

## Supplementary material to:

# Beyond tail median and conditional tail expectation: extreme risk estimation using tail $L^p$ -optimisation

Laurent Gardes<sup>(1)</sup>, Stéphane Girard<sup>(2)</sup> & Gilles Stupfler<sup>(3)</sup>

<sup>(1)</sup> Université de Strasbourg & CNRS, IRMA, UMR 7501, France

<sup>(2)</sup> Univ. Grenoble Alpes, Inria, CNRS, Grenoble INP, LJK, France

<sup>(3)</sup> School of Mathematical Sciences, University of Nottingham, UK

Section A gives the proofs of our main results, Section B gives our auxiliary results and their proofs and Section C presents additional figures referred to in the main article.

Before that, let us recall that our second-order condition  $\mathcal{C}_2(\gamma, \rho, A)$  is actually equivalent to the convergence

$$\forall x > 0, \lim_{t \rightarrow \infty} \frac{1}{A(t)} \left[ \frac{U(tx)}{U(t)} - x^\gamma \right] = x^\gamma \frac{x^\rho - 1}{\rho}. \quad (1)$$

In (1), the right-hand side should be read as  $x^\gamma \log x$  when  $\rho = 0$ .

## A Proofs of the main results

*Proof of Proposition 1.* Throughout this proof, we let, for a given  $\alpha \in (0, 1)$ ,  $X_\alpha$  be a random variable having the same distribution as  $X$  given  $X > q(\alpha)$ .

A tail  $L^p$ -median with order  $\alpha$  is nothing but an  $L^p$ -quantile of order  $1/2$  for  $X_\alpha$ , and we know that (for  $p > 1$ ) an  $L^p$ -quantile exists and is unique from *e.g.* [3], proving the first statement. The second statement is an immediate consequence of the equivariance of quantiles with respect to continuous and increasing transformations. To show the third statement, apply (A.1) in the proof of Proposition 1 in [4] to the  $L^p$ -median of  $X_\alpha$ , to obtain

$$\mathbb{E}[(m_p(\alpha) - X_\alpha)^{p-1} \mathbb{1}_{\{X_\alpha < m_p(\alpha)\}}] = \mathbb{E}[(X_\alpha - m_p(\alpha))^{p-1} \mathbb{1}_{\{X_\alpha > m_p(\alpha)\}}].$$

Because  $X_\alpha \stackrel{d}{=} X \mid X > q(\alpha)$ , this gives

$$\mathbb{E}[(m_p(\alpha) - X)^{p-1} \mathbb{1}_{\{q(\alpha) < X < m_p(\alpha)\}} \mid X > q(\alpha)] = \mathbb{E}[(X - m_p(\alpha))^{p-1} \mathbb{1}_{\{X > m_p(\alpha)\}} \mid X > q(\alpha)].$$

Multiply both sides of this identity by  $\mathbb{P}(X > q(\alpha))$  to conclude the proof of the third statement. The fourth statement is obvious for  $p = 1$ , and is a consequence of a straightforward modification of Proposition 5(c) in [2] for  $p > 1$ . To prove the fifth assertion, let  $\beta \in (0, 1)$  be such that  $\alpha < \beta$ . Then  $\mathbb{P}(X_\beta \leq x) = 0 \leq \mathbb{P}(X_\alpha \leq x)$  if  $x \leq q(\beta)$  and otherwise,

$$\mathbb{P}(X_\alpha \leq x) - \mathbb{P}(X_\beta \leq x) = \frac{F(x) - \alpha}{1 - \alpha} - \frac{F(x) - \beta}{1 - \beta} = \frac{\beta - \alpha}{(1 - \alpha)(1 - \beta)}(1 - F(x)) \geq 0.$$

In other words,  $X_\alpha$  is first-order stochastically dominated by  $X_\beta$ . Since

$$m_p(\alpha) = \arg \min_{m \in \mathbb{R}} \mathbb{E}(|X_\alpha - m|^p - |X_\alpha|^p) \quad \text{and} \quad m_p(\beta) = \arg \min_{m \in \mathbb{R}} \mathbb{E}(|X_\beta - m|^p - |X_\beta|^p)$$

it follows that  $m_p(\alpha) \leq m_p(\beta)$  by the fourth statement, completing the proof.  $\blacksquare$

*Proof of Proposition 2.* We consider first the case  $p = 1$ . Recall that  $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$  and  $\Gamma(x + 1) = x\Gamma(x)$ , for all  $x, y > 0$ . Therefore

$$B(1, \gamma^{-1}) = \frac{\Gamma(1)\Gamma(\gamma^{-1})}{\Gamma(1 + \gamma^{-1})} = \gamma.$$

Moreover

$$g_{1,\gamma}(t) = \int_t^1 u^{-1/\gamma-1} du = \gamma(t^{-1/\gamma} - 1).$$

It immediately follows that  $g_{1,\gamma}^{-1}(u) = (1 + u/\gamma)^{-\gamma}$  and therefore  $\kappa(1, \gamma) = 2^{-\gamma}$ . Under  $\mathcal{C}_1(\gamma)$ , the ratio  $m_1(\alpha)/q(\alpha) = q((1 + \alpha)/2)/q(\alpha) = U(2(1 - \alpha)^{-1})/U((1 - \alpha)^{-1})$  converges to  $2^\gamma$  as  $\alpha \rightarrow 1^-$  and thus Proposition 2 is true for  $p = 1$ .

We now assume that  $p > 1$ . Rewrite Proposition 1(iii) as  $J_1(m; p) = J_2(m; p)$  with

$$J_1(m; p) := \mathbb{E} \left[ \left( \frac{X}{m} - 1 \right)^{p-1} \mathbb{1}_{\{X > m\}} \right] \quad \text{and} \quad J_2(m; p) := \mathbb{E} \left[ \left( 1 - \frac{X}{m} \right)^{p-1} \mathbb{1}_{\{q(\alpha) < X < m\}} \right].$$

By Lemma 1(i) with  $a = 1$ ,

$$J_1(m; p) = \frac{\overline{F}(m)}{\gamma} B(p, \gamma^{-1} - p + 1)(1 + o(1)) \tag{2}$$

as  $m \rightarrow \infty$ . We now work on the expectation  $J_2(m; p)$ , and we are first going to prove that

$$\liminf_{\alpha \rightarrow 1^-} \frac{q(\alpha)}{m_p(\alpha)} > 0, \tag{3}$$

when  $m_p(\alpha)$  is solution of the characteristic equation in Proposition 1(iii). Suppose that this is not the case, so that there is a sequence  $\alpha_n \rightarrow 1^-$  with  $q(\alpha_n)/m_p(\alpha_n) \rightarrow 0$ . By this characteristic equation and (2), we get

$$\limsup_{n \rightarrow \infty} \frac{1}{\overline{F}(m_p(\alpha_n))} \mathbb{E} \left[ \left( 1 - \frac{X}{m_p(\alpha_n)} \right)^{p-1} \mathbb{1}_{\{q(\alpha_n) < X < m_p(\alpha_n)\}} \right] < \infty.$$

Now for large enough  $n$ ,  $q(\alpha_n) < m_p(\alpha_n)/2$  and so

$$\begin{aligned}
& \frac{1}{\overline{F}(m_p(\alpha_n))} \mathbb{E} \left[ \left( 1 - \frac{X}{m_p(\alpha_n)} \right)^{p-1} \mathbb{1}_{\{q(\alpha_n) < X < m_p(\alpha_n)\}} \right] \\
& \geq 2^{-(p-1)} \frac{1}{\overline{F}(m_p(\alpha_n))} \mathbb{P}(q(\alpha_n) < X < m_p(\alpha_n)/2) \\
& = 2^{-(p-1)} \frac{\overline{F}(m_p(\alpha_n)/2)}{\overline{F}(m_p(\alpha_n))} \left( \frac{\overline{F}(q(\alpha_n))}{\overline{F}(m_p(\alpha_n)/2)} - 1 \right) \\
& = 2^{-(p-\gamma^{-1}-1)} \left( \frac{\overline{F}(q(\alpha_n))}{\overline{F}(m_p(\alpha_n)/2)} - 1 \right) (1 + o(1)).
\end{aligned}$$

Since  $q(\alpha_n)/m_p(\alpha_n) \rightarrow 0$ , by Potter bounds (cf. Proposition B.1.9.5 p.367 in [6]) the lower bound should tend to  $\infty$ , contradicting our working assumption and proving (3). Apply now Lemma 1(ii) with  $a = 1$  to get

$$\frac{J_2(m_p(\alpha); p)}{\overline{F}(m_p(\alpha))} = \frac{1}{\gamma} g_{p,\gamma} \left( \frac{q(\alpha)}{m_p(\alpha)} \right) + o(1), \quad (4)$$

as  $\alpha \rightarrow 1^-$ . Use finally (2) with  $m = m_p(\alpha)$  together with (4) to get

$$\lim_{\alpha \rightarrow 1^-} g_{p,\gamma} \left( \frac{q(\alpha)}{m_p(\alpha)} \right) = B(p, \gamma^{-1} - p + 1).$$

Since  $g_{p,\gamma}$  is a continuous and strictly decreasing function, the conclusion follows.  $\blacksquare$

*Proof of Proposition 3.* Again, we consider the case  $p = 1$  first. Here  $m_1(\alpha) = q((1 + \alpha)/2)$ ; moreover,  $B(1, \gamma^{-1}) = \gamma$  for any  $\gamma > 0$  and  $\kappa(1, \gamma) = 2^{-\gamma}$ , see the first part of the proof of Proposition 2. This entails:

- $B(1, (1 - \rho)\gamma^{-1}) = \frac{\gamma}{1 - \rho}$ ;
- $g_{1,\gamma/(1-\rho)}(\kappa(1, \gamma)) = \frac{\gamma}{1 - \rho} (2^{1-\rho} - 1)$ .

Combining all these identities gives  $R(1, \gamma, \rho) = (2^\rho - 1)/\rho$ . Besides, condition (1) with  $t = (1 - \alpha)^{-1}$  and  $x = 2$  yields

$$\frac{m_1(\alpha)}{q(\alpha)} = \frac{q((1 + \alpha)/2)}{q(\alpha)} = \frac{U(2(1 - \alpha)^{-1})}{U((1 - \alpha)^{-1})} = 2^\gamma + A((1 - \alpha)^{-1}) 2^\gamma \frac{2^\rho - 1}{\rho} (1 + o(1))$$

which, recalling that  $\kappa(1, \gamma) = 2^{-\gamma}$  and  $R(1, \gamma, \rho) = (2^\rho - 1)/\rho$ , shows the result for  $p = 1$ .

We now turn to the case  $p > 1$ . As in the proof of Proposition 2, we consider the characteristic equation  $J_1(m_p(\alpha); p) = J_2(m_p(\alpha); p)$ , and we compute asymptotic expansions of both sides as  $\alpha \rightarrow 1^-$ . We use first (A.8) in the proof of Proposition 2 in [4] along with the change of variables  $u = 1/x$  to get

$$\frac{J_1(m_p(\alpha); p)}{\overline{F}(m_p(\alpha))} = \frac{B(p, \gamma^{-1} - p + 1)}{\gamma} + A \left( \frac{1}{\overline{F}(m_p(\alpha))} \right) K_1(p, \gamma, \rho) (1 + o(1)) \quad (5)$$

$$\begin{aligned}
\text{with } K_1(p, \gamma, \rho) & := \int_1^{+\infty} (p-1)(x-1)^{p-2} x^{-1/\gamma} \frac{x^{\rho/\gamma} - 1}{\gamma\rho} dx \\
& = \frac{1}{\gamma^2\rho} [(1-\rho)B(p, (1-\rho)\gamma^{-1} - p + 1) - B(p, \gamma^{-1} - p + 1)].
\end{aligned}$$

We now work on the expectation  $J_2(m_p(\alpha); p)$ . Recall equation (12) in the proof of Lemma 1(ii), which with  $a = 1$  and  $m(\alpha) = m_p(\alpha)$  reads

$$\begin{aligned} \frac{J_2(m_p(\alpha); p)}{\overline{F}(m_p(\alpha))} &= \frac{1}{\gamma} g_{p,\gamma} \left( \frac{q(\alpha)}{m_p(\alpha)} \right) + \left[ \frac{\overline{F}(q(\alpha))}{\overline{F}(m_p(\alpha))} - \left( \frac{q(\alpha)}{m_p(\alpha)} \right)^{-1/\gamma} \right] \left( 1 - \frac{q(\alpha)}{m_p(\alpha)} \right)^{p-1} \\ &\quad - (p-1) \int_{q(\alpha)/m_p(\alpha)}^1 (1-u)^{p-2} \left( \frac{\overline{F}(m_p(\alpha)u)}{\overline{F}(m_p(\alpha))} - u^{-1/\gamma} \right) du. \end{aligned}$$

By Remark B.3.8 in [6], the convergence in condition  $\mathcal{C}_2(\gamma, \rho, A)$  is uniform on compact subintervals of  $(0, \infty)$ . Because  $q(\alpha)/m_p(\alpha)$  has a finite and positive limit  $x_{p,\gamma} = \kappa(p, \gamma)$  as  $\alpha \rightarrow 1^-$ , we obtain

$$\frac{\overline{F}(q(\alpha))}{\overline{F}(m_p(\alpha))} - \left( \frac{q(\alpha)}{m_p(\alpha)} \right)^{-1/\gamma} = A \left( \frac{1}{\overline{F}(m_p(\alpha))} \right) x_{p,\gamma}^{-1/\gamma} \frac{x_{p,\gamma}^{\rho/\gamma} - 1}{\gamma\rho} (1 + o(1)).$$

