



**HAL**  
open science

# On boundedness of solutions of periodic systems: a multivariable cell structure approach

Denis Efimov, Johannes Schiffer

► **To cite this version:**

Denis Efimov, Johannes Schiffer. On boundedness of solutions of periodic systems: a multivariable cell structure approach. 2018. hal-01755088v1

**HAL Id: hal-01755088**

**<https://inria.hal.science/hal-01755088v1>**

Preprint submitted on 30 Mar 2018 (v1), last revised 12 Sep 2018 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# On boundedness of solutions of periodic systems: a multivariable cell structure approach

Denis Efimov, Johannes Schiffer

**Abstract**—A wide range of practical systems exhibits dynamics, which are periodic with respect to several state variables and which possess multiple invariant solutions. Yet, when analyzing stability of such systems, many classical techniques often fall short in that they only permit to establish *local* stability properties. Motivated by this, we present a new sufficient criterion for *global* stability of such a class of nonlinear systems. The proposed approach is characterized by two main properties. First, it develops the conventional cell structure framework to the case of multiple periodic states. Second, it extends the standard Lyapunov theory by relaxing the usual definiteness requirements of the employed Lyapunov functions to sign-indefinite functions. The stability robustness with respect to exogenous perturbations is analyzed. The efficacy of the proposed approach is illustrated via the global stability analysis of a nonlinear system.

## I. INTRODUCTION

Stability of dynamical systems is one of the most fundamental problems studied in control systems theory [7], [9], [19], [21], [22], [23], [27], [32] and related domains, such as mechanics, electric circuits, power systems, systems biology, *etc.* In a general (nonlinear) setting, the main approach employed for stability analysis is based on Lyapunov theory [27]. A key advantage of a Lyapunov-based stability analysis is that boundedness and convergence properties of the system's solutions can be assessed without explicit computation of the latter. Instead, it suffices to verify a set of inequalities for the Lyapunov function and its time derivative, which is derived with respect to the system's equations. More precisely, the existence of a continuously differentiable (or at least Lipschitz continuous) Lyapunov function, which is positive definite with respect to an equilibrium (or an invariant set) and the time derivative of which is non-positive along the solutions of the system under investigation, is equivalent to stability of that equilibrium (or set). Similarly, instability of an equilibrium can be studied using the Chetaev function approach [9], [14]. A Chetaev function may be sign-indefinite<sup>1</sup> with a negative or positive definite derivative.

D. Efimov is with Inria, Non-A team, Parc Scientifique de la Haute Borne, 40 avenue Halley, 59650 Villeneuve d'Ascq, France and with CRISTAL (UMR-CNRS 9189), Ecole Centrale de Lille, Avenue Paul Langevin, 59651 Villeneuve d'Ascq, France and with Department of Control Systems and Informatics, University ITMO, 49 avenue Kronverkskiy, 197101 Saint Petersburg, Russia.

J. Schiffer is with School of Electronic and Electrical Engineering, University of Leeds, Leeds LS2 9JT, UK.

This work was supported in part by the Government of Russian Federation (Grant 074-U01) and the Ministry of Education and Science of Russian Federation (Project 14.Z50.31.0031).

<sup>1</sup>A function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is called sign-definite if  $V(0) = 0$  and  $V(x) > 0$  for all  $x \in \mathbb{R}^n \setminus \{0\}$  or  $V(x) < 0$  for all  $x \in \mathbb{R}^n \setminus \{0\}$ ; and it is called sign-indefinite if  $V(x)$  takes both, positive and negative, values.

Classical stability theory is mainly concerned with the analysis of a single equilibrium. However, in numerous applications, such as biological and power systems or distributed optimization, there exist several equilibria or invariant sets (including hidden attractors [12]). Hence, a rigorous analysis of such systems with several disjoint invariant sets represents an important special case of stability theory, which requires suitable methods [4], [28], [19], [31], [6], [35], [17], [13]. For this case the stability notions have to be significantly modified and relaxed as, in particular, it has been done in [13] and further in [2], [3] for the input-to-state stability (ISS) property. See also [1], [5], [8] for other results on robust stability analysis of systems with multiple equilibria.

Unfortunately, the application of these existing results to periodic systems is, in many cases, not straightforward. The main reason for this is the technical difficulty of constructing Lyapunov functions. For example, when some of the states of the system are periodic (*e.g.*, they evolve on the circle), the corresponding Lyapunov function of [3] also has to be periodic with respect to these states, which is a severe requirement. Paramount examples of such systems are the forced nonlinear pendulum [16], [18], power systems [30], [33], [38], [39] and microgrids [34], distributed or centralized optimization [10], [37], phase-locked loops [24], [25], and complex networks of oscillators [35], [11], [36].

Motivated by this wide range of potential applications, we consider a special class of systems, which possess periodic right-hand sides with respect to several state components. The presented analysis builds upon our previous work [15], where an extension of the ISS theory from [3] to periodic systems has been proposed. However, that extension has been derived for systems on manifolds and, thus, does not allow to establish boundedness of trajectories in  $\mathbb{R}^n$ . The latter is possible by using the cell structure approach proposed in [26] (and later in [29]) and further developed in [19], [40]. Yet, it is only applicable to systems with a scalar periodic variable, which is a severe restriction when considering complex networks. By seeking to overcome the aforementioned limitations of the existing approaches, in the present paper the cell structure framework is extended to autonomous systems, the state of which evolves in  $\mathbb{R}^n$  and which possess multiple periodic variables. As in [15], our developments are inspired by the conventional cell structure approach [26]. The presented results are illustrated via application to a complex nonlinear system.

The outline of this paper is as follows. Preliminaries and the theory from [19], [40] are given in Section II. The problem statement is presented in Section III with the main results in Section IV. Robustness with respect to external inputs is investigated in Section V. The efficiency of the presented

robust stability conditions is illustrated in Section VI.

## II. PRELIMINARIES

Denote the real and integer numbers by  $\mathbb{R}$  and  $\mathbb{Z}$ , respectively, and  $\mathbb{R}_+ = \{s \in \mathbb{R} : s \geq 0\}$ ,  $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$ .

Let the map  $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be of class  $\mathcal{C}^1$ ,  $f(0) = 0$ , and consider a nonlinear system of the following form:

$$\dot{x}(t) = f(x(t)) \quad \forall t \geq 0 \quad (1)$$

with the state  $x(t) \in \mathbb{R}^n$ . We denote by  $X(t, x)$  the uniquely defined solution of (1) at time  $t$  fulfilling  $X(0, x) = x$ . A set  $S \subset M$  is invariant for the system (1) if  $X(t, x) \in S$  for all  $t \in \mathbb{R}$  and for all  $x \in S$ ; for  $x \in \mathbb{R}^n$  the point  $y \in \mathbb{R}^n$  belongs to its  $\omega$ -limit ( $\alpha$ -limit) set if there is a sequence  $t_i$ ,  $\lim_{i \rightarrow +\infty} t_i = +\infty$ , such that  $\lim_{i \rightarrow +\infty} X(t_i, x) = y$  ( $\lim_{i \rightarrow +\infty} X(-t_i, x; 0) = y$ ); for any  $x \in \mathbb{R}^n$  its  $\alpha$ - and  $\omega$ -limit sets are invariant [20]. Define the distance from a point  $x \in \mathbb{R}^n$  to the set  $S \subset \mathbb{R}^n$  as  $|x|_S = \inf_{a \in S} |x - a|$ , where  $|x| = |x|_{\{0\}}$  for  $x \in \mathbb{R}^n$  is a usual Euclidean norm of a vector  $x \in \mathbb{R}^n$ , and denote

$$|x|_\infty = \max_{1 \leq i \leq n} |x_i|, \quad |x|_1 = \sum_{i=1}^n |x_i|,$$

then

$$|x|_\infty \leq |x| \leq |x|_1 \leq \sqrt{n}|x| \leq n|x|_\infty.$$

A continuous function  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to class  $\mathcal{K}$  if  $\alpha(0) = 0$  and the function is strictly increasing. The function  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to the class  $\mathcal{K}_\infty$  if  $\alpha \in \mathcal{K}$  and it is increasing to infinity. A continuous function  $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to the class  $\mathcal{KL}$  if  $\beta(\cdot, t) \in \mathcal{K}_\infty$  for each fixed  $t \in \mathbb{R}_+$  and  $\lim_{t \rightarrow +\infty} \beta(s, t) = 0$  for each fixed  $s \in \mathbb{R}_+$ .

