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Maximum likelihood drift estimation for a threshold diffusion

Antoine Lejay* and Paolo Pigato†

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We study the maximum likelihood estimator of the drift parameters of a stochastic differential equation, with both drift and diffusion coefficients constant on the positive and negative axis, yet discontinuous at zero. This threshold diffusion is called drifted Oscillating Brownian motion.

For this continuously observed diffusion, the maximum likelihood estimator coincide with a quasi-likelihood estimator with constant diffusion term. We show that this estimator is the limit, as observations become dense in time, of the (quasi)-maximum likelihood estimator based on discrete observations.

In long time, the asymptotic behaviors of the positive and negative occupation times rule the ones of the estimators.

Differently from most known results in the literature, we do not restrict ourselves to the ergodic framework: indeed, depending on the signs of the drift, the process may be ergodic, transient or null recurrent. For each regime, we establish whether or not the estimators are consistent; if they are, we prove the convergence in long time of the properly rescaled difference of the estimators towards a normal or mixed normal distribution. These theoretical results are backed by numerical simulations.

Keywords: Threshold diffusion, Oscillating Brownian motion, maximum likelihood estimator, null recurrent process, ergodic process, transient process, mixed normal distribution

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1 Introduction

We consider the process, called a *drifted Oscillating Brownian motion* (DOBM), which is the solution to the Stochastic Differential equation (SDE)

$$\xi_t = \xi_0 + \int_0^t \sigma(\xi_s) dW_s + \int_0^t b(\xi_s) ds, \quad (1)$$

with

$$\sigma(x) = \begin{cases} \sigma_+ > 0 & \text{if } x \geq 0, \\ \sigma_- > 0 & \text{if } x < 0 \end{cases} \quad \text{and} \quad b(x) = \begin{cases} b_+ \in \mathbb{R} & \text{if } x \geq 0, \\ b_- \in \mathbb{R} & \text{if } x < 0. \end{cases} \quad (2)$$

The strong existence to (1) follows for example from the results of [29]. Separately on \mathbb{R}_+ and \mathbb{R}_- , the dynamics of such process is the one of a Brownian motion with drift, with threshold and regime-switch at 0, consequence of the discontinuity of the coefficients.

This model can be seen as an alternative to the model studied in [40], which is a continuous time version of the Self-Exciting Threshold Autoregressive models (SETAR), a subclass of the TAR models [49, 50].

The practical applications of such processes are numerous. In finance, we show in [31] that an exponential form of this process generalizes the Black & Scholes model in a way to model leverage effects. Moreover, the introduction of a piecewise constant drift such as the one in (2) is a straightforward way to produce a mean-reverting process, if $b_+ < 0$ and $b_- > 0$. In [31], we find some empirical evidence on financial data that this may be the case. This corroborates other studies with different models [39, 42, 47, 48]. Option pricing for this model may be performed efficiently using analytic techniques as in [13, 36, 37].

Still in finance, the solution to (1) models other quantities than stocks. In [14], Eq. (1) with constant volatility serves as a model for the surplus of a company after the payment of dividends, which are paid only if the profits of the company are higher than a certain threshold. Similar threshold dividend pay-out strategies are considered in [2]. In these works, the behavior of the process at the discontinuity is referred to as “refraction”. SETAR models have also applications to deal with transaction costs or regulator interventions [53] and to interest and exchange rates [7, 10].

More general discontinuous drifts and volatilities arise in presence of Atlas models and other ranks based models [18]. SDE with discontinuous coefficients have also numerous applications in physics [43, 46], meteorology [16] and many other domains.

In [47, 48], F. Su and K.-S. Chan study the asymptotic behavior of the quasi-likelihood estimator of a diffusion with piecewise regular diffusivity and piecewise affine drift with an unknown threshold. The quasi-likelihood they use is based on the Girsanov density where the diffusivity is replaced by 1. In particular, they provide a hypothesis test to decide whether or not the drift is affine or piecewise affine in the ergodic situation.

In [27], Y. Kutoyants considers the estimation of a threshold r of a diffusion with a known or unknown drift switching at r . His results are then specialized to Ornstein-Uhlenbeck type processes. Also this framework assumes that the diffusion is ergodic.

In the present paper, we first derive the maximum likelihood estimators for the drift parameters b_- and b_+ from continuous-time observations. We then write a quasi-maximum likelihood estimator, as in [47, 48], and show that, for a process ξ as in (1), these two estimators are actually equal. Then, we derive the corresponding estimators based on discrete observations, and show that as the observations become dense in time, the estimator converges to the one based on continuous observations.

We study the asymptotic behavior of the maximum likelihood estimators as the time tends to infinity, in order to derive some confidence intervals when available. This article completes [32], where we estimate (σ_-, σ_+) from high-frequency data. We use our estimators on financial historical data in [31].

Based on the maximum and quasi-maximum likelihood, our estimators of b_{\pm} can be expressed as

$$\beta_T^{\pm} = \pm \frac{(\pm \xi_T) \vee 0 - (\pm \xi_0) \vee 0 - L_T(\xi)/2}{Q_T^{\pm}},$$

where Q_T^+ (resp. Q_T^-) is the occupation time of the positive (resp. negative) side of the real axis, and $L_T(\xi)$ is the symmetric local time of ξ at 0. The occupation times Q_T^{\pm} can be estimated from discrete observations of a trajectory of ξ using Riemann sums. We also propose a new estimator of $L_T(\xi)$ from discrete observations, which can be implemented without any previous knowledge of σ_{\pm} . Substituting in the formula above Q^{\pm} and $L_T(\xi)$ with these discrete counterparts, we obtain the (quasi)-maximum likelihood estimators from discrete observations. Since all these quantities converge to their continuous-time versions as the observations become dense in time, the estimators for the drift coefficients converge as well.

The long time asymptotic regime of the process depends on the respective signs of the coefficients (b_-, b_+) . Using symmetries, this leads 5 different cases in which the process may be ergodic, null recurrent or transient and the estimators have different asymptotic behaviors. In some situations, the estimators are not convergent. In others, we establish consistency as well as Central Limit Theorems, with speed $T^{1/2}$ or $T^{1/4}$, depending again on the signs of b_{\pm} . We summarize in Table 1 the various asymptotic behaviors. We are in a situation close to the one encountered by M. Ben Alaya and A. Kebaier in [4, 5] for estimating square-root diffusions, where several situations shall be treated. The works [27, 47, 48] mentioned above only consider ergodic situations. Non-parametric estimation of the drift in the recurrent case is considered in [3].

Finally, we develop in Section 7.1 a hypothesis test for the value of the drift which is based on the Wilk's theorem, which relates asymptotically the log-likelihood to a χ^2 distribution with 2 degrees of freedom. Besides, we show in Section 7.2 the Local Asymptotic Normality (LAN [28, 30]) and the Local Asymptotic Mixed Normality (LAMN [23]) in the ergodic case and the null recurrent case with non vanishing drift. These LAN/LAMN properties are related to the efficiency of the operators. The Wilk's as well as the LAN/LAMN properties are proved by combining the quadratic nature of the log-likelihood with our martingale central limit theorems.

Outline. In Section 2, we present the maximum and quasi-maximum likelihood estimators, which are based on the Girsanov transform, and their discrete-time versions. In Section 3, we characterize the different regimes of the process accordingly

			$\beta_T^+ - b_+$	$\beta_T^- - b_-$
(E)	$b_+ < 0, b_- > 0$	ergodic	$\approx \frac{1}{\sqrt{T}} \sqrt{1 - \frac{b_+}{b_-}} \sigma_+ \mathcal{N}$	$\approx \frac{1}{\sqrt{T}} \sqrt{1 - \frac{b_-}{b_+}} \sigma_- \mathcal{N}$
(N0)	$b_+ = 0, b_- = 0$	null recurrent	$\frac{1}{\sqrt{T}} \beta_1^+$	$\frac{1}{\sqrt{T}} \beta_1^-$
(N1)	$b_+ = 0, b_- > 0$	null recurrent	$\approx \frac{1}{\sqrt{T}} \sigma_+ \mathcal{N}^+$	$\approx \frac{1}{T^{1/4}} \sigma_- \frac{\sqrt{b_-}}{\sqrt{\sigma_+}} \frac{\mathcal{N}^-}{\sqrt{ \mathcal{N} }}$
(T0)	$b_+ > 0, b_- \geq 0$	transient	$\approx \frac{1}{\sqrt{T}} \sigma_+ \mathcal{N}$	\mathcal{R}_{T0} as $T \rightarrow \infty$
(T1)	$b_+ > 0, b_- < 0$	transient	$\approx \frac{1}{\sqrt{T}} \sigma_+ \mathcal{N}$	\mathcal{R}_{T1}^+ as $T \rightarrow \infty$

Table 1: Asymptotic behavior of estimators, where \mathcal{N} , \mathcal{N}^+ and \mathcal{N}^- are independent, unit Gaussian variables. The law of (β_1^-, β_1^+) in case **(N0)** is given in (36). The r.v.s \mathcal{R}_{T0} and \mathcal{R}_{T1}^+ follow the law in (33). Results of both sides in **(T1)** are wrt to \mathbb{P}_+ (cf. Proposition 7), which intuitively can be thought as conditioning to the process diverging towards positive infinity.

to the signs of the drifts. Our main results are presented in Section 4. The limit theorems that we use are presented in Section 5. The proofs for each cases are detailed in Section 6. We present the Wilk theorem and the LAN/LAMN property in Section 7. Finally, in Section 8, we conclude this article with numerical experiments.

2 Maximum and quasi-maximum likelihood estimators

In this section, we construct two estimators for the parameters (b_-, b_+) of the drift coefficient of ξ . The maximum likelihood estimator based on continuous time observations can be constructed if we assume to observe a continuous time path $(\xi_t)_{t \in [0, T]}$. A quasi-maximum likelihood estimator can also be constructed, and from the fact that $b(\cdot)$ and $\sigma(\cdot)$ are constant above and below the threshold, it follows that these estimators are in fact the same one. Then, we construct the analogous estimator based on discrete time observations of the process. In the end we will see that, as observations become dense in time, the discrete time estimator converges to the one in continuous time.

2.1 The maximum and quasi-maximum likelihood estimator based on continuous time observations

First, we propose and discuss the maximum likelihood estimator from continuous time observations.

Data 1. We observe a path $(\xi_t)_{t \in [0, T]}$ on the time interval $[0, T]$ of one solution to (1), together with its negative and positive occupation times

$$Q_T^- = \int_0^T 1_{\xi_s < 0} ds \text{ and } Q_T^+ = \int_0^T 1_{\xi_s > 0} ds, \quad (3)$$

as well as its symmetric local time

$$L_T(\xi) = \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_0^T 1_{-\epsilon \leq \xi_s \leq \epsilon} d\langle \xi \rangle_s.$$

The question of the discretization of Q_T^\pm as well as $L_T(\xi)$ is detailed in Sections 2.2 and 8.1.