Use now a uniform second-order control inequality such as Theorem B.3.10 in [6] together with the dominated convergence theorem to get

$$\begin{aligned} &\int_{q(\alpha)/m_p(\alpha)}^1 (1-u)^{p-2} \left( \frac{\overline{F}(m_p(\alpha)u)}{\overline{F}(m_p(\alpha))} - u^{-1/\gamma} \right) du \\ &= A \left( \frac{1}{\overline{F}(m_p(\alpha))} \right) \int_{x_{p,\gamma}}^1 (1-u)^{p-2} u^{-1/\gamma} \frac{u^{\rho/\gamma} - 1}{\gamma\rho} du (1 + o(1)). \end{aligned}$$

Integrating by parts then entails

$$-(p-1) \int_{x_{p,\gamma}}^1 (1-u)^{p-2} u^{-1/\gamma} \frac{u^{\rho/\gamma} - 1}{\gamma\rho} du = -(1-x_{p,\gamma})^{p-1} x_{p,\gamma}^{-1/\gamma} \frac{x_{p,\gamma}^{\rho/\gamma} - 1}{\gamma\rho} + K_2(p, \gamma, \rho)$$

with

$$K_2(p, \gamma, \rho) = \frac{1}{\gamma^2\rho} \left[ (1-\rho)g_{p,\gamma/(1-\rho)}(x_{p,\gamma}) - B(p, \gamma^{-1} - p + 1) \right].$$

Therefore

$$\frac{J_2(m_p(\alpha); p)}{\overline{F}(m_p(\alpha))} = \frac{1}{\gamma} g_{p,\gamma} \left( \frac{q(\alpha)}{m_p(\alpha)} \right) + A \left( \frac{1}{\overline{F}(m_p(\alpha))} \right) K_2(p, \gamma, \rho) (1 + o(1)). \quad (6)$$

Letting

$$K(p, \gamma, \rho) := \frac{1-\rho}{\gamma\rho} \left[ B(p, (1-\rho)\gamma^{-1} - p + 1) - g_{p,\gamma/(1-\rho)}(\kappa(p, \gamma)) \right]$$

we then get by combining (5) and (6) that

$$\begin{aligned} \frac{q(\alpha)}{m_p(\alpha)} &= g_{p,\gamma}^{-1} \left( B(p, \gamma^{-1} - p + 1) + A \left( \frac{1}{\overline{F}(m_p(\alpha))} \right) (K(p, \gamma, \rho) + o(1)) \right) \\ &= \kappa(p, \gamma) + A \left( \frac{1}{\overline{F}(m_p(\alpha))} \right) (K(p, \gamma, \rho) [g_{p,\gamma}^{-1}]'(B(p, \gamma^{-1} - p + 1)) + o(1)). \end{aligned}$$

Since the function  $A \circ (1/\overline{F})$  is regularly varying with index  $\rho/\gamma$  and  $g'_{p,\gamma}(t) = -(1-t)^{p-1} t^{-1/\gamma-1}$ , this yields

$$\frac{q(\alpha)}{m_p(\alpha)} = \kappa(p, \gamma) - A \left( (1-\alpha)^{-1} \right) \left[ \frac{1}{\kappa(p, \gamma)} \right]^{-1+(\rho-1)/\gamma} \frac{1}{[1-\kappa(p, \gamma)]^{p-1}} (K(p, \gamma, \rho) + o(1)).$$

Factor now  $\kappa(p, \gamma)$  out and use a Taylor expansion to complete the proof. ■

*Proof of Theorem 1.* We first show that if  $k = k(n) := \lfloor n(1 - \alpha_n) \rfloor$ , then

$$\sqrt{n(1 - \alpha_n)} \left( \frac{\widehat{m}_p(\alpha_n)}{m_p(\alpha_n)} - 1 \right) \text{ and } \sqrt{n(1 - \alpha_n)} \left( \frac{\widehat{m}_p(1 - k/n)}{m_p(1 - k/n)} - 1 \right)$$

share the same asymptotic distribution. To this end, remark that  $k/n \sim 1 - \alpha_n$  and thus, by assumption,  $k \rightarrow \infty$  and  $k/n \rightarrow 0$ . Furthermore, we have  $\widehat{m}_p(\alpha_n) = \widehat{m}_p(1 - k/n)$  and, under condition  $\mathcal{C}_2(\gamma, \rho, A)$ , Proposition 3 and the regular variation of the function  $|A|$  entail

$$\frac{m_p(1 - k/n)}{m_p(\alpha_n)} - 1 = o\left(A((1 - \alpha_n)^{-1})\right) = o\left(1/\sqrt{n(1 - \alpha_n)}\right),$$

since  $\sqrt{n(1 - \alpha_n)}A((1 - \alpha_n)^{-1}) = O(1)$ . The aforementioned conclusion then easily follows, and it is therefore enough to consider the case when  $n(1 - \alpha_n)$  defines a sequence of positive integers.

We start by the case  $p = 1$ , for which the estimator  $\widehat{m}_1(1 - k/n)$  is, sample-wise, the tail  $L^1$ -median related to the empirical distribution function  $\widehat{F}_n$ , whose quantile function is  $\widehat{q}_n(\alpha) = X_{\lceil n\alpha \rceil, n}$ . Since (by continuity of  $F$ )  $\widehat{F}_n(\widehat{q}_n(1 - k/n)) = 1 - k/n$ , it follows that

$$\widehat{m}_1(1 - k/n) = \widehat{q}_n \left( \frac{1}{2} \left[ 1 + \left\{ 1 - \frac{k}{n} \right\} \right] \right) = X_{\lceil n - k/2 \rceil, n} = X_{n - \lfloor k/2 \rfloor, n}.$$

Applying Theorem 2.4.8 p.52 in [6] gives

$$\sqrt{n(1 - \alpha_n)} \left( \frac{\widehat{m}_1(\alpha_n)}{m_1(\alpha_n)} - 1 \right) = \sqrt{2} \sqrt{\frac{k}{2}} \left( \frac{X_{n - \lfloor k/2 \rfloor, n}}{q(1 - k/2n)} - 1 \right) \xrightarrow{d} \mathcal{N}(0, 2\gamma^2).$$

Besides, we have by Lemma 2(i) and (ii) that  $V_1(1, \gamma) = 2$ . Since  $V_2(1, \gamma) = 1/\gamma^2$ , we get indeed

$$\sqrt{n(1 - \alpha_n)} \left( \frac{\widehat{m}_1(\alpha_n)}{m_1(\alpha_n)} - 1 \right) \xrightarrow{d} \mathcal{N}(0, V(1, \gamma))$$

as required.

In the case  $p > 1$ , use Lemma 6 and then Lemmas 7, 9 and 11 to get, for any  $u$ ,

$$\psi_n(u; p) \xrightarrow{d} -uZ \sqrt{V_1(p, \gamma)} + \frac{u^2}{2} \sqrt{V_2(p, \gamma)} \text{ as } n \rightarrow \infty$$

(with  $Z$  being standard Gaussian) in the sense of finite-dimensional convergence. As a function of  $u$ , this limit is almost surely finite and defines a convex function which has a unique minimum at

$$u^* = \sqrt{\frac{V_1(p, \gamma)}{V_2(p, \gamma)}} Z = \sqrt{V(p, \gamma)} Z \stackrel{d}{=} \mathcal{N}(0, V(p, \gamma)).$$

Applying the convexity lemma of [5] completes the proof of the required convergence.

It only remains to show that  $V_2(\cdot, \gamma)$  and  $V(\cdot, \gamma)$  are right-continuous at 1. We get by using the relationship  $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$  that

$$\forall p > 1, (p - 1)B(p - 1, \gamma^{-1} - p + 2) = \frac{\Gamma(p)\Gamma(\gamma^{-1} - p + 2)}{\Gamma(\gamma^{-1} + 1)}$$

and thus  $(p-1)B(p-1, \gamma^{-1} - p + 2) \rightarrow 1$  as  $p \downarrow 1$ . Furthermore, it is a consequence of an integration by parts that

$$\begin{aligned} \forall p > 1, \forall t \in (0, 1), (p-1)g_{p-1, \gamma}(t) &= (p-1) \int_t^1 (1-u)^{p-2} u^{-1/\gamma-1} du \\ &= (1-t)^{p-1} t^{-1/\gamma-1} - \left(\frac{1}{\gamma} + 1\right) \int_t^1 (1-u)^{p-1} u^{-1/\gamma-2} du. \end{aligned}$$

It follows that if  $t_p \rightarrow t \in (0, 1)$  as  $p \downarrow 1$ , then by the dominated convergence theorem,

$$(p-1)g_{p-1, \gamma}(t_p) \rightarrow t^{-1/\gamma-1} - \left(\frac{1}{\gamma} + 1\right) \int_t^1 u^{-1/\gamma-2} du = 1 \quad \text{as } p \downarrow 1$$

and therefore,  $(p-1)g_{p-1, \gamma}(\kappa(p, \gamma)) \rightarrow 1$  as  $p \downarrow 1$ . All in all, using Lemma 2(ii),

$$\lim_{p \downarrow 1} V_2(p, \gamma) = \left(2 \frac{[\kappa(1, \gamma)]^{1/\gamma}}{\gamma}\right)^2 = \frac{1}{\gamma^2} = V_2(1, \gamma).$$

This completes the proof. ■

*Proof of Theorem 2.* Let us consider the function  $G_p$  defined by  $G_p(t) = 1/\kappa(p, t)$  for all  $0 < t < 1/(p-1)$ . Using Proposition 3, we get:

$$\begin{aligned} \log \left( \frac{\tilde{m}_p(\alpha_n)}{m_p(\alpha_n)} \right) &= \log G_p(\hat{\gamma}_n) - \log G_p(\gamma) + \log \left( \frac{X_{\lceil n\alpha_n \rceil, n}}{q(\alpha_n)} \right) \\ &\quad - \log [1 + (R(p, \gamma, \rho) + o(1))A((1 - \alpha_n)^{-1})]. \end{aligned}$$

Now

$$\sqrt{n(1 - \alpha_n)} \left( \log G_p(\hat{\gamma}_n) - \log G_p(\gamma) + \log \left( \frac{X_{\lceil n\alpha_n \rceil, n}}{q(\alpha_n)} \right) \right) \xrightarrow{d} (\log G_p)'(\gamma)\xi_1 + \xi_2$$

by the delta-method. We also have

$$\sqrt{n(1 - \alpha_n)} \log [1 + (R(p, \gamma, \rho) + o(1))A((1 - \alpha_n)^{-1})] \rightarrow \lambda R(p, \gamma, \rho)$$

by a Taylor expansion. Applying the delta-method again, we get

$$\sqrt{n(1 - \alpha_n)} \left( \frac{\tilde{m}_p(\alpha_n)}{m_p(\alpha_n)} - 1 \right) \xrightarrow{d} (\log G_p)'(\gamma)\xi_1 + \xi_2 - \lambda R(p, \gamma, \rho).$$

It remains to show that  $(\log G_p)'(\gamma) = \sigma(p, \gamma)$ . Note that

$$(\log G_p)'(\gamma) = -\frac{1}{\kappa(p, \gamma)} \times \frac{\partial}{\partial t} [\kappa(p, t)] \Big|_{t=\gamma}$$

and let us focus on the derivative appearing in the right-hand side. To this end, we define

$$\mathfrak{g}(x, t) := \int_x^1 (1-u)^{p-1} u^{-1/t-1} du = g_{p,t}(x)$$

and, for a fixed  $t$ , we let  $y \mapsto \mathfrak{h}(y, t)$  be the inverse of the mapping  $x \mapsto \mathfrak{g}(x, t)$  so that  $\mathfrak{h}(y, t) = g_{p,t}^{-1}(y)$ . The derivative we should compute is then

$$\left. \frac{\partial}{\partial t} [\kappa(p, t)] \right|_{t=\gamma} = \left. \frac{\partial}{\partial t} [\mathfrak{h}(B(p, t^{-1} - p + 1), t)] \right|_{t=\gamma}.$$

Therefore

$$\begin{aligned} & \left. \frac{\partial}{\partial t} [\kappa(p, t)] \right|_{t=\gamma} \\ &= \left. \frac{d}{dt} [B(p, t^{-1} - p + 1)] \right|_{t=\gamma} \times (\partial_1 \mathfrak{h})(B(p, \gamma^{-1} - p + 1), \gamma) + (\partial_2 \mathfrak{h})(B(p, \gamma^{-1} - p + 1), \gamma) \end{aligned} \quad (7)$$

where  $\partial_j$  denotes the partial derivative w.r.t. the  $j$ th variable. Clearly

$$\left. \frac{d}{dt} [B(p, t^{-1} - p + 1)] \right|_{t=\gamma} = \frac{B(p, \gamma^{-1} - p + 1)}{\gamma^2} (\Psi(\gamma^{-1} + 1) - \Psi(\gamma^{-1} - p + 1)) \quad (8)$$

where  $\Psi(x) = \Gamma'(x)/\Gamma(x)$  is Euler's digamma function. To compute the partial derivatives of  $\mathfrak{h}$ , we differentiate the obvious identity  $\mathfrak{g}(\mathfrak{h}(y, t), t) = y$  to get

$$(\partial_1 \mathfrak{h})(y, t) = \frac{1}{(\partial_1 \mathfrak{g})(\mathfrak{h}(y, t), t)} \text{ and } (\partial_2 \mathfrak{h})(y, t) = -\frac{(\partial_2 \mathfrak{g})(\mathfrak{h}(y, t), t)}{(\partial_1 \mathfrak{g})(\mathfrak{h}(y, t), t)}.$$