### A. Boundedness of solutions of periodic systems

As outlined in Section I, the present paper is dedicated to the stability analysis of periodic systems [19], [40]. More precisely, we assume in the following that for the system (1) there exists  $\xi \in \mathbb{R}^n$ ,  $\xi \neq 0$ , such that for all  $x \in \mathbb{R}^n$

$$f(x) = f(x + \xi).$$

Next, we recall a sufficient criterion derived in [26], [19], [40], which allows to establish *boundedness* of solutions of periodic systems. To this end consider a special case of the system (1) given by

$$f(x) = Px + b\varphi(c^\top x),$$

where  $P \in \mathbb{R}^{n \times n}$  is a singular matrix,  $c \in \mathbb{R}^n$ ,  $b \in \mathbb{R}^n$ ,  $\varphi : \mathbb{R} \rightrightarrows \mathbb{R}$  is a  $\Delta$ -periodical set-valued function, which is upper semicontinuous, with a nonempty, convex and closed set of values for any value of its argument. We note that a time-varying version of  $\varphi$  has been considered in [19], [40], but we restrict ourselves to the autonomous version of  $\varphi$ . Then under these restrictions and for any initial condition  $x_0 \in \mathbb{R}^n$  the system (1) has a solution  $X(t, x_0)$ . Assume also that for all  $\sigma \in \mathbb{R} \setminus \{0\}$  and all  $\phi \in \varphi(\sigma)$

$$\mu_1 \leq \frac{\phi}{\sigma} \leq \mu_2; \quad \mu_1^{-1} \mu_2^{-1} \varphi(0) = 0$$

for some  $\mu_1 \in \mathbb{R} \cup \{-\infty\}$  and  $\mu_2 \in \mathbb{R} \cup \{+\infty\}$ . The periodicity of  $\varphi$  implies that either  $\mu_1 < 0$ ,  $\mu_2 > 0$  or  $\mu_1 = \mu_2 = 0$ , and the latter case is excluded from consideration due to its triviality.

**Theorem 1.** [26], [19], [40] Assume that there exists  $\lambda > 0$  such that:

- 1) the matrix  $P + \lambda I_n$ , where  $I_n \in \mathbb{R}^{n \times n}$  is the identity matrix, has  $n - 1$  eigenvalues with negative real parts;
- 2) for all  $\omega \in \mathbb{R}$

$$\mu_1^{-1} \mu_2^{-1} + (\mu_1^{-1} + \mu_2^{-1}) \operatorname{Re} \chi(i\omega - \lambda) + |\chi(i\omega - \lambda)|^2 \leq 0,$$

where  $\chi(s) = c^\top (P - sI_n)^{-1} b$ .

Then, for any initial condition  $x_0 \in \mathbb{R}^n$  the solution  $X(t, x_0)$  of the periodic system (1) is bounded for  $t \in [0, +\infty)$ .

*Remark 1.* Under the conditions of this theorem the system (1) admits only one periodic coordinate, i.e. the vector  $\xi$  has only one non-zero element and the remaining  $n - 1$  state variables are not periodic.

The proof of Theorem 1 (see Theorem 4.3.1 in [19], or Theorem 4.7 in [40]) is based on the fact that under the introduced conditions there is  $H = H^\top \in \mathbb{R}^{n \times n}$  (which has one negative and  $n - 1$  positive eigenvalues) such that for  $V_0(x) = x^\top Hx$  we have that  $dV_0(x(t))/dt \leq -2\lambda V_0(x(t))$  for all  $t \in [0, +\infty)$ , which implies that the set  $\Omega_0 = \{x \in \mathbb{R}^n : V_0(x) \leq 0\}$  is forward invariant for (1), i.e.  $X(t, x_0) \in \Omega_0$  for all  $t \in [0, +\infty)$  provided that  $x_0 \in \Omega_0$ . Next, introducing the functions  $V_j(x) = V_0(x - j\delta)$  and sets  $\Omega_j = \{x \in \mathbb{R}^n : V_j(x) \leq 0\}$ , where  $j$  is any integer and the vector  $\delta \in \mathbb{R}^n$  satisfies the conditions  $\delta \neq 0$ ,  $P\delta = 0$  and  $c^\top \delta = \Delta$ , by periodicity of  $f$  in (1) we obtain that  $dV_j(x(t))/dt \leq -2\lambda V_j(x(t))$  for all  $t \in [0, +\infty)$ , then the sets  $\Omega_j$  are forward invariant for (1). Finally, it is shown in [19], [40] that for any  $x_0 \in \mathbb{R}^n$  there is an index  $j_0$  such that  $x_0 \in \Gamma_{j_0}$ , where  $\Gamma_j = \Omega_j \cap \Omega_{-j} \cap \{x \in \mathbb{R}^n : |h^\top x| \leq j|h^\top \delta|\}$  with  $h \in \mathbb{R}^n$  being the eigenvector of the matrix  $H$  corresponding to the negative eigenvalue. As it has been shown above  $X(t, x_0) \in \Gamma_{j_0}$  for all  $t \in [0, +\infty)$  (since it is true for  $\Omega_{j_0} \cap \Omega_{-j_0}$ ). In addition the set  $\Gamma_{j_0}$  is bounded, which was necessary to prove. In other words, an important observation of [26], [19], [40] is that any intersection of the sets  $\Omega_j$  for all integers  $j$  forms a kind of cell cover of  $\mathbb{R}^n$ , where each cell is bounded and forward invariant. Therefore, this framework is commonly known as cell structure approach.

The following example illustrates the main idea of the proof of Theorem 1.

**Example 1.** Suppose (1) is given by  $x = (z, \theta) \in \mathbb{R}^2$  and  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ . Furthermore, suppose  $f$  is periodic in  $\theta$  with period  $T > 0$ , i.e., for any integer  $j \in \mathbb{Z}$ , we have that  $f(z, \theta + Tj) = f(z, \theta)$ . Following Theorem 1, suppose that there exists a function  $V : \mathbb{R}^2 \rightarrow \mathbb{R}$ , such that

$$V(0, 0) = 0, \quad V(z, 0) > 0 \quad \forall z \in \mathbb{R} \setminus \{0\},$$

$$\frac{\partial V(x)}{\partial x} f(x) \leq -\lambda V(x) \quad \forall x \in \mathbb{R}^2.$$

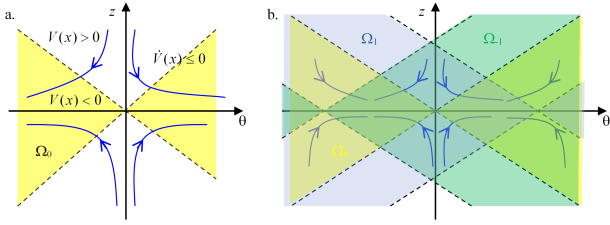


Figure 1. Illustration of the sets  $\Omega_j$  for  $j \in \{-1, 0, 1\}$

By periodicity of  $f$  in  $T$  for any integer  $j$

$$\dot{V}(z, \theta - Tj) \leq -\lambda V(z, \theta - Tj) \quad \forall (z, \theta) \in \mathbb{R}^2.$$

Denote  $V_j(x) = V(z, \theta - Tj)$ , then the invariant sets  $\Omega_j = \{x \in \mathbb{R}^2 : V_j(x) \leq 0\}$  are employed in Theorem 1 to establish boundedness of solutions for the system and they are illustrated in Fig. 1. The set  $\Omega_0$  and a possible behavior of trajectories in its neighborhood are shown in Fig. 1,a; the intersection of  $\Omega_j$  for  $j = -1, 0, 1$  is plotted in Fig. 1,b. By construction, the common points in  $\cap_{j \in \mathbb{Z}} \Omega_j$  form isolated invariant cells (the darkest rectangles in Fig. 1,b), to which all trajectories converge. Note that a passage from one cell to another is impossible since the individual cells are only connected through equilibria. In general, the boundary at which  $V_j$  changes sign is not a straight line, but can be of an arbitrary shape. The straight lines are chosen here to ease the illustration of the main idea of Theorem 1 and also to motivate the notion of cell structure.

### III. PROBLEM STATEMENT

The functions proposed in [26] for the analysis of boundedness of trajectories of the system (1) with a scalar periodic state are sign-indefinite with a sign-indefinite derivative. Clearly, such a relaxation of the definiteness of a Lyapunov function might simplify significantly its construction. Usually sign-indefinite functions with a sign-definite derivative are used to establish instability of an equilibrium of (1), *e.g.* Chetaev functions [9], [14]. An important observation of [26] is that the combination of “instability” and periodicity leads to boundedness of trajectories: under periodicity the existence of invariant solutions separating the domain of periodic variables, with probably repulsing trajectories around those invariants, implies the existence of a certain cell structure created by periodically repeated invariant solutions, which bounds the admissible behavior of the trajectories (see Example 1). A major restriction of the cell structure approach in [26], [19], [40] is that it can only be applied in the case of a scalar periodic component in  $x(t)$ . As a consequence, the main objective of this work is to extend that approach to a generic multidimensional case.