We define

$$M^\pm := \int_0^T \sigma_\pm 1_{\pm \xi_s \geq 0} dW_s \quad (4)$$

$$\text{and } R_T^\pm := \int_0^T 1_{\pm \xi_s \geq 0} d\xi_s. \quad (5)$$

The quantities M^\pm are continuous time martingales with $\langle M^\pm \rangle = \sigma_\pm^2 Q^\pm$ and $\langle M^+, M^- \rangle = 0$. Moreover,

$$R_T^\pm = M_T^\pm + b_\pm Q_T^\pm. \quad (6)$$

Noticing that the occupation time is non decreasing, for the sake of simplicity we write

$$Q_\infty^\pm := \lim_{T \rightarrow \infty} Q_T^\pm \in \mathbb{R}_+ \cup \{\infty\}.$$

The Girsanov weight of the distribution of (1) with respect the driftless ($b_+ = b_- = 0$) SDE is

$$G_T(b_+, b_-) = \exp \left(\int_0^T \frac{b(\xi_s)}{\sigma^2(\xi_s)} d\xi_s - \frac{1}{2} \int_0^T \frac{b^2(\xi_s)}{\sigma^2(\xi_s)} ds \right). \quad (7)$$

With the expression of Q^\pm and R^\pm given by (3) and (5), this Girsanov weight can be expressed simply as

$$G_T(b_+, b_-) = \exp \left(\frac{b_+}{\sigma_+^2} R_T^+ + \frac{b_-}{\sigma_-^2} R_T^- - \frac{b_+^2}{\sigma_+^2} Q_T^+ - \frac{b_-^2}{\sigma_-^2} Q_T^- \right).$$

A reasonable way to set up an estimator of (b_-, b_+) is to consider $G_T(b_+, b_-)$ as a likelihood and to optimize this quantity. This is how estimators for the drift are classically constructed [26, 35]. We also set the following function, defined similarly as in [47] as

$$\begin{aligned} \Lambda_T(b_+, b_-) &= \int_0^T b(\xi_s) d\xi_s - \frac{1}{2} \int_0^T b(\xi_s)^2 ds \\ &= b_+ R_T^+ + b_- R_T^- - \frac{1}{2} b_+^2 Q_T^+ - \frac{1}{2} b_-^2 Q_T^-. \end{aligned}$$

Such function can be interpreted as a quasi-likelihood function, and has the advantage wrt (7) of not involving the diffusion coefficient. This approach can be used also when a specific functional form for the diffusion coefficient σ is not specified.

Notation 1. To avoid confusion with the $+$ and $-$ used as indices, we write $(x)^+ := \max\{x, 0\}$ for the positive part of x and $(x)^- := \max\{-x, 0\} \geq 0$ for the negative part.

Proposition 1. *Both the likelihood $G_T(b_+, b_-)$ and the quasi-likelihood $\Lambda_T(b_+, b_-)$ are maximal at (β_T^+, β_T^-) given by*

$$\beta_T^+ = \frac{R_T^+}{Q_T^+} = b_+ + \frac{M_T^+}{Q_T^+} \text{ and } \beta_T^- = \frac{R_T^-}{Q_T^-} = b_- + \frac{M_T^-}{Q_T^-}.$$

Estimators (β_T^+, β_T^-) can also be expressed as

$$\beta_T^\pm = \pm \frac{(\xi_T)^\pm - (\xi_0)^\pm - L_T(\xi)/2}{Q_T^\pm}. \quad (8)$$

Proof. The equation $\nabla \Lambda_T(b_+, b_-) = 0$ is solved for (β_T^+, β_T^-) as above. Since $Q_T^\pm \geq 0$, $\Lambda_T(b_+, b_-)$ is not only critical but also maximal at (β_T^+, β_T^-) with $\beta_T^\pm = R_T^\pm / Q_T^\pm$. With (6), we obtain $\beta_T^\pm = b_\pm + M_T^\pm / Q_T^\pm$. The same holds for $\log G_T(b_+, b_-)$.

By the Itô-Tanaka formula,

$$(\xi)_T^+ = (\xi)_0^+ + \int_0^t 1_{\xi_s \geq 0} d\xi_s + \frac{1}{2} L_T(\xi) \text{ and } (\xi)_T^- = (\xi)_0^- - \int_0^t 1_{\xi_s \leq 0} d\xi_s + \frac{1}{2} L_T(\xi).$$

Hence (8). \square

2.2 An estimator based on discrete observations

Within the framework of Data 1, we constructed estimator β in Proposition (1). In practice, however, one cannot observe the whole trajectory $(\xi_t)_{t \in [0, T]}$. We explain in Section 8 how the quantities involved in the estimator expressed as in (8) can be approximated from discrete observations of the process. Hence, an approximation of the estimators β_T^\pm may be constructed from the discrete observations $(\xi_{iT/N})_{i=0, \dots, N}$.

We now consider an alternative framework, in which we suppose from the beginning that the path is observed only on a discrete time grid.

Data 2. *We observe of a path $(\xi_{iT/N})_{i=0, \dots, N}$ at discrete times.*

Owing to the expressions for $G_T(b_+, b_-)$ and $\Lambda_T(b_+, b_-)$, we discretize the stochastic integrals as well as the occupation times. With $\xi_i := \xi_{iT/N}$ and $\Delta_i \xi = \xi_{i+1} - \xi_i$, we set

$$Q_{T,N}^\pm := \frac{T}{n} \sum_{i=0}^{N-1} 1_{\xi_i \geq 0} \text{ and } R_{T,N}^\pm := \sum_{i=0}^{N-1} 1_{\pm \xi_i \geq 0} \Delta_i \xi \quad (9)$$

which are discretized versions of Q_T^\pm and R_T^\pm . We also define the *discretized likelihood* as

$$G_{T,N}(b_+, b_-) = \exp \left(\sum_{i=0}^{N-1} \frac{b(\xi_i)}{\sigma^2(\xi_i)} \Delta_i \xi - \frac{1}{2N} \sum_{i=0}^{N-1} \frac{b^2(\xi_i)}{\sigma^2(\xi_i)} \right).$$

and the *discretized quasi-likelihood* as

$$\Lambda_{T,N}(b_+, b_-) = \sum_{i=0}^{N-1} b(\xi_i) \Delta_i \xi - \frac{1}{2N} \sum_{i=0}^{N-1} b(\xi_i)^2.$$

We also define the following discrete-time approximation of the local time:

$$L_{T,N} := 2 \sum_{i=0}^{N-1} 1_{\xi_i \xi_{i+1} < 0} |\xi_{i+1}|. \quad (10)$$

We have a result similar to the one of Proposition 1.

Proposition 2. *The likelihood $\mathbf{G}_{T,N}(b_+, b_-)$ and the quasi-likelihood $\Lambda_{T,N}(b_+, b_-)$ are maximal at $(b_+, b_-) = (\beta_{T,N}^+, \beta_{T,N}^-)$, with*

$$\beta_{T,N}^+ := \frac{R_{T,N}^+}{Q_{T,N}^+} \text{ and } \beta_{T,N}^- := \frac{R_{T,N}^-}{Q_{T,N}^-}. \quad (11)$$

We also have the following discrete version of (8):

$$\beta_{T,N}^\pm = \pm \frac{(\xi)_T^\pm - (\xi)_0^\pm - L_{T,N}/2}{Q_{T,N}^\pm}. \quad (12)$$

Proof. The fact that $\mathbf{G}_{T,N}(b_+, b_-)$ and $\Lambda_{T,N}(b_+, b_-)$ are maximal at $\beta_{T,N}^\pm$ is trivial. We note that

$$1_{\xi_i \geq 0} \Delta_i \xi = (\xi_{i+1})^+ - (\xi_i)^+ - 1_{\xi_i \xi_{i+1} < 0} |\xi_{i+1}|$$

so that

$$R_{T,N}^+ = (\xi)_T^+ - (\xi)_0^+ - \frac{1}{2} L_{T,N}. \quad (13)$$

Similarly, one can show that

$$R_{T,N}^- = -(\xi)_T^- + (\xi)_0^- + \frac{1}{2} L_{T,N}. \quad (14)$$

This proves (12). \square

Lemma 1. *For any $T > 0$, $Q_{T,N}^\pm$ converges in probability to Q_T^\pm and*

$$R_{T,N}^\pm \xrightarrow[N \rightarrow \infty]{\mathbb{P}} R_T^\pm = \int_0^T 1_{\pm \xi_s \geq 0} d\xi_s = \pm \left((\xi_T)^\pm - (\xi_0)^\pm - \frac{1}{2} L_T(\xi) \right). \quad (15)$$

In particular, $\beta_{T,N}^\pm$ converges in probability to β_T^\pm as $N \rightarrow \infty$.

Proof. As we are considering a convergence in probability, due to the Girsanov theorem, we may assume that $b_+ = b_- = 0$. For the sake of simplicity, we assume that $\xi_0 = 0$. The convergence of $Q_{T,N}^\pm$ to Q_T^\pm follows from [33, Theorem 4.14, p. 3587].

We consider now only the convergence of $R_{T,N}^+$ to R_T^+ , the computations being the same for the “negative” part.

Besides, we know that $\xi_t = \sigma(X_t)X_t$, where X is a Skew Brownian motion, *i.e.*, the solution to the SDE $X_t = W_t + \theta L_t(X)$, where W is a Brownian motion, $L_t(X)$ is the local time of X at 0 and $\theta = (\sigma_- - \sigma_+)/(\sigma_- + \sigma_+)$. The local times $L(\xi)$ and $L(X)$ are linked by (see [32])

$$L(X) = \frac{\sigma_+ + \sigma_-}{2\sigma_+\sigma_-} L(\xi). \quad (16)$$

The density of the skew Brownian motion is

$$p_\theta(t, x, y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-y)^2}{2t}\right) + \theta \operatorname{sgn}(y) \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(|x| + |y|)^2}{2t}\right).$$

Using this transform, with $f(x, y) := \sigma(y)1_{xy < 0}|y|$,

$$\frac{\mathbf{L}_{T,N}}{2} = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} 1_{NX_i X_{i+1} < 0} \sqrt{N} |X_{i+1}| \sigma(X_{i+1}) = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} f(\sqrt{N} X_i, \sqrt{N} X_{i+1}).$$

Following [33], which adapts to the Skew Brownian motion some results of [21], we define

$$F(x) := \int_{-\infty}^{+\infty} p_\theta(1, x, y) f(x, y) dy.$$

Hence, for $x < 0$,

$$\begin{aligned} F(x) &= \sigma_+(1 + \theta) \int_0^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-y)^2}{2}\right) y dy \\ &= \sigma_+ \frac{(1 + \theta)}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) + \sigma_+(1 + \theta)x \bar{\Phi}(-x) \end{aligned}$$

with $\bar{\Phi}(x) = \int_x^{+\infty} \exp(-z^2)/\sqrt{2\pi} dz$. Similarly, for $x > 0$,

$$F(x) = +\sigma_- \frac{(1 - \theta)}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) - \sigma_-(1 - \theta)x \bar{\Phi}(x).$$

Let us define

$$C := (1 + \theta) \int_0^{+\infty} F(x) dx + (1 - \theta) \int_{-\infty}^0 F(x) dx = (1 - \theta^2) \frac{\sigma_- + \sigma_+}{4}.$$

Injecting the value of θ ,

$$C = \frac{\sigma_- \sigma_+}{\sigma_- + \sigma_+}. \quad (17)$$

From [33, Proposition 2], using (16) and (17),

$$\frac{1}{2} \mathbf{L}_{T,N} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} CL_T(X) = \frac{1}{2} L_T(\xi).$$

This, with (13), shows (15). \square

3 Analytic characterization of the regime of the process

3.1 Scale function and speed measure

A well known fact [20, 24, 45] states that the infinitesimal generator $(\mathcal{L}, \text{Dom}(\mathcal{L}))$ of the process ξ solution to (1) may be written as

$$\begin{aligned} \mathcal{L}f &= \frac{1}{2} \sigma^2(x) e^{-h(x)} \frac{d}{dx} \left(e^{h(x)} \frac{df(x)}{dx} \right) \quad \text{with } h(x) = \int_0^x \frac{2b(y)}{\sigma^2(y)} dy \\ &\text{for all } f \in \text{Dom}(\mathcal{L}) = \{f \in \mathcal{C}_0(\mathbb{R}) \mid \mathcal{L}f \in \mathcal{C}_0(\mathbb{R})\}. \end{aligned}$$

The process X is fully characterized by its *speed measure* M with a density m and its *scale function* S with

$$m(x) := \frac{2}{\sigma(x)^2} \exp(h(x)) \quad \text{and} \quad S(x) := \int_0^x \exp(-h(y)) dy. \quad (18)$$

	$b_+ > 0$	$b_+ = 0$	$b_+ < 0$
$b_- > 0$	transient T0	null recurrent N1	ergodic E
$b_- = 0$	transient T0	null recurrent N0	null recurrent N1
$b_- < 0$	transient T1	transient T0	transient T0

Table 2: Recurrence and transience properties of ξ .

