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Exact Distance Oracles Using Hopsets

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Abstract

For fixed $h \geq 2$, we consider the task of adding to a graph G a set of weighted shortcut edges on the same vertex set, such that the length of a shortest h -hop path between any pair of vertices in the augmented graph is exactly the same as the original distance between these vertices in G . A set of shortcut edges with this property is called an *exact h -hopset* and may be applied in processing distance queries on graph G . In particular, a 2-hopset directly corresponds to a distributed distance oracle known as a *hub labeling*. In this work, we explore centralized distance oracles based on 3-hopsets and display their advantages in several practical scenarios. In particular, for graphs of constant highway dimension, and more generally for graphs of constant skeleton dimension, we show that 3-hopsets require *exponentially* fewer shortcuts per node than any previously described distance oracle while incurring only a *quadratic* increase in the query decoding time, and actually offer a speedup when compared to simple oracles based on a direct application of 2-hopsets. Finally, we consider the problem of computing minimum-size h -hopset (for any $h \geq 2$) for a given graph G , showing a polylogarithmic-factor approximation for the case of unique shortest path graphs. When $h = 3$, for a given bound on the space used by the distance oracle, we provide a construction of hopsets achieving polylog approximation both for space and query time compared to the optimal 3-hopset oracle given the space bound.

1 Introduction

An exact h -hopset for a weighted graph G is a weighted edge set, whose addition to the graph guarantees that every pair of vertices has a path between them with at most h edges (*hops*) and whose length is exactly the length of shortest path between the vertices.

The concept of a hopset was first explicitly described by Cohen [20] in its *approximate* setting, in which the length of h -hop path between a pair of vertices in the hopset should approximate the length of the shortest path in G . Hopsets were introduced in the context of parallel computation of approximate shortest paths. In this paper, we study hopsets in their exact version, with the general objective of optimizing exact shortest path queries.

Data structure which allow for querying distance between any pair of vertices of a graph have been intensively studied under the name of *distance oracles*. The efficiency of an exact distance oracle is typically measured by the interplay between the *space* requirement of the representation of the data structure and its *decoding time*. It is a well-established empirical fact that many real-world networks admit efficient (i.e., low-space and fast) distance oracles [6, 23]. A key example here concerns transportation networks, and specifically road networks, which are empirically known [33, 32, 5] to be augmentable by carefully tailored sets of shortcut edges, allowing for shortest-path computation. These sets of shortcuts may be hopsets (as is the case for the hub-labeling approach which effectively implements a 2-hopset), but may also be considered

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Distance oracle	Treewidth t		Skeleton dimension k	
	<i>Size</i>	<i>Time</i>	<i>Size</i>	<i>Time</i>
2-hopset (hubs):	$n \cdot O(t \log n)$	$O(t + \log \log n)$	$n \cdot O(k \log n)$	$O(k \log n)$ ¹
3-hopset:	$n \cdot O(t \log \log n)$	$O(t^2 \log^2 \log n)$	$n \cdot O(k \log k \log \log n)$	$O(k^2 \log^2 k \log^2 \log n)$

Table 1: Comparison of distance oracles based on 2-hopsets (hub labeling [20, 28, 36]) and 3-hopsets (this paper). Size represents the number of shortcut edges in the hopset, i.e., the number of $O(\log n)$ -bitsize words when measuring oracle size. The main results concern skeleton dimension and are stated in simplified form, assuming unique shortest path graphs with average edge length at most $O(\text{poly } \log n)$, with expected query times given for both types of oracles.

in some related (and frequently more involved) framework, such as contraction hierarchies [31] or transit-node routing [11].

An interesting theoretical insight due to Abraham et al. [4, 2, 5] provides theoretical bounds on the number of shortcuts required in all of the above-mentioned frameworks. They introduce a parameter describing the structure of shortest paths within ball neighborhoods of a graph, called *highway dimension* \tilde{h} , and express the number of shortcuts that need to be added for each node so as to achieve shortest-path queries in a graph of n nodes with weighted diameter D as a polynomial of \tilde{h} , $\log n$, and $\log D$; this approach has been extended in subsequent work [3, 36]. The value of \tilde{h} is known to be small in practice (e.g., typically $\tilde{h} < 100$ for continental-sized road networks [2]), and does indeed appear to be inherently linked to the size of the required shortcut sets. In fact, empirical tests have suggested that the (average) number of necessary shortcuts per node is in fact very close to \tilde{h} , laying open the question of whether the additional dependence of the number of shortcuts on logarithmic factors in n and D may be an artifact of the theoretical analysis of the oracles, which for each node require a separate shortcut for every “scale” of distance.

In this work, we provide strong evidence that the dependence of the number of shortcuts on such logarithmic factors in n and D is indeed not essential, and we design a simple distance oracle based on a 3-hopset in which the number of shortcuts per node depends only on \tilde{h} , $\log \log n$, and the logarithm of the average edge length. This result is in fact shown in the framework of a strictly broader class of graphs, namely, graphs with a bounded value of a parameter known as *skeleton dimension* k ($k \leq \tilde{h}$), describing the width of the shortest-path tree of a node after pruning all branches at a constant fraction of their depth. We also show a similar result for 3-hopsets in graphs of treewidth t , obtaining a distance oracle in which the number of shortcuts per node is a function of t and $\log \log n$. (For the case of bounded treewidth, different constructions with comparable performance were previously known, cf. [15]). The space and time-bounds of oracles based on 3-hopsets are presented in Table 1, and compared with the corresponding parameters of oracles based on 2-hopsets. For the case of constant skeleton dimension or constant treewidth, we remark that using a 3-hopset instead of a 2-hopset reduces the number of shortcuts per node from $O(\log n)$ to $O(\log \log n)$ while achieving a query time of $O(\log^2 \log n)$.

1.1 Results and Organization of the Paper

The rest of the paper is organized as follows. In Section 2, we introduce the necessary notions related to h -hopset and give a general approach for how a h -hopset can be used as a distance oracle, focusing on the special case of $h = 3$. As a warmup to the main results, in Section 3, we

¹ The query time of a simple 2-hopset data structure is $O(k \log n)$. In a centralized setting, it can be reduced to $O(k \log \log n)$ by application in combination of a 2-hopset with a given $O(\log n)$ -factor approximation of the distance, which can be provided by an auxiliary fast approximate distance oracle (e.g. [17]).

show how to construct efficient h -hopsets for bounded treewidth graphs. We also consider the query time for 3-hopsets for weighted trees and, more generally, for bounded treewidth graphs. Then, in Section 4, we provide the first of our main results, using 3-hopsets to obtain improved (smaller) distance oracles in USP graphs with bounded skeleton dimension.

In the second part of the paper, we consider LP-based approximation algorithms for constructing 3-hopsets in unique shortest path graphs. A *unique shortest path graph* (USP) is a graph such that, given any two nodes u and v , there is a unique shortest path P_{uv} between them. In practice, this common assumption can be made without loss of generality, as one can perturb the input to ensure uniqueness; however such a perturbation may significantly change the size of the required distance oracle. Our construction builds on and significantly extends the LP-formulation of 2-hopsets and the framework of prehub labelings introduced in [9].

In Section 5, we provide a ILP formulation for the problem of finding a h -hopset of minimal size. For the case of USP graphs, we then show how this ILP formulation is related to its LP relaxation, namely, that the problem has an at most polylogarithmic integrality gap. We extend the same approach to provide an algorithm which constructs 3-hopsets which are ready to use as distance oracles in USP graphs, with (approximate) optimality guarantees on size and query time of the oracle.

Our work is presented in the context of weighted undirected graphs, but all results can easily be extended to weighted directed graphs.

1.2 Other Related Work

Hopsets. Exact hopsets were implicitly constructed in the context of single-source shortest paths parallel computation [42, 35, 19, 39]. Such works study the work versus time trade-offs of such computation. Cohen [20] explicitly introduced the notion of (h, ϵ) -hopset of G as set H of weighted edges such that paths of h hops at most in $G \cup H$ have length within $(1 + \epsilon)$ of the corresponding shortest path in G . The parameter h is called the *hopbound*. For any graph G and $\epsilon, \epsilon' > 0$, she proposed a construction of $(O(\text{poly } \log n), \epsilon)$ -hopset of G with size $O(n^{1+\epsilon'})$. More recently, Elkin et al. [25] proposed the construction of $(O(\epsilon^{-1} \log \kappa)^{\log \kappa}, \epsilon)$ -hopset with $O(n^{1+1/\kappa} \log n \log \kappa)$ edges for any $\epsilon > 0$ and integral $\kappa \geq 1$. Abboud et al. [1] recently showed the optimality of the Elkin et al. [25] result. In particular, they showed that for any $\delta > 0$ and integer k , any hopset of size less than $n^{1+\frac{1}{2^{k+1}-1}-\delta}$ must have hop bound $h = \Omega(c_k/\epsilon^{k+1})$, where c_k is a constant depending only on k . As far as we know, exact hopsets (with $\epsilon = 0$) have not been explicitly studied. However, they are related to the following well studied notion.

Hopsets vs. TC-spanners. In directed graphs, a hopset can be seen as a special case of an *h -transitive-closure spanner* (*h -TC-spanner*), i.e., an unweighted directed graph with the same transitive closure as a given unweighted directed graph G , having hop diameter at most h . Hopsets and TC-spanners are a fundamental graph-theoretic objects and are widely used in various settings from distance oracles to pre-processing for range queries in sequential or parallel setting or even in property testing.

The concept of adding transitive arcs to a digraph in order to reduce its diameter was introduced by Thorup [40] in the context of parallel processing. Bhattacharyya et al. [12] define a *h -transitive-closure spanner* (*h -TC-spanner* for short) of an unweighted digraph G as a digraph H with same transitive closure as G and diameter at most h . They note that this is a central concept in a long line of work around pre-processing a tree for range queries [7, 16, 41]. A TC-spanner can also be defined as a spanner (for the classical spanner definition [37]) of the transitive closure of a graph that has bounded diameter. We will see that an exact h -hopset defines a h -TC-spanner but that the converse is not necessarily true. Bhattacharyya et al. [12]

proposed a construction of h -TC-spanner of size $O(n \log n \lambda_h(n))$ for H -minor-free graphs (where λ_h denotes the h th-row inverse Ackermann function, cf. Section 3).