Inspired by [26], [3], an equivalent characterization of the ISS property for a periodic system on *manifolds* has been proposed in [15] in terms of ISS-Leonov functions, which are sign-indefinite. In the present work the concept of Leonov functions is further developed for (1) with  $x \in \mathbb{R}^n$ . More precisely, we assume the following:

**Assumption 1.** Let  $x = (z, \theta) \in \mathbb{R}^n$ , where  $z \in \mathbb{R}^k$  and  $\theta \in \mathbb{R}^q$  are two subsets of the state vector,  $n = k + q$ ,  $k > 0$  and  $q > 0$ . The vector field  $f$  in (1) is  $2\pi$ -periodic with respect to  $\theta$ .

In other words, we suppose that the system (1) can be embedded into a manifold  $M = \mathbb{R}^k \times \mathbb{S}^q$  by a simple projection of the variables  $\theta$  on the sphere  $\mathbb{S}^q$ . For our subsequent derivations, it is convenient to introduce auxiliary sets:  $\mathcal{W} = (\mathbb{R}^k \times S)$  where  $S$  is a  $q$ -th sphere (a set topologically equivalent to  $\mathbb{S}^q$ ) such that

$$\begin{aligned} \pi \leq |\theta|_\infty < 2\pi \quad \forall x = (z, \theta) \in \mathcal{W}, \\ \mathcal{U} = \bigcup_{r \in \mathbb{Z}_+} \mathcal{U}_r, \end{aligned} \quad (2)$$

$$\mathcal{U}_r = \{x = (z, \theta) \in \mathbb{R}^n : z = 0, |\theta|_\infty = 2r\pi, f(x) = 0\}.$$

Therefore, the set  $\mathcal{U}$  includes all equilibria obtained by shifting the one at the origin with the use of  $2\pi$ -periodicity in  $\theta$ . However, in general, the system (1) may possess other equilibria that do not belong to  $\mathcal{U}$ .

**Definition 1.** We say that a  $C^1$  function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is a Leonov function for (1) if there exist  $g \geq 0$ , functions  $\alpha \in \mathcal{K}_\infty$ ,  $\psi \in \mathcal{K}$  and a continuous function  $\lambda : \mathbb{R} \rightarrow \mathbb{R}$ ,  $\lambda(0) = 0$  and  $\lambda(s)s > 0$  for all  $s \neq 0$ , such that for  $x = (z, \theta) \in \mathbb{R}^k \times \mathbb{R}^q$ :

$$\begin{aligned} \alpha(|z|) - \psi(|\theta|) - g \leq V(x) \quad \forall x \in \mathbb{R}^n, \\ \inf_{x \in \mathcal{W}} V(x) > 0, \quad \sup_{x \in \mathcal{U}} V(x) \leq 0 \end{aligned} \quad (3)$$

and the following dissipation holds:

$$\frac{\partial V(x)}{\partial x} f(x) \leq -\lambda(V(x)) \quad \forall x \in \mathbb{R}^n. \quad (4)$$

Roughly speaking, the function  $V$  is radially unbounded with respect to the non-periodic variable  $z$  and negative definite with respect to the distance to the set  $\mathcal{W}$  (it takes positive values on the set  $\mathcal{W}$  and negative ones at the origin and on the set  $\mathcal{U}$ ), *i.e.* the conditions (3) can be equivalently formulated as

$$\begin{aligned} \alpha(|z|) - \psi'(|\theta|_S) + g' \leq V(x) \quad \forall x \in \mathbb{R}^n, \\ \sup_{x \in \mathcal{U}} V(x) \leq 0 \end{aligned}$$

for some  $\psi' \in \mathcal{K}$  and  $g' \geq 0$  with  $\psi'(|\theta|_S) \geq g'$  (since  $V(0) \leq 0$ ). Note that the set  $\mathcal{W}$  cannot contain an equilibrium at all since in such a case the derivative of  $V$  would be strictly negative at an equilibrium.

**Remark 2.** Chetaev’s theorem about instability of a set can be formulated as follows [14]: for a  $C^1$  function  $U : \mathbb{R}^n \rightarrow \mathbb{R}$ , with  $U(x) = 0$  for all  $x \in \mathcal{Q}$ , where  $\mathcal{Q} \subset \mathbb{R}^n$  is a compact invariant set of (1), if there exists  $\epsilon_0 > 0$  such that  $\mathcal{U}^+ \cap \mathcal{B}_{\mathcal{Q}}(\epsilon) \neq \emptyset$  for any  $\epsilon \in (0, \epsilon_0]$ , where  $\mathcal{U}^+ = \{x \in \mathcal{B}_{\mathcal{Q}}(\epsilon_0) : U(x) > 0\}$  and  $\mathcal{B}_{\mathcal{Q}}(\epsilon) = \{x \in \mathbb{R}^n : |x|_{\mathcal{Q}} < \epsilon\}$ , and if

$$\frac{\partial U(x)}{\partial x} f(x) > 0 \quad \forall x \in \mathcal{U}^+, \quad (5)$$

then (1) is unstable with respect to  $\mathcal{Q}$  with the region of repulsion  $\mathcal{B}_{\mathcal{Q}}(\epsilon_0)$ . Therefore, take  $U(x) = -V(x)$ , where

$V : \mathbb{R}^n \rightarrow \mathbb{R}$  is a Leonov function for (1), due to (4) the inequality  $\frac{\partial V(x)}{\partial x} f(x) < 0$  may satisfy for  $V(x) < 0$ , in such a case the property (5) can be verified, and if the set  $\{x \in \mathbb{R}^n : V(x) = 0\}$  (since  $V$  is continuous and sign indefinite according to (3) or (6) below, then this set is not empty) contains a compact and invariant set of (1), e.g. an equilibrium, then existence of a Leonov function  $V$  implies instability of (1) with the region of repulsion belonging to  $\{x \in \mathbb{R}^n : V(x) \leq 0\}$ .

*Remark 3.* The appearance of the norm  $|\cdot|_\infty$  in formulation of Leonov function is originated by the topology induced by periodicity of  $f$  in  $\theta$ : replicating the equilibrium at the origin using periodicity in  $\theta \in \mathbb{R}^q$  creates a set of equilibria located on intersections of the levels  $|\theta|_\infty = 2j\pi$  for  $j \in \mathbb{Z}$ .

The following condition on existence of a Leonov function is formulated using conventional upper and lower estimates in terms of functions of class  $\mathcal{K}$ .

**Proposition 1.** *Let for a  $C^1$  function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  there exist functions  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty, \sigma_1, \sigma_2 \in \mathcal{K}$ , and a continuous function  $\lambda : \mathbb{R} \rightarrow \mathbb{R}, \lambda(0) = 0$  and  $\lambda(s)s > 0$  for all  $s \neq 0$ , and scalars  $0 < g_1 \leq g_2 \leq \sigma_2(\pi)$ , such that for all  $x = (z, \theta) \in \mathbb{R}^n$*

$$\begin{aligned} \alpha_1(|z|) - \sigma_1(|\theta|_\infty - \pi) + g_1 &\leq V(x) \\ &\leq \alpha_2(|z|) - \sigma_2(|\theta|_\infty - \pi) + g_2, \end{aligned} \quad (6)$$

and (4) holds. Then  $V$  is a Leonov function for (1).

*Proof.* By assumption,  $g_1 \leq g_2 \leq \sigma_2(\pi)$ . Furthermore, note that at the origin the relations (6) take the form:

$$0 \geq g_2 - \sigma_2(\pi) \geq V(0) \geq g_1 - \sigma_1(\pi).$$

Thus,  $g_1 \leq \sigma_1(\pi)$ . The restriction  $g_1 > 0$  ensures that on the set  $\mathcal{W} = \{x = (z, \theta) \in \mathbb{R}^n : |\theta|_\infty = \pi\}$  (and in some of its vicinity) the function  $V$  takes positive values:

$$V(x) \geq \alpha_1(|z|) + g_1 > 0 \quad \forall x \in \mathcal{W},$$

while  $g_2 \leq \sigma_2(\pi)$  implies that  $V(0) \leq 0$ , and that

$$V(x) \leq g_2 - \sigma_2(\pi) \leq 0 \quad \forall x \in \mathcal{U}_1.$$

Next, since  $\sigma_2 \in \mathcal{K}$ , it is monotonically increasing, and

$$V(x) \leq g_2 - \sigma_2((2r-1)\pi) \leq g_2 - \sigma_2(\pi) \leq 0 \quad \forall x \in \mathcal{U}_r,$$

for all  $r \geq 1$ . For any  $x \in \mathbb{R}^n$  we have from (6):

$$\begin{aligned} V(x) &\geq \alpha_1(|z|) - \sigma_1(|\theta|_\infty - \pi) + g_1 \\ &\geq \alpha_1(|z|) - \tilde{\sigma}_1(|\theta|) - g, \end{aligned}$$

where existence of some  $\tilde{\sigma}_1 \in \mathcal{K}$  and  $g \geq 0$  follows from the relations of the norms  $|\cdot|$  and  $|\cdot|_\infty$ , and the demonstrated property  $g_1 \leq \sigma_1(\pi)$  (i.e.  $V(0) \leq 0$ ). Therefore, all properties of (3) are verified.  $\square$

#### IV. MAIN RESULT

In this section, at first, it is shown that the existence of a Leonov function implies boundedness of trajectories for a periodic system (1). Second, a particular scenario is analyzed.