3.2 The regimes of the process

The diffusion X is either recurrent or transient. If $\lim_{x \rightarrow +\infty} S(x) = +\infty$ and $\lim_{x \rightarrow -\infty} S(x) = -\infty$, then the process is (positively or null) *recurrent*. Otherwise, it is *transient* [20, 24].

When $b(x) = b_+$ for $x \geq 0$,

$$S(x) = \begin{cases} x & \text{if } b_+ = 0, \\ \frac{\sigma_+^2}{2b_+} \left(1 - \exp\left(-\frac{2b_+x}{\sigma_+^2}\right) \right) & \text{if } b_+ > 0, \\ \frac{\sigma_+^2}{2|b_+|} \left(\exp\left(\frac{2|b_+|x}{\sigma_+^2}\right) - 1 \right) & \text{if } b_+ < 0. \end{cases}$$

Similar formulas hold for b_- . Hence, the process ξ is transient if only if $b_+ > 0$ or $b_- < 0$.

A recurrent process is either *null recurrent* or *positive recurrent*. The process is positive recurrent if and only if $M(\mathbb{R}) := \int_{\mathbb{R}} m(x) dx < +\infty$, in which case it is actually *ergodic*. Therefore, the process ξ is ergodic if and only if $b_+ < 0$ and $b_- > 0$. Otherwise, the process ξ is only null recurrent.

When the process is ergodic ($b_+ < 0$, $b_- > 0$), its invariant measure is

$$\frac{m(x)}{M(\mathbb{R})} dx = \begin{cases} \frac{1}{\sigma_+^2} \times \frac{|b_+|b_-}{b_- + |b_+|} e^{-\frac{2x|b_+|}{\sigma_+^2}} & \text{if } x \geq 0, \\ \frac{1}{\sigma_-^2} \times \frac{b_-|b_+|}{b_- + |b_+|} e^{\frac{2xb_-}{\sigma_-^2}} & \text{if } x < 0. \end{cases}$$

Therefore, the regime of ξ depends only on the respective signs of b_+ and b_- . Nine combinations are possible. As some cases are symmetric, we actually consider five cases exhibiting different asymptotic behaviors of Q_T^\pm , hence of the estimators. This is summarized in Table 2.

These cases are:

- E)** Ergodic case $b_+ < 0$, $b_- > 0$.
- N0)** Null recurrent case $b_+ = 0$, $b_- = 0$.
- N1)** Null recurrent case $b_+ = 0$, $b_- > 0$.
- T0)** Transient case $b_+ > 0$, $b_- \geq 0$.
- T1)** Transient case $b_+ > 0$, $b_- < 0$.

Case **T0** corresponds to two entries of table 2. The case $b_+ < 0$, $b_- = 0$ is symmetric to **N1**. Case $b_+ \leq 0$, $b_- < 0$ is symmetric to **T0**.

4 Asymptotic behavior of the estimators

In this section, we state our main results on the asymptotic behavior of the occupation times of the process and the corresponding ones of the estimators, for each of the 5 cases.

Proposition 3 (Ergodic case **E**). *If $b_+ < 0, b_- > 0$, then*

$$\left(\frac{Q_T^+}{T}, \frac{Q_T^-}{T} \right) \xrightarrow[T \rightarrow \infty]{\text{a.s.}} \left(\frac{|b_-|}{|b_-| + |b_+|}, \frac{|b_+|}{|b_-| + |b_+|} \right).$$

In addition,

$$\begin{aligned} & (\beta_T^+, \beta_T^-) \xrightarrow[T \rightarrow \infty]{\text{a.s.}} (b_+, b_-) \\ \text{and } & \frac{\sqrt{T}}{\sqrt{|b_-| + |b_+|}} (\beta_T^+ - b_+, \beta_T^- - b_-) \xrightarrow[T \rightarrow \infty]{\text{law}} \left(\frac{\sigma_+}{\sqrt{|b_-|}} \mathcal{N}^+, \frac{\sigma_-}{\sqrt{|b_+|}} \mathcal{N}^- \right), \end{aligned}$$

where \mathcal{N}^+ and \mathcal{N}^- are two independent, unit Gaussian random variables.

Proposition 4 (Null recurrent case with vanishing drift **N0**). *Assume $b_+ = b_- = 0$. Assume $\xi_0 = 0$. Then*

$$\left(\frac{Q_T^+}{T}, \frac{Q_T^-}{T} \right) \stackrel{\text{law}}{=} (\Lambda, 1 - \Lambda) \text{ for all } T > 0,$$

where Λ follows a law of arcsine type with density

$$p_\Lambda(u) := \frac{1}{\pi} \frac{1}{\sqrt{u(1-u)}} \frac{\sigma_+/\sigma_-}{1 - (1 - (\sigma_+/\sigma_-)^2)u} \quad \text{for } 0 < u < 1.$$

Besides,

$$\sqrt{T}(\beta_T^+, \beta_T^-) \stackrel{\text{law}}{=} (\beta_1^+, \beta_1^-) \quad (19)$$

where the explicit joint density of (β_1^+, β_1^-) is given by (36) below. In particular, (β_T^+, β_T^-) converges almost surely to $(b_+, b_-) = (0, 0)$.

Proposition 5 (Null recurrent case with non-vanishing drift **N1**). *Assume $b_+ = 0, b_- > 0$. Then*

$$\frac{Q_T^+}{T} \xrightarrow[T \rightarrow \infty]{\text{a.s.}} 1 \text{ and } (\beta_T^+, \beta_T^-) \xrightarrow[T \rightarrow \infty]{\text{a.s.}} (b_+, b_-).$$

In addition, there exists three independent unit Gaussian random variables $\mathcal{N}^-, \mathcal{N}^+$ and \mathcal{N} such that

$$\left(\frac{Q_T^-}{\sqrt{T}}, \sqrt{T}(\beta_T^+ - b_+), T^{1/4}(\beta_T^- - b_-) \right) \xrightarrow[T \rightarrow \infty]{\text{law}} \left(\frac{\sigma_+}{b_-} |\mathcal{N}|, \sigma_+ \mathcal{N}^+, \sigma_- \frac{\sqrt{b_-}}{\sqrt{\sigma_+}} \cdot \frac{\mathcal{N}^-}{\sqrt{|\mathcal{N}|}} \right). \quad (20)$$

Remark 1. This case exhibits two different rates of convergence. This is due to the fact below the threshold that the particle is pushed up, but only behave like a Brownian motion when above. Below the threshold, we are in a similar situation as for the positive recurrent case, while above it is like the null recurrent case.

Proposition 6 (Transient case for upward drift **T0**). Assume $b_+ > 0$, $b_- \geq 0$ so that the process ξ is transient and $\lim_{T \rightarrow \infty} \xi_T = +\infty$. Then Q_T^+/T converges almost surely to 1 as $T \rightarrow \infty$ and

$$\beta_T^+ \xrightarrow[T \rightarrow \infty]{\text{a.s.}} b_+ \text{ and } \sqrt{T}(\beta_T^+ - b_+) \xrightarrow[T \rightarrow \infty]{\text{law}} \sigma_+ \mathcal{N}^+ \quad (21)$$

for a unit Gaussian random variable \mathcal{N}^+ . Let ℓ_0 be the last passage time to 0, which is almost surely finite. Assume $\xi_0 = 0$. We have

$$\beta_T^- 1_{T > \ell_0} = \mathcal{R}_{\mathbf{T0}} 1_{T > \ell_0} \text{ and } \lim_{T \rightarrow \infty} \beta_T^- = \mathcal{R}_{\mathbf{T0}} \text{ a.s. with } \mathcal{R}_{\mathbf{T0}} := \frac{L_\infty(\xi)}{2Q_{\ell_0}^-} = \frac{L_\infty(\xi)}{2Q_\infty^-}. \quad (22)$$

The density of $\mathcal{R}_{\mathbf{T0}}$ is given by (33) below. The case $b_+ \leq 0$, $b_- < 0$ is treated by symmetry.

Proposition 7 (Transient case for diverging drift **T1**). Assume $b_+ > 0$, $b_- < 0$ so that the process ξ is transient. Assume that $\xi_0 = 0$. Then there exists a Bernoulli random variable $\mathcal{B} \in \{0, 1\}$ such that

$$\begin{aligned} \mathbb{P}(\mathcal{B} = 1) &= 1 - \mathbb{P}(\mathcal{B} = 0) = \frac{\sigma_- b_+}{\sigma_+ b_- + \sigma_- b_+}, \\ \mathbb{P}_+ \left(\frac{Q_T^+}{T} \xrightarrow[T \rightarrow \infty]{} 1 \right) &= 1 \text{ and } \mathbb{P}_- \left(\frac{Q_T^-}{T} \xrightarrow[T \rightarrow \infty]{} 1 \right) = 1 \\ \text{with } \mathbb{P}_+(\cdot) &= \mathbb{P}(\cdot | \mathcal{B} = 1) \text{ and } \mathbb{P}_-(\cdot) = \mathbb{P}(\cdot | \mathcal{B} = 0). \end{aligned}$$

On the event $\{\mathcal{B} = 1\}$ (resp. $\{\mathcal{B} = 0\}$), β_T^+ (resp. β_T^-) converges almost surely to b_+ (resp. b_-) while $\beta_T^- - b_-$ (resp. $\beta_T^+ - b_+$) is the ratio of two a.s. finite random variables.

In addition, for unit Gaussian random variables \mathcal{N}^+ and \mathcal{N}^- ,

$$\sqrt{T}(\beta_T^+ - b_+) \xrightarrow[T \rightarrow \infty]{\text{law}} \sigma_+ \mathcal{N}^+ \text{ under } \mathbb{P}_+, \quad (23)$$

$$\sqrt{T}(\beta_T^- - b_-) \xrightarrow[T \rightarrow \infty]{\text{law}} \sigma_- \mathcal{N}^- \text{ under } \mathbb{P}_-. \quad (24)$$

In addition,

$$\lim_{T \rightarrow \infty} \beta_T^- = \mathcal{R}_{\mathbf{T1}}^- \text{ a.s. under } \mathbb{P}_+ \text{ with } \mathcal{R}_{\mathbf{T1}}^- = \frac{L_\infty(\xi)}{2Q_\infty^-}, \quad (25)$$

$$\lim_{T \rightarrow \infty} \beta_T^+ = -\mathcal{R}_{\mathbf{T1}}^+ \text{ a.s. under } \mathbb{P}_- \text{ with } \mathcal{R}_{\mathbf{T1}}^+ = \frac{L_\infty(\xi)}{2Q_\infty^+}. \quad (26)$$

The distribution of $\mathcal{R}_{\mathbf{T1}}^-$ is that of \mathcal{R} given (33) below. That of $\mathcal{R}_{\mathbf{T1}}^+$ is found by symmetry.

5 Auxiliary tools

In this section, we give first some results on a martingale central limit theorem that will be used constantly. To deal with the transient or null recurrent cases, we make use of some analytic properties of one-dimensional diffusions.

5.1 Limit theorems on martingales

The following result follows immediately from [34, Proposition 1, p. 148; Theorem 1, p. 150].

Proposition 8 (A criterion for convergence). *Under the true probability \mathbb{P} ,*

- (i) *as $T \rightarrow \infty$, β_T^+ (resp. β_T^-) converges a.s. to b_+ (resp. b_-) on the event $\{Q_\infty^+ = +\infty\}$ (resp. $\{Q_\infty^- = +\infty\}$).*
- (ii) *as $T \rightarrow \infty$, M_T^+ (resp. M_T^-) converges a.s. to a finite value on the event $\{Q_\infty^+ < +\infty\}$ (resp. $\{Q_\infty^- < +\infty\}$). In other words, β_T^\pm is not a consistent estimator on $\{Q_\infty^\pm < +\infty\}$.*

We now state an instance of a Central Limit theorem for martingales which follows from [8]. This theorem will be used to deal with the cases **E**, **N1** and **T1**. Let us start by recalling the notion of stable convergence introduced by A. Rényi [22, 44].