We now differentiate  $\mathfrak{g}(x, t) = \int_x^1 (1-u)^{p-1} u^{-1/t-1} du$  and we wrap up to obtain

$$(\partial_1 \mathfrak{h})(y, t) = -\frac{1}{[1 - \mathfrak{h}(y, t)]^{p-1} [\mathfrak{h}(y, t)]^{-1/t-1}} \text{ and } (\partial_2 \mathfrak{h})(y, t) = \frac{\int_{\mathfrak{h}(y, t)}^1 (1-u)^{p-1} u^{-1/t-1} \log(u) du}{t^2 [1 - \mathfrak{h}(y, t)]^{p-1} [\mathfrak{h}(y, t)]^{-1/t-1}}. \quad (9)$$

Reporting (8) and (9) into (7) and recalling that  $\mathfrak{h}(B(p, \gamma^{-1} - p + 1), \gamma) = \kappa(p, \gamma)$  yields

$$\begin{aligned} & \left. \frac{\partial}{\partial t} [\kappa(p, t)] \right|_{t=\gamma} \\ &= \frac{\int_{\kappa(p, \gamma)}^1 (1-u)^{p-1} u^{-1/\gamma-1} \log(u) du - B(p, \gamma^{-1} - p + 1) [\Psi(\gamma^{-1} + 1) - \Psi(\gamma^{-1} - p + 1)]}{\gamma^2 [1 - \kappa(p, \gamma)]^{p-1} [\kappa(p, \gamma)]^{-1/\gamma-1}} \end{aligned}$$

and therefore

$$(\log G_p)'(\gamma) = -\frac{1}{\kappa(p, \gamma)} \times \left. \frac{\partial}{\partial t} [\kappa(p, t)] \right|_{t=\gamma} = \sigma(p, \gamma)$$

as required. The proof is complete. ■

*Proof of Theorem 3.* To show (i), write

$$\log \left( \frac{\widehat{m}_p^W(\beta_n)}{m_p(\beta_n)} \right) = (\widehat{\gamma}_n - \gamma) \log \left( \frac{1 - \alpha_n}{1 - \beta_n} \right) + \log \left( \frac{\widehat{m}_p(\alpha_n)}{m_p(\alpha_n)} \right) - \log \left( \left[ \frac{1 - \beta_n}{1 - \alpha_n} \right]^\gamma \frac{m_p(\beta_n)}{m_p(\alpha_n)} \right).$$

The convergence  $\log[(1 - \alpha_n)/(1 - \beta_n)] \rightarrow \infty$  yields

$$\frac{\sqrt{n(1 - \alpha_n)}}{\log[(1 - \alpha_n)/(1 - \beta_n)]} \log \left( \frac{\widehat{m}_p(\alpha_n)}{m_p(\alpha_n)} \right) = O_{\mathbb{P}} \left( \frac{1}{\log[(1 - \alpha_n)/(1 - \beta_n)]} \right) = o_{\mathbb{P}}(1), \quad (10)$$

$$\begin{aligned} \text{and } & \frac{\sqrt{n(1 - \alpha_n)}}{\log[(1 - \alpha_n)/(1 - \beta_n)]} \log \left( \left[ \frac{1 - \beta_n}{1 - \alpha_n} \right]^{\gamma} \frac{m_p(\beta_n)}{m_p(\alpha_n)} \right) \\ &= \frac{\sqrt{n(1 - \alpha_n)}}{\log[(1 - \alpha_n)/(1 - \beta_n)]} \left( \log \left( \frac{m_p(\beta_n)}{q(\beta_n)} \right) - \log \left( \frac{m_p(\alpha_n)}{q(\alpha_n)} \right) + \log \left( \left[ \frac{1 - \beta_n}{1 - \alpha_n} \right]^{\gamma} \frac{q(\beta_n)}{q(\alpha_n)} \right) \right) \\ &= O \left( \frac{\sqrt{n(1 - \alpha_n)}}{\log[(1 - \alpha_n)/(1 - \beta_n)]} [|A((1 - \alpha_n)^{-1})| + |A((1 - \beta_n)^{-1})|] \right) \\ &= O \left( \frac{\sqrt{n(1 - \alpha_n)}}{\log[(1 - \alpha_n)/(1 - \beta_n)]} |A((1 - \alpha_n)^{-1})| \right) \\ &= o(1). \end{aligned} \quad (11)$$

Convergence (10) is a consequence of our Theorem 1. Convergence (11) follows from a combination of Proposition 3, of Theorem 2.3.9 in [6] (resulting in

$$\frac{1}{A((1 - \alpha_n)^{-1})} \left( \left[ \frac{1 - \beta_n}{1 - \alpha_n} \right]^{\gamma} \frac{q(\beta_n)}{q(\alpha_n)} - 1 \right) \rightarrow -\frac{1}{\rho} \text{ as } n \rightarrow \infty$$

because  $\rho < 0$ ), and of the regular variation property of  $|A|$ . A combination of these elements with the delta-method leads to the desired conclusion.

The proof of (ii) is very similar: just replace  $\widehat{m}_p^W(\beta_n)$  by  $\widetilde{m}_p^W(\beta_n)$ ,  $\widehat{m}_p(\alpha_n)$  by  $\widetilde{m}_p(\alpha_n)$  and apply Theorem 2 instead of Theorem 1 to prove an analogue of (10). We omit the details for the sake of brevity. ■

## B Auxiliary results and proofs

Lemma 1 below contains an equivalent of certain moments upon which the proofs of Propositions 2 and 3 rely. It is also useful when it comes to proving Theorem 1.

**Lemma 1.** *Suppose that  $\mathcal{C}_1(\gamma)$  holds. Let  $a \geq 1$  and  $p > 1$  with  $\gamma < 1/(a[p - 1])$ . Then:*

(i) *As  $m \rightarrow \infty$ ,*

$$\mathbb{E} \left[ \left( \frac{X}{m} - 1 \right)^{a(p-1)} \mathbb{1}_{\{X > m\}} \right] = \overline{F}(m) \frac{B(a(p-1) + 1, \gamma^{-1} - a(p-1))}{\gamma} (1 + o(1)).$$

(ii) *If moreover  $m(\alpha) > q(\alpha)$  is such that  $\liminf_{\alpha \rightarrow 1^-} q(\alpha)/m(\alpha) > 0$  then, as  $\alpha \rightarrow 1^-$ :*

$$\mathbb{E} \left[ \left( 1 - \frac{X}{m(\alpha)} \right)^{a(p-1)} \mathbb{1}_{\{q(\alpha) < X < m(\alpha)\}} \right] = \frac{\overline{F}(m(\alpha))}{\gamma} g_{a(p-1)+1, \gamma} \left( \frac{q(\alpha)}{m(\alpha)} \right) (1 + o(1)).$$



*Proof of Lemma 1.* The proof of (i) relies on an integration by parts, which yields

$$\mathbb{E} \left[ \left( \frac{X}{m} - 1 \right)^{a(p-1)} \mathbb{1}_{\{X > m\}} \right] = \frac{1}{m} \int_m^\infty a(p-1) \left( \frac{x}{m} - 1 \right)^{a(p-1)-1} \bar{F}(x) dx.$$

Use now a linear change of variables and rewrite the right-hand side as

$$\bar{F}(m) \left( \int_1^\infty a(p-1)(y-1)^{a(p-1)-1} y^{-1/\gamma} dy + \int_1^\infty a(p-1)(y-1)^{a(p-1)-1} \left[ \frac{\bar{F}(my)}{\bar{F}(m)} - y^{-1/\gamma} \right] dy \right).$$

A uniform bound such as Theorem B.2.18 in [6] thus entails

$$\mathbb{E} \left[ \left( \frac{X}{m} - 1 \right)^{a(p-1)} \mathbb{1}_{\{X > m\}} \right] = \bar{F}(m) \int_1^{+\infty} a(p-1)(y-1)^{a(p-1)-1} y^{-1/\gamma} dy (1 + o(1)).$$

Another integration by parts and the change of variables  $z = 1/y$  finally yield

$$\begin{aligned} \mathbb{E} \left[ \left( \frac{X}{m} - 1 \right)^{a(p-1)} \mathbb{1}_{\{X > m\}} \right] &= \frac{\bar{F}(m)}{\gamma} \int_1^{+\infty} (y-1)^{a(p-1)} y^{-1/\gamma-1} dy (1 + o(1)) \\ &= \frac{\bar{F}(m)}{\gamma} \int_0^1 (1-z)^{a(p-1)} z^{1/\gamma-a(p-1)-1} dz (1 + o(1)) \\ &= \frac{\bar{F}(m)}{\gamma} B(a(p-1) + 1, \gamma^{-1} - a(p-1)) (1 + o(1)) \end{aligned}$$

as required.

We now show (ii). Integrating by parts leads to

$$\begin{aligned} &\mathbb{E} \left[ \left( 1 - \frac{X}{m(\alpha)} \right)^{a(p-1)} \mathbb{1}_{\{q(\alpha) < X < m(\alpha)\}} \right] \\ &= (1-\alpha) \left( 1 - \frac{q(\alpha)}{m(\alpha)} \right)^{a(p-1)} - \frac{a(p-1)}{m(\alpha)} \int_{q(\alpha)}^{m(\alpha)} \left( 1 - \frac{x}{m(\alpha)} \right)^{a(p-1)-1} \bar{F}(x) dx. \end{aligned}$$

Using a change of variables, we get

$$\begin{aligned} &\frac{1}{\bar{F}(m(\alpha))} \mathbb{E} \left[ \left( 1 - \frac{X}{m(\alpha)} \right)^{a(p-1)} \mathbb{1}_{\{q(\alpha) < X < m(\alpha)\}} \right] \\ &= \frac{\bar{F}(q(\alpha))}{\bar{F}(m(\alpha))} \left( 1 - \frac{q(\alpha)}{m(\alpha)} \right)^{a(p-1)} - a(p-1) \int_{q(\alpha)/m(\alpha)}^1 (1-u)^{a(p-1)-1} u^{-1/\gamma} du \\ &\quad - a(p-1) \int_{q(\alpha)/m(\alpha)}^1 (1-u)^{a(p-1)-1} \left( \frac{\bar{F}(m(\alpha)u)}{\bar{F}(m(\alpha))} - u^{-1/\gamma} \right) du. \end{aligned}$$

By an integration by parts again,

$$a(p-1) \int_{q(\alpha)/m(\alpha)}^1 (1-u)^{a(p-1)-1} u^{-1/\gamma} du = \left( \frac{q(\alpha)}{m(\alpha)} \right)^{-1/\gamma} \left( 1 - \frac{q(\alpha)}{m(\alpha)} \right)^{a(p-1)} - \frac{1}{\gamma} g_{a(p-1)+1, \gamma} \left( \frac{q(\alpha)}{m(\alpha)} \right)$$

so that we obtain the following decomposition:

$$\begin{aligned}
& \frac{1}{\overline{F}(m(\alpha))} \mathbb{E} \left[ \left(1 - \frac{X}{m(\alpha)}\right)^{a(p-1)} \mathbb{1}_{\{q(\alpha) < X < m(\alpha)\}} \right] - \frac{1}{\gamma} g_{a(p-1)+1, \gamma} \left( \frac{q(\alpha)}{m(\alpha)} \right) \\
&= \left[ \frac{\overline{F}(q(\alpha))}{\overline{F}(m(\alpha))} - \left( \frac{q(\alpha)}{m(\alpha)} \right)^{-1/\gamma} \right] \left(1 - \frac{q(\alpha)}{m(\alpha)}\right)^{a(p-1)} \\
&- a(p-1) \int_{q(\alpha)/m(\alpha)}^1 (1-u)^{(a-1)p-1} \left( \frac{\overline{F}(m(\alpha)u)}{\overline{F}(m(\alpha))} - u^{-1/\gamma} \right) du. \tag{12}
\end{aligned}$$

By Proposition B.1.10 in [6], the right-hand side of this equality converges to 0 as  $\alpha \rightarrow 1^-$  and the conclusion follows.  $\blacksquare$

The next two auxiliary results compute the constants appearing in our asymptotic results (Propositions 2 and 3, Theorems 1, 2 and Corollary 1) in the cases  $p = 1$  and  $p = 2$ . We start by examining the case  $p = 1$ .