**Theorem 2.** *If for the system (1) under Assumption 1 there exists a Leonov function, then for all  $x_0 \in \mathbb{R}^n$  the corresponding trajectories  $X(t, x_0)$  are bounded for all  $t \geq 0$ .*

*Proof.* Denote the set of negative values of  $V$  as

$$\Omega = \{x \in \mathbb{R}^n : V(x) \leq 0\}.$$

From the definition of the set  $\Omega$  and the lower bound of the function  $V$  in (3) we obtain:

$$\alpha(|z|) \leq g + \psi(|\theta|) \quad \forall x = (z, \theta) \in \Omega.$$

Recall the sets  $\mathcal{U}$  and  $\mathcal{W}$  defined in (2), and that  $|\theta| \leq q|\theta|_\infty$ . Then from the definition of a Leonov function the set  $\mathcal{U}$  belongs to  $\Omega$ , while the set  $\mathcal{W} \notin \Omega$ . Furthermore, the set  $\Omega$  can be decomposed into two parts:

$$\Omega = \Omega' \cup \Omega'',$$

where

$$\Omega' \subset \{x \in \mathbb{R}^n : |z| \leq \alpha^{-1}(\tilde{g}), |\theta|_\infty < c\}$$

is a compact set containing the origin for some  $c \in (\pi, 2\pi)$ , i.e.  $\mathcal{U}_0 \subset \Omega'$ , with  $g \leq \tilde{g} \leq g + \psi(qc)$  (the constant  $\tilde{g}$  evaluates the upper bound on  $|z|$  after which the growing terms in  $z$  will force  $V$  to take positive values for any  $|\theta|_\infty < c$ ), and

$$\Omega'' \subset \{x \in \mathbb{R}^n : |\theta|_\infty > c\}.$$

Both these sets are non-empty since  $0 \in \Omega'$  and  $\mathcal{U}_r \in \Omega''$  for  $r \geq 1$  due to (3), and the set  $\Omega''$  is unbounded. In addition, the subsets  $\Omega'$  and  $\Omega''$  are separated by the set  $\mathcal{W}$  (which determines the value of  $c$ ).

Denote  $V(t) = V(X(t, x_0))$  for any  $x_0 \in \mathbb{R}^n$ , then under the conditions of the theorem we have:

$$\dot{V}(t) + \lambda(V(t)) \leq 0 \quad \forall t \geq 0,$$

and, clearly,  $V(t)$  is strictly decreasing while  $X(t, x_0) \in \mathbb{R}^n \setminus \Omega$ . Therefore, if  $x_0 \in \Omega$  then  $X(t, x_0) \in \Omega$  for all  $t \geq 0$  and the set  $\Omega$  is forward invariant for (1). Conversely, if  $x_0 \in \mathbb{R}^n \setminus \Omega$  then there exists  $0 \leq T_{x_0} < +\infty$  such that  $X(T_{x_0}, x_0) \in \Omega$  and, by invariance, the trajectory remains in this set for all  $t \geq T_{x_0}$ . Thus, the set  $\Omega$  is globally attractive and forward invariant. Note, that this property does not imply any kind of stability since  $\Omega$  may be unbounded (it could also be interpreted as *instability* of a set close to  $\mathcal{W}$  with the domain of repulsion  $\mathbb{R}^n \setminus \Omega$ ).

To establish stability, we exploit the *periodicity* of (1). Denote by  $j = [j_1, \dots, j_q]$  a multi-index vector, where  $j_i \in \mathbb{Z}$  for all  $i = 1, \dots, q$ . Introduce the new variable  $x_j = x - \begin{bmatrix} 0_k \\ 2\pi j \end{bmatrix}$  and the function  $V_j(x) = V\left(x - \begin{bmatrix} 0_k \\ 2\pi j \end{bmatrix}\right)$  for any such multi-index vector  $j$  (i.e.  $V_0(x) = V(x)$ ), where  $0_k$  is the zero vector of dimension  $k$ . Hence

$$V_j(x) = V_j\left(x_j + \begin{bmatrix} 0_k \\ 2\pi j \end{bmatrix}\right) = V(x_j),$$

and by  $2\pi$ -periodicity of  $f$  in  $\theta$ ,

$$\dot{x}_j(t) = f(x(t)) = f(x_j(t)) \quad \forall t \geq 0.$$

Therefore, taking into account the properties substantiated for (1) and  $V$ , we obtain that the set  $\{x_j \in \mathbb{R}^n : V(x_j) \leq 0\}$  is globally attractive and forward invariant, which in the original coordinates  $x$  implies these properties for the set

$$\begin{aligned}\Omega_j &= \{x \in \mathbb{R}^n : V_j(x) \leq 0\} \\ &= \left\{x \in \mathbb{R}^n : V \left( x - \begin{bmatrix} 0_k \\ 2\pi j \end{bmatrix} \right) \leq 0 \right\}.\end{aligned}$$

Using similar arguments as for  $\Omega$  it is possible to show that  $\Omega_j = \Omega'_j \cup \Omega''_j$ , where

$$\begin{aligned}\Omega'_j &\subset \{x \in \mathbb{R}^n : |z| \leq \alpha^{-1}(\tilde{g}), |\theta - 2\pi j|_\infty < c\}, \\ \Omega''_j &\subset \{x \in \mathbb{R}^n : |\theta - 2\pi j|_\infty > c\}.\end{aligned}$$

The sets  $\Omega_j$  are forward invariant, then a trajectory  $X(t, x_0)$  has to belong asymptotically to the intersection of these sets:

$$\bigcap_j \Omega_j = \bigcap_j (\Omega'_j \cup \Omega''_j) = \bigcap_j \Omega'_j \cup \bigcap_j \Omega''_j \cup \Xi,$$

where  $\Xi$  is a union of all possible intersections of the form  $\Omega'_{j_1} \cap \Omega''_{j_2}$  with  $j_1 \neq j_2$ . Since  $c \in (\pi, 2\pi)$ , then  $\bigcap_j \Omega'_j = \bigcup_j \Theta_j$ , where  $\Theta_j \subset \Omega'_j$  is formed by compact isolated sets due to all  $\Omega'_j$  possess the same properties. By definition  $\bigcap_j \Omega''_j = \emptyset$  due to the fact that  $\Omega''_j \subset \mathbb{R}^n \setminus \Upsilon_j$  with  $\Upsilon_j = \{x \in \mathbb{R}^n : |\theta - 2\pi j|_\infty \leq c\}$ , and thus  $\bigcap_j \Omega''_j \subset \bigcap_j \mathbb{R}^n \setminus \Upsilon_j = \mathbb{R}^n \setminus \bigcup_j \Upsilon_j = \mathbb{R}^n \setminus \mathbb{R}^n = \emptyset$ . Since  $(0_k, 2\pi j_1) \in \mathcal{U} \cap \Omega'_{j_1}$  for any  $j_1$  by definition of both sets, and  $\mathcal{U} \setminus \Omega'_{j_2} \subset \Omega''_{j_2}$  due to  $c \in (\pi, 2\pi)$ , then  $(0_k, 2\pi j_1) \in \Omega''_{j_2}$  for any  $j_1 \neq j_2$ . Consequently,  $\Omega'_{j_1} \cap \Omega''_{j_2} \neq \emptyset$  for any multi-index vectors  $j_1 \neq j_2$ , and  $\Xi \subset \bigcup_j \Omega'_j \cap \mathcal{R}_j$  since  $c \in (\pi, 2\pi)$ , where  $\mathcal{R}_j = \{x \in \mathbb{R}^n : |\theta - 2\pi j|_\infty < \pi\}$ . Therefore,

$$\bigcap_j \Omega_j \subset \left( \bigcup_j \Omega'_j \cap \mathcal{R}_j \right) \cup \left( \bigcup_j \Theta_j \right),$$

for each  $j$  the sets  $\Omega'_j \cap \mathcal{R}_j$  and  $\Theta_j$  are compact and isolated; for any neighboring  $\Omega'_{j_1} \cap \mathcal{R}_{j_1} \subset \Omega''_{j_2}$  and  $\Theta_{j_2} \subset \Omega'_{j_2}$  there is no intersection by construction ( $\Omega'_{j_2} \cap \Omega''_{j_2} = \emptyset$ ). Then the compact sets in  $\bigcap_j \Omega_j$  form a kind of cell cover of  $\{x \in \mathbb{R}^n : |z| \leq \alpha^{-1}(\tilde{g})\}$ . Recall that for all multi-index vectors  $j$  the corresponding sets  $\Omega_j$  are globally attracting and forward invariant. Take any  $x_0 \in \mathbb{R}^n$ , then  $X(t, x_0)$  asymptotically enters and stays in a cell  $\Omega'_j$  for some  $j$ . Hence, for any  $x_0 \in \mathbb{R}^n$  the corresponding solution  $X(t, x_0)$  is bounded for all  $t \geq 0$ .  $\square$

Theorem 2 provides a very general result on application of the theory of Leonov functions. To illustrate and clarify the idea of the approach, equivalent conditions for a special case are developed below.