Exact Distance Oracles. A long line of research studies the interplay between data structure space and query decoding time. A lot of attention has been given to distance oracles for planar graphs [24, 10, 18, 14, 26, 22, 30], and it has recently been shown that a distance oracle with $O(n^{1.5})$ space and $O(\log n)$ query-time is possible [30]. In the context of weighted directed graphs with treewidth t , Chaudhuri and Zaroliagis [15] propose a distance oracle using $O(t^2 n \lambda_h(n))$ space and $O(t^3 h + \lambda_h(n))$ query time for integral $h > 1$ where λ_h is the h th-row inverse Ackermann function (as defined in Subsection 3). In the context of unweighted graphs with treewidth t , Farzan and Kamali [27] obtain distance oracles with $O(t^3 \log^3 t)$ query time using optimal space (within low order terms). This construction heavily relies on the unweighted setting as exhaustive look-up tables are constructed for handling graphs with polylogarithmic size.

Distance Labelings and 2-Hopsets. The distance labeling problem is a special case of a distributed distance oracle, and consists of assigning labels to the nodes of a graph such that the distance between two nodes s and t can be computed from the labels of s and t (see, e.g., [28]).

The notion of 2-hopset studied in this work coincides with the special case of two-hop distance labeling (also called *hub-labeling*), where labels are constructed from hub sets: in hub-labeling, a small hub set $S(u) \subseteq V(G)$ is assigned to each node of a graph G such that for any pair u, v of nodes, the intersection of hub sets $S(u) \cap S(v)$ contains a node on a shortest $u - v$ path. Such a construction is proposed by Gavoille et al. [28] and applies to graphs of treewidth t with labels of $O(t \log n)$ size and allows to answer distance queries in $O(t \log n)$ time; the hub sets have a hierarchical structure, which allows for an improvement of query time to $O(t \log \log n)$ time by a binary search over levels. Hub labelings are the best currently known distance labelings for sparse graphs, achieving sublinear node label size [8, 29], and may also be used to provide a 2-additive-approximation for distance labeling in general graphs using sublinear-space labels [29].

In graphs of bounded highway dimension, hub labels were among the first identified distance oracles to provide label size and query time polynomial in the highway dimension and polylogarithmic in other graph parameters [5]. This result was then extended to the more general class of graphs with bounded skeleton dimension [36].

Hub sets with near to optimal size can be constructed in polynomial time. A greedy setcover-type $O(\log n)$ -approximation algorithm (with respect to average size of a hub set) was proposed by Cohen et al. [21]. For the case of USP graphs, this approximation ratio was improved by Angelidakis et al. [9] to the logarithm of the graph hop-diameter, i.e., the maximum number of hops of a shortest path in G .

2 Preliminaries

2.1 Definitions

We are given a weighted undirected graph $G = (V, E, \omega)$ where $\omega : E \rightarrow \mathbb{R}^+$ associates a weight with each edge of G . For a positive integer parameter h and a pair $u, v \in V$, the h -limited distance between u and v , denoted $d_G^h(u, v)$, is defined as the length of the shortest uv path that contains at most h edges (aka *hops*). The usual shortest path distance can be defined as $d_G(u, v) = d_G^{m-1}(u, v)$. For the sake of brevity, we often let uv denote the pair $\{u, v\}$ representing an edge from u to v .

Definition 1. An (exact) h -hopset for a weighted graph G is a set of edges H such that $d_{G \cup H}^h(u, v) = d_G(u, v)$ for all u, v in $V(G)$ where $G \cup H = (V, E \cup H, w')$ is the graph augmented with edges of the hopset with weights $w'(u, v) = d_G(u, v)$ for $uv \in H$ and $w'(u, v) = w(u, v)$ for $uv \in E \setminus H$. The parameter h is called the hopbound of the hopset. Edges from set H are called shortcuts in G .

By convention, we will assume that all self-loops at nodes of V are included in H . Thus, $G \cup H$ is a graph whose h -th power in the $(\min, +)$ algebra on $n \times n$ matrices of edge weights corresponds to the transitive closure of the weight matrix of graph G .

Equivalently, a h -hopset can be defined as a set H of edges such that for any pair s, t , there exists a path P of h edges at most from s to t in $G \cup H$ and a shortest path Q from s to t in G such that all nodes of P belong to Q and appear in the same order. Note that a h -hopset is completely specified by its set H of edges as the associated weights are deduced from distances in the graph.

2.2 Using a Hopset as a Distance Oracle

Hopsets may be used to answer shortest-path queries in a graph $G = (V, E)$. In general, given a hopset H , the naïve way to approach a query for $d_G(u, v)$ for a given node pair u, v is to perform a bidirectional Dijkstra search in graph $G \cup H$ from this node pair, limited to a maximum of $\lceil h/2 \rceil$ hops distance from each of these nodes. We have, in particular for any pair $u, v \in V$:

$$d_G(u, v) = \min_{w \in V} (d_{G \cup H}^{\lceil h/2 \rceil}(u, w) + d_{G \cup H}^{\lfloor h/2 \rfloor}(w, v)).$$

Different optimizations of this technique are possible.

In this paper, we focus only on the time complexity of the case of $h = 3$, where we perform the following optimization of query execution. We represent set H as the union of two (not necessarily disjoint) sets of shortcuts, $H = H_1 \cup H_2$, where an edge belongs to H_1 if it is used as the first or third (last) hop on a shortest path in $G \cup H$, and it belongs to $G \cup H_2$ if it is used as the second hop on such a path. By convention, we assume that self-loops at nodes are added to H_1 , thus e.g. a 3-hop path between a pair of adjacent nodes in G is constructed by taking a self-loop from H_1 , the correct edge from $G \subseteq G \cup H_2$, and another self-loop from H_1 . (Note that we never directly use edges of G as first or last hops in the hopset; if such an edge is required for correctness of construction, it should be explicitly added to set H_1 .) We further apply an orientation to the shortcuts in H_1 , constructing a corresponding set of arcs \vec{H}_1 , such that, for any node pair $u, v \in V$, there exist $x, y \in V$ such that $(u, x) \in \vec{H}_1$, $\{x, y\} \in H_2$, $(v, y) \in \vec{H}_1$, and:

$$d_G(u, v) = d_G(u, x) + d_G(x, y) + d_G(v, y).$$

We note that $|H_1| \leq |\vec{H}_1| \leq 2|H_1|$, since each shortcut from H_1 corresponds to at most a pair of symmetric arcs in \vec{H}_1 . For a node $w \in V$, let $N_1(w) = |\{x \in V : (w, x) \in \vec{H}_1\}|$ represent the out-neighborhood of w in the graph (V, \vec{H}_1) . To perform shortest path queries on G , we now store for each node w the lists $(\{x, d_G(w, x)\} : x \in N_1(w))$. We also store a hash map, mapping all node pairs $\{x, y\} \in H_2$ to the length of the respective link, $d_G(x, y)$. Now, we answer the distance query for a node pair $u, v \in G$ as follows:

$$d_G(u, v) = \min_{x \in N_1(u), y \in N_1(v): \{x, y\} \in H_2} (d_G(u, x) + d_G(x, y) + d_G(v, y)).$$

Using the given data structures, the query is then processed using $|N_1(u)| \cdot |N_1(v)|$ hashmap look-ups, one for each pair $(x, y) \in N_1(u) \times N_1(v)$, i.e., in time $\mathcal{T}_{uv} = O(|N_1(u)| \cdot |N_1(v)|)$. Time \mathcal{T}_{uv} is simply referred to as the *query time* for the considered node pair in the 3-hopset oracle

H . Assuming uniform query density over all node pairs, the *uniform-average query time* $\mathcal{T}(H)$ is given as: $\mathcal{T}(H) \equiv \mathbf{E}_{uv} \mathcal{T}_{uv} = O\left(\frac{1}{n^2} \left(\sum_{u \in V} |N_1(u)|\right)^2\right) = O(|H_1|^2/n^2)$. Thus, in the uniform density setting (which we refer to in Section 5 only), the average time of processing a query is proportional to the square of the average degree of a node with respect to edge set H_1 .

The size of set H_2 affects only the size of the data structure required by the distance oracle, which is given as at most $\mathcal{S} = O(|E| + |H_1| + |H_2|)$ edges, with each edge represented using $O(\log n)$ bits.

In the 3-hopset distance oracles described in the following sections, we will confine ourselves to describing shortcut sets H_1 and H_2 , noting that the correct orientation \vec{H}_1 of H_1 will follow naturally from the details of the provided constructions.

3 Warmup: Bounded Treewidth Graphs

As a warm-up, we provide a h -hopset construction for bounded treewidth graphs and use it to design a distance oracle in the case $h = 3$.

We first consider the case of (weighted) trees. The construction of h -hopsets for trees is classical. It is implicit in [7, 16], explicit for unweighted trees in [13] and directed trees in [41]. We provide a short construction which fine-grains the dependence of the hopset size on h (e.g., replacing $2h$ by h with respect to the asymptotic analysis in [7]). The construction is based on the following lemma for splitting the tree into smaller sub-trees.

Lemma 1. *Given a tree T with n nodes and a value $p > 1$, there exists a set P of $2p$ nodes at most such that each connected component of $T \setminus P$ contains less than n/p nodes and is connected to at most two nodes in P . Set P can be computed in linear time through a bottom-up traversal of the tree.*

Proof. Start with $P' = \emptyset$ and root T at some arbitrary node r . As long as the connected component of $T \setminus P'$ containing root r has n/p nodes or more, add to P' a node u from this component such that the subtree $T(u)$ rooted as u has size n/p or more while $|T(v)| < n/p$ for all descendants v of u . This results in a set P' of at most p nodes such that the connected components of $T \setminus P$ have size less than n/p . Define P'' as the set of lowest common ancestors of any two nodes $u, v \in P'$. The size of P'' is at most $p - 1$ since its nodes correspond to the internal nodes with two children or more in the minimal sub-tree containing P which has at most p leaves. Let $P = P' \cup P''$ be the union of P and P' . For any connected component T' of $T \setminus P$, there exist at most two nodes in P that are connected to nodes of T' in T : at most one is connected to the root r' of T' (T' is considered as a sub-tree of T) and at most one has its parent in T' (if there were two such nodes, their lowest common ancestor would be in P and not in T' , contradicting the connectivity of T'). \square

h -hopset construction for trees. A 1-hopset in a tree T is obtained by adding all pairs as edges with appropriate weight. For $h > 1$, we recursively define a h -hopset of T as follows. Select a set P of $2p$ nodes at most with $p = \frac{n}{\lambda_{h-2}(n)}$ according to Lemma 1. (The number $\lambda_{h-2}(n)$ is suitably chosen according to the $(h - 2)$ -row inverse Ackerman function defined next.) When $h = 2$, we add an edge from each node u of T to each node in P . When $h > 2$, we consider the forest T' induced by nodes in P : it has node set P and edges xy such that y is the closest ancestor of x in T that belongs to P . The weight of such an edge is defined as $w'(x, y) = d_T(x, y)$. We then add a $(h - 2)$ -hopset of T' to the construction. Additionally, we add one or two edges per node not in P : for each connected component C of $T \setminus P$, add an edge ux for each node $u \in C$ and each $x \in P$ connected to C . Note that Lemma 1 ensures that there are at most two such

nodes x for a given component C . In both cases ($h \geq 2$), we construct recursively a h -hopset of each sub-tree induced by a connected component C of $T \setminus P$. In the special case of $h = 3$, the $(h - 2)$ -hopsets contribute to H_2 while all edges connecting to a node in some selected set P contribute to H_1 according to the $H = H_1 \cup H_2$ convention introduced in the Preliminaries.