Definition 1 (Stable convergence). A sequence $(X_n)_{n \in \mathbb{N}}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to *converge stably* with respect to a σ -algebra $\mathcal{G} \subset \mathcal{F}$ if for any bounded, continuous function f and any bounded, \mathcal{G} -measurable random variable Y ,

$$\mathbb{E}(f(X_n)Y) \xrightarrow{n \rightarrow \infty} \mathbb{E}(f(X)Y).$$

Proposition 9 (A central limit theorem for martingales). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the underlying probability space of the process ξ with a filtration $(\mathcal{F}_t)_{t \geq 0}$. If for some constants $c_+, c_- > 0$,*

$$\frac{Q_T^+}{T} \xrightarrow[T \rightarrow +\infty]{\mathbb{P}} c_+ \text{ and } \frac{Q_T^-}{T} \xrightarrow[T \rightarrow +\infty]{\mathbb{P}} c_-,$$

then for the martingales M^\pm defined by (4),

$$\left(\frac{M_T^+}{\sqrt{T}}, \frac{M_T^-}{\sqrt{T}} \right) \xrightarrow[T \rightarrow \infty]{\mathcal{F}_\infty\text{-stably}} (\sigma_+ \sqrt{c_+} \mathcal{N}^+, \sigma_- \sqrt{c_-} \mathcal{N}^-), \quad (27)$$

on a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ extending $(\Omega, \mathcal{F}, \mathbb{P})$ and containing two independent unit Gaussian random variables $\mathcal{N}^+, \mathcal{N}^-$, themselves independent from ξ . In addition,

$$\sqrt{T} \left(\frac{M_T^+}{Q_T^+}, \frac{M_T^-}{Q_T^-} \right) \xrightarrow[T \rightarrow \infty]{\mathcal{F}_\infty\text{-stably}} \left(\frac{\sigma_+}{\sqrt{c_+}} \mathcal{N}^+, \frac{\sigma_-}{\sqrt{c_-}} \mathcal{N}^- \right), \quad (28)$$

Proof. Set

$$a_T := \begin{bmatrix} 1/\sqrt{T} & 0 \\ 0 & 1/\sqrt{T} \end{bmatrix} \text{ and } q_T = \langle M, M \rangle_T = \begin{bmatrix} \sigma_+^2 Q_T^+ & 0 \\ 0 & \sigma_-^2 Q_T^- \end{bmatrix}.$$

Thus,

$$a_T q_T a_T' = \begin{bmatrix} \sigma_+^2 \frac{Q_T^+}{T} & 0 \\ 0 & \sigma_-^2 \frac{Q_T^-}{T} \end{bmatrix} \xrightarrow[T \rightarrow \infty]{\mathbb{P}} \begin{bmatrix} c_+ \\ c_- \end{bmatrix}. \quad (29)$$

Theorem 2.2 in [8] yields (27). Besides,

$$\sqrt{T} \frac{M_T^\pm}{Q_T^\pm} = \frac{T}{Q_T^\pm} \times \frac{M_T^\pm}{\sqrt{T}}. \quad (30)$$

If a sequence $(X_n)_n$ converges \mathcal{F}_∞ -stably and a sequence $(Y_n)_n$ of \mathcal{F}_∞ -measurable random variables converges in probability, then $(X_n, Y_n)_n$ converges \mathcal{F}_∞ -stably. Using the property in (30) and (29) yields (28). \square

5.2 The fundamental system

Along with the characterization through the scale function and the speed measure, much information on the process can be read from the so-called *fundamental system* [12, 20, 45]: For any $\lambda > 0$, there exists some functions ϕ_λ and ψ_λ such that

- ψ_λ and ϕ_λ are continuous, positive from \mathbb{R} to \mathbb{R} with $\phi_\lambda(0) = \psi_\lambda(0) = 1$.
- ψ_λ is increasing with $\lim_{x \rightarrow -\infty} \psi_\lambda(x) = 0$, $\lim_{x \rightarrow \infty} \psi_\lambda(x) = +\infty$.
- ϕ_λ is decreasing with $\lim_{x \rightarrow -\infty} \phi_\lambda(x) = +\infty$, $\lim_{x \rightarrow \infty} \phi_\lambda(x) = 0$.
- ϕ_λ and ψ_λ are solutions to $\mathcal{L}f = \lambda f$.

In the case of piecewise constant coefficients with one discontinuity at 0, these solutions may be computed as linear combinations of the minimal functions for constant coefficients. Using the fact that ϕ , ϕ' , ψ and ψ' are continuous at 0,

$$\psi_\lambda(x) = \begin{cases} \exp\left(x \frac{-b_- + \sqrt{b_-^2 + 2\sigma_-^2 \lambda}}{\sigma_-^2}\right) & \text{if } x < 0 \\ \kappa_+ \exp\left(x \frac{-b_+ + \sqrt{b_+^2 + 2\sigma_+^2 \lambda}}{\sigma_+^2}\right) + \delta_+ \exp\left(x \frac{-b_+ - \sqrt{b_+^2 + 2\sigma_+^2 \lambda}}{\sigma_+^2}\right) & \text{if } x \geq 0, \end{cases} \quad (31)$$

$$\phi_\lambda(x) = \begin{cases} \kappa_- \exp\left(x \frac{-b_- - \sqrt{b_-^2 + 2\sigma_-^2 \lambda}}{\sigma_-^2}\right) + \delta_- \exp\left(x \frac{-b_- + \sqrt{b_-^2 + 2\sigma_-^2 \lambda}}{\sigma_-^2}\right) & \text{if } x < 0, \\ \exp\left(x \frac{-b_+ - \sqrt{b_+^2 + 2\sigma_+^2 \lambda}}{\sigma_+^2}\right) & \text{if } x \geq 0 \end{cases} \quad (32)$$

with

$$\begin{aligned} \kappa_+ &:= \frac{-b_- \sigma_+^2 + b_+ \sigma_-^2 + \sigma_-^2 \sqrt{b_+^2 + 2\lambda \sigma_+^2} + \sigma_+^2 \sqrt{b_-^2 + 2\lambda \sigma_-^2}}{2\sigma_-^2 \sqrt{b_+^2 + 2\lambda \sigma_+^2}}, \\ \delta_+ &:= \frac{b_- \sigma_+^2 - b_+ \sigma_-^2 + \sigma_-^2 \sqrt{b_+^2 + 2\lambda \sigma_+^2} - \sigma_+^2 \sqrt{b_-^2 + 2\lambda \sigma_-^2}}{2\sigma_-^2 \sqrt{b_+^2 + 2\lambda \sigma_+^2}}, \\ \kappa_- &:= \frac{-b_- \sigma_+^2 + b_+ \sigma_-^2 - \sigma_-^2 \sqrt{b_+^2 + 2\lambda \sigma_+^2} + \sigma_+^2 \sqrt{b_-^2 + 2\lambda \sigma_-^2}}{2\sigma_+^2 \sqrt{b_-^2 + 2\lambda \sigma_-^2}}, \\ \delta_- &:= \frac{b_- \sigma_+^2 - b_+ \sigma_-^2 + \sigma_-^2 \sqrt{b_+^2 + 2\lambda \sigma_+^2} + \sigma_+^2 \sqrt{b_-^2 + 2\lambda \sigma_-^2}}{2\sigma_+^2 \sqrt{b_-^2 + 2\lambda \sigma_-^2}}. \end{aligned}$$

We also define the quantities [41]

$$\begin{aligned}\widehat{\psi}(\lambda) &:= \frac{1}{2} \frac{\psi'_\lambda(0)}{\psi_\lambda(0)} = \frac{-b_- + \sqrt{b_-^2 + 2\sigma_-^2 \lambda}}{2\sigma_-^2} \geq 0 \\ \text{and } \widehat{\phi}(\lambda) &:= -\frac{1}{2} \frac{\phi'_\lambda(0)}{\phi_\lambda(0)} = \frac{b_+ + \sqrt{b_+^2 + 2\sigma_+^2 \lambda}}{2\sigma_+^2} \geq 0.\end{aligned}$$

In particular,

$$\widehat{\psi}(0) = 0 \text{ and } \widehat{\phi}(0) = \frac{b_+}{\sigma_+^2} \text{ when } b_- \geq 0 \text{ and } b_+ \geq 0.$$

5.3 Last passage time and occupation time for the transient process

When ξ is a transient process, the last passage time $\ell_0 = \sup\{t \geq 0 \mid \xi_t = 0\}$ of ξ at 0 is almost surely finite. Its Laplace transform is (See (53) in [41]):

$$\mathbb{E}_0[\exp(-\lambda \ell_0)] = \frac{\widehat{\psi}(0) + \widehat{\phi}(0)}{\widehat{\psi}(\lambda) + \widehat{\phi}(\lambda)}.$$

Let us now assume that $b_+ > 0$ and $b_- \geq 0$. This is the transient case **T0** where the process ends up almost surely in the positive semi-axis. Thus, $Q_{\ell_0}^- = Q_\infty^-$ and $L_{\ell_0}(\xi) = L_\infty(\xi)$.

Let us write $\widehat{b}_\pm := b_\pm / \sigma_\pm^2$. From Corollary 5 in [41],

$$\mathbb{E}_0[\exp(-\alpha L_\infty(\xi) - \lambda Q_\infty^-)] = \frac{\widehat{\psi}(0) + \widehat{\phi}(0)}{\alpha + \widehat{\phi}(0) + \widehat{\psi}(\lambda)} = \frac{\widehat{b}_+}{\alpha + \widehat{b}_+ - \frac{\widehat{b}_-}{2} + \frac{1}{2\sigma_-^2} \sqrt{b_-^2 + 2\sigma_-^2 \lambda}}.$$

Let $p_{L_\infty(\xi)}(t)$ be the density of $L_\infty(\xi)$. Setting $\lambda = 0$, we see that $L_\infty(\xi)$ is distributed according to an exponential distribution of rate \widehat{b}_+ . Thus, $p_{L_\infty(\xi)}(t) = \widehat{b}_+ \exp(-\widehat{b}_+ t)$. A conditioning shows that

$$\mathbb{E}_0[\exp(-\alpha L_\infty(\xi) - \lambda Q_\infty^-)] = \int_0^{+\infty} \exp(-\alpha t) \mathbb{E}_0(\exp(-\lambda Q_\infty^-) \mid L_\infty(\xi) = t) p_{L_\infty}(t) dt.$$

By inverting the Laplace transform with respect to α , since $p_{L_\infty}(t) = \widehat{b}_+ \exp(-\widehat{b}_+ t)$,

$$\mathbb{E}_0(\exp(-\lambda Q_\infty^-) \mid L_\infty(\xi) = t) = \exp\left(-t \left(-\frac{\widehat{b}_-}{2} + \frac{1}{\sqrt{2}\sigma_-} \sqrt{\frac{b_-^2}{2\sigma_-^2} + \lambda}\right)\right).$$

Inverting the latter Laplace transform with respect to λ , the density $p_{Q_\infty^-}(s|t)$ of Q_∞^- given $\{L_\infty(\xi) = t\}$ is

$$p_{Q_\infty^-}(s|t) = \frac{t}{\sigma_- 2\sqrt{2\pi} s^{3/2}} \exp\left(\frac{tb_-}{2\sigma_-^2} - \frac{b_-^2 s}{2\sigma_-^2} - \frac{t^2}{8\sigma_-^2 s}\right).$$

Hence, the distribution $p_{(Q_\infty^-, L_\infty(\xi))}(s, t)$ of the pair $(Q_\infty^-, L_\infty(\xi))$ is

$$p_{(Q_\infty^-, L_\infty(\xi))}(s, t) = \frac{tb_+}{\sigma_+^2 \sigma_- \sqrt{2\pi} s^{3/2}} \exp\left(\left(\frac{b_-}{2\sigma_-^2} - \frac{b_+}{\sigma_+^2}\right)t - \frac{b_-^2 s}{2\sigma_-^2} - \frac{t^2}{8\sigma_-^2 s}\right).$$

The density $p_{\mathcal{R}_{\mathbf{T}_0}}(r)$ of the random variable $\mathcal{R}_{\mathbf{T}_0} := L_\infty(\xi)/2Q_\infty^-$ is then

$$\begin{aligned} p_{\mathcal{R}_{\mathbf{T}_0}}(r) &= 2 \int_0^{+\infty} s \cdot p_{(Q_\infty^-, L_\infty(\xi))}(s, 2rs) \, ds \\ &= \frac{rb_+}{\sigma_+^2 \sigma_- \sqrt{2}} \left(\frac{2rb_+}{\sigma_+^2} + \frac{(r - b_-)^2}{2\sigma_-^2} \right)^{-3/2}, \quad r > 0. \end{aligned}$$

The distribution of $L_\infty(\xi)/2Q_\infty^+$ when $b_- < 0$, $b_+ \leq 0$ is found by symmetric arguments.

