v_1, v_2, \dots, v_i\}$, then ρ is called a maximum adjacency ordering (MA-ordering) of S.

For a symmetric submodular system S = (V, f), Queyranne [17] showed that the last two elements $(v_{|V|-1}, v_{|V|})$ of an MA-ordering of S, is a pendant pair of this system.

Let S = (V, f) be a system and X be an arbitrary subset of V. By $\varphi(X)$, we mean a single element obtained by unifying all elements of X. The contraction of S with respect to a subset $A \subseteq V$ is denoted by $S_A = (V_A, f_A)$ and defined by $V_A = (V \setminus A) + \varphi(A)$ and

$$f_A(X) = \begin{cases} f(X) & \text{if } \varphi(A) \notin X\\ f((X - \varphi(A)) \cup A) & \text{if } \varphi(A) \in X. \end{cases}$$
(6)

Suppose that A and B are two nonempty disjoint subsets of V. We denote by $(S_A)_B$, the contraction of S_A with respect to B.

3 Obtaining Pendant Pairs from a Gomory Hu tree

Stoer and Wagner [19] obtained a pendant pair of a weighted undirected graph G = (V, E) by using MA-ordering in $O(|E| + |V| \log |V|)$ time. By generalizing their algorithm to a symmetric submodular system S = (V, f), Queyranne [17] obtained a pendant pair of this system in $O(|V|^2)$ time.

In this section, by using the fact that there exists a Gomory Hu tree for a symmetric submodular system S = (V, f), we obtain some pendant pairs of it from a Gomory Hu tree of this system. Also, we show that a Gomory Hu tree of a symmetric submodular system can be constructed by pendant pairs. Firstly, we prove the following lemma.

Lemma 1. Let T = (V, F) be a flow equivalent tree of a symmetric submodular system S = (V, f) with the weight function $w : E \mapsto \mathbb{R}^+$. If e = uv is an arbitrary edge of T, then for every $A \in \mathcal{F}_u(T - uv)$ and $B \in \mathcal{F}_v(T - uv)$, we have $w(e) \leq \min\{f(X) | X \in \Gamma(A, B)\}$.

Proof. Since T is a flow equivalent tree of S, then the value of a minimum uv-separator in S is equal to w(e). In other words $w(e) = min\{f(X)|X \in \Gamma(u,v)\}$. Since $\mathcal{F}_u(T - uv)$ and $\mathcal{F}_v(T - uv)$ are two subsets of $\Gamma(u,v)$, then $w(e) \leq min\{f(X)|X \in \Gamma(A,B)\}$.

Theorem 1. Let T = (V, F) be a Gomory Hu tree of a symmetric submodular system S = (V, f) with the weight function $w : E \mapsto \mathbb{R}^+$. If e = uv is an arbitrary edge of T, then for every $A \in \mathcal{F}_u(T - uv)$ and $B \in \mathcal{F}_v(T - uv)$, $V_v(T - uv)$ is a minimum AB-separator in S.

Proof. Since T has the cut property, then $w(e) = f(V_v(T - uv))$. Now, according to Lemma 1 we have $f(V_v(T - uv)) \leq min\{f(X)|X \in \Gamma(A, B)\}$. On the other hand, $V_v(T - uv)$ is one of the elements of $\Gamma(A, B)$ then $f(V_v(T - uv)) = min\{f(X)|X \in \Gamma(A, B)\}$, and the proof is completed.

The following theorem is immediate from Theorem 1.

Theorem 2. Let T = (V, F) be a Gomory Hu tree of a symmetric submodular system S = (V, f). If e = uv is an arbitrary edge of T, then $(\varphi(A), \varphi(V_v(T - uv))$ for every $A \in \mathcal{F}_u(T - uv)$ is a pendant pair of $(S_A)_{V_v(T - uv)}$.

Theorem 2 shows that every Gomory Hu tree of a symmetric submodular system can be obtained by using pendant pairs. We know that every Gomory Hu tree of a symmetric submodular system S is a flow equivalent tree having the cut property. We show by an example that Theorem 2 is not necessarily true for every flow equivalent tree of S. Let $V = \{1, 2, 3, 4\}$. Consider the symmetric submodular system S = (V, f) presented in Table 1.

Table 1. A symmetric submodular system S = (V, f).

A	{1}	$\{2\}$	{3}	$\{4\}$	$\{1, 2\}$	$\{1, 3\}$	$\{1, 4\}$	V
$f(A) = f(V \setminus A)$	4	6	3	5	4	5	9	0

It can be shown that the tree T = (V, f) depicted in Figure 1 is a flow equivalent tree of S. Suppose that $w : E \mapsto \mathbb{R}^+$ is the weight function of T. By considering the edge e = 24 of T, we have $V_4(T - 24) = \{4\}$. Now, choose the element $\{2\}$ from $\mathcal{F}_2(T - 24)$. According to Figure 1, $\{4\}$ is a minimum 24-cut and w(24) = 4. Since T is a flow equivalent tree of S, then the value of minimum 24-separator in S is equal to 4. However, in the given system we have $f(\{4\}) = 5$. Therefore, (2, 4) cannot be a pendant pair of S.

$$3 \overline{1} \overline{4} 2 \overline{4} 4$$

Fig. 1. Flow Equivalent tree of S.

Theorem 3. Let T = (V, F) be a flow equivalent tree of a symmetric submodular system S = (V, f) with the weight function $w : F \mapsto \mathbb{R}^+$. If for every edge e = uv

of T, there exists a set $A \in \mathcal{F}_u(T - uv)$ such that $(\varphi(A), \varphi(V_v(T - uv))$ is a pendant pair of $(\mathcal{S}_A)_{V_v(T-uv)}$, then T is a Gomory Hu tree.