Notation: Ackermann function. To analyze the construction, following [7], we introduce the following variants of the Ackermann function:

$$\left\{ \begin{array}{l} A(0, j) = 2j, \text{ for } j \geq 0 \\ A(i, 0) = 1, \text{ for } i \geq 1 \\ A(i, j) = A(i - 1, A(i, j - 1)), \text{ for } i, j \geq 1; \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} B(0, j) = j^2, \text{ for } j \geq 0 \\ B(i, 0) = 2, \text{ for } i \geq 1 \\ B(i, j) = B(i - 1, B(i, j - 1)), \text{ for } i, j \geq 1. \end{array} \right.$$

The k th-row inverse Ackermann function $\lambda_k(\cdot)$ is defined by $\lambda_{2i}(n) = \min\{j \mid A(i, j) \geq n\}$ and $\lambda_{2i+1}(n) = \min\{j \mid B(i, j) \geq n\}$ for $i \geq 0$. Equivalently, we have $\lambda_0(n) = \frac{n}{2}$, $\lambda_1(n) = \sqrt{n}$ and $\lambda_k(n) = \lambda_{k-2}^*(n)$ where we define for any function f : $f^{(0)}(n) = n$, $f^{(i)}(n) = f(f^{(i-1)}(n))$ for $i > 0$, and $f^*(n) = \min\{j \mid f^{(j)} \leq 1\}$. Note that $\lambda_2(n) = \log n$, $\lambda_3(n) = \log \log n$, $\lambda_4(n) = \log^* n$ and $\lambda_5(n) = \frac{1}{2} \log^* n$.

The inverse Ackermann function is defined as $\alpha(n) = \min\{j \mid A(j, j) \geq n\}$. Note that we have $\lambda_{2\alpha(n)}(n) = \alpha(n)$. We are now ready to state the parameters of the designed hopset.

Proposition 1. *For any integer $h > 1$ and weighted tree T with n nodes, a h -hopset H of T with $O(n\lambda_h(n))$ edges can be computed in $O(n\lambda_h(n))$ time. A linear size $2(\alpha(n) + 1)$ -hopset can be computed in $O(n\alpha(n))$ time. In the case $h = 3$, the constructed hopset allows to obtain a distance oracle using space of $O(n \log \log n)$ edges of $O(\log n)$ bits and having query time $O(\log^2 \log n)$.*

Correctness. The correctness of the constructed h -hopset H comes from the fact two nodes u, v in two different connected components of $T \setminus P$ both have an hopset edge to a node in P on the path P_{uv} from u to v according to Lemma 1. Let x and y denote the nodes in $P \cap P_{uv}$ that are linked to u and v respectively ($ux, vy \in H$). For $h > 2$, the $h - 2$ -hopset added in the construction implies that a path of $h - 2$ hops at most links x to y in $T \cup H$ and we thus have $d_{T \cup H}^h(u, v) = d_T(u, v)$. For $h = 2$, we also have $vx \in H$ (and $uy \in H$), and $x \in P_{uv}$ implies $d_{T \cup H}^2(u, v) = d_T(u, v)$.

Analysis. We claim that the resulting h -hopset has $O(n\lambda_h(n))$ edges for $h > 1$. Recall that a 1-hopset has $\Theta(n^2)$ edges. Note that the choice of $p = \frac{n}{\lambda_{h-2}(n)}$ in our construction implies that connected components created by Lemma 1 have size $\lambda_{h-2}(n)$ at most. The components created in a recursive call with recursion depth j will have size $\lambda_{h-2}^{(j)}(n)$. The number of recursion levels is thus $\min\{j \mid \lambda_{h-2}^{(j)}(n) \leq 1\} = \lambda_h(n)$. We now show that $O(n)$ edges are added to the construction at each recursion level. For $h = 2$, we have $p = O(1)$ and the number of edges added at each recursion level is thus at most $O(n)$. For $h = 3$, we have $p = \frac{n}{\lambda_1(n)} = \sqrt{n}$ and the $h - 2$ -hopset constructed on $2p$ nodes at most has $O(n)$ edges. For $h > 3$, we proceed by induction on h : we assume that the $(h - 2)$ -hopset constructed for a tree with $2p$ nodes at most has $O(2p\lambda_{h-2}(2p))$ edges that is $O(n)$ edges for $p = \frac{n}{\lambda_{h-2}(n)}$ (note that λ_{h-2} is non-decreasing for any $h > 0$).

Query time for 3-hopsets. For the special case of $h = 3$, we have $\lambda_3(n) = \log \log n$, and the size required to represent the 3-hop data structure is $\mathcal{S} = O(n \log \log n)$ edges. Moreover, following the convention $H = H_1 \cup H_2$ introduced in the Preliminaries, we note that in the

adopted construction $\deg_{H_1}(v) = O(\log \log n)$ for all $v \in V$. A bound of $O(\log^2 \log n)$ query time follows the above analysis.

Linear size hopset. We can obtain a linear size $2(\alpha(n) + 1)$ -hopset by splitting T into subtrees of size $\alpha(n)$ at most using Lemma 1 with $p = \frac{n}{\alpha(n)}$. Two nodes in a connected component of $T \setminus P$ are thus obviously linked by a path of length $\alpha(n)$ at most in T . Similarly as before, we link every node to the (at most 2) nodes in P connected to its component and add a $2\alpha(n)$ -hopset for the forest induced by nodes in P . We thus obtain a $2(\alpha(n) + 1)$ -hopset with $O(n + \frac{n}{\alpha(n)}\lambda_{2\alpha(n)}(n)) = O(n)$ edges.

Lower bound. We note that the $O(n\lambda_h(n))$ hopset size is indeed tight for some trees. If P is a path with nodes from 1 to n , any h -hopset can be seen as a covering of intervals in $[1, n]$ where $[i, j]$ denotes the interval $i, i + 1, \dots, j$ of integers. More precisely, a set I of intervals h -covers $[1, n]$ when every interval $[i, j] \subseteq [1, n]$ is the union of at most h intervals in I [7]. We can easily obtain a h -covering from any h -hopset H of the path P by associating each edge uv of $P \cup H$ to the interval $[u, v]$. A lower bound of $\Omega(n\lambda_h(n))$ for the size of a h -covering of $[1, n]$ is proved in [7].

Treewidth definition. Recall that a graph G has treewidth t if there exists a tree T whose nodes are subsets of $V(G)$ called *bags* such that: $|X| \leq t + 1$ for all $X \in V(T)$; for all edges $uv \in E(G)$, there exists a bag $X \in V(T)$ containing both u and v ($u, v \in X$); and for all nodes $u \in V(G)$, the bags containing u form a sub-tree of T . Without loss of generality, we assume that each bag contains exactly $t + 1$ nodes, and that two neighboring bags share exactly t nodes (the decomposition is standard). This implies $|V(T)| \leq n$ as each bag brings one new node. Note that removing a non-leaf bag separates the graph into several connected components. We consider that all edges of T have weight 1. For convenience, we assume that T is root at some bag R and define for each node $u \in V(G)$ the root bag of u as the bag $R_u \in V(T)$ containing u which is closest to the root.

h -hopset construction for bounded treewidth graphs. Consider a graph G with treewidth t and an associated tree T . The general idea is to follow the construction of a h -hopset of T with slight modifications. Similarly to the tree case, we select a set P of $2p$ bags at most with $p = \frac{n/t}{\lambda_{h-2}(n/t)}$ according to Lemma 1. We then construct a $(h - 2)$ -hopset $H_{T'}$ of the forest T' induced by bags in P according to the tree construction. For each edge XY in $H_{T'}$, we add an edge xy to the graph hopset for all $x \in X$ and $y \in Y$. Such edges are called *tree-hopset* edges. Now for each node u such that its root bag R_u falls in a connected component of $T \setminus P$, we consider the (at most 2) bags $Y \in P$ that are connected to that component and add an edge uy to the graph hopset for all $y \in Y$. Such edges are called *separator* edges. We then recurse on each component of $T \setminus P$ until we reach subtrees of size $n' \leq t$. We then pursue with $p = \frac{n'}{\lambda_{h-2}(n')}$ and so on recursively until reaching components of size 1 at most. Finally, for each node u , we add an edge ux to the graph hopset for all $x \in R_u$. Such edges are called *bag* edges. To construct a linear size hopset, we use a single step with $p = \frac{n}{\alpha(n)}$ and a $2\alpha(n)$ -hopset of T' . For each tree edge XY inside components of $T \setminus P$ we add an edge xy to the construction for all $x \in X$ and $y \in Y$ such that $x \notin Y$ and $y \notin X$. Such edges are also considered as tree-hopset edges.