**Lemma 2.** *Suppose that  $\mathcal{C}_1(\gamma)$  holds for some  $\gamma > 0$ .*

(i) *We have  $B(1, \gamma^{-1}) = \gamma$ ,  $g_{1, \gamma}(t) = \gamma(t^{-1/\gamma} - 1)$  and  $g_{1, \gamma}^{-1}(u) = (1 + u/\gamma)^{-\gamma}$ .*

(ii) *We have  $\kappa(1, \gamma) = 2^{-\gamma}$ .*

*Assume further that  $\mathcal{C}_2(\gamma, \rho, A)$  holds for some  $\rho \leq 0$ .*

(iii) *We have*

$$R(1, \gamma, \rho) = \begin{cases} \frac{2^\rho - 1}{\rho} & \text{if } \rho < 0, \\ \log 2 & \text{if } \rho = 0. \end{cases}$$

(iv) *We have  $V(1, \gamma) = 2\gamma^2$ .*

(v) *We have  $\sigma(1, \gamma) = \log 2$ .*

*Proof of Lemma 2.* Statements (i) and (ii) are proven at the very beginning of the proof of Proposition 2, while statement (iii) is shown in the first part of the proof of Proposition 3. Part (iv) is a direct consequence of (i) and (ii). To prove (v), note that  $\Psi(\gamma^{-1} + 1) - \Psi(\gamma^{-1}) = \gamma$ , see Formula 6.3.5 in [1]. Thus, by (i) and (ii):

$$\sigma(1, \gamma) = \frac{\gamma^2 - \int_{2^{-\gamma}}^1 u^{-1/\gamma-1} \log(u) du}{2\gamma^2}.$$

Besides, a straightforward calculation gives

$$\int_{2^{-\gamma}}^1 u^{-1/\gamma-1} \log(u) du = \gamma^2(1 - 2 \log 2).$$

This yields  $\sigma(1, \gamma) = \log 2$  as required and completes the proof.  $\blacksquare$

The following lemma examines the case  $p = 2$ .

**Lemma 3.** *Suppose that  $\mathcal{C}_1(\gamma)$  holds for some  $\gamma \in (0, 1)$ .*

(i) *We have  $B(2, \gamma^{-1} - 1) = \gamma^2/(1 - \gamma)$  and*

$$g_{2,\gamma}(t) = \frac{\gamma t^{-\gamma^{-1}}}{1 - \gamma} (1 - \gamma - t) + B(2, \gamma^{-1} - 1).$$

(ii) *We have  $\kappa(2, \gamma) = 1 - \gamma$ .*

*Assume further that  $\mathcal{C}_2(\gamma, \rho, A)$  holds for some  $\rho \leq 0$ .*

(iii) *We have  $R(2, \gamma, \rho) = 1/(1 - \gamma - \rho)$ .*

(iv) *If moreover  $\gamma < 1/2$ , then  $V(2, \gamma) = 2\gamma^2(1 - \gamma)/(1 - 2\gamma)$ .*

(v) *We have  $\sigma(2, \gamma) = 1/(1 - \gamma)$ .*

*Proof of Lemma 3.* To show (i) note that

$$B(2, \gamma^{-1} - 1) = \frac{\Gamma(2)\Gamma(\gamma^{-1} - 1)}{\Gamma(\gamma^{-1} + 1)} = \frac{\gamma^2}{1 - \gamma}.$$

Consequently

$$g_{2,\gamma}(t) = \int_t^1 (1 - u)u^{-1/\gamma-1} du = \frac{\gamma t^{-\gamma^{-1}}}{1 - \gamma} (1 - \gamma - t) + \frac{\gamma^2}{1 - \gamma} = \frac{\gamma t^{-\gamma^{-1}}}{1 - \gamma} (1 - \gamma - t) + B(2, \gamma^{-1} - 1)$$

which is (i). Part (ii) follows by noting that from (i),  $\kappa(2, \gamma) = g_{2,\gamma}^{-1}(B(2, \gamma^{-1} - 1))$  is such that  $1 - \gamma - \kappa(2, \gamma) = 0$ , *i.e.*  $\kappa(2, \gamma) = 1 - \gamma$ . To show (iii), we use part (i) to get

$$g_{2,\gamma/(1-\rho)}(t) = \frac{\gamma t^{-(1-\rho)/\gamma}}{1 - \rho - \gamma} \left( 1 - \frac{\gamma}{1 - \rho} - t \right) + B(2, (1 - \rho)\gamma^{-1} - 1).$$

Combined with the equality  $\kappa(2, \gamma) = 1 - \gamma$  from (ii), this yields indeed  $R(2, \gamma, \rho) = 1/(1 - \gamma - \rho)$ . To prove (iv), use parts (i) and (ii) to obtain

$$V_1(2, \gamma) = \frac{(1 - \gamma)^{1/\gamma}}{\gamma} (B(3, \gamma^{-1} - 2) + g_{3,\gamma}(1 - \gamma)) + \gamma^2.$$

Now

$$B(3, \gamma^{-1} - 2) = \frac{\Gamma(3)\Gamma(\gamma^{-1} - 2)}{\Gamma(\gamma^{-1} + 1)} = \frac{2\gamma^3}{(1 - \gamma)(1 - 2\gamma)}.$$

It is also easy to obtain, by expanding the integrand in  $g_{3,\gamma}$ , that

$$g_{3,\gamma}(1 - \gamma) = \int_{1-\gamma}^1 (1 - u)^2 u^{-1/\gamma-1} du = (1 - \gamma)^{-1/\gamma} \frac{\gamma^3}{1 - 2\gamma} - \frac{2\gamma^3}{(1 - \gamma)(1 - 2\gamma)}.$$

Consequently

$$V_1(2, \gamma) = \frac{\gamma^2}{1-2\gamma} + \gamma^2 = 2\gamma^2 \frac{1-\gamma}{1-2\gamma}.$$

Meanwhile, using again parts (i) and (ii) together with Lemma 2(i) entails

$$V_2(2, \gamma) = \left( \frac{(1-\gamma)^{1/\gamma}}{\gamma} [B(1, \gamma^{-1}) + g_{1,\gamma}(\kappa(2, \gamma))] \right)^2 = 1.$$

Therefore  $V(2, \gamma) = V_2(1, \gamma)/V_2(2, \gamma) = 2\gamma^2(1-\gamma)/(1-2\gamma)$ , which is (iv). Finally, to show (v), we remark that  $\Psi(\gamma^{-1}+1) - \Psi(\gamma^{-1}-1) = \gamma(2-\gamma)/(1-\gamma)$ , see Formula 6.3.5 in [1]. Therefore, using (i) and (ii), we get

$$\sigma(2, \gamma) = \frac{\gamma^3(2-\gamma) - (1-\gamma)^2 \int_{1-\gamma}^1 (1-u)u^{-1/\gamma-1} \log(u) du}{\gamma^3(1-\gamma)^{-1/\gamma+2}}.$$

Furthermore, it is straightforward to get

$$(1-\gamma)^2 \int_{1-\gamma}^1 (1-u)u^{-1/\gamma-1} \log(u) du = \gamma^3(2-\gamma - (1-\gamma)^{-1/\gamma+1})$$

and thus  $\sigma(2, \gamma) = 1/(1-\gamma)$  as required. The proof is complete.  $\blacksquare$

Our final series of lemmas is devoted to the proof of Theorem 1. In the rest of this section,  $k = k(n)$  is a sequence of integers with  $k \rightarrow \infty$  and  $k/n \rightarrow 0$ . It will be convenient to consider a complementary quantile function and a complementary tail  $L^p$ -median function defined by

$$\forall \alpha \in (0, 1), \quad Q(\alpha) = q(1-\alpha) \quad \text{and} \quad M_p(\alpha) = m_p(1-\alpha).$$

Pick a fixed real number  $u$  and let

$$I_n(u) := \left[ 0, \frac{|u|M_p(k/n)}{\sqrt{k}} \right].$$

We also define  $\varphi(x; p) = |x|^{p-1} \text{sign}(x)$ . Finally, throughout this section,  $[a, b]$  denotes the proper interval  $[a, b]$  if  $a \leq b$  and the interval  $[b, a]$  otherwise.

The next lemma gathers a few useful results about the interactions between  $Q(k/n)$ ,  $M_p(k/n)$  and the function  $\varphi$ .

**Lemma 4.** *Suppose that  $\mathcal{C}_2(\gamma, \rho, A)$  holds with  $\gamma < 1/(p-1)$  and  $p > 1$ , and  $\sqrt{k}A(n/k) = O(1)$ . Then:*

(i) *The function  $y \mapsto \varphi(y - M_p(k/n) - t; p)$  is continuously differentiable on  $[Q(k/n), X_{n-k,n}]$  with arbitrarily large probability as  $n \rightarrow \infty$ , irrespectively of  $t$  such that  $|t| \in I_n(u)$ .*

(ii) *We have*

$$\begin{aligned} & \varphi(X_{n-k,n} - M_p(k/n) - t; p) - \varphi(Q(k/n) - M_p(k/n) - t; p) \\ &= (p-1) [Q(k/n)]^{p-1} \left[ \frac{1}{\kappa(p, \gamma)} - 1 \right]^{p-2} \left( \frac{X_{n-k,n}}{Q(k/n)} - 1 \right) (1 + o_{\mathbb{P}}(1)) \\ &= O_{\mathbb{P}} \left( \frac{[M_p(k/n)]^{p-1}}{\sqrt{k}} \right) \end{aligned}$$

*uniformly in  $|t| \in I_n(u)$ .*

(iii) For  $n$  large enough, the function  $y \mapsto \varphi(Q(k/n) - M_p(k/n) - y; p)$  is continuously differentiable on the interval  $[-|u|M_p(k/n)/\sqrt{k}, |u|M_p(k/n)/\sqrt{k}]$ .

(iv) We have

$$\varphi(Q(k/n) - M_p(k/n) - t; p) - \varphi(Q(k/n) - M_p(k/n); p) = O\left(\frac{[M_p(k/n)]^{p-1}}{\sqrt{k}}\right)$$

uniformly in  $|t| \in I_n(u)$ .

*Proof of Lemma 4.* To prove (i), note that the function  $\varphi$  is continuously differentiable on  $\mathbb{R} \setminus \{0\}$  with derivative  $x \mapsto (p-1)|x|^{p-2}$ . Using Proposition 2 (with the fact that  $0 < \kappa(p, \gamma) < 1$ ) and Theorem 2.4.8 p.52 in [6] it is then easy to get that

$$\begin{aligned} Q(k/n) - M_p(k/n) + \frac{|u|M_p(k/n)}{\sqrt{k}} &< 0 \text{ for } n \text{ large enough} \\ \text{and } X_{n-k,n} - M_p(k/n) + \frac{|u|M_p(k/n)}{\sqrt{k}} &< 0 \text{ with probability tending to 1.} \end{aligned}$$

It is straightforward to see that this entails (i). To show (ii), use the mean value theorem and the definition of the function  $\varphi$  to get that, with probability tending to 1, there is  $C_{n,t} \in [Q(k/n), X_{n-k,n}]$  such that:

$$\begin{aligned} &\varphi(X_{n-k,n} - M_p(k/n) - t; p) - \varphi(Q(k/n) - M_p(k/n) - t; p) \\ &= Q(k/n) \left( \frac{X_{n-k,n}}{Q(k/n)} - 1 \right) (p-1) |C_{n,t} - M_p(k/n) - t|^{p-2}. \end{aligned} \quad (13)$$

Because  $C_{n,t} \in [Q(k/n), X_{n-k,n}]$ , we can use Theorem 2.4.8 p.52 in [6] and Proposition 2 to get

$$|C_{n,t} - M_p(k/n) - t|^{p-2} = [Q(k/n)]^{p-2} \left[ \frac{1}{\kappa(p, \gamma)} - 1 \right]^{p-2} (1 + o_{\mathbb{P}}(1)) \quad (14)$$

uniformly in  $|t| \in I_n(u)$ . Combining first (13) and (14), and using then Proposition 2 completes the proof of (ii). The proof of (iii) is similar to that of (i) and is therefore omitted. To show (iv), use the mean value theorem again: there is  $c_{n,t} \in [0, t]$  such that

$$|\varphi(Q(k/n) - M_p(k/n) - t; p) - \varphi(Q(k/n) - M_p(k/n); p)| = -t(p-1) |Q(k/n) - M_p(k/n) - c_{n,t}|^{p-2}.$$

Use then Proposition 2 and take into account that  $|t| \leq |u|M_p(k/n)/\sqrt{k}$  to conclude the proof.  $\blacksquare$

The next result makes it possible to replace the expression of the empirical criterion  $\psi_n(u; p)$ , written in terms of order statistics, essentially by a sum of independent and identically distributed random variables. It is therefore a key step in the theoretical analysis of the direct tail  $L^p$ -median estimator.

**Lemma 5.** Suppose that  $\mathcal{C}_2(\gamma, \rho, A)$  holds with  $\gamma < 1/(p-1)$  and  $p > 1$ , and  $\sqrt{k}A(n/k) = O(1)$ . Then:

$$\begin{aligned} & \frac{1}{\sqrt{k}[M_p(k/n)]^{p-1}} \sum_{i=1}^k \varphi(X_{n-i+1,n} - M_p(k/n) - t; p) \\ &= \frac{1}{\sqrt{k}[M_p(k/n)]^{p-1}} \sum_{i=1}^n \varphi(X_i - M_p(k/n) - t; p) \mathbb{1}_{\{X_i > Q(k/n)\}} \\ &+ \frac{\sqrt{k}}{[M_p(k/n)]^{p-1}} \varphi(Q(k/n) - M_p(k/n) - t; p) \left[ 1 - \frac{\sum_{i=1}^n \mathbb{1}_{\{X_i > Q(k/n)\}}}{k} \right] + o_{\mathbb{P}}(1) \end{aligned}$$

uniformly in  $|t| \in I_n(u)$ .