**Corollary 1.** *Suppose that there exist  $C^1$  functions  $W : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ , functions  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ , constants  $\lambda > 0$*

*and  $\bar{a}_i \geq \underline{a}_i > 0$ ,  $\bar{c}_i \geq \underline{c}_i \geq 0$ ,  $b_i \geq \bar{b}_i > 0$ ,  $i = 1, \dots, q$ , such that for all  $x \in \mathbb{R}^n$*

$$\begin{aligned}\alpha_1(|z|) &\leq W(x) \leq \alpha_2(|z|), \\ \sum_{i=1}^q \underline{a}_i \theta_i^2 - b_i |\theta_i| + \underline{c}_i &\leq \varphi(x) \leq \sum_{i=1}^q \bar{a}_i \theta_i^2 - \bar{b}_i |\theta_i| + \bar{c}_i, \\ \frac{\partial V(x)}{\partial x} f(x) + \lambda V(x) &\leq 0,\end{aligned}$$

*where  $V(x) = W(x) - \varphi(x)$  and  $x = (z, \theta)$ . Then  $X(t, x_0)$  of (1) is bounded for any  $x_0 \in \mathbb{R}^n$  and  $t \geq 0$  provided that*

$$\begin{aligned}\underline{a}_{\min} \pi + \sum_{i=1}^q \underline{c}_i \pi^{-1} &< q \underline{b}_{\max} < 2 \underline{a}_{\min} \pi + 0.5 \sum_{i=1}^q \underline{c}_i \pi^{-1}, \\ q^2 \bar{a}_{\max} \pi + \sum_{i=1}^q \bar{c}_i \pi^{-1} &< \bar{b}_{\min} < 2q^2 \bar{a}_{\max} \pi + 0.5 \sum_{i=1}^q \bar{c}_i \pi^{-1},\end{aligned}$$

*where  $\underline{a}_{\min} = \min_{1 \leq i \leq q} \underline{a}_i$ ,  $\underline{b}_{\max} = \max_{1 \leq i \leq q} b_i$ ,  $\bar{a}_{\max} = \max_{1 \leq i \leq q} \bar{a}_i$  and  $\bar{b}_{\min} = \min_{1 \leq i \leq q} \bar{b}_i$ .*

*Proof.* The claim is established by applying Theorem 2 and Proposition 1. To this end, it is only necessary to show that  $V$  is a Leonov function for (1), i.e.

$$\sigma_2(|\theta|_\infty - \pi) - g_2 \leq \varphi(x) \leq \sigma_1(|\theta|_\infty - \pi) - g_1$$

for some  $\sigma_1, \sigma_2 \in \mathcal{K}$  and some  $0 < g_1 \leq g_2 \leq \sigma_2(\pi)$ , since all the remaining conditions are already satisfied under the standing assumptions.

Note that

$$\begin{aligned}\sum_{i=1}^q \bar{a}_i \theta_i^2 - \bar{b}_i |\theta_i| + \bar{c}_i &\leq \sum_{i=1}^q \bar{a}_{\max} \theta_i^2 - \bar{b}_{\min} |\theta_i| + \bar{c}_i \\ &= \bar{a}_{\max} |\theta|^2 - \bar{b}_{\min} |\theta|_1 + \sum_{i=1}^q \bar{c}_i \\ &\leq q^2 \bar{a}_{\max} |\theta|_\infty^2 - \bar{b}_{\min} |\theta|_\infty + \sum_{i=1}^q \bar{c}_i = \bar{f}(|\theta|_\infty)\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^q \underline{a}_i \theta_i^2 - b_i |\theta_i| + \underline{c}_i &\geq \sum_{i=1}^q \underline{a}_{\min} \theta_i^2 - b_{\max} |\theta_i| + \underline{c}_i \\ &= \underline{a}_{\min} |\theta|^2 - b_{\max} |\theta|_1 + \sum_{i=1}^q \underline{c}_i \\ &\geq \underline{a}_{\min} |\theta|_\infty^2 - q b_{\max} |\theta|_\infty + \sum_{i=1}^q \underline{c}_i = \underline{f}(|\theta|_\infty).\end{aligned}$$

Under the introduced restrictions the following properties are satisfied for  $\underline{f}$ :

$$\begin{aligned}\sum_{i=1}^q \underline{c}_i &= \underline{f}(0) \geq 0, \\ \underline{a}_{\min} \pi + \sum_{i=1}^q \underline{c}_i \pi^{-1} &< q b_{\max} \Rightarrow 0 > \underline{f}(\pi), \\ 2 \underline{a}_{\min} \pi + 0.5 \sum_{i=1}^q \underline{c}_i \pi^{-1} &> q b_{\max} \Rightarrow \underline{f}(2\pi) > 0\end{aligned}$$

and  $2\pi^2 \underline{a}_{\min} > \sum_{i=1}^q \underline{c}_i$  is a sufficient condition for existence of  $\underline{b}_{\max}$ . Similarly for  $\bar{f}$ :

$$\begin{aligned} \sum_{i=1}^q \bar{c}_i &= \bar{f}(0) \geq 0, \\ q^2 \bar{a}_{\max} \pi + \sum_{i=1}^q \bar{c}_i \pi^{-1} &< \bar{b}_{\min} \Rightarrow 0 > \bar{f}(\pi), \\ 2q^2 \bar{a}_{\max} \pi + 0.5 \sum_{i=1}^q \bar{c}_i \pi^{-1} &> \bar{b}_{\min} \Rightarrow \bar{f}(2\pi) > 0 \end{aligned}$$

and the inequality  $2\pi^2 q^2 \bar{a}_{\max} > \sum_{i=1}^q \bar{c}_i$  has to be satisfied. From these properties of  $\underline{f}$  and  $\bar{f}$  we can conclude that the required functions  $\sigma_1, \sigma_2 \in \mathcal{K}$  and constants  $0 < g_1 \leq g_2 \leq \sigma_2(\pi)$  exist, completing the proof (for example, these functions can be selected in the form  $\sigma_1(s) = \ell_1 \max\{\ell_2 s, s^2\}$  and  $\sigma_2(s) = \ell_3 \min\{\ell_4 s, s^2\}$  for some  $\ell_i > 0$  for  $i = 1, 2, 3, 4$ ).  $\square$

Another possibility of relaxation of the conditions of Theorem 2 is based on the observation that in the proof of the main result, the property (4) is used only for  $V(x) \geq 0$ :

**Corollary 2.** *If for the system (1) under Assumption 1 there exists a Leonov function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  such that the inequality (4) is verified only for  $x \in (\mathbb{R}^n \setminus \Omega) \cap \mathcal{Z}$ , where  $\Omega = \{x \in \mathbb{R}^n : V(x) \leq 0\}$  and  $\mathcal{Z} = \{x = (z, \theta) \in \mathbb{R}^n : |z| \geq \zeta\}$  for some  $\zeta \in \mathbb{R}_+$ , then for all  $x_0 \in \mathbb{R}^n$  the corresponding trajectories  $X(t, x_0)$  are bounded for all  $t \geq 0$ .*