Theorem 1. *For all $h > 1$, any graph with treewidth t has a h -hopset with $O(tn\lambda_h(n))$ edges and a $2(\alpha(n) + 1)$ -hopset with $O(t^2n)$ edges.*

Correctness of construction. Let H denote the hopset constructed for a graph G with treewidth t and associated tree T . Consider a shortest path $Q = u_0, \dots, u_k$ for some integer $k \geq 1$. First consider the case where a bag X of T contains both u_0 and u_k . We can assume without loss of generality that R_{u_0} is an ancestor of R_{u_k} . As R_{u_k} lies on the path from X to R_{u_0} , it must contain u_0 and edge $u_k u_0$ is in H according to the last step of the above construction. Now suppose that no bag contains both u_0 and u_k . Consider the first recursion call where splitting a subtree with a set P of bags separates R_{u_0} and R_{u_k} . Consider the path from R_{u_0} to R_{u_k} in T . Let X (resp. Y) be the first (resp. last) bag in P on that path. Either u_0 is in X or H contains separator edges from u_0 to all nodes in X . Similarly, either u_k is in Y or H contains a separator edge from u_k to all nodes in Y . The $(h-2)$ -hopset considered during that recursion call contains a path P' of $h' \leq h-2$ hops from X to Y . If two consecutive bags contain u_0 and u_k respectively, then H contains edge $u_0 u_k$ as a tree-hopset edge. Otherwise, let X' (resp. Y') be the first bag in P' not containing u_0 (resp. u_k). By treewidth definition, there exist bags $X_1, \dots, X_k \in V(T)$ containing edges $u_0 u_1, \dots, u_{k-1} u_k$ respectively (i.e., X_i contains u_{i-1} and u_i for all $i \in \{1, \dots, k\}$). The shortest path Q corresponds to a walk in T from X_1 to X_2 , then to X_3 and so on. All bags on the path (in T) from X_i to X_{i+1} must contain u_i . As that walk must go through X' , we can define the highest index $i_0 > 0$ such that $u_{i_0} \in X'$. Similarly, we can define the smallest index $j_0 \geq i_0$ such that $u_{j_0} \in Y'$. Our construction H then contains separator edges $u_0 u_{i_0}$ and $u_k u_{j_0}$. When $i_0 = j_0$, H contains a path of 2 hops at most with same length as Q . If two consecutive bags of P' contain u_{i_0} and u_{j_0} respectively, then H contains a tree-hopset edge $u_{i_0} u_{j_0}$. Otherwise, we can similarly define indexes $i_1, \dots, i_{h''}$ and $j_1, \dots, j_{h'''}$ with $i_0 < i_1 < \dots < i_{h''} < j_{h'''} < \dots < j_1 < j_0$ and $h'' + h''' + 1 \leq h' \leq h-2$. Our construction H then contains tree-hopset edges $u_{i_0} u_{i_1}, \dots, u_{i_{h''-1}} u_{i_{h''}}, u_{i_{h''}} u_{j_{h'''}} , u_{j_{h'''-1}} u_{j_{h'''}} , \dots, u_{j_0} u_{j_1}$. In all cases, H contains a path of h hops at most and same length as Q .

Analysis. In the first recursion levels, a subtree of size n' is split into subtrees smaller than $t\lambda_{h-2}(n'/t) \leq t\lambda_{h-2}(n')$. At recursion depth $\lambda_{h-2}^*(n) = \lambda_h(n)$, we thus obtain subtrees of size t at most. Deeper recursion calls are similar to the tree case. The total number of recursion levels is thus $\lambda_h(n) + \lambda_h(t) = O(\lambda_h(n))$. When processing a subtree of size n' , we build a $(h-2)$ -hopset for a forest of $2p$ bags at most using $O(2p\lambda_{h-2}(2p))$ edges according to Proposition 1. For $n' > t$, we use $p = \frac{n'/t}{\lambda_{h-2}(n'/t)}$ and thus produce $O(t^2 \frac{n'}{t}) = O(tn)$ tree-hopset edges at most. For $n' \leq t$, we use $p = \frac{n'}{\lambda_{h-2}(n')}$. However, for a given bag X , there are at most $n' \leq t$ nodes not in X among the other $n' - 1$ bags. We thus produce at most t tree-hopset edges per bag. In both cases, each recursion level thus brings $O(tn)$ tree-hopset edges as well as $O(tn)$ separator edges. There are tn bag edges at most in total. We can thus obtain a h -hopset with $O(tn\lambda_h(n))$ edges for any graph of treewidth t . In the linear size construction, we use a single step using a $2\alpha(n)$ -hopset for T' with $O(\frac{n}{\alpha(n)}\lambda_{2\alpha(n)}(n)) = O(n)$ edges. We thus have $O(t^2n + tn)$ tree-hopset edges and $O(tn)$ separator edges.

Query time for 3-hopsets. For the special case of $h = 3$, we have $\lambda_3(n) = \log \log n$, and the size required to represent the 3-hop data structure is $\mathcal{S} = O(tn \log \log n)$ edges. Following the convention $H = H_1 \cup H_2$, we classify tree-hopset edges in H_2 while both separator edges and bag edges are classified in H_1 . For any $v \in V$, we thus have $\deg_{H_1}(v) = O(t \log \log n)$. The following bound on the query time follows.

Theorem 2. *Any graph with treewidth t admits a 3-hopset distance oracle represented on $O(tn \log \log n)$ edges of $O(\log n)$ bits, with a query time of $O(t^2 \log^2 \log n)$. \square*

4 Bounded Skeleton Dimension

In this Section we consider graphs with unique shortest paths (USP), only. A formal definition of the notion of skeleton dimension relies on the concept of the geometric realization of a graph, cf. [36]. The *geometric realization* \tilde{G} of G can be seen as the “continuous” graph where each edge is seen as infinitely many vertices of degree two with infinitely small edges, such that for any $uv \in E(G)$ and $t \in [0, 1]$, there is a node in \tilde{G} at distance $td_G(u, v)$ from u on edge uv . We define the *reach* of $v \in V(\tilde{T})$ as $\text{Reach}_{\tilde{T}}(v) := \max_{x \in V(\tilde{T})} d_{\tilde{T}}(v, x)$. We then define the *skeleton* T^* of T as the subtree of \tilde{T} induced by nodes with reach at least half their distance from the root. More precisely, T^* is the subtree of \tilde{T} induced by $\{v \in V(\tilde{T}) \mid \text{Reach}_{\tilde{T}}(v) \geq \frac{1}{2}d_{\tilde{T}}(u, v)\}$. The *width* of a tree T with root u is defined as the maximum number of nodes (points) in \tilde{T} at a given distance from its root. More precisely, the width of T is $\text{Width}(T) = \max_{r>0} |\text{Cut}_r(\tilde{T})|$ where $\text{Cut}_r(\tilde{T})$ is the set of nodes $v \in V(\tilde{T})$ with $d_{\tilde{T}}(u, v) = r$.

The *skeleton dimension* k of a graph G is now defined as the maximum width of the skeleton of a shortest path tree, that is $k = \max_{u \in V(G)} \text{Width}(T_u^*)$, where T_u denotes the shortest path tree of u obtained as the union of shortest paths from u to all $v \in V(G)$.

For the definition of the related concept of *highway dimension* we refer the reader to [5]. We note that if the geometric realization \tilde{G} of a graph G has highway dimension \tilde{h} , then G has skeleton dimension $k \leq \tilde{h}$; hence, in all subsequent asymptotic analyses, upper bounds expressed in terms of skeleton dimension can be replaced by analogous bounds in terms of highway dimension.

4.1 Construction of the 3-Hopset

We denote by L_{\max} the maximum length of an edge in graph G . The construction of the 3-hopset H is obtained by taking a union of sets of shortcuts, each of which covers sets of node pairs within a given distance range. The first shortcut set H' covers all node pairs $u, v \in V$ with $d_G(u, v) \leq D'$, for some choice of distance bound D' , whereas each of the subsequent shortcut sets $H^{(D)}$ covers nodes at a distance in an exponentially increasing distance range, $d_G(u, v) \in [D, D^{1+\epsilon}]$, where $\epsilon := \frac{1}{2 \log_2 k}$ is suitably chosen. We then put:

$$H = H' \cup \bigcup_{i=1,2,\dots} H^{(D'^{i(1+\epsilon)})}.$$

Construction of set H' . We note that a construction of 2-hopsets for graphs of skeleton dimension k was performed in [36]. As a direct corollary of [36][Lem. 2, Cor. 1,2], given a distance bound D' , there exists a randomized polynomial-time construction of a set of shortcuts H' for graph G with the property that for any pair of nodes $u, v \in V$ with $d_G(u, v) \leq D'$, we have $d_{G \cup H'}^2 = d_G(u, v)$, such that $|H'| = O(nk \log D')$, and moreover for all $u \in V$, we have $\mathbf{E} \deg_{H'}(u) = O(k \log D')$ and $\deg_{H'}(u) = O(k \log D' \log \log n + \log n)$. We use set H' directly for the value $D' := L_{\max}^4 k^6 \log^{12} n$, considering H' as a 3-hopset for node pairs $u, v \in V$ with $d_G(u, v) \leq D'$. We have:

$$|H'| = O(nk(\log \log n + \log L_{\max} + \log k)),$$

and for all $u \in V$:

$$\begin{aligned} \mathbf{E} \deg_{H'}(u) &= O(k(\log \log n + \log L_{\max} + \log k)), \\ \deg_{H'}(u) &= O(k \log \log n (\log \log n + \log L_{\max} + \log k) + \log n). \end{aligned}$$

We remark that, without loss of generality, in asymptotic analysis one may assume that $L_{\max} \leq kL$, where L is the *average* edge length in G , noting that edges longer than kL can be subdivided into edges of length at most kL by inserting additional vertices, increasing the number of nodes of the graph only by a multiplicative constant. Thus, in the above bounds, we can replace $(\log \log n + \log L_{\max} + \log k)$ by $(\log \log n + \log L + \log k)$.

Construction of set $H^{(D)}$. We now proceed to construct a 3-hopset for node pairs u, v with $d_G(u, v) \in [D, D^{1+\epsilon}]$. The construction of set $H^{(D)}$ is randomized and completely determined by assignment of real values $\rho(u) \in [0, 1]$ to each node $u \in V$, uniformly and independently at random. We condition all subsequent considerations on the event that all values ρ are distinct, $|\rho(V)| = |V|$, which holds with probability 1.

Now, hopset $H^{(D)}$ is defined as $H^{(D)} := H_1^{(D)} \cup H_2^{(D)}$, where following our usual notation, $H_1^{(D)}$ is the set of first and last hops, and $H_2^{(D)}$ is the set of middle hops.

Set of first and last hops. For $u \in V$, let $R^{(D)}(u)$ be the set of nodes which lie on a shortest path of length at least D which has one of its endpoints at u , and which have minimum value of ρ among all vertices on this path at distance in $[D/4, D/2]$ from u :

$$R^{(D)}(u) = \bigcup_{v \in V: d_G(u, v) \geq D} \left\{ \operatorname{argmin}_{r \in P_{uv}, d_G(u, r) \in [D/4, D/2]} \rho(r) \right\}.$$

We now put: $H_1^{(D)} := \{ur : u \in V, r \in R(u)\}$.