Let us now assume that $b_+ > 0$ and $b_- < 0$. This is the transient case **T1** where the process can end up in both semi-axis. The Laplace transform is

$$\mathbb{E}_0[\exp(-\alpha L_\infty(\xi) - \lambda Q_\infty^-)] = \frac{\hat{\psi}(0) + \hat{\phi}(0)}{\alpha + \hat{\phi}(0) + \hat{\psi}(\lambda)} = \frac{\hat{b}_+ - \hat{b}_-}{\alpha + \hat{b}_+ - \frac{\hat{b}_-}{2} + \frac{1}{2\sigma_-^2} \sqrt{b_-^2 + 2\sigma_-^2 \lambda}}.$$

With analogous computations as before we get that the density $p_{\mathcal{R}_{\mathbf{T}_1}^-}(r)$ of $\mathcal{R}_{\mathbf{T}_1}^-$ is

$$p_{\mathcal{R}_{\mathbf{T}_1}^-}(r) = \frac{r}{\sigma_- \sqrt{2}} \left(\frac{b_+}{\sigma_+^2} - \frac{b_-}{\sigma_-^2} \right) \left(\frac{2rb_+}{\sigma_+^2} + \frac{(r - b_-)^2}{2\sigma_-^2} \right)^{-3/2}, \quad r > 0.$$

Considering also the previous case, we can write the following formula, holding for the density of $\mathcal{R} = \mathcal{R}_{\mathbf{T}_0}$ or $\mathcal{R} = \mathcal{R}_{\mathbf{T}_1}^-$ in both cases **T0** and **T1**:

$$p_{\mathcal{R}}(r) = \frac{r}{\sigma_- \sqrt{2}} \left(\frac{b_+}{\sigma_+^2} + \frac{(|b_-|^-)}{\sigma_-^2} \right) \left(\frac{2rb_+}{\sigma_+^2} + \frac{(r - b_-)^2}{2\sigma_-^2} \right)^{-3/2}, \quad r > 0. \quad (33)$$

Notice that this is the density of a positive random variable which is not integrable. This gives the limit behavior of the estimator β_T^- of b_- . The behavior of β_T^+ in the corresponding cases can be found by symmetric arguments.

6 Proofs of the asymptotic behavior of the estimators

6.1 Asymptotic behavior for the ergodic case (E)

The ergodic case is the most favorable one. The process ξ is ergodic, so that for any bounded, measurable function f , $\frac{1}{T} \int_0^T f(\xi_s) \, ds$ converges almost surely to $\int f(x) \frac{m(x)}{M(\mathbb{R})} \, dx$.

With the explicit expression of M that follows from (18),

$$M(\mathbb{R}_+) = \frac{-1}{b_+}, \quad M(\mathbb{R}_-) = \frac{1}{b_-} \quad \text{and} \quad M(\mathbb{R}) = \frac{|b_+ b_-|}{|b_-| + |b_+|}.$$

From the ergodic theorem, since $Q_t^\pm = \int_0^t 1_{\pm \xi_s \geq 0} ds$,

$$\frac{Q_T^\pm}{T} \xrightarrow[T \rightarrow \infty]{\text{a.s.}} \frac{M(\mathbb{R}_\pm)}{M(\mathbb{R})}$$

so that

$$\frac{Q_T^+}{T} \xrightarrow[T \rightarrow \infty]{\text{a.s.}} \frac{|b_-|}{|b_-| + |b_+|} \text{ and } \frac{Q_T^-}{T} \xrightarrow[T \rightarrow \infty]{\text{a.s.}} \frac{|b_+|}{|b_-| + |b_+|}. \quad (34)$$

The rate of convergence follows from Proposition 9 and (34).

6.2 Asymptotic behavior for the null recurrent case with vanishing drift (N0)

When $b_- = b_+ = 0$, the process ξ is an Oscillating Brownian motion (OBM, introduced first in [25], see also [32]). Supposing $\xi_0 = 0$, using the scaling relation [32, Remark 3.7], for any $T > 0$,

$$\left(\frac{(\xi_T)^+}{\sqrt{T}}, \frac{(\xi_T)^-}{\sqrt{T}}, \frac{L_T(\xi)}{\sqrt{T}}, \frac{Q_T^+}{T} \right) \stackrel{\text{law}}{=} \left((\xi_1)^+, (\xi_1)^-, L_1(\xi), Q_1^+ \right).$$

Therefore,

$$\sqrt{T} \begin{pmatrix} \beta_T^+ \\ \beta_T^- \end{pmatrix} = \begin{pmatrix} \frac{(\xi_T)^+/\sqrt{T} - \frac{1}{2}L_T(\xi)/\sqrt{T}}{Q_T^+/T} \\ \frac{\frac{1}{2}L_T(\xi)/\sqrt{T} - (\xi_T)^-/\sqrt{T}}{Q_T^-/T} \end{pmatrix} \stackrel{\text{law}}{=} \begin{pmatrix} \beta_1^+ \\ \beta_1^- \end{pmatrix} = \begin{pmatrix} \frac{(\xi_1)^+ - \frac{1}{2}L_1(\xi)}{Q_1^+} \\ \frac{\frac{1}{2}L_1(\xi) - (\xi_1)^-}{Q_1^-} \end{pmatrix}.$$

We recall now that $X = \Phi(\xi) := \xi/\sigma(\xi)$ is a Skew Brownian motion [11, 32]. An explicit formula for the density for the position a Skew Brownian motion, its local time and its occupation time is known [1, 13]. Since the transform Φ is piecewise linear, one easily recover the one of an OBM, its local and occupation times. Hence, the density of $(\xi_1, L_1(\xi), Q_1^+(\xi))$ is

$$p_{(\xi_1, L_1(\xi), Q_1^+)}(\rho, \lambda, \tau) = \begin{cases} \frac{(\lambda/2 + \rho)\lambda/2}{2\pi\sigma_- \sigma_+^3 (1-\tau)^{3/2} \tau^{3/2}} \exp\left(-\frac{(\lambda/2)^2}{2\sigma_-^2(1-\tau)} - \frac{(\lambda/2 + \rho)^2}{2\sigma_+^2 \tau}\right) & \text{for } \rho \geq 0, \\ \frac{(\lambda/2 - \rho)\lambda/2}{2\pi\sigma_+ \sigma_-^3 (1-\tau)^{3/2} \tau^{3/2}} \exp\left(-\frac{(\lambda/2)^2}{2\sigma_+^2 \tau} - \frac{(\lambda/2 - \rho)^2}{2\sigma_-^2(1-\tau)}\right) & \text{for } \rho < 0. \end{cases} \quad (35)$$

The change of variable in the density suggested by

$$(\beta_1^+, \beta_1^-, Q_1^+) = \left(\frac{(\xi_1)^+ - L_1(\xi)/2}{Q_1^+}, \frac{L_1(\xi)/2 - (\xi_1)^-}{1 - Q_1^+}, Q_1^+ \right)$$

gives

$$p_{(\beta_1^+, \beta_1^-, Q_1^+)}(a, b, \delta) = 2\delta(1-\delta)p_{(\xi_1, L_1(\xi), Q_1^+)}(a\delta + b(1-\delta), |a\delta + b(1-\delta)| - a\delta + b(1-\delta), \delta)$$

and then, since $Q_1^+ \in [0, 1]$,

$$\begin{aligned} p_{(\beta_1^+, \beta_1^-)}(a, b) \\ = \int_0^1 2\delta(1 - \delta) p_{(\xi_1, L_1(\xi), Q_1^+)}(a\delta + b(1 - \delta), |a\delta + b(1 - \delta)| - a\delta + b(1 - \delta), \delta) d\delta. \end{aligned} \quad (36)$$

6.3 Asymptotic behavior for the null recurrent case with non-vanishing drift (N1)

We consider $b_+ = 0$, $b_- > 0$. The particle is then pushed upward when its position is negative. Yet the process is only null recurrent. The measure M satisfies

$$M(\mathbb{R}_-) = \frac{1}{b_-} \text{ and } M(\mathbb{R}_+) = +\infty. \quad (37)$$

Using 9) in [20, Section 6.8, p. 228] or [38, 51], with (37),

$$\frac{Q_T^-}{T} \xrightarrow[T \rightarrow \infty]{\text{a.s.}} \frac{M(\mathbb{R}_-)}{M(\mathbb{R})} = 0.$$

Since $Q_T^+ + Q_T^- = T$, it holds that Q_T^+/T converges almost surely to 1.

Using Proposition 9 on M^+ only, we obtain that

$$\frac{M_T^+}{\sqrt{T}} \xrightarrow[T \rightarrow \infty]{\text{law}} \sigma^+ \mathcal{N}^+ \quad (38)$$

for a Gaussian random variable $\mathcal{N}^+ \sim \mathcal{N}(0, 1)$ and then that $\sqrt{T}(\beta_T^+ - b_+) \xrightarrow[T \rightarrow \infty]{\text{law}} \sigma^+ \mathcal{N}^+$.

The asymptotic behavior of Q_T^- is more delicate to deal with as the process ξ is only null recurrent. For this, we use the results of [15] which extends the one of D.A. Darling and M. Kac [9] on additive and martingale additive functionals.

The Green kernel with respect to the invariant measure M of \mathcal{L} is given by [12, 20, 45]

$$g_\lambda(x, y) := \frac{1}{W_\lambda} \begin{cases} \psi_\lambda(x)\phi_\lambda(y) & \text{if } x < y, \\ \phi_\lambda(x)\psi_\lambda(y) & \text{if } x \geq y \end{cases} \text{ with } W_\lambda = \frac{\psi'_\lambda(0)\phi_\lambda(0) - \psi_\lambda(0)\phi'_\lambda(0)}{S'(0)}.$$

Using (31)-(32),

$$W_\lambda = \frac{\sqrt{2\lambda}}{\sigma_+} + \frac{\sqrt{b_-^2 + 2\lambda\sigma_-^2}}{\sigma_-^2} - \frac{b_-}{\sigma_-^2} \xrightarrow[\lambda \rightarrow 0]{} \frac{\sqrt{2}}{\sigma_+}. \quad (39)$$

On the other hand, it follows from (31)-(32) that

$$\psi_\lambda(x) \xrightarrow[\lambda \rightarrow 0]{} \psi_0(x) := 1 \text{ and } \phi_\lambda(x) \xrightarrow[\lambda \rightarrow 0]{} \phi_0(x) := 1 \text{ for any } x \in \mathbb{R}. \quad (40)$$

For a measurable function $f : \mathbb{R} \rightarrow \mathbb{R}_+$ such that $\int_{\mathbb{R}} f \, dM < +\infty$, combining (39) and (40) yields that

$$\sqrt{\lambda} \int_{\mathbb{R}} g_{\lambda}(x, y) f(x) m(y) \, dy \xrightarrow{\lambda \rightarrow 0} \frac{\sigma_+}{\sqrt{2}} \int_{\mathbb{R}} f(y) m(y) \, dy, \quad \forall x \in \mathbb{R}.$$

The above convergence allows one to identify the parameters to use in the limit theorem we will use on (M^-, Q^-) . We then define $\ell(\lambda) := \sqrt{2}/\sigma_+$ which is a constant function, and $\alpha := 1/2$, the exponent of λ .