*Proof of Lemma 5.* Let first

$$\varphi_{n,t}(x) = \varphi(x - M_p(k/n) - t; p).$$

With this notation,

$$\begin{aligned} & \frac{1}{\sqrt{k}[M_p(k/n)]^{p-1}} \sum_{i=1}^k \varphi(X_{n-i+1,n} - M_p(k/n) - t; p) \\ &= \frac{\sqrt{k}}{[M_p(k/n)]^{p-1}} \left( \varphi_{n,t}(X_{n-k,n}) + \frac{1}{k} \sum_{i=1}^k [\varphi_{n,t}(X_{n-i+1,n}) - \varphi_{n,t}(X_{n-k,n})] \right) \\ &= \frac{\sqrt{k}}{[M_p(k/n)]^{p-1}} \left( \varphi_{n,t}(X_{n-k,n}) - \frac{n}{k} \int_{X_{n-k,n}}^{\infty} [\varphi_{n,t}(u) - \varphi_{n,t}(X_{n-k,n})] d\widehat{F}_n(u) \right). \end{aligned}$$

An integration by parts now entails

$$\begin{aligned} & \frac{1}{\sqrt{k}[M_p(k/n)]^{p-1}} \sum_{i=1}^k \varphi(X_{n-i+1,n} - M_p(k/n) - t; p) \\ &= \frac{\sqrt{k}}{[M_p(k/n)]^{p-1}} \left( \varphi_{n,t}(X_{n-k,n}) + \frac{n}{k} \int_{X_{n-k,n}}^{\infty} \widehat{F}_n(u) d\varphi_{n,t}(u) \right). \end{aligned} \quad (15)$$

Besides, we have that

$$\begin{aligned} & \frac{1}{\sqrt{k}[M_p(k/n)]^{p-1}} \sum_{i=1}^n \varphi(X_i - M_p(k/n) - t; p) \mathbb{1}_{\{X_i > Q(k/n)\}} \\ &= \frac{\sqrt{k}}{[M_p(k/n)]^{p-1}} \left( \frac{\sum_{i=1}^n \mathbb{1}_{\{X_i > Q(k/n)\}}}{k} \varphi_{n,t}(Q(k/n)) + \frac{1}{k} \sum_{i=1}^n [\varphi_{n,t}(X_i) - \varphi_{n,t}(Q(k/n))] \mathbb{1}_{\{X_i > Q(k/n)\}} \right) \\ &= \frac{\sqrt{k}}{[M_p(k/n)]^{p-1}} \left( \frac{\sum_{i=1}^n \mathbb{1}_{\{X_i > Q(k/n)\}}}{k} \varphi_{n,t}(Q(k/n)) - \frac{n}{k} \int_{Q(k/n)}^{\infty} [\varphi_{n,t}(u) - \varphi_{n,t}(Q(k/n))] d\widehat{F}_n(u) \right). \end{aligned}$$

Again, an integration by parts gives

$$\begin{aligned} & \frac{1}{\sqrt{k}[M_p(k/n)]^{p-1}} \sum_{i=1}^n \varphi(X_i - M_p(k/n) - t; p) \mathbb{1}_{\{X_i > Q(k/n)\}} \\ &= \frac{\sqrt{k}}{[M_p(k/n)]^{p-1}} \left( \frac{\sum_{i=1}^n \mathbb{1}_{\{X_i > Q(k/n)\}}}{k} \varphi_{n,t}(Q(k/n)) + \frac{n}{k} \int_{Q(k/n)}^{\infty} \widehat{F}_n(u) d\varphi_{n,t}(u) \right). \end{aligned} \quad (16)$$

Combining (15) and (16) yields

$$\begin{aligned}
& \frac{1}{\sqrt{k}[M_p(k/n)]^{p-1}} \sum_{i=1}^k \varphi(X_{n-i+1,n} - M_p(k/n) - t; p) \\
= & \frac{1}{\sqrt{k}[M_p(k/n)]^{p-1}} \sum_{i=1}^n \varphi(X_i - M_p(k/n) - t; p) \mathbb{1}_{\{X_i > Q(k/n)\}} \\
& + \frac{\sqrt{k}}{[M_p(k/n)]^{p-1}} \varphi_{n,t}(Q(k/n)) \left[ 1 - \frac{\sum_{i=1}^n \mathbb{1}_{\{X_i > Q(k/n)\}}}{k} \right] \\
& + \frac{\sqrt{k}}{[M_p(k/n)]^{p-1}} \left( \varphi_{n,t}(X_{n-k,n}) - \varphi_{n,t}(Q(k/n)) + \frac{n}{k} \int_{X_{n-k,n}}^{Q(k/n)} \widehat{F}_n(u) d\varphi_{n,t}(u) \right).
\end{aligned}$$

The result will therefore be shown provided we can prove that the expression

$$\begin{aligned}
& \frac{\sqrt{k}}{[M_p(k/n)]^{p-1}} \left( \varphi_{n,t}(X_{n-k,n}) - \varphi_{n,t}(Q(k/n)) + \frac{n}{k} \int_{X_{n-k,n}}^{Q(k/n)} \widehat{F}_n(u) d\varphi_{n,t}(u) \right) \\
= & \frac{\sqrt{k}}{[M_p(k/n)]^{p-1}} \int_{X_{n-k,n}}^{Q(k/n)} \left( \frac{n}{k} \widehat{F}_n(u) - 1 \right) d\varphi_{n,t}(u)
\end{aligned}$$

converges in probability to 0 uniformly in  $|t| \in I_n(u)$ . To this end, we use Theorem 5.1.4 p.161 in [6]: there is a sequence of standard Brownian motions  $W_n$  and a function  $A_0$  asymptotically equivalent to  $A$  such that for any  $\varepsilon > 0$  small enough,

$$\frac{n}{k} \widehat{F}_n(xQ(k/n)) - x^{-1/\gamma} = \frac{1}{\sqrt{k}} \left( W_n(x^{-1/\gamma}) + \sqrt{k} A_0(n/k) x^{-1/\gamma} \frac{x^{\rho/\gamma} - 1}{\gamma\rho} + x^{(\varepsilon-1/2)/\gamma} o_{\mathbb{P}}(1) \right)$$

where the  $o_{\mathbb{P}}(1)$  is uniform in  $x \in [1/2, \infty)$ . Now by Lemma 4(i), the function  $\varphi_{n,t}$  is continuously differentiable on  $[X_{n-k,n}, Q(k/n)]$  with probability converging to 1, so that we can use a change of variables to get

$$\begin{aligned}
& \frac{\sqrt{k}}{[M_p(k/n)]^{p-1}} \int_{X_{n-k,n}}^{Q(k/n)} \left( \frac{n}{k} \widehat{F}_n(u) - 1 \right) d\varphi_{n,t}(u) \\
= & \frac{\sqrt{k}}{[M_p(k/n)]^{p-1}} \int_{X_{n-k,n}/Q(k/n)}^1 \left( \frac{n}{k} \widehat{F}_n(xQ(k/n)) - 1 \right) \varphi'_{n,t}(xQ(k/n)) Q(k/n) dx \\
= & \frac{\sqrt{k}}{[M_p(k/n)]^{p-1}} \int_{X_{n-k,n}/Q(k/n)}^1 (x^{-1/\gamma} - 1) \varphi'_{n,t}(xQ(k/n)) Q(k/n) dx \\
& + \frac{1}{[M_p(k/n)]^{p-1}} \int_{X_{n-k,n}/Q(k/n)}^1 W_n(x^{-1/\gamma}) \varphi'_{n,t}(xQ(k/n)) Q(k/n) dx \\
& + \frac{\sqrt{k} A_0(n/k)}{[M_p(k/n)]^{p-1}} \int_{X_{n-k,n}/Q(k/n)}^1 x^{-1/\gamma} \frac{x^{\rho/\gamma} - 1}{\gamma\rho} \varphi'_{n,t}(xQ(k/n)) Q(k/n) dx \\
& + \frac{1}{[M_p(k/n)]^{p-1}} o_{\mathbb{P}} \left( \int_{X_{n-k,n}/Q(k/n)}^1 x^{(\varepsilon-1/2)/\gamma} \varphi'_{n,t}(xQ(k/n)) Q(k/n) dx \right).
\end{aligned}$$

Using the convergence  $X_{n-k,n}/Q(k/n) \xrightarrow{\mathbb{P}} 1$  and the fact that a standard Brownian motion is stochastically bounded on the interval  $[1/2, 2]$ , we obtain, irrespectively of  $|t| \in I_n(u)$ ,

$$\begin{aligned} & \frac{\sqrt{k}}{[M_p(k/n)]^{p-1}} \int_{X_{n-k,n}}^{Q(k/n)} \left( \frac{n}{k} \widehat{F}_n(u) - 1 \right) d\varphi_{n,t}(u) \\ &= \frac{\sqrt{k}}{[M_p(k/n)]^{p-1}} \int_{X_{n-k,n}/Q(k/n)}^1 (x^{-1/\gamma} - 1) \varphi'_{n,t}(xQ(k/n)) Q(k/n) dx \\ &+ \text{O}_{\mathbb{P}} \left( \frac{|\varphi_{n,t}(Q(k/n)) - \varphi_{n,t}(X_{n-k,n})|}{[M_p(k/n)]^{p-1}} \right). \end{aligned}$$

Finally, use together Lemma 4(ii), the convergence  $X_{n-k,n}/Q(k/n) \xrightarrow{\mathbb{P}} 1$  and Proposition 2 to get

$$\begin{aligned} & \frac{\sqrt{k}}{[M_p(k/n)]^{p-1}} \int_{X_{n-k,n}}^{Q(k/n)} \left( \frac{n}{k} \widehat{F}_n(u) - 1 \right) d\varphi_{n,t}(u) \\ &= \text{O}_{\mathbb{P}} \left( \frac{\sqrt{k}}{[M_p(k/n)]^{p-1}} |\varphi_{n,t}(Q(k/n)) - \varphi_{n,t}(X_{n-k,n})| \right) = \text{O}_{\mathbb{P}}(1) \end{aligned}$$

uniformly in  $|t| \in I_n(u)$ , which concludes the proof. ■

Recall that

$$\sqrt{k} \left( \frac{\widehat{m}_p(1 - k/n)}{m_p(1 - k/n)} - 1 \right) = \arg \min_{u \in \mathbb{R}} \psi_n(u; p)$$

with

$$\psi_n(u; p) = \frac{1}{p[M_p(k/n)]^p} \sum_{i=1}^k \left( \left| X_{n-i+1,n} - M_p(k/n) - \frac{uM_p(k/n)}{\sqrt{k}} \right|^p - |X_{n-i+1,n} - M_p(k/n)|^p \right).$$

The next lemma is the initial step necessary to the computation of an asymptotic expansion of  $\psi_n(u; p)$ .

**Lemma 6.** *Suppose that  $\mathcal{C}_2(\gamma, \rho, A)$  holds with  $\gamma < 1/(p-1)$  and  $p > 1$ , and  $\sqrt{k}A(n/k) = \text{O}(1)$ . Define:*

$$Y_{n,i}(v; p) = \varphi(X_i - v; p) \mathbb{1}_{\{X_i > Q(k/n)\}} \quad \text{and} \quad Z_{n,i}(v; p) = Y_{n,i}(v; p) - \mathbb{E}(Y_{n,i}(v; p)).$$

Then we have, for any  $u \in \mathbb{R}$ :

$$\psi_n(u; p) = -uT_{1,n} + T_{2,n}(u) + T_{3,n}(u) + \text{O}_{\mathbb{P}}(1)$$

with

$$\begin{aligned} T_{1,n} &= \frac{1}{\sqrt{k}[M_p(k/n)]^{p-1}} \sum_{i=1}^n Y_{n,i}(M_p(k/n); p) \\ &+ \frac{\sqrt{k}}{[M_p(k/n)]^{p-1}} \varphi(Q(k/n) - M_p(k/n); p) \left[ 1 - \frac{\sum_{i=1}^n \mathbb{1}_{\{X_i > Q(k/n)\}}}{k} \right] \\ T_{2,n}(u) &= -\frac{1}{[M_p(k/n)]^p} \sum_{i=1}^n \int_0^{uM_p(k/n)/\sqrt{k}} [\mathbb{E}(Y_{n,i}(M_p(k/n) + t; p)) - \mathbb{E}(Y_{n,i}(M_p(k/n); p))] dt \\ \text{and } T_{3,n}(u) &= -\frac{1}{[M_p(k/n)]^p} \sum_{i=1}^n \int_0^{uM_p(k/n)/\sqrt{k}} [Z_{n,i}(M_p(k/n) + t; p) - Z_{n,i}(M_p(k/n); p)] dt. \end{aligned}$$



*Proof of Lemma 6.* Note, for any continuously differentiable function  $H$ , the identity

$$H(x - y) - H(x) = -yH'(x) - \int_0^y [H'(x - t) - H'(x)]dt.$$

Apply this to the function  $H : x \mapsto p^{-1}|x|^p$ , with derivative  $x \mapsto |x|^{p-1} \text{sign}(x) = \varphi(x; p)$ , to get

$$\begin{aligned} & \psi_n(u; p) \\ = & -u \left\{ \frac{1}{\sqrt{k}[M_p(k/n)]^{p-1}} \sum_{i=1}^k \varphi(X_{n-i+1,n} - M_p(k/n); p) \right\} \\ & - \frac{1}{[M_p(k/n)]^p} \sum_{i=1}^k \int_0^{uM_p(k/n)/\sqrt{k}} [\varphi(X_{n-i+1,n} - M_p(k/n) - t; p) - \varphi(X_{n-i+1,n} - M_p(k/n); p)] dt. \end{aligned} \quad (17)$$