Set of middle hops. We put in $H_2^{(D)}$ links between all pairs of nodes which have a small value of ρ , satisfy the natural upper bound of $D^{1+\epsilon}$ on distance between them, and have sufficiently large reach, i.e., the shortest path between them can be extended by at least $D/4$:

$$H_2^{(D)} := \left\{ qr : q, r \in \bigcup_{u \in V} R^{(D)}(u) \wedge d_G(q, r) \leq D^{1+\epsilon} - D/2 \wedge (\exists v \in V r \in P_{qv} \wedge d_G(r, v) \geq D/4) \right\}.$$

The validity of H as a 3-hopset is immediate to verify from the construction.

4.2 Bound on 3-Hopset Size and Oracle Time

Lemma 2. *Fix $u \in V$ and $D > 0$. We have: $|R^{(D)}(u)| \leq k$.*

Proof. By the fact that the size of the cut of the skeleton tree for node u at distance $D/2$ from u is upper-bounded by the skeleton dimension k , we have that the set of paths $\mathcal{P} := \{\Pi_v : v \in V \wedge d_G(u, v) \geq D\}$, where $\Pi_v := \{w \in P_{uv} : d_G(u, w) \in [D/4, D/2]\}$, has at least at most k distinct paths, $|\mathcal{P}| \leq k$. The bound on the size of set $|R^{(D)}(u)|$ now follows directly from its definition. \square

From the above Lemma, it follows that for any $u \in V$ we have $\deg_{H_1^{(D)}}(u) \leq k$, thus summing over the $O(\log \log(nL_{\max})/\log(1 + \epsilon)) = O(\log \log(nL_{\max}) \log k)$ levels of the construction, we successively obtain:

$$\deg_{H_1}(u) \leq \deg_{H'}(u) + k \cdot O(\log \log(nL_{\max}) \log k) = O(k \log \log n \log k (\log \log n + \log L) + \log n), \quad (1)$$

$$\mathbf{E} \deg_{H_1}(u) \leq \mathbf{E} \deg_{H'}(u) + k \cdot O(\log \log(nL_{\max}) \log k) = O(k \log k (\log \log n + \log L)), \quad (2)$$

$$|H_1| \leq |H'| + nk \cdot O(\log \log(nL_{\max}) \log k) = O(nk \log k (\log \log n + \log L)). \quad (3)$$

We now proceed to bound the size of the set H_2 of middle hops.

Lemma 3. Fix $D \geq D'$. With probability $1 - O(1/n^2)$, it holds that for all $u \in V$, for all $r \in R^{(D)}(u)$, we have $\rho(r) \leq L_{\max}/D$.

Proof. As noted in the proof of Lemma 2, to be included in $R^{(D)}(u)$, a node r must be the minimum element along one of the at most k possible paths Π_v . Each such path includes all nodes on the path P_{uv} at distance in the range $[D/4, D/2]$ from u , where we recall that $D \geq D' = CL_{\max}^4 k^6 \log^{12} n > CL_{\max} \ln^2 n$, for some sufficiently large choice of constant $C > 0$. It follows that each path Π_v contains $|\Pi_v| \geq \max\{\ln^2 n, \frac{CD}{8L_{\max}}\}$ nodes. Now, taking note of the independence of the choice of random variables $(\rho(w) : w \in \Pi_v)$, we have by a simple concentration bound that $\Pr[\min \rho(\Pi_v) > L_{\max}/D] \leq O(1/n^4)$, for a suitable choice of constant C . By taking a union bound over all paths Π_v in \mathcal{P} , and then another union bound over all $u \in V$, the claim follows. \square

We now proceed under the assumption that the event from the claim of the Lemma holds. We now consider an arbitrary node $q \in R^{(D)}(u)$ for some $u \in V$, and look at $\deg_{H_2^{(D)}}(q)$. We now have that if $qr \in H_2^{(D)}$, then by the definition of $H_2^{(D)}$ and the above Lemma, the following conditions jointly hold:

- $\rho(r) \leq L_{\max}/D$
- $r \in \{w \in V : \exists v \in V \ D^{1+\epsilon} \geq d_G(q, v) \geq d_G(q, w) + D/4 \wedge P_{qw} \subseteq P_{gv}\} =: W(q)$.

We note that $W(q)$ is the subset of the vertex set of the shortest path tree of node q , pruned to contain only those paths which have reach at least $D/4$ at depth less than $D^{1+\epsilon}$. This tree has depth bounded by $D^{1+\epsilon}$, and width bounded by a generalized skeleton dimension parameter denoted as k_α (following [36]), with parameter $\alpha = \frac{D/4}{D^{1+\epsilon}} = D^{-\epsilon}/4$. Following [36][Section 6], k_α can be easily expressed using skeleton dimension k as:

$$k_\alpha \leq k^{\lceil \log_2(1+1/\alpha) \rceil} < k^{1+\log_2(4D^\epsilon)} = k^3 D^\epsilon \log_2 k.$$

We then have $|W(q)| \leq D^{1+\epsilon} k_\alpha < k^3 D^{1+\epsilon(1+\log_2 k)}$. Moreover, by an easy concentration bound, we have that for all $q \in V$, $|\{r \in W(q) : \rho(r) \leq L_{\max}/D\}| = O(\log n) + \frac{2L_{\max}}{D} |W(q)|$, with probability $1 - O(1/n^2)$. It follows that with probability $1 - O(1/n^2)$, we have for all $q \in \bigcup_{u \in V} R^{(D)}(u)$:

$$\deg_{H_2^{(D)}}(q) \leq O(\log n) + \frac{2L_{\max}}{D} |W(q)| \leq O(\log n + L_{\max} k^3 D^\epsilon \log_2 k).$$

Noting that with probability $1 - O(1/n^2)$:

$$\left| \bigcup_{u \in V} R^{(D)}(u) \right| \leq |\{w \in V : \rho(w) \leq L_{\max}/D\}| \leq O(\log n + nL_{\max}/D)$$

we finally obtain that with probability $1 - O(1/n^2)$:

$$\begin{aligned} |H_2^{(D)}| &\leq O(\log n + nL_{\max}/D) O(\log n + L_{\max} k^3 D^\epsilon \log_2 k) = O(\log^2 n + nL_{\max}^2 k^3 D^{\epsilon \log_2 k - 1}) \\ &\leq O(nL_{\max}^2 k^3 D^{-1/2}) \leq O(nD'^{-1/4}) \leq O(n/\log^3 n), \end{aligned}$$

where in the last two transformations we used the fact that $\epsilon = \frac{1}{2\log_2 k}$ and that $D \geq D' \geq L_{\max}^4 k^6 \log^{12} n$. Using a union bound and summing over all levels of the construction, we eventually obtain that with probability $1 - O(1/n)$:

$$|H_2| \leq O(n/\log_2 n). \quad (4)$$

Thus, the set of middle links is sparse and does not contribute to the asymptotic size of the overall representation of the 3-hopset.

Overall, considering a randomized construction which rejects random choices of ρ for which any of the considered w.h.p. events fail, by combining Eq. (1)–(4) with the hopset-based distance oracle framework described in the Preliminaries, we obtain the following Theorem.

Theorem 3. *For a unique shortest path graph with skeleton dimension k and average link length $L \geq 1$, there exists a randomized construction of a 3-hopset distance oracle of size $|H| = O(nk \log k (\log \log n + \log L))$, which for an arbitrary queried node pair performs distance queries in expected time $O(k^2 \log^2 k (\log^2 \log n + \log^2 L))$ (where the expectation is taken over the randomized construction of the oracle), and in time $O(k^2 \log^2 k \log^2 \log n (\log^2 \log n + \log^2 L) + \log^2 n)$ with certainty. \square*

In particular, for graphs with constant-length edges and small skeleton dimension ($k = O(\log n)$), the 3-hopset has size $|H| = O(nk \log k \log \log n)$, with expected time of any query given as $O(k^2 \log^2 k \log^2 \log n)$.

5 LP-based Approximation Algorithm

In this Section, we propose an Integer Linear Programming (ILP) formulation for h -hopsets with a minimum number of edges, which we then relax to a LP formulation. Whereas both formulations are applicable to the general case, we prove relations between them only for USP graphs.

5.1 ILP and LP Formulations

A necessary and sufficient condition for H to be a h -hopset for G is that for every pair of vertices s, t there exists a path $P_{st} = (s = v_0, v_1, \dots, v_{l_{st}} = t)$ in $G \cup H$ such that $l_{st} \leq h$ and in graph G there exists some shortest $s - t$ path passing through all of the vertices $v_0, \dots, v_{l_{st}}$, in the given order. For a fixed pair s, t , we consider the directed graph H^{st} with vertex set $V \times \{0, \dots, h\} \equiv V_h$ (by convention, elements of V_h will be denoted compactly as v_i , where $v \in V, i \in \{0, \dots, h\}$) and with an arc set defined as follows. For $i \in \{0, \dots, h-1\}$, we add arc (u_i, v_{i+1}) to H^{st} if and only if $\{u, v\} \in G \cup H$ and u, v lie on some shortest $s - t$ path in the given order, i.e., if $d_G(s, u) + d_G(u, v) + d_G(v, t) = d_G(s, t)$. In particular, all arcs of the form (u_i, u_{i+1}) , for $u \in V$, belong to H^{st} . Now, we have that H is a h -hopset for G if and only if there exists a path from s_0 to t_h in H^{st} . This is equivalent to saying that for all $s, t \in V$, the flow value from s_0 to t_h is at least 1 in H^{st} . Given graph G , we thus have the following ILP formulation for the minimum h -hopset problem, using indicator variables x_{uv} for $G \cup H$ (given as 1 if $\{u, v\} \in G \cup H$ and 0 otherwise) and variables $f_{u_i v_j}^{st}$, representing the flow value along arc (u_i, v_j) in H^{st} :

$$\text{Minimize:} \quad \sum_{u \neq v, \{u, v\} \notin E} x_{uv} \quad (5)$$

Subject to:

$$x_{uv} \in \{0, 1\} \quad (6)$$

$$0 \leq f_{u_i v_j}^{st} \leq \begin{cases} x_{uv}, & \text{if } j = i + 1 \text{ and } d_G(s, u) + d_G(u, v) + d_G(v, t) = d_G(s, t), \\ 0, & \text{otherwise.} \end{cases} \quad (7)$$

$$\sum_{u_i} f_{v_j u_i}^{st} - \sum_{u_i} f_{u_i v_j}^{st} = \begin{cases} 0, & \text{for } v_j \in V_h \setminus \{s_0, t_h\} \\ +1, & \text{for } v_j = s_0 \\ -1, & \text{for } v_j = t_h \end{cases}, \quad (8)$$

where indices s, t, u, v traverse V and indices i, j traverse $\{0, \dots, h\}$.