Let $(\mathcal{M}_t)_{t \geq 0}$ be a Mittag-Leffler process of index $\alpha = 1/2$ (it is the inverse of an increasing stable process of index $1/2$). The process $2^{-1/2} \mathcal{M}$ is equal in distribution to the running maximum of a Brownian motion, or equivalently, to the local time of a Brownian motion [15, Remark 2.9, p. 21].

From Theorem 3.1 and Corollary 3.2 in [15, p. 26], since $\langle M^- \rangle_t = \sigma_-^2 Q_t^-$, $t \geq 0$ and M is continuous,

$$\left(\sqrt{\ell(n)} \frac{M_{nt}^-}{n^{1/4}}, \ell(n) \frac{Q_{nt}^-}{n^{1/2}} \right)_{t \in [0,1]} \xrightarrow[n \rightarrow \infty]{\text{law}} \left(\sigma_- \sqrt{\nu} B^-(\mathcal{M}_t), \nu \mathcal{M}_t \right)_{t \in [0,1]} \quad (41)$$

with respect to the uniform topology, where B is a Brownian motion independent from \mathcal{M} and

$$\nu = \int_{\mathbb{R}} m(y) \mathbb{E}_y \left(\int_0^1 1_{\xi_s \leq 0} \, ds \right) \, dy = M(\mathbb{R}_-) = \frac{1}{b_-}.$$

From the reflection principle, the distribution of \mathcal{M}_1 is the same as the one of a truncated normal distribution $\sqrt{2} \mathcal{T}$ where $\mathcal{T} := |\mathcal{G}|$ with $\mathcal{G} \sim \mathcal{N}(0, 1)$. Setting $t = 1$ in (41) and using the scaling property of the Brownian motion B^- ,

$$\left(\frac{M_T^-}{T^{1/4}}, \frac{Q_T^-}{T^{1/2}} \right) \xrightarrow[T \rightarrow \infty]{\text{law}} \left(\sigma_- \frac{\sqrt{\sigma_+}}{\sqrt{b_-}} \sqrt{\mathcal{T}} \cdot \mathcal{N}^-, \frac{\sigma_+}{b_-} \mathcal{T} \right)$$

for a Gaussian random variable $\mathcal{N}^- \sim \mathcal{N}(0, 1)$ independent from \mathcal{T} .

It remains to show the independence of \mathcal{N}^+ , \mathcal{N}^- and \mathcal{T} .

Since $\langle M^{\pm} \rangle_t = \sigma_{\pm}^2 Q_t^{\pm}$ and $\langle M^+, M^- \rangle_t = 0$ for any $t \geq 0$, the Knight theorem [19, Theorem 7.3', p. 92] implies that there exists on an extension of $(\Omega, \mathcal{F}, \mathbb{P})$ a 2-dimensional Brownian motion (B^+, B^-) such that $M_t^{\pm} = B^{\pm}(\sigma_{\pm}^2 Q_t^{\pm})$ for any $t \geq 0$. Let us set $B_n^+(t) = n^{-1/2} B^+(nt)$ and $B_n^-(t) = n^{-1/4} B^-(\sqrt{nt})$ for any integer n and any $t \geq 0$. From the scaling property, (B_n^+, B_n^-) is still a 2-dimensional Brownian motion which converges in distribution to a 2-dimensional Brownian motion $(B_{\infty}^+, B_{\infty}^-)$ in the space $\mathcal{C}([0, 1], \mathbb{R}^2)$ of continuous functions.

For any $0 \leq s \leq t \leq 1$, $Q_{nt}^+ - Q_{ns}^+ \leq n(t - s)$ so that $\{(Q_{nt}^+/n)_{t \in [0,1]}\}_{n \geq 1}$ is tight in the space of continuous functions. Hence, $\{(Q_{nt}^+/n)_{t \in [0,1]}\}_{n \geq 1}$ converges in probability to the identity map $t \mapsto t$ in the space of continuous function $\mathcal{C}([0, 1], \mathbb{R})$.

Combining this result with (41), it holds that $\left\{ \left(B_n^+(t), B_n^-(t), n^{-1} Q_{nt}^+, n^{-1/2} Q_{nt}^- \right)_{t \in [0,1]} \right\}_{n \geq 1}$ is tight in $\mathcal{C}([0, 1], \mathbb{R}^4)$ and then necessarily

$$\left(B_n^+(t), B_n^-(t), \frac{Q_{nt}^+}{n}, \frac{Q_{nt}^-}{\sqrt{n}} \right)_{t \in [0,1]} \xrightarrow[n \rightarrow \infty]{\text{law}} \left(B_{\infty}^+(t), B_{\infty}^-(t), t, \nu \mathcal{M}_t \right)_{t \in [0,1]} \quad (42)$$

in the space of continuous functions $\mathcal{C}([0, 1], \mathbb{R}^4)$. Being the inverse of a 1/2-stable process, hence a pure jump process, \mathcal{M} is independent from (B_∞^+, B_∞^-) for the arguments presented in [15, p. 38] or [19, Theorem 6.3, p. 77].

For any $t \in [0, 1]$, owing to the definition of B_n^+ and B_n^- ,

$$\begin{aligned} \frac{M_{nt}^+}{\sqrt{n}} &= \frac{1}{n^{1/2}} B^+(\sigma_+^2 Q_{nt}^+) = B_n^+ \left(\sigma_+^2 \frac{Q_{nt}^+}{n} \right) \\ \text{and } \frac{M_{nt}^-}{n^{1/4}} &= \frac{1}{\sqrt{n}} B^-(\sigma_-^2 Q_{nt}^-) = B_n^- \left(\sigma_-^2 \frac{Q_{nt}^-}{\sqrt{n}} \right). \end{aligned}$$

Using $(n^{-1}Q_{nt}^+)_{t \in [0,1]}$ and $(n^{-1/2}Q_{nt}^-)_{t \in [0,1]}$ as random time changes, we deduce from (42) and the results in [6, p. 144] that

$$\left(\frac{M_{nt}^+}{n^{1/2}}, \frac{M_{nt}^-}{n^{1/4}}, \frac{Q_{nt}^+}{n}, \frac{Q_{nt}^-}{\sqrt{n}} \right)_{t \in [0,1]} \xrightarrow[n \rightarrow \infty]{\text{law}} (B_\infty^+(\sigma_+^2 t), B_\infty^-(\sigma_-^2 \nu \mathcal{M}_t), t, \nu \mathcal{M}_t)_{t \in [0,1]}.$$

By setting $\mathcal{N}^+ := B_\infty^+(\sigma_+^2)/\sigma_+ \sim \mathcal{N}(0, 1)$ in (38), $\mathcal{T} := \mathcal{M}_1/\sqrt{2}$ and $\mathcal{N}^- := B_\infty^-(\sigma_-^2 \nu)/\sigma_- \sqrt{\nu} \sim \mathcal{N}(0, 1)$, this proves (20) using $t = 1$ in the above limit.

6.4 Asymptotic behavior for the transient case (T0)

We recall that we consider only $b_+ > 0$ and $b_- \geq 0$.

It is known that the last passage time ℓ_0 to 0 is almost surely finite so that $Q_\infty^- < \infty$ almost surely. Since $Q_T^+ = T - Q_T^-$, we obtain that Q_T^+/T converges almost surely to 1. The convergence results regarding β_T^+ follows from Proposition 9 applied only to one component.

The asymptotic behavior of β_T^- follows from Proposition 8(ii).

When $T > \ell_0$ and $\xi_0 = 0$, then $\xi_T > 0$ and thus $\beta_T^- = L_T(\xi)/2Q_T^-$. Yet the local time $L_T(\xi)$ and the occupation times Q_T^- are constant when $T > \ell_0$. The result follows by the computations of Section 5.3.

6.5 Asymptotic behavior for the transient case generated by diverging drift (T1)

When $b_- < 0$ and $b_+ > 0$, the process is also transient as $S(+\infty) < +\infty$ and $S(-\infty) > -\infty$. The scale function S map \mathbb{R} to $(-\gamma_-, \gamma_+)$ with $\gamma_\pm = |\sigma_\pm^2|2b_\pm$. From the Feller test [24, Theorem 5.29], the process does not explode. Thus, as $\xi_0 = 0$, it follows from [52] or [24, Proposition 5.5.22, p. 354] that

$$p := \mathbb{P}\left(\lim_{T \rightarrow \infty} \xi_T = +\infty\right) = 1 - \mathbb{P}\left(\lim_{T \rightarrow \infty} \xi_T = -\infty\right) = \frac{\gamma_-}{\gamma_- + \gamma_+} = \frac{\sigma_-^2 b_+}{\sigma_+^2 b_- + \sigma_-^2 b_+}.$$

Then event that $\{\lim_{T \rightarrow \infty} |\xi_T| = +\infty\}$ arise when the process starts an excursion with infinite lifetime, thus after the last passage time to 0. We denote by A^\pm the event $\{\lim_{T \rightarrow \infty} Q_T^\pm/T = 1\}$, so that $A^+ \cap A^- = \emptyset$. Hence, $\mathbb{P}(A^+) = 1 - \mathbb{P}(A^-) = p$.

Using the same arguments as in the case **T0**, given A^\pm , β_T^\pm converges almost surely to b_\pm while $\beta_T^\mp = \pm L_\infty(\xi)/Q_{\ell_0}^\mp$.

For the Central Limit Theorem, we apply Corollary 2.3 in [8] on M_T^+/\sqrt{T} . As $Q_T^+ > 0$ a.s. as soon as $T > 0$ since $\xi_0 = 0$ and

$$\{\mathcal{B} = p\} = A^+, \text{ a.s. for } p = 0, 1,$$

it follows that for a normal distribution \mathcal{N} ,

$$\sqrt{\frac{T}{\sigma_+^2 Q_T^+}} \cdot \frac{M_T}{\sqrt{T}} \xrightarrow[N \rightarrow \infty]{\mathcal{F}_\infty\text{-stably}} \mathcal{N} \text{ under } \mathbb{P}(\cdot \mid \mathcal{B} = 1).$$

It follows that

$$\beta_T^+ - b_+ = \sqrt{T} \frac{M_T}{Q_T^+} = \sigma_+ \sqrt{\frac{T}{Q_T^+}} \sqrt{\frac{T}{\sigma_+^2 Q_T^+}} \frac{M_T}{\sqrt{T}} \xrightarrow[N \rightarrow \infty]{\mathcal{F}_\infty\text{-stably}} \sigma_+ \mathcal{N} \text{ under } \mathbb{P}(\cdot \mid \mathcal{B} = 1).$$

Hence the result.

7 Wilk's theorem and LAN property

Owing to the quadratic nature of the log-likelihood, we easily deduce both a Wilk theorem, on which a hypothesis test may be developed, as well as the Local Asymptotic Normality (LAN) property, which proves that our estimators are asymptotically efficient.

7.1 Wilk's theorem and a hypothesis test

The log-likelihood $\log G_T(b_+, b_-)$ can be computed from the data using (7). Moreover, this function is quadratic in b_+ and b_- . We have then

$$\nabla \log G_T(b_+, b_-) = \begin{bmatrix} \frac{(\xi_T)^+}{\sigma_+^2} - \frac{(\xi_0)^+}{\sigma_+^2} - \frac{1}{2\sigma_+^2} L_T(\xi) - \frac{b_+}{\sigma_+^2} Q_T^+ \\ -\frac{(\xi_T)^-}{\sigma_-^2} + \frac{(\xi_0)^-}{\sigma_-^2} + \frac{1}{2\sigma_-^2} L_T(\xi) - \frac{b_-}{\sigma_-^2} Q_T^- \end{bmatrix}$$

and $\text{Hess } \log G_T(b_+, b_-) = \begin{bmatrix} -\frac{Q_T^+}{\sigma_+^2} & 0 \\ 0 & -\frac{Q_T^-}{\sigma_-^2} \end{bmatrix}$.