By Lemma 5, we have

$$\begin{aligned} & \frac{1}{\sqrt{k}[M_p(k/n)]^{p-1}} \sum_{i=1}^k \varphi(X_{n-i+1,n} - M_p(k/n); p) \\ = & \frac{1}{\sqrt{k}[M_p(k/n)]^{p-1}} \sum_{i=1}^n \varphi(X_i - M_p(k/n); p) \mathbb{1}_{\{X_i > Q(k/n)\}} \\ & + \frac{\sqrt{k}}{[M_p(k/n)]^{p-1}} \varphi(Q(k/n) - M_p(k/n); p) \left[ 1 - \frac{\sum_{i=1}^n \mathbb{1}_{\{X_i > Q(k/n)\}}}{k} \right] + o_{\mathbb{P}}(1). \end{aligned} \quad (18)$$

Besides, it is a consequence of, first, Lemmas 4(iv) and 5, and then of Theorem 5.1.4 p.161 in [6] that

$$\begin{aligned} & \frac{1}{\sqrt{k}[M_p(k/n)]^{p-1}} \sum_{i=1}^k [\varphi(X_{n-i+1,n} - M_p(k/n) - t; p) - \varphi(X_{n-i+1,n} - M_p(k/n); p)] \\ = & \frac{1}{\sqrt{k}[M_p(k/n)]^{p-1}} \sum_{i=1}^n [\varphi(X_i - M_p(k/n) - t; p) - \varphi(X_i - M_p(k/n); p)] \mathbb{1}_{\{X_i > Q(k/n)\}} + o_{\mathbb{P}}(1) \end{aligned}$$

uniformly in  $|t| \in I_n(u)$ . Consequently

$$\begin{aligned} & \frac{1}{[M_p(k/n)]^p} \sum_{i=1}^k \int_0^{uM_p(k/n)/\sqrt{k}} [\varphi(X_{n-i+1,n} - M_p(k/n) - t; p) - \varphi(X_{n-i+1,n} - M_p(k/n); p)] dt \\ = & \frac{1}{[M_p(k/n)]^p} \sum_{i=1}^n \int_0^{uM_p(k/n)/\sqrt{k}} [\varphi(X_i - M_p(k/n) - t; p) - \varphi(X_i - M_p(k/n); p)] \mathbb{1}_{\{X_i > Q(k/n)\}} dt \\ & + o_{\mathbb{P}}(1). \end{aligned} \quad (19)$$

Combining (17), (18) and (19) completes the proof. ■

The next set of lemmas is dedicated to the asymptotic analysis of  $\psi_n(u; p)$ , using its asymptotic expansion provided in Lemma 6. We first show that  $T_{1,n}$  is asymptotically Gaussian and centred.

**Lemma 7.** *Suppose that  $\mathcal{C}_2(\gamma, \rho, A)$  holds with  $\gamma < 1/(2[p-1])$  and  $p > 1$ , and  $\sqrt{k}A(n/k) = O(1)$ . We have, as  $n \rightarrow \infty$ :*

$$T_{1,n} \xrightarrow{d} \sqrt{V_1(p, \gamma)} Z$$

where  $Z$  is a standard Gaussian random variable and  $V_1(p, \gamma)$  is defined in Theorem 1.

*Proof of Lemma 7.* Write

$$T_{1,n} = \frac{\varphi(Q(k/n) - M_p(k/n); p)}{\sqrt{k}[M_p(k/n)]^{p-1}} \sum_{i=1}^n \left( \left[ \frac{\varphi(X_i - M_p(k/n); p)}{\varphi(Q(k/n) - M_p(k/n); p)} - 1 \right] \mathbb{1}_{\{X_i > Q(k/n)\}} + \frac{k}{n} \right).$$

Remark now that Proposition 1(iii) entails

$$\begin{aligned} & \mathbb{E}(\varphi(X - M_p(k/n); p) \mathbb{1}_{\{X > Q(k/n)\}}) \\ &= \mathbb{E}([X - M_p(k/n)]^{p-1} \mathbb{1}_{\{X > M_p(k/n)\}}) - \mathbb{E}([M_p(k/n) - X]^{p-1} \mathbb{1}_{\{Q(k/n) < X < M_p(k/n)\}}) = 0. \end{aligned} \quad (20)$$

It follows that  $T_{1,n}$  is a sum of independent and identically distributed random variables with zero mean. We then examine the convergence of  $T_{1,n}$  by using Lyapunov's central limit theorem, for the application of which it is sufficient to show that for some  $\delta > 0$ ,

$$\begin{aligned} & \frac{n \mathbb{E}|R_n|^{2+\delta}}{[n \text{Var}(R_n)]^{1+\delta/2}} \rightarrow 0 \quad \text{as } n \rightarrow \infty \\ & \text{with } R_n = \frac{\varphi(Q(k/n) - M_p(k/n); p)}{\sqrt{k}[M_p(k/n)]^{p-1}} \left( \left[ \frac{\varphi(X - M_p(k/n); p)}{\varphi(Q(k/n) - M_p(k/n); p)} - 1 \right] \mathbb{1}_{\{X > Q(k/n)\}} + \frac{k}{n} \right). \end{aligned} \quad (21)$$

We start by looking for an equivalent of  $\text{Var}(R_n)$ , and for this we write:

$$\begin{aligned} & \text{Var}(R_n) \\ &= \left( \frac{\varphi(Q(k/n) - M_p(k/n); p)}{\sqrt{k}[M_p(k/n)]^{p-1}} \right)^2 \mathbb{E} \left( \left[ \left\{ \frac{\varphi(X - M_p(k/n); p)}{\varphi(Q(k/n) - M_p(k/n); p)} - 1 \right\} \mathbb{1}_{\{X > Q(k/n)\}} + \frac{k}{n} \right]^2 \right). \end{aligned}$$

Expanding and using (20) yields

$$\text{Var}(R_n) = \left( \frac{\varphi(Q(k/n) - M_p(k/n); p)}{\sqrt{k}[M_p(k/n)]^{p-1}} \right)^2 \left( \mathbb{E} \left( \left[ \left\{ \frac{\varphi(X - M_p(k/n); p)}{\varphi(Q(k/n) - M_p(k/n); p)} \right\}^2 + 1 \right] \mathbb{1}_{\{X > Q(k/n)\}} \right) - \frac{k^2}{n^2} \right).$$

Hence the identity

$$\begin{aligned} & n \text{Var}(R_n) \\ &= \frac{n}{k[M_p(k/n)]^{2(p-1)}} \left( \mathbb{E}([\varphi(X - M_p(k/n); p)]^2 \mathbb{1}_{\{X > Q(k/n)\}}) + \frac{k}{n} [\varphi(Q(k/n) - M_p(k/n); p)]^2 (1 + o(1)) \right). \end{aligned}$$

Now

$$\begin{aligned} & \frac{1}{[M_p(k/n)]^{2(p-1)}} \mathbb{E}([\varphi(X - M_p(k/n); p)]^2 \mathbb{1}_{\{X > Q(k/n)\}}) \\ &= \mathbb{E} \left[ \left( \frac{X}{M_p(k/n)} - 1 \right)^{2(p-1)} \mathbb{1}_{\{X > M_p(k/n)\}} \right] + \mathbb{E} \left[ \left( 1 - \frac{X}{M_p(k/n)} \right)^{2(p-1)} \mathbb{1}_{\{Q(k/n) < X < M_p(k/n)\}} \right] \end{aligned}$$

and therefore a combined use of Proposition 2 and Lemma 1 entails

$$\begin{aligned} & \frac{1}{[M_p(k/n)]^{2(p-1)}} \mathbb{E} \left( [\varphi(X - M_p(k/n); p)]^2 \mathbb{1}_{\{X > Q(k/n)\}} \right) \\ &= \frac{\bar{F}(M_p(k/n))}{\gamma} (B(2p-1, \gamma^{-1} - 2(p-1)) + g_{2p-1, \gamma}(\kappa(p, \gamma)) + o(1)). \end{aligned}$$

Use Proposition 2 once again together with the regular variation property of  $\bar{F}$  to get, as  $n \rightarrow \infty$ :

$$\begin{aligned} & \frac{n}{k[M_p(k/n)]^{2(p-1)}} \mathbb{E} \left( [\varphi(X - M_p(k/n); p)]^2 \mathbb{1}_{\{X > Q(k/n)\}} \right) \\ & \rightarrow \frac{[\kappa(p, \gamma)]^{1/\gamma}}{\gamma} (B(2p-1, \gamma^{-1} - 2(p-1)) + g_{2p-1, \gamma}(\kappa(p, \gamma))). \end{aligned}$$

A final use of Proposition 2 gives, as  $n \rightarrow \infty$ :

$$\begin{aligned} n \text{Var}(R_n) & \rightarrow \frac{[\kappa(p, \gamma)]^{1/\gamma}}{\gamma} (B(2p-1, \gamma^{-1} - 2(p-1)) + g_{2p-1, \gamma}(\kappa(p, \gamma))) + [1 - \kappa(p, \gamma)]^{2(p-1)} \\ &= V_1(p, \gamma). \end{aligned} \tag{22}$$

The second step is to control  $\mathbb{E}|R_n|^{2+\delta}$ , where we choose  $\delta > 0$  so small that we have  $\gamma < [(2+\delta)(p-1)]^{-1}$ . It is a consequence of Hölder's inequality that  $|a+b+c|^{2+\delta} \leq 3^{1+\delta} (|a|^{2+\delta} + |b|^{2+\delta} + |c|^{2+\delta})$  for any  $a, b, c$  and therefore

$$\mathbb{E}|R_n|^{2+\delta} = O \left( \left( \frac{\varphi(Q(k/n) - M_p(k/n); p)}{\sqrt{k}[M_p(k/n)]^{p-1}} \right)^{2+\delta} \left( \mathbb{E} \left( \left| \frac{\varphi(X - M_p(k/n); p)}{\varphi(Q(k/n) - M_p(k/n); p)} \right|^{2+\delta} \mathbb{1}_{\{X > Q(k/n)\}} \right) + \frac{k}{n} \right) \right).$$

By Proposition 2,

$$n\mathbb{E}|R_n|^{2+\delta} = O \left( \frac{1}{k^{\delta/2}} \times \left[ \frac{n}{k} \mathbb{E} \left( \left| \frac{\varphi(X - M_p(k/n); p)}{\varphi(Q(k/n) - M_p(k/n); p)} \right|^{2+\delta} \mathbb{1}_{\{X > Q(k/n)\}} \right) + 1 \right] \right).$$

Finally,

$$\begin{aligned} & \mathbb{E} \left( \left| \frac{\varphi(X - M_p(k/n); p)}{\varphi(Q(k/n) - M_p(k/n); p)} \right|^{2+\delta} \mathbb{1}_{\{X > Q(k/n)\}} \right) \\ &= O \left( \frac{1}{[M_p(k/n)]^{(2+\delta)(p-1)}} \mathbb{E} \left( |\varphi(X - M_p(k/n); p)|^{2+\delta} \mathbb{1}_{\{X > Q(k/n)\}} \right) \right) \\ &= O \left( \mathbb{E} \left[ \left( \frac{X}{M_p(k/n)} - 1 \right)^{(2+\delta)(p-1)} \mathbb{1}_{\{X > M_p(k/n)\}} \right] + \mathbb{E} \left[ \left( 1 - \frac{X}{M_p(k/n)} \right)^{(2+\delta)(p-1)} \mathbb{1}_{\{Q(k/n) < X < M_p(k/n)\}} \right] \right) \\ &= O(\bar{F}(M_p(k/n))) = O\left(\frac{k}{n}\right) \end{aligned}$$

by a combined use of Proposition 2 and Lemma 1. All in all,

$$n\mathbb{E}|R_n|^{2+\delta} = O\left(\frac{1}{k^{\delta/2}}\right) \rightarrow 0 \tag{23}$$

as  $n \rightarrow \infty$ , so that a combination of (22) and (23) entails (21) as required. Lyapunov's central limit theorem therefore applies and yields:

$$\frac{T_{1,n}}{\sqrt{n \operatorname{Var}(R_n)}} = \frac{T_{1,n}}{\sqrt{\operatorname{Var}(T_{1,n})}} \xrightarrow{d} Z$$

where  $Z$  is a standard Gaussian random variable, which, taking (22) into account, is the desired result.  $\blacksquare$

Lemma 8, which was proven in [4], is a crucial tool in the asymptotic control of  $T_{2,n}(u)$ .

**Lemma 8.** *Let  $(x_n)$  be a positive sequence tending to infinity,  $(T_n)$  be a sequence of subsets of  $\mathbb{R}$ , and  $(h_{t,n})$ ,  $t \in T_n$  be a class of functions such that*

$$\sup_{t \in T_n} \sup_{x \geq x_n} |h_{t,n}(x)| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

*Assume that the survival function  $\bar{F}$  satisfies condition  $\mathcal{C}_2(\gamma, \rho, A)$ . Then:*

$$\sup_{t \in T_n} \sup_{x \geq x_n} [\max(|h_{t,n}(x)|, |A(1/\bar{F}(x))|)]^{-1} \left| \frac{\bar{F}(x(1+h_{t,n}(x)))}{\bar{F}(x)} - \left(1 - \frac{h_{t,n}(x)}{\gamma}\right) \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

*Proof of Lemma 8.* See the proof of Lemma 5(ii) in [4].  $\blacksquare$

The next result shows that the nonrandom term  $T_{2,n}(u)$  converges to a multiple of  $u^2$ .