To obtain an LP relaxation of the above problem, we replace the integral condition $x_{uv} \in \{0, 1\}$ by the fractional one $x_{uv} \in [0, 1]$. We look at the connection between the integral and fractional forms for the special case of unique shortest path graphs.

We remark that the above formulation can be seen as a generalization of the LP and ILP statement of Angelidakis et al. [9] proposed for the special case of 2-hop labeling. In the case of 2-hop labeling, Angelidakis et al. do not rely on an explicit flow formulation but use a single constraint of the simpler form $\sum_{w \in P^{st}} \min\{x_{sw}, x_{wt}\} \geq 1$, where P^{st} represents the set of nodes on some shortest $s - t$ path in G . However, the analysis of the integrality gap does not carry over from the case of $h = 2$ to $h > 2$, i.e., as soon as there exist internal shortcuts which have neither s nor t as one of their endpoints.

5.2 Bounding Integrality Gap for Unique Shortest Path Graphs

We analyze the integrality gap of the above LP formulation for the case of *unique shortest path (USP) graphs*, i.e., graphs in which each pair of nodes $s, t \in V$ is connected by a unique shortest path P^{st} in G . We will occasionally identify P^{st} with its set of nodes, and we will introduce a linear order on its vertices, writing for $u, v \in P^{st}$ that $u <^{st} v$ if $d_G(s, u) < d_G(s, v)$; we will denote the order simply as “ $<$ ” when the path P^{st} is clear from the context. Observe that in the LP formulation, we may have $f_{u_i v_j}^{st} \neq 0$ only if $u <^{st} v$ and $j = i + 1$. Thus, fixing $s, t \in V$, the flow $f^{st} = (f_{u_i v_j}^{st} : u_i, v_j \in V_h)$ is non-zero between vertices of $\{P^{st}\} \times \{0, 1, \dots, h\}$ only, and the flow is oriented towards t on this path.

Let $(x_{uv}, f_{u_i v_j}^{st})$ be a fixed solution to the LP problem in a USP graph, with cost $COST_{LP} = \sum_{u \neq v, \{u, v\} \notin E} x_{uv}$. We will show how to use this set to construct a valid hopset H'' for G (thus, equivalently, also solving the ILP formulation). We first apply a randomized rounding procedure following the classical scheme of Raghavan and Thomson [38]. We define the family of independent random variables $(x'_{u_i v_{i+1}} : u, v \in V, i \in \{0, \dots, h\})$, with $x'_{u_i v_{i+1}} \in [0, 1]$. For $u \neq v, \{u, v\} \notin E$ we put $\Pr[x'_{u_i v_{i+1}} = 1] = \min\{C x_{uv}, 1\}$, where $C \geq 1$ is a suitably chosen probability amplification parameter (we put $C = 8h \ln n$). We will assume, without affecting the validity or cost of the solution, that $x_{uv} = x'_{u_i v_{i+1}} = 1$, when $u = v$ or $\{u, v\} \in E$.

We denote $H' = \{\{u, v\} : u, v \in V \wedge u \neq v \wedge \{u, v\} \notin E \wedge \exists_{i \in \{0, \dots, h-1\}} x'_{u_i v_{i+1}} = 1\}$. Let $\pi : V \rightarrow \{1, \dots, n\}$ be a bijection (informally, a permutation) picked uniformly at random. We define the set of shortcuts $S(\{u, v\})$ associated with each pair $\{u, v\} \in H'$ as the set of all pairs of nodes on path P^{uv} , one of which is a prefix minimum on this path with respect to π , and the other of which is a suffix minimum with respect to π :

$$S(\{u, v\}) := \left\{ \{u^*, v^*\} : u^*, v^* \in P^{uv} \wedge \pi(u^*) = \min_{z \in P^{uv}, z \leq^{uv} u^*} \pi(z) \wedge \pi(v^*) = \min_{z \in P^{uv}, z \geq^{uv} v^*} \pi(z) \right\}.$$

The obtained solution is given as the set of all such shortcuts:

$$H'' := \bigcup_{\{u,v\} \in H'} S(\{u,v\}).$$

Proposition 2. *With probability $1 - O(1/n)$, set H'' is a hopset for G of size $O(h^2 \log^3 n \cdot \text{COST}_{\text{LP}})$.*

The rest of the section is devoted to the proof of Proposition 2.

5.2.1 Size of Hopset H''

Proposition 3. *We have $|H''| = O(h^2 \log^3 n \cdot \text{COST}_{\text{LP}})$, with probability $1 - O(1/n)$.*

Proof. We first remark that for any $i \in \{0, \dots, h-1\}$, we have by a standard application of a multiplicative Chernoff bound that the following bound holds with probability $1 - O(1/n^2)$:

$$\sum_{u \neq v, \{u,v\} \notin E} x'_{u_i v_{i+1}} \leq 2C \sum_{u \neq v, \{u,v\} \notin E} x_{u_i v_{i+1}} = O(h \log n \cdot \text{COST}_{\text{LP}})$$

It follows by a union bound over $i \in \{0, \dots, h-1\}$ that $|H'| = O(h^2 \log n \cdot \text{COST}_{\text{LP}})$, with probability $1 - O(1/n)$.

We now proceed to bound the size of each set $S(\{u,v\})$, for $\{u,v\} \in H'$. This is given precisely by the product of the size of the set of prefix minima and suffix minima of permutation π on path P^{uv} . Denoting the random variable describing the number of prefix minima on a path as $X^{st} := |\{u^* \in P^{st} : \pi(u^*) = \min_{z \in P^{st}, z \leq^{st} u^*}\}|$, we have:

$$|S(\{u,v\})| = X^{uv} \cdot X^{vu}.$$

It is well-known the number of prefix minima has expectation $\mathbf{E}X^{uv} = \ln |P^{uv}| + O(1) \leq \ln n + O(1)$ and that the distribution of X^{uv} is concentrated around its expectation; in particular, we have by a simple multiplicative Chernoff bound that $\Pr[X^{uv} \leq 4 \ln n] \geq 1 - n^{-3}$. Applying a union bound over all $\{u,v\}$, we have:

$$\Pr[\forall \{u,v\} \in H' |S(\{u,v\})| \leq 16 \ln^2 n] \geq 1 - n^{-1}.$$

Overall, we thus have that $|H''| = O(h^2 \log n \cdot \text{COST}_{\text{LP}} \cdot \log^2 n) = O(h^2 \log^3 n \cdot \text{COST}_{\text{LP}})$, with probability $1 - O(1/n)$. \square

5.2.2 Correctness of Hopset H''

For fixed $s, t \in V$, the choice of $x'_{u_i v_{i+1}}$ is performed iteratively over i , as a random process. Each step $i = 0, 1, \dots, h-1$ of this process determines the vertex $v^{(i+1)st} \in P^{st}$, given inductively as:

$$v^{(i+1)st} = \max_{(<^{st})} \{v \in P^{st} : \exists_{u \in P^{st}} u \leq v^{(i)st} \wedge x'_{u_i v_{i+1}} = 1\},$$

where we denote $v^{(0)st} := s$.

First of all, observe that we have the following sufficient condition for the validity of a h -hopset for the pair s, t .

Lemma 4. *If $v^{(h)st} = t$, then there exists a $s - t$ path in $G \cup H''$ with at most h hops whose vertices form an increasing subsequence on P^{st} according to the order $<^{st}$.*

Proof. For $i \in \{0, \dots, h-1\}$, denote by $u^{(i+1)st}$ the vertex u used in the definition of $v^{(i+1)st}$, i.e.:

$$u^{(i+1)st} = \max_{(<st)} \{u \in P^{st} : x'_{u_i v_{i+1}^{(i+1)st}} = 1\}.$$

Note that $u^{(i+1)st} \leq v^{(i)st} \leq v^{(i+1)st}$. For some $l \leq h$, let $(\phi_0, \dots, \phi_l) \subseteq (0, \dots, h)$, with $\phi_0 = 0$ and $\phi_l = h$, denote a minimal subsequence of indices such that $u^{(\phi_i)st} \leq v^{(\phi_{i-1})st} \leq u^{(\phi_{i+1})st} \leq v^{(\phi_i)st}$, for all $i \in \{1, \dots, l-1\}$. Note that each path $P^{u^{(\phi_{i+1})st} v^{(\phi_i)st}}$ is a subpath of P^{st} by the unique shortest path condition, and consider the minimum vertex $z^{(i)st}$ according to permutation π on this subpath: $z^{(i)st} := \arg \min \{\pi(z) : z \in P^{st} \wedge u^{(\phi_{i+1})st} \leq z \leq v^{(\phi_i)st}\}$. Note that $z^{(0)st} = s$, $z^{(l)st} = t$, and for all $i \in \{0, \dots, l-1\}$, we have:

$$u^{(\phi_{i+1})st} \leq z^{(i)st} \leq v^{(\phi_i)st} \leq u^{(\phi_{i+2})st} \leq z^{(i+1)st} \leq v^{(\phi_{i+1})st}.$$

We have $\{u^{(\phi_{i+1})st}, v^{(\phi_{i+1})st}\} \in H'$, and moreover $z^{(i)st}$ is a prefix minimum with respect to π on $P^{u^{(\phi_{i+1})st} v^{(\phi_{i+1})st}}$ (for the subpath $P^{u^{(\phi_{i+1})st} v^{(\phi_i)st}}$), whereas $z^{(i+1)st}$ is a prefix maximum with respect to π on $P^{u^{(\phi_{i+1})st} v^{(\phi_{i+1})st}}$ (for the subpath $P^{u^{(\phi_{i+2})st} v^{(\phi_{i+1})st}}$). It follows from the definition of H'' that $\{z^{(i)st}, z^{(i+1)st}\} \in H''$. Recalling that $z^{(i)st} \leq^{st} z^{(i+1)st}$, $z^{(0)st} = s$ and $z^{(l)st} = t$ for some $l \leq h$, the claim follows from the existence of the path $(z^{(0)st}, z^{(1)st}, \dots, z^{(l)st})$. \square

The rest of the proof of correctness is devoted to showing that the event “ $v^{(h)st} = t$ ” holds with high probability. We have the following claim.