Therefore, around any point (b_+, b_-) ,

$$\begin{aligned} \log G_T(b_+ + \Delta b_+, b_- + \Delta b_-) &= \log G_T(b_+, b_-) \\ &+ \nabla \log G_T(b_+, b_-) \cdot \begin{bmatrix} \Delta b_+ \\ \Delta b_- \end{bmatrix} - \frac{Q_T^+}{2\sigma_+^2} \Delta b_+^2 - \frac{Q_T^-}{2\sigma_-^2} \Delta b_-^2. \end{aligned} \quad (43)$$

In particular, we prove a result in the asymptotic behavior of the log-likelihood for the ergodic case **E** or the null recurrent case **N1**. A similar result can be given for the null recurrent cases **N0** with a different limit distribution that can be identified with the density given in Section 6.2.

Proposition 10 (Wilk's theorem; Ergodic case **E** or null recurrent case **N1**). *Denote by $(b_+^{\text{true}}, b_-^{\text{true}})$ be the real parameters. Then*

$$-2 \log \frac{G_T(b_+^{\text{true}}, b_-^{\text{true}})}{G_T(\beta_T^+, \beta_T^-)} \xrightarrow[T \rightarrow \infty]{\text{law}} \chi_2 := (\mathcal{N}^+)^2 + (\mathcal{N}^-)^2 \text{ under } \mathbb{P}_{(b_+^{\text{true}}, b_-^{\text{true}})} \quad (44)$$

for two independent, unit Gaussian random variables \mathcal{N}^+ and \mathcal{N}^- . Besides, when $(b_+, b_-) \neq (b_+^{\text{true}}, b_-^{\text{true}})$, then

$$- \log \frac{G_T(b_+, b_-)}{G_T(\beta_T^+, \beta_T^-)} \xrightarrow[T \rightarrow \infty]{\text{a.s.}} + \infty \text{ under } \mathbb{P}_{(b_+^{\text{true}}, b_-^{\text{true}})}. \quad (45)$$

Proof. Considering (43) at $(b_+, b_-) = (\beta_T^+, \beta_T^-)$ since $\nabla \log G_T(\beta_T^+, \beta_T^-) = (0, 0)$, for any parameter (b_+, b_-) and any $\alpha_+, \alpha_- > 0$,

$$-2 \log G_T(b_+, b_-)/G_T(\beta_T^+, \beta_T^-) = \frac{Q_T^+}{T^{\alpha_+ \sigma_+^2}} \left(T^{\frac{\alpha_+}{2}} (b_+ - \beta_T^+) \right)^2 + \frac{Q_T^-}{T^{\alpha_- \sigma_-^2}} \left(T^{\frac{\alpha_-}{2}} (b_- - \beta_T^-) \right)^2.$$

When the process is ergodic (Case **E**), we set $(\alpha_+, \alpha_-) = (1, 1)$. It follows from Proposition 3 that (44) holds. When $(b_+, b_-) \neq (b_+^{\text{true}}, b_-^{\text{true}})$, then $\beta_T^\pm - b_\pm$ does not converge to 0 while Q_T^\pm converges a.s. to infinity. This proves (45). The result is similar in the case **N1** with $(\alpha_+, \alpha_-) = (1, 1/2)$. \square

A hypothesis test can be developed from Proposition 10. The null hypothesis is $(b_+, b_-) = (b_+^0, b_-^0)$ for a given drift (b_+^0, b_-^0) while the alternative hypothesis is $(b_+, b_-) \neq (b_+^0, b_-^0)$.

Using (43), we compute $w = -2 \log G_T(b_+^0, b_-^0)/G_T(\beta_T^+, \beta_T^-)$. The null hypothesis is rejected with a confidence level α if $w > q_\alpha$ where q_α is the α -quantile $\mathbb{P}[\chi_2 \leq q_\alpha] = \alpha$ while χ_2 follows a χ^2 distribution with 2 degrees of freedom.

By using the quasi-likelihood instead of the likelihood, we obtain in a similar way for the ergodic case (Case **E**) that

$$-2(\Lambda_T(b_+^{\text{true}}, b_-^{\text{true}}) - \Lambda_T(\beta_T^+, \beta_T^-)) \xrightarrow[T \rightarrow \infty]{\text{law}} \sigma_+^2 (\mathcal{N}^+)^2 + \sigma_-^2 (\mathcal{N}^-)^2 \text{ under } \mathbb{P}_{(b_+^{\text{true}}, b_-^{\text{true}})},$$

where \mathcal{N}^+ and \mathcal{N}^- are two unit Gaussian random variables. This could also be used for an hypothesis test. When using either the log-likelihood of the quasi-likelihood, σ_+ and σ_- shall be known.

7.2 The LAN property

The LAN (local asymptotic normal) property, introduced by L. Le Cam in [30], characterizes the efficiency of the estimator (See also [17, 28] among many other references). It was extended as the LAMN (local asymptotic mixed normal) to deal with a mixed normal limits by P. Jeganathan in [23].

The quadratic nature of the log-likelihood as well as our limit theorems implies that the LAN (resp. LAMN) property is verified in the ergodic case **E** (resp. null recurrent case **N1**).

Proposition 11 (LAN property; Ergodic case **E**). *In the ergodic case **E**, the LAN property holds for the likelihood at $(b_+^{\text{true}}, b_-^{\text{true}})$ with rate of convergence $(\sigma_+^2/\sqrt{T}, \sigma_-^2/\sqrt{T})$ and asymptotic Fisher information*

$$\Gamma := \frac{1}{|b_-| + |b_+|} \begin{bmatrix} \sigma_+^2 |b_-| & 0 \\ 0 & \sigma_-^2 |b_+| \end{bmatrix}.$$

Proof. At $(b_+^{\text{true}}, b_-^{\text{true}})$, the gradient may be written

$$D(b_+^{\text{true}}, b_-^{\text{true}}) = \begin{bmatrix} \frac{Q_T^+}{\sqrt{T}\sigma_+^2} \sqrt{T}(\beta_T^+ - b_+^{\text{true}}) \\ \frac{Q_T^-}{\sqrt{T}\sigma_-^2} \sqrt{T}(\beta_T^- - b_-^{\text{true}}) \end{bmatrix}.$$

Using (43),

$$\begin{aligned} R(T) &:= \log \frac{G\left(b_+^{\text{true}} + \sigma_+^2 \frac{\Delta b_+}{\sqrt{T}}, b_-^{\text{true}} + \sigma_-^2 \frac{\Delta b_-}{\sqrt{T}}\right)}{G_T(b_+^{\text{true}}, b_-^{\text{true}})} \\ &= \begin{bmatrix} Q_T^+(\beta_T^+ - b_+^{\text{true}}) \\ Q_T^-(\beta_T^- - b_-^{\text{true}}) \end{bmatrix} \cdot \begin{bmatrix} \Delta b_+ \\ \Delta b_- \end{bmatrix} - \frac{\sigma_+^2 Q_T^+}{2T} \Delta b_+^2 - \frac{\sigma_-^2 Q_T^-}{2T} \Delta b_-^2 \\ &= \frac{1}{\sqrt{T}} \begin{bmatrix} M_T^+ \\ M_T^- \end{bmatrix} \cdot \begin{bmatrix} \Delta b_+ \\ \Delta b_- \end{bmatrix} - \frac{1}{2T} \begin{bmatrix} \Delta b_+ \\ \Delta b_- \end{bmatrix} \cdot \langle M^+, M^- \rangle_T \begin{bmatrix} \Delta b_+ \\ \Delta b_- \end{bmatrix}. \end{aligned} \quad (46)$$

With $c = |b_+| + |b_-|$, Proposition 3 implies that $(T^{-1/2}M^+, -1/2 M^-)$ converges in distribution to $\mathcal{G} \sim \mathcal{N}(0, \Gamma)$ and $T^{-1}\langle M^+, M^- \rangle_T$ converges to the diagonal, definite positive matrix Γ . This proves the LAN property. \square

Proposition 12 (LAMN property; Null recurrent case **N1**). *In the null recurrent case **N1**, the LAMN property holds for the likelihood at $(b_+^{\text{true}}, b_-^{\text{true}})$ with rate of convergence $(\sigma_+^2/T^{1/2}, \sigma_-^2/T^{1/4})$ and asymptotic (random) Fisher information*

$$\Gamma := \begin{bmatrix} \sigma_+^2 & 0 \\ 0 & \sigma_-^2 \frac{\sigma_{\pm}}{b_{\pm}} |\mathcal{N}| \end{bmatrix}$$

for a normal random variable $\mathcal{N} \sim \mathcal{N}(0, 1)$.

8 Simulation study

8.1 From continuous to discrete data

In this section, we apply estimator (8) and (11) to simulated processes. We test whether the results are good or not, depending on sign and magnitude of involved quantities.

Within the framework of Data 2, $\{\xi_t\}_{t \in [0, T]}$ is observed on a discrete time grid $\{kT/N; k = 0, \dots, N\}$. The time step between two observations is $\Delta t := T/N$.

In (9), we have defined $\mathbf{Q}_{T, N}^{\pm}$ and $\mathbf{R}_{T, N}^{\pm}$ which are easily computed from the observations $\{\xi_{kT/N}\}_{k=0, \dots, N}$.

The convergence of $Q_{T,N}^\pm$ was discussed in [32] for the OBM (see also Lemma 1). We prove that the speed of convergence is strictly better than \sqrt{N} , meaning that $\sqrt{N}(Q_{T,N}^+ - Q_T^+) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0$. This result can be extended to the drifted process ξ via the Girsanov theorem.

Alternatively to computing $R_{T,N}^\pm$, we may approximate the local time:

- The approximation $L_{T,N}$ to the local time given by (10) is easily implemented.
- In [32, 33], we have also considered two other consistent estimators for the local time. For the OBM, they are

$$L_{T,N}^\dagger := \frac{-3}{2} \sqrt{\frac{\pi}{2\Delta t}} \frac{\sigma_+ + \sigma_-}{\sigma_+ \sigma_-} \sum_{k=1}^N \left((|\xi_{kT/N}|^+ - |\xi_{(k-1)T/N}|^+) \cdot (|\xi_{kT/N}|^- - |\xi_{(k-1)T/N}|^-) \right)$$

and

$$L_{T,N}^\times := \frac{4}{\sqrt{2\pi}} \times \frac{1}{\sigma_+ + \sigma_-} \times \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} 1_{\xi_{kT/N} \xi_{(k+1)T/N+1} < 0}.$$

- The expressions of $L_{T,N}^\dagger$ and $L_{T,N}^\times$ require σ_+ and σ_- , unlike the one of $L_{T,N}$. To the best of our knowledge, this is the first time that such estimator is considered.
- For the Brownian motion, these estimators converge at rate $N^{1/4}$, thanks to the results of [21]. The proof has not been adapted to OBM because of the technical difficulties due to the discontinuity of the coefficients in 0. Anyway we conjecture a rate of $1/4$.
- The three different estimators for the local time seem to have all comparable accuracy on numerical simulations if the volatility coefficients σ_\pm are *known*. It looks like estimator $L_{T,N}^\times$ is the best one when $\sigma_+ \approx \sigma_-$, whereas $L_{T,N}$ and $L_{T,N}^\dagger$ look more accurate when $\sigma_+ \neq \sigma_-$.
- Anyway, if these coefficients are not known, only $L_{T,N}$ can be implemented directly, whereas to implement the other estimators we have first to estimate σ_\pm from observations of ξ . This problem has been thoroughly investigated in [32]. As a consequence, a good estimation of the local time relies on the estimation of σ_\pm . Therefore, we choose to show here the implementation of the estimator using $L_{T,N}$, which does not suffer of this possible additional problem.

In practice, this means that we are actually showing estimator $\beta_{T,N}^\pm$ in (11) in place of β_T^\pm . The numerical results using an approximation of β_T^\pm using $Q_{T,N}^\pm$ and $L_{T,N}^\dagger$ for the local time are very similar, if σ_\pm is supposed to be known.