*Proof.* Note that the inequality (4) does not allow to make a conclusion about behavior of the Leonov function  $V(x)$  while  $x \in \Omega$ , hence, limiting the validity of (4) to the set  $\mathbb{R}^n \setminus \Omega$  is not restrictive at all. Next, it is necessary to add into consideration the set  $\mathcal{C} \subset \mathbb{R}^n \setminus (\Omega \cup \mathcal{Z})$  where (4) is not satisfied, but since  $\Omega \cap \mathcal{C} = \emptyset$  it means that the set  $\Omega = \Omega' \cup \Omega''$  preserves its property of forward invariance for (1) and local attractiveness. Only the local attractiveness can be claimed for  $\Omega$  since the inequality (4) is not satisfied on  $\mathcal{C}$  for  $V(x) > 0$ , then the system behavior is not defined there and a trajectory may stay in  $\mathcal{C}$  (or travel between isolated subsets of  $\mathcal{C}$ ). If  $\mathcal{C} \subset \mathbb{R}^n \setminus \Omega$ , then  $\mathcal{C} \subset \{x = (z, \theta) \in \mathbb{R}^n : \xi \leq |\theta|_\infty \leq c\}$  for some  $\xi \in (0, \pi)$  and  $c \in (\pi, 2\pi)$  by definition of the set  $\Omega$  (see the proof of Theorem 2), consequently,

$$\mathcal{C} \subset \{x = (z, \theta) \in \mathbb{R}^n : |z| < \zeta, \xi \leq |\theta|_\infty \leq c\},$$

which is a bounded set. Note that a trajectory, being trapped in  $\mathcal{C}$  or even traveling between isolated subsets of  $\mathcal{C}$ , cannot leave a bounded set containing  $\mathcal{C}$ . Indeed, if a trajectory reaches the set with  $|\theta|_\infty > c$ , then it is in the set  $\Omega$ , which is forward invariant, then assume that  $|\theta|_\infty$  stays bounded by  $c$ , and from the definition of a Leonov function and its lower bound ( $\alpha(|z|) - \psi(|\theta|) - g \leq V(x)$  for all  $x = (z, \theta) \in \mathbb{R}^n$  and some  $g \geq 0$ ,  $\alpha \in \mathcal{K}_\infty$ ,  $\psi \in \mathcal{K}$ ) there is a finite constant  $\Xi \geq \max\{\xi, \alpha^{-1}(\psi(qc) + g)\}$  such that

$$V(x) > 0, \quad \frac{\partial V(x)}{\partial x} f(x) \leq -\lambda(V(x))$$

provided that  $x \in \{x = (z, \theta) \in \mathbb{R}^n : |z| \geq \Xi, |\theta|_\infty \leq c\}$ , which implies upper boundedness of  $V(x)$  in this set and

boundedness of  $z$ . Therefore, the union  $\Omega' \cup \Omega'' \cup \mathcal{C}$  contains forward invariant and globally attracting sets for (1), all these sets are isolated, and being in  $\mathcal{C}$  implies boundedness of  $X(t, x_0)$ , then the set  $\Omega = \Omega' \cup \Omega''$  saves its meaning and repeating the arguments of the proof of Theorem 2 (with introduction of the sets  $\Omega_j = \Omega'_j \cup \Omega''_j$ ) leads to the conclusion about boundedness of  $X(t, x_0)$  for all  $x_0 \in \mathbb{R}^n$ .  $\square$

**Corollary 3.** *If for the system (1) under Assumption 1 there exists a Leonov function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  such that*

$$\underline{V}(x) \leq V(x) \quad \forall x \in \mathbb{R}^n$$

for some continuous function  $\underline{V} : \mathbb{R}^n \rightarrow \mathbb{R}$  with  $\inf_{x \in \mathcal{W}} \underline{V}(x) > 0$  and (4) is weakened to

$$\frac{\partial \underline{V}(x)}{\partial x} f(x) \leq -\underline{\lambda}(V(x)) \quad \forall x \in \mathcal{X}, \quad (7)$$

where  $\mathcal{X} = \{x \in \mathbb{R}^n : \underline{V}(x) > 0\}$  and a continuous function  $\underline{\lambda} : \mathbb{R} \rightarrow \mathbb{R}$  satisfies  $\underline{\lambda}(0) = 0$  and  $\underline{\lambda}(s)s > 0$  for all  $s > 0$ , then for all  $x_0 \in \mathbb{R}^n$  the corresponding trajectories  $X(t, x_0)$  are bounded for all  $t \geq 0$  provided that there exists  $\underline{g} \in \mathbb{R}_+$  such that

$$\mathcal{W} \subset \{x \in \mathbb{R}^n : V(x) > \underline{g}\} \subseteq \mathcal{X}.$$

*Proof.* Due to the properties of  $\underline{V}$ , the inequality (7) can be interpreted as

$$\frac{\partial V(x)}{\partial x} f(x) < 0, \quad V(x) > 0 \quad \forall x \in \mathcal{X},$$

and since there exists  $\underline{g} \in \mathbb{R}_+$  ( $0 \leq \underline{g} \leq \inf_{x \in \mathcal{W}} \underline{V}(x)$ ) such that  $\mathcal{W} \subset \{x \in \mathbb{R}^n : V(x) > \underline{g}\} \subseteq \mathcal{X}$ , then forward invariance of the set  $\underline{\Omega} = \{x \in \mathbb{R}^n : V(x) \leq \underline{g}\}$  with its decomposition  $\underline{\Omega} = \underline{\Omega}' \cup \underline{\Omega}''$  can be substantiated in a way as before, where

$$\underline{\Omega}' \subset \{x \in \mathbb{R}^n : |z| \leq \alpha^{-1}(\underline{g}), |\theta|_\infty < \underline{c}\}$$

is a compact set containing the origin for some  $\underline{c} \in (\pi, 2\pi)$ ,  $\underline{g} \leq \underline{\tilde{g}} \leq \underline{g} + g + \psi(qc)$ , and

$$\underline{\Omega}'' \subset \{x \in \mathbb{R}^n : |\theta|_\infty > \underline{c}\},$$

these sets are separated by  $\mathcal{W}$ . Next, all arguments of the proof of Theorem 2 can be repeated.  $\square$

The applicability of the proposed theory is illustrated via an example in Section VI.

## V. A RELATION WITH INPUT-TO-STATE STABILITY

In addition to stability, robustness properties with respect to exogenous disturbances are also of great relevance in many control applications. For the class of periodic systems considered in the present paper, robustness in terms of the ISS property has been established in [15] via the notion of an ISS-Leonov function. However, in [15] it is assumed that the states evolve on a manifold  $M$ , while in the present paper it is assumed that they evolve in  $\mathbb{R}^n$ .

We next show that a Leonov function as proposed in the present paper is closely related to an ISS-Leonov function as introduced in [15] and, thus, with minor additional requirements, existence of a Leonov function also allows to establish the ISS property for (1). To this end, it is more convenient

to characterize a Leonov function via the conditions (4) and (6) rather than the more generic conditions in Definition 1. Furthermore, we consider the following system with an external input:

$$\dot{x}(t) = F(x(t), d(t)), \quad (8)$$

where  $x(t) \in \mathbb{R}^n$  and  $d(t) \in \mathbb{R}^m$  (the input  $d(\cdot)$  is a locally essentially bounded and measurable signal) for  $t \geq 0$ ,  $f(x) = F(x, 0)$  and Assumption 1 is satisfied. In such a case any solution  $X(t, x_0, d)$  of (8) for an initial condition  $x_0 \in \mathbb{R}^n$  and an input  $d(\cdot)$  can be embedded in a manifold  $M = \mathbb{R}^k \times \mathbb{S}^q$  by a  $2\pi$ -projection of the part of the state  $\theta$ .

Recall from [15] that for a set  $\mathcal{A} \subset \mathbb{R}^n$ , a  $\mathcal{C}^1$  function  $\mathcal{V} : \mathbb{R}^n \rightarrow \mathbb{R}$  is a *practical ISS-Leonov* function for (8) if there exist functions  $\vartheta_1, \vartheta_2, \sigma, \gamma \in \mathcal{K}_\infty$ , a continuous function  $\lambda : \mathbb{R} \rightarrow \mathbb{R}$ ,  $\lambda \in \mathcal{K}_\infty$  for nonnegative arguments, and scalars  $y \geq 0$ ,  $u \geq 0$  such that for all  $x = (z, \theta) \in \mathbb{R}^n$  and  $d \in \mathbb{R}^m$

$$\vartheta_1(|x|_{\mathcal{A}}) - \sigma(|\theta|) \leq \mathcal{V}(x) \leq \vartheta_2(|x|_{\mathcal{A}} + u), \quad (9)$$

and the following dissipation inequality holds:

$$\frac{\partial \mathcal{V}(x)}{\partial x} F(x, d) \leq -\lambda(\mathcal{V}(x)) + \gamma(|d|) + y. \quad (10)$$

If the latter inequality holds for  $y = 0$ , then  $\mathcal{V}$  is said to be an *ISS-Leonov* function.

**Assumption 2.** *Let a set  $\mathcal{A} \subset \mathbb{R}^n$  contain all  $\alpha$ - and  $\omega$ -limit sets of (1) and its projection on  $M$  be compact and decomposable (in the sense of [3]).*

It has been shown in [15] that under the hypothesis given in Assumption 2, the existence of a practical ISS-Leonov function is necessary and sufficient for ISS of (8) with respect to the set  $\mathcal{A}$ . Then the following conjecture can be formulated for a Leonov function.