Proof. Since for every e = uv of T, there exists a subset A in $\mathcal{F}_u(T - uv)$, that $(\varphi(A), \varphi(V_v(T - uv))$ is a pendant pair of $(\mathcal{S}_A)_{V_v(T - uv)}$ then $w(e) = f(V_v(T - uv))$. Thus, T has the cut property. Therefore, T is a Gomory Hu tree of \mathcal{S} .

In the rest of this section, we will prove some properties of pendant pairs of a system.

Theorem 4. Let (s,l) be a pendant pair of a system S = (V, f). For every $A \subseteq V \setminus \{s, l\}, (s, l)$ is a pendant pair of S_A .

Proof. Since (s, l) is a pendant pair of S, then $f(l) = min\{f(X)|X \in \Gamma(s, l)\}$. From (6) we have $f(l)=f_A(l) = min\{f_A(X)|X \in \Gamma(s, l)\}$. Thus, (a, l) is a pendant pair of S. The proof is completed

Thus, (s, l) is a pendant pair of \mathcal{S}_A . The proof is completed.

Note that, the converse of Theorem 4 is not generally true. Consider the given system in Table 1. Table 2 contains MA-orderings of S and $S_{\{1,3\}}$ and also pendant pairs, obtained from these MA-orderings.

Table 2. MA-orderings of S and $S_{\{1,3\}}$.

system	MA-ordering	Pendant Pair
S	4, 2, 1, 3	(1, 3)
$S_{\{1,3\}}$	13, 2, 4	(2,4)

It can be observed that (2,4) is a pendant pair of $S_{\{1,3\}}$; however, it is not a pendant pair of S.

Proposition 1. If (s,l) is a pendant pair of a system S = (V, f), then (l, s) is a pendant pair of S iff f(l) = f(s).

Proof. Let (l, s) be a pendant pair of S. Thus, $f(l) = \min\{f(X) | X \in \Gamma(s, l)\}$. Since (s, l) is also a pendant pair of S, then f(s) = f(l). Now, suppose that f(s) = f(l). Since (s, l) is a pendant pair of S, then (l, s) is also a pendant pair of S.

4 Gomoru Hu Tree of the Contraction of a System

Let T = (V, F) be a tree. A subset $X \subseteq V$ is called a *T*-connected subset of V, if the graph induced by X in T is a subtree. The following theorem shows that by having a flow equivalent tree T of a symmetric submodular system S = (V, f), we can easily obtain a flow equivalent tree of S_X for every T-connected subset of V. **Theorem 5.** Let T = (V, F) be a flow equivalent tree of a symmetric submodular system S = (V, f). If X is a T-connected subset of V, then $T_{>X<}$ is a flow equivalent tree of $S_X = (V_X, f_X)$.

Proof. Let u and v be two distinct elements of V_X . Consider the path P_{uv} connecting u and v in T. Suppose that T' = (V', F') is the induced subtree by X in T. Let E' be the set of edges in P_{uv} with the minimum weight. If $E' \not\subseteq F'$, then there is nothing to prove. Now, suppose that E' is a subset of F'. Assume that $m_1 = \min\{f_X(A) | A \in \Gamma(u, \varphi(X))\}$, $m_2 = \min\{f_X(A) | A \in \Gamma(\varphi(X), v)\}$ and $m^* = \min\{f_X(A) | A \in \Gamma(u, v)\}$. Obviously, the values of the minimum $u\varphi(X)$ -cut, the minimum $\varphi(X)v$ -cut and the minimum uv-cut in tree $T_{>X<}$ are equal to m_1, m_2 and $\min\{m_1, m_2\}$, respectively. Thus, to show that $T_{>X<}$ is a flow equivalent tree of \mathcal{S}_X and P'_{uv} be a path connecting u and v in T'. Now, we have two cases: case (i), $\varphi(X)$ is appeared in P'_{uv} . Therefore, the value of uv-separator in \mathcal{S}_X is equal to $\min\{m_1, m_2\}$ which is equal to the value of minimum uv-cut in $T_{>X<}$. Case (ii), if $\varphi(X)$ is not appeared in P'_{uv} , then we can easily conclude that the value of minimum uv-cut in $T_{>X<}$ are equal. The proof is completed.

Theorem 6. Let T = (V, F) be a Gomory Hu tree of a symmetric submodular system S = (V, f). If X is a T-connected subset of V, then $T_{>X<}$ is a Gomory Hu tree of $S_X = (V_X, f_X)$.

Proof. According to Theorem 5, $T_{>X<}$ is a flow equivalent of S_X . Furthermore, from (6), $T_{>X<}$ has the cut property. Then, $T_{>X<}$ is a Gomory Hu tree of S_X .

Then, by having a Gomory Hu tree of a symmetric submodular system S = (V, f), we can find a Gomory Hu tree of the contracted system, with respect to a connected set X, without finding any minimum st-separators in S_X . Also, we can deduce that $T_{>V_v(T-uv)<}$ is a Gomory Hu tree of $S_{V_v(T-uv)}$.

Corollary 1. Let T = (V, f) be a Gomory Hu tree of a symmetric submodular system S = (V, f) and uv be an arbitrary edge of T. For every $A \in \mathcal{F}_u(T - uv)$, for which A is a connecting set in T, the tree $T_{>A\cup V_v(T-uv)<}$ is a Gomory Hu tree of $(S_A)_{V_v(T-uv)}$.

5 Conclusion

In this paper, we obtained some pendant pairs of a symmetric submodular system by using its Gomory Hu tree. Furthermore, for a contraction of S with respect to a connected set, we constructed a Gomory Hu tree only by contracting the connected set in Gomory Hu tree of S.

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