**Lemma 9.** *Suppose that  $\mathcal{C}_2(\gamma, \rho, A)$  holds with  $\gamma < 1/(p-1)$  and  $p > 1$ , and  $\sqrt{k}A(n/k) = O(1)$ . We have, as  $n \rightarrow \infty$ :*

$$T_{2,n}(u) \rightarrow \sqrt{V_2(p, \gamma)} \frac{u^2}{2}$$

where  $V_2(p, \gamma)$  is defined in Theorem 1.

*Proof of Lemma 9.* The  $Y_{n,i}(v; p)$  introduced in Lemma 6 are independent copies of the random variable  $Y_n(v; p) := \varphi(X - v; p) \mathbb{1}_{\{X > Q(k/n)\}}$ . Thus

$$T_{2,n}(u) = -\frac{n}{[M_p(k/n)]^p} \int_0^{uM_p(k/n)/\sqrt{k}} [\mathbb{E}(Y_n(M_p(k/n) + t; p)) - \mathbb{E}(Y_n(M_p(k/n); p))] dt.$$

Note that, if  $\varphi'(x) = (p-1)|x|^{p-2}$  denotes the derivative of  $\varphi(\cdot; p)$  on  $\mathbb{R} \setminus \{0\}$ ,

$$\mathbb{E}(Y_n(M_p(k/n) + t; p)) - \mathbb{E}(Y_n(M_p(k/n); p)) = \mathbb{E} \left( \int_{X-M_p(k/n)}^{X-M_p(k/n)-t} \varphi'(v) \mathbb{1}_{\{X > Q(k/n)\}} dv \right)$$

and write then

$$\begin{aligned} & \mathbb{E}(Y_n(M_p(k/n) + t; p)) - \mathbb{E}(Y_n(M_p(k/n); p)) \\ &= \mathbb{E} \left( \int_{X-M_p(k/n)}^{X-M_p(k/n)-t} \varphi'(v) \mathbb{1}_{\{X > M_p(k/n)\}} dv \right) \\ &+ \mathbb{E} \left( \int_{X-M_p(k/n)}^{X-M_p(k/n)-t} \varphi'(v) \mathbb{1}_{\{Q(k/n) < X \leq M_p(k/n)\}} dv \right). \end{aligned} \tag{24}$$

The first term in the right-hand side of (24) is exactly term  $T_{2,2,n}(t)$  of the proof of Lemma 8 in [4], only with  $M_p(k/n)$  in place of the sequence denoted by  $q_{r_n}(p)$  therein. Using the equivalent of  $T_{2,2,n}(t)$  on p.21 of the supplementary material of [4] and then Proposition 2, we get, uniformly in  $|t| \in I_n(u)$ ,

$$\begin{aligned}
& \mathbb{E} \left( \int_{X-M_p(k/n)}^{X-M_p(k/n)-t} \varphi'(v) \mathbb{1}_{\{X > M_p(k/n)\}} dv \right) \\
&= \frac{-t}{\gamma} (p-1) [M_p(k/n)]^{p-2} \overline{F}(M_p(k/n)) B(p-1, \gamma^{-1} - p + 2) (1 + o(1)) \\
&+ o \left( [M_p(k/n)]^{p-1} \overline{F}(M_p(k/n)) A \left( \frac{1}{\overline{F}(M_p(k/n))} \right) \right) \\
&= \frac{-t}{\gamma} (p-1) [M_p(k/n)]^{p-2} \frac{k}{n} [\kappa(p, \gamma)]^{1/\gamma} B(p-1, \gamma^{-1} - p + 2) (1 + o(1)) \\
&+ o \left( [M_p(k/n)]^{p-1} \frac{\sqrt{k}}{n} \right). \tag{25}
\end{aligned}$$

To work on the second term in the right-hand side of (24), hereby denoted by  $E_n(t)$ , we first write

$$\begin{aligned}
E_n(t) &= \begin{cases} \mathbb{E} \left( \int_{\mathbb{R}} \varphi'(v) \mathbb{1}_{\{X-M_p(k/n) < v < X-M_p(k/n)-t, Q(k/n) < X \leq M_p(k/n)\}} dv \right) & \text{if } t < 0 \\ -\mathbb{E} \left( \int_{\mathbb{R}} \varphi'(v) \mathbb{1}_{\{X-M_p(k/n)-t < v < X-M_p(k/n), Q(k/n) < X \leq M_p(k/n)\}} dv \right) & \text{if } t > 0 \end{cases} \\
&= \begin{cases} \int_{\mathbb{R}} \varphi'(v) \mathbb{P}(\max(Q(k/n), M_p(k/n) + v + t) < X \leq M_p(k/n) + \min(0, v)) dv & \text{if } t < 0 \\ -\int_{\mathbb{R}} \varphi'(v) \mathbb{P}(\max(Q(k/n), M_p(k/n) + v) < X \leq M_p(k/n) + \min(0, v + t)) dv & \text{if } t > 0. \end{cases}
\end{aligned}$$

We start by considering the case  $t < 0$ . In that case,

$$\begin{aligned}
E_n(t) &= \int_{-\infty}^0 \varphi'(v) \mathbb{P}(\max(Q(k/n), M_p(k/n) + v + t) < X \leq M_p(k/n) + v) dv \\
&+ \int_0^{+\infty} \varphi'(v) \mathbb{P}(\max(Q(k/n), M_p(k/n) + v + t) < X \leq M_p(k/n)) dv.
\end{aligned}$$

Now  $|t| \in I_n(u) = \left[0, |u| M_p(k/n) / \sqrt{k}\right]$ , and  $\lim_{n \rightarrow \infty} M_p(k/n) / Q(k/n) > 1$ , so that for  $n$  large enough, we have  $M_p(k/n) + v + t > Q(k/n)$  for any  $|t| \in I_n(u)$  and any  $v > 0$ . Moreover, in that case, the double inequality  $M_p(k/n) + v + t < X \leq M_p(k/n)$  can only be true for  $0 < v < -t$ . Consequently, for  $n$  large enough,

$$\begin{aligned}
E_n(t) &= \int_{-\infty}^0 \varphi'(v) \mathbb{P}(\max(Q(k/n), M_p(k/n) + v + t) < X \leq M_p(k/n) + v) dv \\
&+ \int_0^{+\infty} \varphi'(v) \mathbb{P}(M_p(k/n) + v + t < X \leq M_p(k/n)) dv \\
&= \int_{-\infty}^{Q(k/n)-M_p(k/n)-t} \varphi'(v) \mathbb{P}(Q(k/n) < X \leq M_p(k/n) + v) dv \\
&+ \int_{Q(k/n)-M_p(k/n)-t}^0 \varphi'(v) \mathbb{P}(M_p(k/n) + v + t < X \leq M_p(k/n) + v) dv \\
&+ \int_0^{-t} \varphi'(v) \mathbb{P}(M_p(k/n) + v + t < X \leq M_p(k/n)) dv.
\end{aligned}$$

For negative  $v$ , the double inequality  $Q(k/n) < X \leq M_p(k/n) + v$  can only be true for  $Q(k/n) - M_p(k/n) < v$ . Therefore, for  $n$  large enough:

$$\begin{aligned}
E_n(t) &= \int_{Q(k/n)-M_p(k/n)}^{Q(k/n)-M_p(k/n)-t} \varphi'(v) \mathbb{P}(Q(k/n) < X \leq M_p(k/n) + v) dv \\
&+ \int_{Q(k/n)-M_p(k/n)-t}^0 \varphi'(v) \mathbb{P}(M_p(k/n) + v + t < X \leq M_p(k/n) + v) dv \\
&+ \int_0^{-t} \varphi'(v) \mathbb{P}(M_p(k/n) + v + t < X \leq M_p(k/n)) dv \\
&=: E_{1,n}^{(-)}(t) + E_{2,n}^{(-)}(t) + E_{3,n}^{(-)}(t).
\end{aligned} \tag{26}$$

We start by controlling  $E_{1,n}^{(-)}(t)$ . We write

$$E_{1,n}^{(-)}(t) = -\frac{k}{n} \int_0^{-t} \varphi'(Q(k/n) - M_p(k/n) + w) \left( \frac{\bar{F}(Q(k/n) + w)}{\bar{F}(Q(k/n))} - 1 \right) dw.$$

Because  $|t| \in I_n(u)$ , we can use Lemma 8 to get

$$E_{1,n}^{(-)}(t) = \frac{k}{n} \int_0^{-t} \varphi'(Q(k/n) - M_p(k/n) + w) \left( \frac{w}{\gamma Q(k/n)} (1 + o(1)) + o\left(A\left(\frac{n}{k}\right)\right) \right) dw.$$

Here and in the remainder of this proof, the  $o(\cdot)$  and  $O(\cdot)$  are to be understood as uniform in  $|t| \in I_n(u)$ . Using Proposition 2 and again the fact that  $|t| \in I_n(u)$ , this gives, for  $n$  large enough, that for some (generic) constant  $C > 0$  whose value may change throughout this proof,

$$\begin{aligned}
E_{1,n}^{(-)}(t) &\leq C \frac{k}{n} [M_p(k/n)]^{p-2} \left( \int_0^{-t} w dw \frac{1}{Q(k/n)} (1 + o(1)) + o\left(|t| A\left(\frac{n}{k}\right)\right) \right) \\
&\leq C \frac{k}{n} [M_p(k/n)]^{p-2} \left( (-t) \times \frac{-t}{Q(k/n)} (1 + o(1)) + o\left(\frac{M_p(k/n)}{\sqrt{k}}\right) \right) \\
&= o\left([M_p(k/n)]^{p-2} \frac{k}{n} |t|\right) + o\left([M_p(k/n)]^{p-1} \frac{\sqrt{k}}{n}\right).
\end{aligned} \tag{27}$$

We now work on  $E_{2,n}^{(-)}(t)$ : we have

$$E_{2,n}^{(-)}(t) = \int_{Q(k/n)-M_p(k/n)-t}^0 \varphi'(v) \bar{F}(M_p(k/n) + v) \left( \frac{\bar{F}(M_p(k/n) + v + t)}{\bar{F}(M_p(k/n) + v)} - 1 \right) dv.$$

Now  $M_p(k/n) + v \geq Q(k/n) - t \rightarrow \infty$  uniformly in  $|t| \in I_n(u)$  in view of Proposition 2, so we can apply Lemma 8 to obtain

$$\begin{aligned}
&E_{2,n}^{(-)}(t) \\
&= \int_{Q(k/n)-M_p(k/n)-t}^0 \varphi'(v) \bar{F}(M_p(k/n) + v) \times \left( \frac{-t}{\gamma(M_p(k/n) + v)} (1 + o(1)) + o\left(A\left(\frac{1}{\bar{F}(M_p(k/n) + v)}\right)\right) \right) dv.
\end{aligned}$$

Note then that, by Proposition 2,

$$\begin{aligned}
& \int_{Q(k/n)-M_p(k/n)-t}^0 \varphi'(v) \overline{F}(M_p(k/n) + v) \times \frac{-t}{\gamma(M_p(k/n) + v)} dv \\
&= \int_{-1+[Q(k/n)-t]/M_p(k/n)}^0 \varphi'(M_p(k/n)w) \overline{F}(M_p(k/n)(1+w)) \times \frac{-t}{\gamma(1+w)} dw \\
&= \frac{-t}{\gamma} (p-1) [M_p(k/n)]^{p-2} \int_{-1+\kappa(p,\gamma)}^0 |w|^{p-2} \overline{F}(M_p(k/n)(1+w)) (1+w)^{-1} dw (1+o(1)).
\end{aligned}$$

Use now the regular variation property of  $\overline{F}$  with index  $-1/\gamma < 0$ , a corollary of which is the uniform convergence of  $\overline{F}(M_p(k/n)(1+w))/\overline{F}(M_p(k/n))$  to  $(1+w)^{-1/\gamma}$  when  $1+w \geq \kappa(p,\gamma) > 0$ , to get:

$$\begin{aligned}
& \int_{Q(k/n)-M_p(k/n)-t}^0 \varphi'(v) \overline{F}(M_p(k/n) + v) \times \frac{-t}{\gamma(M_p(k/n) + v)} dv \\
&= \frac{-t}{\gamma} (p-1) [M_p(k/n)]^{p-2} \overline{F}(M_p(k/n)) \int_{-1+\kappa(p,\gamma)}^0 |w|^{p-2} (1+w)^{-1-1/\gamma} dw (1+o(1)) \\
&= \frac{-t}{\gamma} (p-1) [M_p(k/n)]^{p-2} \overline{F}(M_p(k/n)) \int_{\kappa(p,\gamma)}^1 (1-z)^{p-2} z^{-1-1/\gamma} dz (1+o(1)) \\
&= \frac{-t}{\gamma} (p-1) [M_p(k/n)]^{p-2} \frac{k}{n} [\kappa(p,\gamma)]^{1/\gamma} g_{p-1,\gamma}(\kappa(p,\gamma)) (1+o(1))
\end{aligned}$$