**Proposition 2.** *Consider a Leonov function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  satisfying the restrictions of Proposition 1, and for  $\mathcal{V}(x) = V(x) + \tau$  with some  $\tau \geq 0$  assume that (4) can be strengthened to (10) with a radially unbounded function  $\lambda$  for positive argument, and the inequalities*

$$\vartheta_1(|x|_{\mathcal{A}}) - \sigma(|\theta|) \leq \alpha_1(|z|) - \sigma_1(|\theta|_\infty - \pi) + g_1 + \tau, \quad (11)$$

$$\alpha_2(|z|) - \sigma_2(|\theta|_\infty - \pi) + g_2 + \tau \leq \vartheta_2(|x|_{\mathcal{A}} + u) \quad (12)$$

are verified for all  $x = (z, \theta) \in \mathbb{R}^n$ . Then  $\mathcal{V}$  is a *practical ISS-Leonov* function for (8) and a set  $\mathcal{A} \subset \mathbb{R}^n$ .

*Proof.* The result can be deduced by a straightforward application of Definition 1 and the restrictions given in propositions 1 and the current one.  $\square$

Note that if  $\{x = (z, \theta) \in \mathbb{R}^n : z = 0\} \subset \mathcal{A}$  then there is  $\tau \geq 0$  such that (11) is verified, and by taking  $u > 0$  large enough the fulfillment of the inequality (12) can also be guaranteed.

**Corollary 4.** *Let assumptions 1 and 2 be satisfied together with the hypothesis of Proposition 2. Then the system (8) is ISS with respect to the set  $\mathcal{A}$ .*

*Proof.* The result is a direct consequence of Proposition 2 and the main result of [15].  $\square$

It is important to stress here that ISS property is obtained on the manifold  $M$  only (when boundedness of the variable  $\theta$  comes directly from the structure of  $M$ ). The fact that  $V$  is also a Leonov function from Definition 1 implies, via the claim of Theorem 2, that for  $d(t) \equiv 0$  the solutions of (1) are bounded in  $\mathbb{R}^n$ .

## VI. AN ACADEMIC EXAMPLE

We demonstrate the usefulness of the proposed approach by performing a global analysis of boundedness of solutions of the following system:

$$\begin{aligned} \dot{z} &= Fz + R \sin(\theta), \\ \dot{\theta} &= Dz + H \sin(\theta), \end{aligned} \quad (13)$$

where  $x = (z, \theta) \in \mathbb{R}^n$  is the state, and  $F \in \mathbb{R}^{k \times k}$ ,  $R \in \mathbb{R}^{k \times q}$ ,  $D \in \mathbb{R}^{q \times k}$  and  $H \in \mathbb{R}^{q \times q}$  are constant matrices. Note that the system has an equilibrium at the origin.

Let the symbol  $\text{diag}(b)$  denote the diagonal matrix with a vector  $b \in \mathbb{R}_+^q$  on the main diagonal. Then impose the following restrictions on the dynamics of (13):

**Assumption 3.** *There exist matrices  $P = P^\top \in \mathbb{R}^{k \times k}$ ,  $Q = Q^\top \in \mathbb{R}^{k \times k}$ , vector  $m \in \mathbb{R}_+^q$  and a constant  $\lambda > 0$  such that the following linear matrix inequalities are satisfied:*

$$\begin{aligned} F^\top P + PF &= -Q, \quad P > 0, \quad Q > 0, \\ 2PR &= -D^\top \text{diag}(m), \quad \text{diag}(m)H + H^\top \text{diag}(m) < 0. \end{aligned}$$

Since Assumption 3 implies that the matrix  $F$  is stable, it is straightforward to establish boundedness of  $z \in \mathbb{R}^k$ . The result below demonstrates how the approach derived in Section IV can also be used to infer global boundedness of the variables  $\theta \in \mathbb{R}^q$ .

**Proposition 3.** *Let Assumption 3 be satisfied for the system (13). Then for all  $x_0 \in \mathbb{R}^n$  the solutions  $X(t, x_0)$  are bounded for all  $t \geq 0$ .*

*Proof.* Under the made hypothesis let us consider the function

$$V(x) = z^\top Pz + \sum_{i=1}^q m_i (1 - \cos(\theta_i)) - c$$

and let us check the conditions of Corollary 2 for a Leonov function candidate  $V(x)$ . Since  $1 - \cos(\theta) \geq 0$ , select  $\alpha(s) = \lambda_{\min}(P)s^2$  (here  $\lambda_{\min}(P)$  denotes the smallest eigenvalue of the matrix  $P$ ),  $\psi$  as any function from the class  $\mathcal{K}$  and  $g = c$ , then the lower bound on  $V(x)$  is verified for all  $x \in \mathbb{R}^n$ . Since  $\cos(0) = 1$  and it is a  $2\pi$ -periodic function, the condition  $\sup_{x \in \mathcal{U}} V(x) \leq 0$  is satisfied for any  $c \geq 0$ . Select  $\mathcal{W} = \{x = (z, \theta) \in \mathbb{R}^n : |\theta|_\infty = p\}$  for some  $p \in (\pi, 2\pi)$  and  $c < (1 - \cos(p)) \inf_{1 \leq i \leq q} m_i$ , then  $\inf_{x \in \mathcal{W}} V(x) > 0$ , and the properties (3) hold.

It remains to check the derivative of the Leonov function  $V(x)$  and the following upper estimate:

$$\frac{\partial V(x)}{\partial x} f(x) \leq -\lambda V(x) \quad \forall x \in (\mathbb{R}^n \setminus \Omega) \cap \mathcal{Z},$$



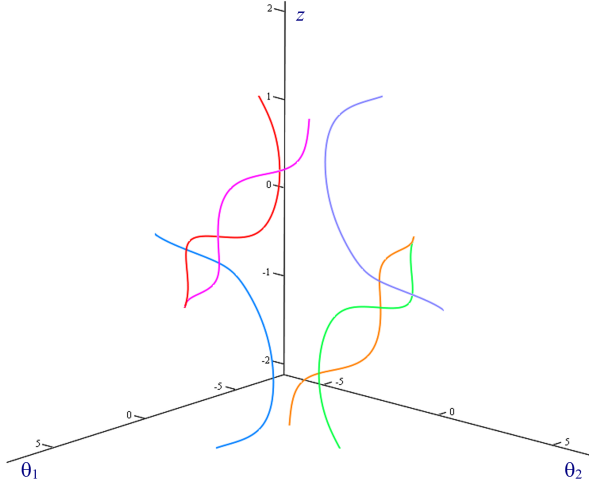


Figure 2. Simulation results for the system (13) for several arbitrarily chosen initial conditions

where  $\Omega = \{x \in \mathbb{R}^n : V(x) \leq 0\}$  and  $\mathcal{Z} = \{x = (z, \theta) \in \mathbb{R}^n : |z| \geq \zeta\}$  for some  $\lambda > 0$  and  $\zeta > 0$ . Lengthy but straightforward calculations show that

$$\begin{aligned} \frac{\partial V(x)}{\partial x} f(x) + \lambda V(x) &\leq -z^\top (Q - \lambda P)z - \sin^\top(\theta) \mathcal{H} \sin(\theta) \\ &\quad + \lambda \left[ \sum_{i=1}^q m_i (1 - \cos(\theta_i)) - c \right], \end{aligned}$$

where

$$\mathcal{H} = -0.5(\text{diag}(m)H + H^\top \text{diag}(m)) > 0.$$

Note that under Assumption 3 there is a value of  $\lambda$  sufficiently small such that  $Q > \lambda P$  and

$$\lambda[m_i(1 - \cos(\theta_i)) - q^{-1}c] > \lambda_{\min}(\mathcal{H}) \sin^2(\theta_i) \quad (14)$$

for a small vicinity of the point  $\theta_i = \pi$  (the point  $\theta_i$  belongs to the set  $\Omega$  by construction above), such that this vicinity has no intersection with  $\Omega$ . Indeed,  $\lambda[m_i(1 - \cos(\theta_i)) - q^{-1}c] \leq \lambda[2m_i - q^{-1}c]$  for all  $\theta_i \in [-\pi, \pi]$ ,  $\lambda_{\min}(\mathcal{H}) > 0$  by Assumption 3 and the inequality (14) is satisfied for  $\theta_i \in (\pi - \rho_i, \pi + \rho_i)$  with  $\rho_i = \arcsin \sqrt{\lambda \frac{2m_i - q^{-1}c}{\lambda_{\min}(\mathcal{H})}}$ , whose value can be selected as small as required by decreasing  $\lambda$ . Then obviously all conditions of Corollary 2 are satisfied.  $\square$

**Example 2.** Select  $k = 1$ ,  $q = 2$  and  $F = -0.1$ ,  $P = 1$ ,  $Q = 0.2$ ,  $R = -[0.2 \ 0.05]$ ,  $D = [1 \ 0.5]^\top$ ,  $m = [0.4 \ 0.2]^\top$ ,  $c = 0.2$ ,

$$H = \begin{bmatrix} -1 & -0.25 \\ 0.5 & -1 \end{bmatrix}.$$

Then all conditions of Assumption 3 are satisfied and  $\lambda = 0.1$ . Illustrative simulation results for this system are given in Fig. 2. Clearly, the simulations confirm the efficiency of the proposed theory.