Lemma 5.

$$\Pr \left[\sum_{u, v \in P^{st}: u \leq v^{(i)st}, v > v^{(i+1)st}} x_{uv} > \frac{1}{2h} \right] < n^{-4}.$$

Proof. Denote the probability from the claim by p . Conditioned on the choice of $v^{(1)st}, \dots, v^{(i)st}$, at the beginning of step i , let w be the right-most (largest) vertex on path P^{st} such that

$$\sum_{u, v \in P^{st}: u \leq v^{(i)st}, v > w} x_{uv} > \frac{1}{2h}.$$

Directly by the definition of $v^{(i+1)st}$, we have:

$$\begin{aligned} p &= \Pr[v^{(i+1)st} \leq w] = \Pr \left[\forall_{u, v \in P^{st}: u \leq v^{(i)st}, v > w} x'_{u_i v_{i+1}} = 0 \right] = \prod_{u, v \in P^{st}: u \leq v^{(i)st}, v > w} \max\{0, 1 - Cx_{uv}\} \\ &\leq \prod_{u, v \in P^{st}: u \leq v^{(i)st}, v > w} (1 - Cx_{uv}) \leq \exp \left[- \sum_{u, v \in P^{st}: u \leq v^{(i)st}, v > w} Cx_{uv} \right] < e^{-C/2h} = n^{-4}. \end{aligned}$$

\square

We now consider the graph H^{st} inferred from the (not necessarily integral) solution to the LP, given on vertex set V as the set of edges uv , such that $f_{uv}^{(i)st} > 0$ for some i .

Each step $i = 0, 1, \dots, h-1$ of the considered process of random choice determines the following $s_0 - t_h$ -flow $F^{(i+1)st}$ on an edge-weighted version of graph H^{st} , described by its flow value $f_{u_j v_{j+1}}^{(i+1)st}$ on each arc (u_j, v_{j+1}) of H^{st} as follows. $F^{(i+1)st}$ is set as a maximum $s_0 - t_h$ flow (with ties broken deterministically in an arbitrary manner) in an edge-weighting of H^{st} such that the capacity of arc (u_j, v_{j+1}) is $f_{u_j v_{j+1}}^{(i)st}$, for all arcs of H^{st} , except for arcs (u_i, v_{i+1}) with

$v > v^{(i+1)st}$, whose capacity is set to 0. By convention, we denote $f_{u_j v_{j+1}}^{(0)st} := f_{u_j v_{j+1}}^{st}$, i.e., as the flow value on the considered arc in the optimal solution to the LP.

Denote by $|F^{(i)st}|$ the value of flow $F^{(i)st}$. The following claim holds.

Lemma 6. $\Pr[|F^{(i+1)st}| \geq |F^{(i)st}| - \frac{1}{2h}] \geq 1 - n^{-4}$.

Proof. We note that for any $u, v \in P^{st}$ such that $u > v^{(i)st}$ we have $f_{u_i v_{i+1}}^{(i+1)st} = 0$, since in the i -th step of the considered process, the in-capacity of vertex u_i is given as $\sum_{w \in P^{st}} f_{w_{i-1} u_i}^{(i)st} = 0$ by the definition of the $(i-1)$ -st step of the process.

Moreover, for any arc (u_j, v_{j+1}) of H^{st} , the values $f_{u_j v_{j+1}}^{(i)st}$ are clearly non-increasing with i , thus in particular:

$$f_{u_j v_{j+1}}^{(i)st} \leq f_{u_j v_{j+1}}^{(i-1)st} \leq \dots \leq f_{u_j v_{j+1}}^{(0)st} \leq x_{uv}.$$

Combining the two above observations, by comparing the size of any two cuts in graph H^{st} for its weightings in successive steps and taking into account the above observations, we obtain the following expression which is used to lower-bound $|F^{(i+1)st}|$:

$$\begin{aligned} |F^{(i)st}| - |F^{(i+1)st}| &\leq \sum_{u, v \in P^{st}: v > v^{(i+1)st}} f_{u_i v_{i+1}}^{(i)st} = \sum_{u, v \in P^{st}: u \leq v^{(i)st}, v > v^{(i+1)st}} f_{u_i v_{i+1}}^{(i)st} \\ &\leq \sum_{u, v \in P^{st}: u \leq v^{(i)st}, v > v^{(i+1)st}} x_{uv} \end{aligned}$$

Thus, applying Lemma 5 we obtain the claim. \square

Lemma 7. $\Pr[v^{(h)st} = t] \geq 1 - n^{-3}$.

Proof. First note that if $v^{(h)st} \neq t$, then $v^{(h)st} < t$, and it follows that $|F^{(h)st}| = 0$ because all the capacities of arcs entering node t_h are equal to 0 by definition in the graph in which flow $F^{(h)st}$ is considered.

Now, observe that using Lemma 6 and applying a union bound over i , we obtain: $\Pr[|F^{(h)st}| \geq F^{(0)st} - \frac{h}{2h}] \geq 1 - hn^{-4} \geq 1 - n^{-3}$. Observe next that $F^{(0)st} \geq 1$ by the constraints of the LP solution, hence $|F^{(h)st}|$ is strictly positive with probability at least $1 - n^{-3}$. \square

Applying a union bound over all pairs $s, t \in V$, we obtain $\Pr[\forall_{s, t \in V} v^{(h)st} = t] \geq 1 - n^{-1}$. The correctness of the scheme with probability $1 - n^{-1}$ follows directly from Lemma 4.

We remark that the above Proposition implies that the h -hopset problem can be efficiently approximated by finding an optimal fractional LP solution and constructing set H'' .

Theorem 4. *There exists a randomized polynomial-time $O(\text{poly log } n)$ -approximation algorithm for the h -hopset problem in unique shortest path graphs, for any $h \leq O(\text{poly log } n)$.* \square

5.3 Approximating Average Query Time for 3-Hopsets

In order to design an efficient distance oracle based on 3-hopsets, we follow the framework described in the preliminaries and use an LP-rounding technique to obtain sets $H_1 \cup H_2 =: H$. The obtained claim relies on the notion of uniform-average query time introduced in the Preliminaries.

Theorem 5. *For any feasible bound \mathcal{S} , let $H_{OPT, \mathcal{S}}$ be a 3-hopset for a unique shortest path graph, which satisfies the given bound on the number of edges $|H_{OPT, \mathcal{S}}| \leq \mathcal{S}$ and such that the uniform-average query time $\mathcal{T}(H_{OPT, \mathcal{S}})$ is minimized. Then, there exists a randomized polynomial-time algorithm which finds a 3-hopset H with $|H''| \leq O(\log^3 n)\mathcal{S}$ and $\mathcal{T}(H'') \leq O(\log^4 n)\mathcal{T}(H_{OPT, \mathcal{S}})$.* \square

Proof. In this case, for the ILP statement we associate with each edge uv a binary indicator variable $x_{uv}^{(1)} \in \{0, 1\}$ stating if $uv \in H_1$, and a second indicator variable $x_{uv} \in \{0, 1\}$, with $x_{uv} \geq x_{uv}^{(1)}$, stating if $uv \in H_1 \cup H_2$. The problem of minimizing the query time of the oracle with size bound \mathcal{S} for uniform node-pair query frequencies is now given as (compare with (5)–(8)):

Minimize:

$$\sum_{u \neq v, \{u, v\} \notin E} x_{uv}^{(1)} \quad (9)$$

Subject to:

$$\sum_{u \neq v, \{u, v\} \notin E} x_{uv} \leq \mathcal{S} \quad (10)$$

$$0 \leq f_{u_i v_j}^{st} \leq \begin{cases} x_{uv}^{(1)}, & \text{if } 2 \neq j = i + 1 \text{ and } d_G(s, u) + d_G(u, v) + d_G(v, t) = d_G(s, t), \\ x_{uv}, & \text{if } 2 = j = i + 1 \text{ and } d_G(s, u) + d_G(u, v) + d_G(v, t) = d_G(s, t), \\ 0, & \text{otherwise.} \end{cases} \quad (11)$$

$$\sum_{u_i} f_{v_j u_i}^{st} - \sum_{u_i} f_{u_i v_j}^{st} = \begin{cases} 0, & \text{for } v_j \in V_h \setminus \{s_0, t_h\} \\ +1, & \text{for } v_j = s_0 \\ -1, & \text{for } v_j = t_h \end{cases}, \quad (12)$$

and its LP relaxation on variables $x_{uv}, x_{uv}^{(1)}$ takes the form of the constraint:

$$0 \leq x_{uv}^{(1)} \leq x_{uv} \leq 1,$$

where as usual indices s, t, u, v traverse V and indices i, j traverse $\{0, 1, 2, 3\}$.

The construction of the integral hopset H'' based on the LP solution takes place as in the previous Subsection (for the case of $h = 3$), with the exception that for the first and last (third) hop, variables $x_{uv}^{(1)}$ should be used in place of x_{uv} in the construction. By an analogue of Proposition 3, we have $|H''| = O(\mathcal{S} \log^3 n)$, with high probability. We consider the natural decomposition $H'' := H''_1 \cup H''_2$ according to the number of the used hop along the path, and obtain by a similar (straightforward) concentration analysis that for all $u \in V$:

$$\deg_{H''_1}(u) \leq O(\log^2 n) \sum_{v \in V} x_{uv}^{(1)}.$$

and so, computing the sum of degrees over all u :

$$|H''_1| \leq O(\log^2 n) \sum_{v \in V} x_{uv}^{(1)}.$$

Noting that the sum on the right-hand side is precisely the minimization criterion in the LP formulation (9), we obtain the claim of the theorem. \square

We remark that the above Theorem can be directly generalized to a notion of average query time for non-uniform query densities, in which the goal is to minimize expected query time in a model in which each node $v \in V$ is assigned its relative frequency $f_v \in [0, 1]$, and a node pair uv is queried with frequency $f_u f_v$.