We do not push further in the present paper the theoretical discussion on the quality of these discrete time approximations. Some more insights, based on numerical results, are given in the following section.

8.2 Implementation and simulation

The aim of the following section is to show on figures the numerical evidence of the central limit theorems stated in Section 4. We also mean to say something more on the choice of the step of the time grid in relation to the quality of the estimation of the

local time. The code used for the following simulations have been implemented using the software **R**. We will consider time grids of the type $\{0, T/N, 2T/N, \dots, T\} \subset \mathbb{N}$, with $N \in \mathbb{N}$, and use as approximation of local and occupation time $L_{T,N}$ and $Q_{T,N}^+$. We actually use $\beta_{T,N}^\pm$ instead of β_T^\pm , for large N . In practice, in some cases the estimator does not really depend on the local time, but is essentially determined by the final value of the process and the occupation times. In those cases, the quality of the discrete time approximation of the local time does not really matter, and therefore we can take N small. In most cases, however, a good approximation of the local time is needed in order to observe on simulations the theoretical central limit behavior expected from Section 4. In these cases $N \in \mathbb{N}$ must be taken large enough.

In what follows, the choice of the parameters is detailed for every figure. The diffusion parameter is taken constant $\sigma_+ = \sigma_- = 0.01$, the same for all the different simulations. We indicate with $[+]$ and $[-]$ estimation on positive and negative semiaxis, e.g. **(N1)** $[+]$ stands for “estimation of b_+ ” in case **N1**.

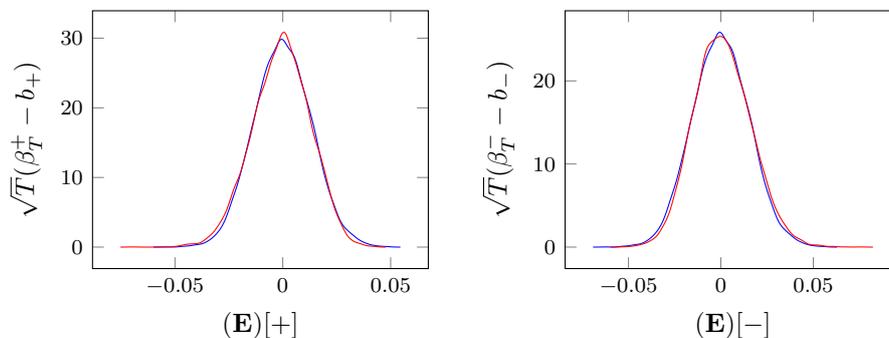


Figure 1: **(E)**. SDE parameters: $\sigma_\pm = 0.01$, $b_- = 0.004$, $b_+ = -0.003$. Simulation parameters: $T = 10^3$, $N = 10^5$. We show both sides (positive and negative) of the estimation, displaying in red the density of $\sqrt{T}(\beta_{T,N}^\pm - b_\pm)$, in blue its theoretical limit. The CLT in Proposition 3 is accurate for large T and time step T/N small, since the quality of the estimation of the local time is key in this case. The limit behavior is Gaussian.

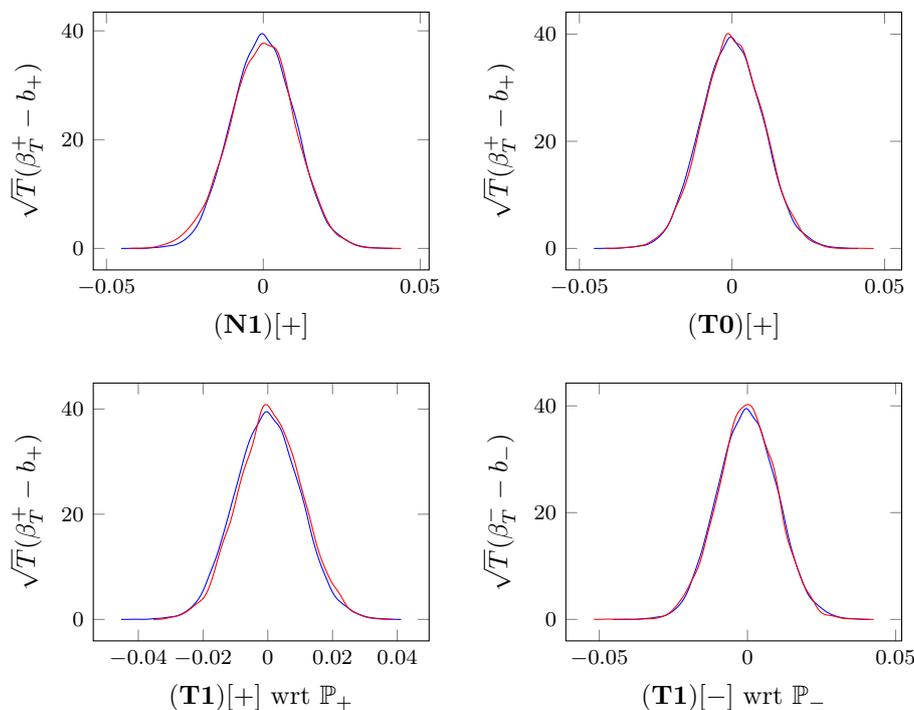


Figure 2: **(N1)[+]**, **(T0)[+]**, **(T1)[+]** w.r.t \mathbb{P}_+ , **(T1)[-]** w.r.t \mathbb{P}_- . SDE parameters: $\sigma_{\pm} = 0.01$; in case **N1**: $b_- = 0.004$, $b_+ = 0$; in case **T0**: $b_- = 0.004$, $b_+ = 0.006$; in case **T1**: $b_- = -0.004$, $b_+ = 0.003$. Simulation parameters: $T = 10^3$, $N = 10^3$. We display in red the density of $\sqrt{T}(\beta_{T,N}^+ - b_+)$, in blue its theoretical limit, in cases **N1** and **T0**. We also show in red $\sqrt{T}(\beta_{T,N}^{\pm} - b_{\pm})$ in case **T1**, but the density is w.r.t \mathbb{P}_{\pm} (cf. (7)). This is approximated computing the estimator on trajectories such that ξ_T is larger or respectively smaller than 0. Again, the blue line shows the theoretical limit density. In all these cases the CLT is Gaussian and we do not need to have a fine discretization/time grid, since the local time is asymptotically negligible and the quantities which matter in the estimator are ξ_T and the occupation times. This accounts of (20)-positive part, (21), (23) and (24).

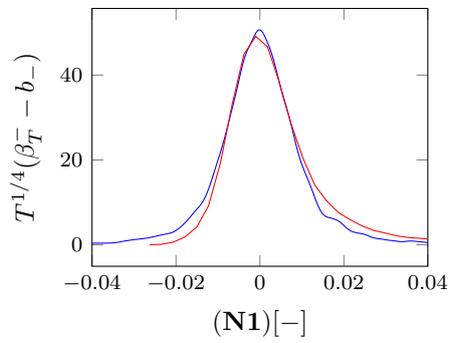


Figure 3: $(\mathbf{N1})[-]$. SDE parameters: $\sigma_{\pm} = 0.01$; in case $\mathbf{N1}$: $b_- = 0.004$; $b_+ = 0$. Simulation parameters: $T = 10^3$, $N = 10^5$. The CLT in (20)-negative part is accurate for large T and time step T/N small, since the quality of the estimation of the local time is key in this case. Remark that in this case (null recurrent), the CLT has speed of convergence $T^{1/4}$ and the limit law is not Gaussian. This accounts of (20)-negative part: we show in red the density of $T^{1/4}(\beta_{T,N}^- - b_-)$, in blue its limit density.

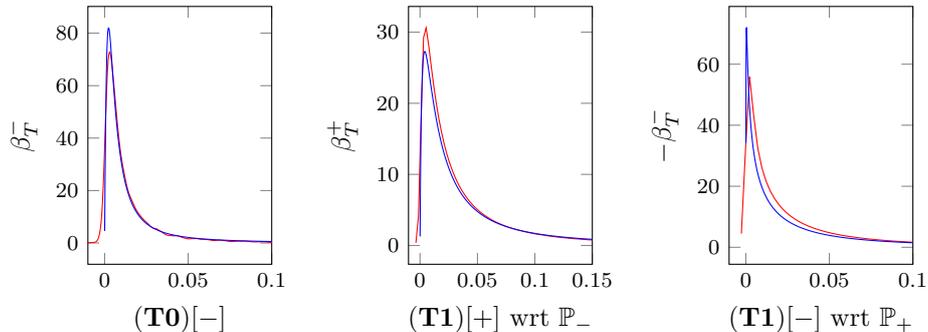


Figure 4: $(\mathbf{T0})[-]$, $(\mathbf{T1})[+]$ w.r.t to \mathbb{P}_- and $(\mathbf{T1})[-]$ w.r.t to \mathbb{P}_+ . SDE parameters: $\sigma_{\pm} = 0.01$; in case $\mathbf{T0}$: $b_- = 0.004$, $b_+ = 0.003$; in case $\mathbf{T1}$: $b_- = -0.003$, $b_+ = 0.01$. In case $\mathbf{T0}$: simulation parameters: $T = 20$, $N = 10^4$. We display in red the density of β_T^- . In case $\mathbf{T1}$: simulation parameters: $T = 20$, $N = 10^5$. We display in red the density of $\beta_{T,N}^{\pm}$ w.r.t \mathbb{P}_{\mp} (cf. (7)). This is approximated computing the estimator on trajectories such that ξ_T is smaller or respectively larger than 0. In these cases the estimator is not consistent, so what we show is not actually a CLT but the convergence of the estimators towards the law (33), whose density is plotted in blue (cf. results (22), (25), (26)). This convergence is accurate for large T but also depends on the time step T/N . Moreover, we see that the theoretical distribution of β_T^- in case $\mathbf{T1}$ is almost singular at the origin, and therefore the exact behavior near the origin is hard to catch on simulated trajectories. This can be improved using different kernels (instead of the Gaussian one) in the estimation of the density. This can be easily done with the function “density” in R. The limit behavior is better approximated when b_+ and b_- have similar magnitude. We choose here to display the case $b_- = -0.003$; $b_+ = 0.01$ to mention this critical behavior. Anyway, this feature does not really matter in statistical application, because in this case the estimator not only is not consistent, but does not even guess the correct sign of the parameter. Indeed, we are here in the very critical case of a transient process generated by diverging drift $\mathbf{T1}$.

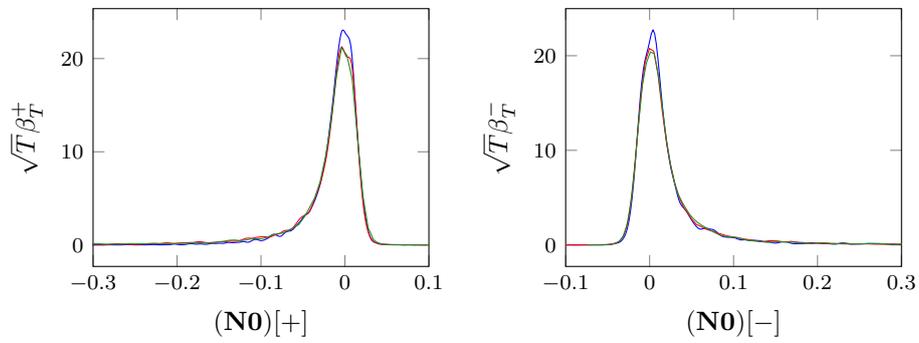


Figure 5: **(NO)**. SDE parameters: $\sigma_{\pm} = 0.01$, $b_- = 0$, $b_+ = 0$. Simulation parameters: $T = 10$, $N = 10^3$; $T = 10^2$, $N = 10^4$; $T = 10^3$, $N = 10^5$. Differently from before, we do not show the convergence to the scaled limit law (36), but the scaling relation (19): the blue, red and green lines represent the density of $\sqrt{T}\beta_{T,N}^{\pm}$, for the three different final times. We show it on both positive and negative semiaxes. Because of the estimation of the local time, this also depends on the choice of N .

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