## VII. CONCLUSIONS

By extending ideas of [26], we have introduced the concept of a Leonov function, and have shown that for a class of periodic dynamical systems the existence of a Leonov function

implies boundedness of all trajectories. Such a function is in general sign-indefinite and not continuously differentiable on the manifold, on which the system dynamics can be projected. These achievements represent significant relaxations compared to the usual requirements on a standard Lyapunov function [3]. The proposed approach is illustrated by providing a global analysis of a nonlinear system. Application of the proposed theory to power systems and microgrids (in particular, to the swing equation) is currently under investigation.

## REFERENCES

- [1] D. Angeli. An almost global notion of input-to-state stability. *IEEE Trans. Automatic Control*, 49:866–874, 2004.
- [2] D. Angeli and D. Efimov. On input-to-state stability with respect to decomposable invariant sets. In *Proc. 52nd IEEE Conference on Decision and Control*, Florence, 2013.
- [3] D. Angeli and D. Efimov. Characterizations of input-to-state stability for systems with multiple invariant sets. *IEEE Trans. Automatic Control*, 60(12):3242–3256, 2015.
- [4] D. Angeli, J.E. Ferrell, and E.D. Sontag. Detection of multistability, bifurcations and hysteresis in a large class of biological positive-feedback systems. *Proc. Natl. Acad. Sci. USA*, 101:1822–1827, 2004.
- [5] D. Angeli and L. Praly. Stability robustness in the presence of exponentially unstable isolated equilibria. *IEEE Trans. Automatic Control*, 56:1582–1592, 2011.
- [6] D. Angeli and E.D. Sontag. Multi-stability in monotone input/output systems. *Systems & Control Lett.*, 51:185–202, 2004.
- [7] N.P. Bhatia and G.P. Szegő. *Stability Theory of Dynamical Systems*. Springer-Verlag, Berlin, 1970.
- [8] M. Chaves, T. Eissing, and F. Allgower. Bistable biological systems: A characterization through local compact input-to-state stability. *IEEE Trans. Automatic Control*, 45:87–100, 2008.
- [9] N.G. Chetaev. *The Stability of Motion*. Pergamon Press, New York, 1961. (English translation).
- [10] Jorge Cortés and Francesco Bullo. Coordination and geometric optimization via distributed dynamical systems. *SIAM Journal on Control and Optimization*, 44(5):1543–1574, 2005.
- [11] Florian Dörfler and Francesco Bullo. Synchronization in complex networks of phase oscillators: A survey. *Automatica*, 50(6):1539–1564, 2014.
- [12] Dawid Dudkowski, Sajad Jafari, Tomasz Kapitaniak, Nikolay V. Kuznetsov, Gennady A. Leonov, and Awadhesh Prasad. Hidden attractors in dynamical systems. *Physics Reports*, 637:1–50, 2016.
- [13] D. Efimov. Global Lyapunov analysis of multi-stable nonlinear systems. *SIAM Journal on Optimization and Control*, 50(5):3132–3154, 2012.
- [14] D. Efimov, W. Perruquetti, and M. Petreczky. On necessary conditions of instability and design of destabilizing controls. In *Proc. 53rd IEEE Conference on Decision and Control*, Los Angeles, US, 2014.
- [15] D. Efimov, J. Schiffer, N. Barabanov, and R. Ortega. A relaxed characterization of iss for periodic systems with multiple invariant sets. *European Journal of Control*, ?(?):?–?, 2017.
- [16] D. Efimov, J. Schiffer, and R. Ortega. Robustness of delayed multistable systems with application to droop-controlled inverter-based microgrids. *International Journal of Control*, 89(5):909–918, 2016.
- [17] D.V. Efimov and A.L. Fradkov. Oscillatory of nonlinear systems with static feedback. *SIAM Journal on Optimization and Control*, 48(2):618–640, 2009.
- [18] F. Forni and R. Sepulchre. Differential analysis of nonlinear systems: Revisiting the pendulum example. In *Proc. 53rd IEEE Conference on Decision and Control*, pages 3848–3859, Los Angeles, US, 2014.
- [19] A.Kh. Gelig, G.A. Leonov, and V.A. Yakubovich. *Stability of nonlinear systems with non unique equilibrium*. Nauka, Moscow, 1978. [in Russian].
- [20] J. Guckenheimer and P.J. Holmes. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, volume 42 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.
- [21] W. Hahn. *Stability of Motion*. Springer-Verlag, 1967.
- [22] H.K. Khalil. *Nonlinear Systems*. Macmillan, 1992.
- [23] V. Lakshmikantham and X.Z. Liu. *Stability analysis in terms of two measures*. World Scientific, 1993.

- [24] G. A. Leonov, N. V. Kuznetsov, M. V. Yuldashev, and R. V. Yuldashev. Analytical method for computation of phase-detector characteristic. *IEEE Transactions on Circuits and Systems II: Express Briefs*, 59(10):633–637, 2012.
- [25] G. A. Leonov, N. V. Kuznetsov, M. V. Yuldashev, and R. V. Yuldashev. Hold-in, pull-in, and lock-in ranges of pll circuits: Rigorous mathematical definitions and limitations of classical theory. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 62(10):2454–2464, 2015.
- [26] G.A. Leonov. On the boundedness of the trajectories of phase systems. *Siberian Mathematical Journal*, 15(3):491–495, 1974.
- [27] A.M. Lyapunov. *The general problem of the stability of motion*. Taylor & Francis, London, 1992.
- [28] Z. Nitecki and M. Shub. Filtrations, decompositions, and explosions. *American Journal of Mathematics*, 97(4):1029–1047, 1975.
- [29] E.J. Noldus. New direct Lyapunov-type method for studying synchronization problems. *Automatica*, 13:139–151, 1977.
- [30] R. Ortega, M. Galaz, A. Astolfi, Y. Sun, and T. Shen. Transient stabilization of multimachine power systems with nontrivial transfer conductances. *IEEE Trans. Automatic Control*, 50(1):60 – 75, jan. 2005.
- [31] A. Rantzer. A dual to Lyapunov’s stability theorem. *Systems&Control Lett.*, 42:161–168, 2001.
- [32] V.V. Rumyantsev and A.S. Oziraner. *Stability and stabilization of motion with respect to part of variables*. Nauka, Moscow, 1987. [in Russian].
- [33] J. Schiffer and F. Dörfler. On stability of a distributed averaging PI frequency and active power controlled differential-algebraic power system model. In *Proc. European Control Conference*, pages 1487–1492, Aalborg, DK, 2016.
- [34] J. Schiffer, R. Ortega, A. Astolfi, J. Raisch, and T. Sezi. Conditions for stability of droop-controlled inverter-based microgrids. *Automatica*, 50(10):2457–2469, 2015.
- [35] G.-B. Stan and R. Sepulchre. Analysis of interconnected oscillators by dissipativity theory. *IEEE Trans. Automatic Control*, 52:256–270, 2007.
- [36] Yang Tang, Feng Qian, Huijun Gao, and Jürgen Kurths. Synchronization in complex networks and its application—a survey of recent advances and challenges. *Annual Reviews in Control*, 38(2):184–198, 2014.
- [37] Sebastian Trip, M Bürger, and Claudio De Persis. An internal model approach to frequency regulation in inverter-based microgrids with time-varying voltages. In *53rd IEEE Conference on Decision and Control*, pages 223–228, Los Angeles, US, 2014.
- [38] Sebastian Trip, Mathias Bürger, and Claudio De Persis. An internal model approach to (optimal) frequency regulation in power grids with time-varying voltages. *Automatica*, 64:240–253, 2016.
- [39] Pravin Varaiya, Felix Fulih Wu, and Rong-Liang Chen. Direct methods for transient stability analysis of power systems: Recent results. *Proceedings of the IEEE*, 73(12):1703–1715, 1985.
- [40] V.A. Yakubovich, G.A. Leonov, and A.Kh. Gel'g. *Stability of stationary sets in control systems with discontinuous nonlinearities*, volume 14 of *Series on stability, vibration, and control of systems*. World Scientific, 2004.