References

- [1] Amir Abboud, Greg Bodwin, and Seth Pettie. A hierarchy of lower bounds for sublinear additive spanners. In Klein [34], pages 568–576.
- [2] Ittai Abraham, Daniel Delling, Amos Fiat, Andrew Goldberg, and Renato Werneck. Highway dimension and provably efficient shortest path algorithms. Technical report, September 2013.
- [3] Ittai Abraham, Daniel Delling, Amos Fiat, Andrew V. Goldberg, and Renato F. Werneck. VC-dimension and shortest path algorithms. In *ICALP*, volume 6755 of *Lecture Notes in Computer Science*, pages 690–699. Springer, 2011.
- [4] Ittai Abraham, Daniel Delling, Amos Fiat, Andrew V. Goldberg, and Renato F. Werneck. Highway dimension and provably efficient shortest path algorithms. *J. ACM*, 63(5):41:1–41:26, 2016.
- [5] Ittai Abraham, Amos Fiat, Andrew V. Goldberg, and Renato F. Werneck. Highway dimension, shortest paths, and provably efficient algorithms. In Moses Charikar, editor, *Proceedings of the Twenty-First Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2010, Austin, Texas, USA, January 17-19, 2010*, pages 782–793. SIAM, 2010.
- [6] Takuya Akiba, Yoichi Iwata, and Yuichi Yoshida. Dynamic and historical shortest-path distance queries on large evolving networks by pruned landmark labeling. In Chin-Wan Chung, Andrei Z. Broder, Kyuseok Shim, and Torsten Suel, editors, *23rd International World Wide Web Conference, WWW '14, Seoul, Republic of Korea, April 7-11, 2014*, pages 237–248. ACM, 2014.
- [7] Noga Alon and Baruch Schieber. Optimal preprocessing for answering on-line product queries. Technical Report 71/87, Tel Aviv University, 1987.
- [8] Stephen Alstrup, Søren Dahlgaard, Mathias Bæk Tejs Knudsen, and Ely Porat. Sublinear distance labeling. In Piotr Sankowski and Christos D. Zaroliagis, editors, *24th Annual European Symposium on Algorithms, ESA 2016, August 22-24, 2016, Aarhus, Denmark*, volume 57 of *LIPICs*, pages 5:1–5:15. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2016.
- [9] Haris Angelidakis, Yury Makarychev, and Vsevolod Oparin. Algorithmic and hardness results for the hub labeling problem. In Philip N. Klein, editor, *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2017, Barcelona, Spain, Hotel Porta Fira, January 16-19*, pages 1442–1461. SIAM, 2017.
- [10] Srinivasa Rao Arikati, Danny Z. Chen, L. Paul Chew, Gautam Das, Michiel H. M. Smid, and Christos D. Zaroliagis. Planar spanners and approximate shortest path queries among obstacles in the plane. In Josep Díaz and Maria J. Serna, editors, *Algorithms - ESA '96, Fourth Annual European Symposium, Barcelona, Spain, September 25-27, 1996, Proceedings*, volume 1136 of *Lecture Notes in Computer Science*, pages 514–528. Springer, 1996.
- [11] H. Bast, Stefan Funke, and Domagoj Matijević. Ultrafast shortest-path queries via transit nodes. In Camil Demetrescu, Andrew V. Goldberg, and David S. Johnson, editors, *The Shortest Path Problem, Proceedings of a DIMACS Workshop, Piscataway, New Jersey, USA, November 13-14, 2006*, volume 74 of *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, pages 175–192. DIMACS/AMS, 2006.

- [12] Arnab Bhattacharyya, Elena Grigorescu, Kyomin Jung, Sofya Raskhodnikova, and David P. Woodruff. Transitive-closure spanners. *SIAM J. Comput.*, 41(6):1380–1425, 2012.
- [13] Hans L. Bodlaender, Gerard Tel, and Nicola Santoro. Trade-offs in non-reversing diameter. *Nord. J. Comput.*, 1(1):111–134, 1994.
- [14] Sergio Cabello. Many distances in planar graphs. *Algorithmica*, 62(1-2):361–381, 2012.
- [15] Shiva Chaudhuri and Christos D. Zaroliagis. Shortest paths in digraphs of small treewidth. part I: sequential algorithms. *Algorithmica*, 27(3):212–226, 2000.
- [16] Bernard Chazelle. Computing on a free tree via complexity-preserving mappings. *Algorithmica*, 2:337–361, 1987.
- [17] Shiri Chechik. Approximate distance oracles with constant query time. In David B. Shmoys, editor, *Symposium on Theory of Computing, STOC 2014, New York, NY, USA, May 31 - June 03, 2014*, pages 654–663. ACM, 2014.
- [18] Danny Z. Chen and Jinhui Xu. Shortest path queries in planar graphs. In F. Frances Yao and Eugene M. Luks, editors, *Proceedings of the Thirty-Second Annual ACM Symposium on Theory of Computing, May 21-23, 2000, Portland, OR, USA*, pages 469–478. ACM, 2000.
- [19] Edith Cohen. Using selective path-doubling for parallel shortest-path computations. *J. Algorithms*, 22(1):30–56, 1997.
- [20] Edith Cohen. Polylog-time and near-linear work approximation scheme for undirected shortest paths. *J. ACM*, 47(1):132–166, 2000.
- [21] Edith Cohen, Eran Halperin, Haim Kaplan, and Uri Zwick. Reachability and distance queries via 2-hop labels. *SIAM J. Comput.*, 32(5):1338–1355, May 2003.
- [22] Vincent Cohen-Addad, Søren Dahlgaard, and Christian Wulff-Nilsen. Fast and compact exact distance oracle for planar graphs. In Chris Umans, editor, *58th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2017, Berkeley, CA, USA, October 15-17, 2017*, pages 962–973. IEEE Computer Society, 2017.
- [23] Daniel Delling, Andrew V. Goldberg, Thomas Pajor, and Renato F. Werneck. Robust distance queries on massive networks. In Andreas S. Schulz and Dorothea Wagner, editors, *Algorithms - ESA 2014 - 22th Annual European Symposium, Wroclaw, Poland, September 8-10, 2014. Proceedings*, volume 8737 of *Lecture Notes in Computer Science*, pages 321–333. Springer, 2014.
- [24] Hristo Djidjev. On-line algorithms for shortest path problems on planar digraphs. In Fabrizio d’Amore, Paolo Giulio Franciosa, and Alberto Marchetti-Spaccamela, editors, *Graph-Theoretic Concepts in Computer Science, 22nd International Workshop, WG ’96, Cadenabbia (Como), Italy, June 12-14, 1996, Proceedings*, volume 1197 of *Lecture Notes in Computer Science*, pages 151–165. Springer, 1996.
- [25] Michael Elkin and Ofer Neiman. Hopsets with constant hopbound, and applications to approximate shortest paths. In Irit Dinur, editor, *IEEE 57th Annual Symposium on Foundations of Computer Science, FOCS 2016, 9-11 October 2016, Hyatt Regency, New Brunswick, New Jersey, USA*, pages 128–137. IEEE Computer Society, 2016.
- [26] Jittat Fakcharoenphol and Satish Rao. Planar graphs, negative weight edges, shortest paths, and near linear time. *J. Comput. Syst. Sci.*, 72(5):868–889, 2006.

- [27] Arash Farzan and Shahin Kamali. Compact navigation and distance oracles for graphs with small treewidth. *Algorithmica*, 69(1):92–116, 2014.
- [28] Cyril Gavoille, David Peleg, Stéphane Pérennes, and Ran Raz. Distance labeling in graphs. *J. Algorithms*, 53(1):85–112, October 2004.
- [29] Pawel Gawrychowski, Adrian Kosowski, and Przemyslaw Uznanski. Sublinear-space distance labeling using hubs. In Cyril Gavoille and David Ilcinkas, editors, *Distributed Computing - 30th International Symposium, DISC 2016, Paris, France, September 27-29, 2016. Proceedings*, volume 9888 of *Lecture Notes in Computer Science*, pages 230–242. Springer, 2016.
- [30] Pawel Gawrychowski, Shay Mozes, Oren Weimann, and Christian Wulff-Nilsen. Better tradeoffs for exact distance oracles in planar graphs. In Artur Czumaj, editor, *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2018, New Orleans, LA, USA, January 7-10, 2018*, pages 515–529. SIAM, 2018.
- [31] Robert Geisberger, Peter Sanders, Dominik Schultes, and Daniel Delling. Contraction hierarchies: Faster and simpler hierarchical routing in road networks. In Catherine C. McGeoch, editor, *Experimental Algorithms, 7th International Workshop, WEA 2008, Provincetown, MA, USA, May 30-June 1, 2008, Proceedings*, volume 5038 of *Lecture Notes in Computer Science*, pages 319–333. Springer, 2008.
- [32] Andrew V. Goldberg, Haim Kaplan, and Renato F. Werneck. Reach for A*: Efficient point-to-point shortest path algorithms. In *ALLENEX*, pages 129–143. SIAM, 2006.
- [33] Ronald J. Gutman. Reach-based routing: A new approach to shortest path algorithms optimized for road networks. In *ALLENEX/ANALCO*, pages 100–111. SIAM, 2004.
- [34] Philip N. Klein, editor. *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2017, Barcelona, Spain, Hotel Porta Fira, January 16-19*. SIAM, 2017.
- [35] Philip N. Klein and Sairam Subramanian. A randomized parallel algorithm for single-source shortest paths. *J. Algorithms*, 25(2):205–220, 1997.
- [36] Adrian Kosowski and Laurent Viennot. Beyond highway dimension: Small distance labels using tree skeletons. In Klein [34], pages 1462–1478.
- [37] David Peleg and Alejandro A. Schäffer. Graph spanners. *Journal of Graph Theory*, 13(1):99–116, 1989.
- [38] Prabhakar Raghavan and Clark D. Thompson. Randomized rounding: a technique for provably good algorithms and algorithmic proofs. *Combinatorica*, 7(4):365–374, 1987.
- [39] Hanmao Shi and Thomas H. Spencer. Time-work tradeoffs of the single-source shortest paths problem. *J. Algorithms*, 30(1):19–32, 1999.
- [40] Mikkel Thorup. Shortcutting planar digraphs. *Combinatorics, Probability & Computing*, 4:287–315, 1995.
- [41] Mikkel Thorup. Parallel shortcutting of rooted trees. *J. Algorithms*, 23(1):139–159, 1997.
- [42] Jeffrey D. Ullman and Mihalis Yannakakis. High-probability parallel transitive closure algorithms. In *SPAA*, pages 200–209, 1990.