(we highlight that  $g_{p-1,\gamma}$  is indeed well-defined and finite since  $p > 1$ ). Similarly, since  $\overline{F} \times A \circ (1/\overline{F})$  is regularly varying with index  $(\rho-1)/\gamma \leq -1/\gamma$ , we have by a uniform convergence argument that

$$\begin{aligned}
& \int_{Q(k/n)-M_p(k/n)-t}^0 \varphi'(v) \overline{F}(M_p(k/n) + v) \times A\left(\frac{1}{\overline{F}(M_p(k/n) + v)}\right) dv \\
&= (p-1) [M_p(k/n)]^{p-1} \overline{F}(M_p(k/n)) A\left(\frac{1}{\overline{F}(M_p(k/n))}\right) \int_{\kappa(p,\gamma)}^1 (1-z)^{p-2} z^{(\rho-1)/\gamma} dz (1+o(1)) \\
&= O\left([M_p(k/n)]^{p-1} \frac{k}{n} A\left(\frac{n}{k}\right)\right) = O\left(\frac{\sqrt{k}}{n} [M_p(k/n)]^{p-1}\right).
\end{aligned}$$

Consequently

$$E_{2,n}^{(-)}(t) = \frac{-t}{\gamma} (p-1) [M_p(k/n)]^{p-2} \frac{k}{n} [\kappa(p,\gamma)]^{1/\gamma} g_{p-1,\gamma}(\kappa(p,\gamma)) (1+o(1)) + o\left([M_p(k/n)]^{p-1} \frac{\sqrt{k}}{n}\right). \quad (28)$$

We now turn to the control of  $E_{3,n}^{(-)}(t)$ . To this end, we write

$$E_{3,n}^{(-)}(t) = \overline{F}(M_p(k/n)) \int_0^{-t} \varphi'(v) \left( \frac{\overline{F}(M_p(k/n) + v + t)}{\overline{F}(M_p(k/n))} - 1 \right) dv.$$

Because  $|t| \in I_n(u)$ , we can use Lemma 8 to get

$$E_{3,n}^{(-)}(t) = \overline{F}(M_p(k/n)) \int_0^{-t} \varphi'(v) \left( -\frac{v+t}{\gamma M_p(k/n)} (1+o(1)) + o\left(A\left(\frac{1}{\overline{F}(M_p(k/n))}\right)\right) \right) dv.$$

Use Proposition 2 and the regular variation property of  $\bar{F}$  to obtain

$$\begin{aligned} E_{3,n}^{(-)}(t) &\leq C \frac{k}{n} \left( \int_0^{-t} v^{p-2} (-t-v) dv \frac{1}{M_p(k/n)} (1 + o(1)) + \int_0^{-t} v^{p-2} dv \times o \left( A \left( \frac{n}{k} \right) \right) \right) \\ &\leq C \frac{k}{n} \left( [M_p(k/n)]^{p-2} \times (-t) \times \frac{(-t)^{p-1}}{[M_p(k/n)]^{p-1}} + [M_p(k/n)]^{p-1} \times \frac{(-t)^{p-1}}{[M_p(k/n)]^{p-1}} \times o \left( \frac{1}{\sqrt{k}} \right) \right). \end{aligned}$$

Since  $|t| \in I_n(u)$ , this entails

$$E_{3,n}^{(-)}(t) = o \left( [M_p(k/n)]^{p-2} \frac{k}{n} |t| \right) + o \left( [M_p(k/n)]^{p-1} \frac{\sqrt{k}}{n} \right). \quad (29)$$

Combining (26), (27), (28) and (29) results in, for  $t < 0$ :

$$E_n(t) = \frac{-t}{\gamma} (p-1) [M_p(k/n)]^{p-2} \frac{k}{n} [\kappa(p, \gamma)]^{1/\gamma} g_{p-1, \gamma}(\kappa(p, \gamma)) (1 + o(1)) + o \left( [M_p(k/n)]^{p-1} \frac{\sqrt{k}}{n} \right). \quad (30)$$

In the case when  $t > 0$ , write instead

$$\begin{aligned} E_n(t) &= - \int_{-\infty}^{-t} \varphi'(v) \mathbb{P}(\max(Q(k/n), M_p(k/n) + v) < X \leq M_p(k/n) + v + t) dv \\ &\quad - \int_{-t}^0 \varphi'(v) \mathbb{P}(\max(Q(k/n), M_p(k/n) + v) < X \leq M_p(k/n)) dv. \end{aligned} \quad (31)$$

Here we used the fact that the double inequality  $\max(Q(k/n), M_p(k/n) + v) < X \leq M_p(k/n) + \min(0, v + t)$  is equivalent to  $M_p(k/n) + v < X \leq M_p(k/n)$  when  $v > 0$ , and thus

$$\forall v > 0, \mathbb{P}(\max(Q(k/n), M_p(k/n) + v) < X \leq M_p(k/n) + \min(0, v + t)) = 0.$$

Breaking the first term of (31) further, we obtain, for  $n$  large enough:

$$\begin{aligned} E_n(t) &= - \int_{-\infty}^{Q(k/n) - M_p(k/n)} \varphi'(v) \mathbb{P}(Q(k/n) < X \leq M_p(k/n) + v + t) dv \\ &\quad - \int_{Q(k/n) - M_p(k/n)}^{-t} \varphi'(v) \mathbb{P}(M_p(k/n) + v < X \leq M_p(k/n) + v + t) dv \\ &\quad - \int_{-t}^0 \varphi'(v) \mathbb{P}(\max(Q(k/n), M_p(k/n) + v) < X \leq M_p(k/n)) dv \\ &= - \int_{Q(k/n) - M_p(k/n) - t}^{Q(k/n) - M_p(k/n)} \varphi'(v) \mathbb{P}(Q(k/n) < X \leq M_p(k/n) + v + t) dv \\ &\quad - \int_{Q(k/n) - M_p(k/n)}^{-t} \varphi'(v) \mathbb{P}(M_p(k/n) + v < X \leq M_p(k/n) + v + t) dv \\ &\quad - \int_{-t}^0 \varphi'(v) \mathbb{P}(M_p(k/n) + v < X \leq M_p(k/n)) dv \\ &=: - \left( E_{1,n}^{(+)}(t) + E_{2,n}^{(+)}(t) + E_{3,n}^{(+)}(t) \right). \end{aligned} \quad (32)$$



In the second equality, we used that  $|t| \in \left[0, |u|M_p(k/n)/\sqrt{k}\right]$  and  $\lim_{n \rightarrow \infty} M_p(k/n)/Q(k/n) > 1$ . Now, arguments similar to those used to control  $E_{1,n}^{(-)}(t)$ ,  $E_{2,n}^{(-)}(t)$  and  $E_{3,n}^{(-)}(t)$  yield:

$$E_{1,n}^{(+)}(t) = o\left([M_p(k/n)]^{p-2} \frac{k}{n} |t|\right) + o\left([M_p(k/n)]^{p-1} \frac{\sqrt{k}}{n}\right), \quad (33)$$

$$\begin{aligned} E_{2,n}^{(+)}(t) &= \frac{t}{\gamma} (p-1) [M_p(k/n)]^{p-2} \frac{k}{n} [\kappa(p, \gamma)]^{1/\gamma} g_{p-1, \gamma}(\kappa(p, \gamma)) (1 + o(1)) \\ &\quad + o\left([M_p(k/n)]^{p-1} \frac{\sqrt{k}}{n}\right), \end{aligned} \quad (34)$$

$$E_{3,n}^{(+)}(t) = o\left([M_p(k/n)]^{p-2} \frac{k}{n} |t|\right) + o\left([M_p(k/n)]^{p-1} \frac{\sqrt{k}}{n}\right). \quad (35)$$

Combining (32), (33), (34) and (35) results in, for  $t > 0$ :

$$E_n(t) = \frac{-t}{\gamma} (p-1) [M_p(k/n)]^{p-2} \frac{k}{n} [\kappa(p, \gamma)]^{1/\gamma} g_{p-1, \gamma}(\kappa(p, \gamma)) (1 + o(1)) + o\left([M_p(k/n)]^{p-1} \frac{\sqrt{k}}{n}\right) \quad (36)$$

which is exactly the same as (30), obtained for negative  $t$ . Combining this with (24) and (25) results in

$$\begin{aligned} &\mathbb{E}(Y_n(M_p(k/n) + t; p)) - \mathbb{E}(Y_n(M_p(k/n); p)) \\ &= \frac{-t}{\gamma} (p-1) [M_p(k/n)]^{p-2} \frac{k}{n} [\kappa(p, \gamma)]^{1/\gamma} \times [B(p-1, \gamma^{-1} - p + 2) + g_{p-1, \gamma}(\kappa(p, \gamma))] (1 + o(1)) \\ &\quad + o\left([M_p(k/n)]^{p-1} \frac{\sqrt{k}}{n}\right). \end{aligned}$$

Therefore

$$\begin{aligned} T_{2,n}(u) &= -\frac{n}{[M_p(k/n)]^p} \int_0^{uM_p(k/n)/\sqrt{k}} [\mathbb{E}(Y_n(M_p(k/n) + t; p)) - \mathbb{E}(Y_n(M_p(k/n); p))] dt \\ &\rightarrow \frac{u^2}{2\gamma} (p-1) [\kappa(p, \gamma)]^{1/\gamma} [B(p-1, \gamma^{-1} - p + 2) + g_{p-1, \gamma}(\kappa(p, \gamma))] = \sqrt{V_2(p, \gamma)} \frac{u^2}{2} \end{aligned}$$

as  $n \rightarrow \infty$ , completing the proof. ■

Lemma 10 below is a Lipschitz property for  $\varphi$ , which will make it possible to control  $T_{3,n}(u)$ .

**Lemma 10.** *For all  $x, h \in \mathbb{R}$ , we have*

$$|\varphi(x-h; p) - \varphi(x; p)| \leq \begin{cases} 2|h|^{p-1} & \text{if } 1 < p < 2 \\ (p-1)(2^{p-2} + 1)(|h|^{p-1} + |x|^{p-2}|h|) & \text{if } p \geq 2. \end{cases}$$

*Proof of Lemma 10.* This is a direct consequence of Lemma 6 in [4] (and its proof) with  $\tau = 1/2$ . ■

Our final lemma shows that  $T_{3,n}(u)$ , which is a sum of independent and identically distributed random variables, converges in probability to zero.

**Lemma 11.** *Suppose that  $\mathcal{C}_2(\gamma, \rho, A)$  holds with  $\gamma < 1/(p-1)$  and  $p > 1$ , and  $\sqrt{k}A(n/k) = O(1)$ . We have, as  $n \rightarrow \infty$ :*

$$T_{3,n}(u) \xrightarrow{\mathbb{P}} 0.$$

*Proof of Lemma 11.* Because  $\mathbb{E}(T_{3,n}(u)) = 0$ , it is enough to show that  $\text{Var}(T_{3,n}(u)) \rightarrow 0$ . The  $Z_{n,i}(v; p) = Y_{n,i}(v; p) - \mathbb{E}(Y_{n,i}(v; p))$  introduced in Lemma 6 are independent copies of the centred variable

$$Z_n(v; p) := Y_n(v; p) - \mathbb{E}(Y_n(v; p)) := \varphi(X - v; p) \mathbb{1}_{\{X > Q(k/n)\}} - \mathbb{E}(\varphi(X - v; p) \mathbb{1}_{\{X > Q(k/n)\}}).$$

Thus

$$\text{Var}(T_{3,n}(u)) = \frac{n}{[M_p(k/n)]^{2p}} \text{Var} \left( \int_0^{uM_p(k/n)/\sqrt{k}} [Z_n(M_p(k/n) + t; p) - Z_n(M_p(k/n); p)] dt \right).$$

Because for any  $v$ ,  $Z_n(v; p)$  has zero mean and is square-integrable, we get

$$\begin{aligned} & \frac{[M_p(k/n)]^{2p}}{n} \text{Var}(T_{3,n}(u)) \\ &= \int_{[0, uM_p(k/n)/\sqrt{k}]^2} \mathbb{E}([Z_n(M_p(k/n) + s; p) - Z_n(M_p(k/n); p)][Z_n(M_p(k/n) + t; p) - Z_n(M_p(k/n); p)]) ds dt. \end{aligned}$$

The Cauchy-Schwarz inequality now yields

$$\frac{[M_p(k/n)]^{2p}}{n} \text{Var}(T_{3,n}(u)) \leq \left( \int_0^{uM_p(k/n)/\sqrt{k}} \sqrt{\mathbb{E}(|Z_n(M_p(k/n) + t; p) - Z_n(M_p(k/n); p)|^2)} dt \right)^2. \quad (37)$$

Apply Lemma 10 to get, for any  $t$  such that  $t \in I_n(u)$ :

$$\begin{aligned} & |Z_n(M_p(k/n) + t; p) - Z_n(M_p(k/n); p)| \\ & \leq |Y_n(M_p(k/n) + t; p) - Y_n(M_p(k/n); p)| + \mathbb{E}|Y_n(M_p(k/n) + t; p) - Y_n(M_p(k/n); p)| \\ & \leq \begin{cases} C|t|^{p-1} \mathbb{1}_{\{X > Q(k/n)\}} & \text{if } 1 < p < 2 \\ C(|t|^{p-1} + |X - M_p(k/n)|^{p-2}|t|) \mathbb{1}_{\{X > Q(k/n)\}} & \text{if } p \geq 2 \end{cases} \\ & + \begin{cases} C|t|^{p-1} \mathbb{P}(X > Q(k/n)) & \text{if } 1 < p < 2 \\ C(|t|^{p-1} \mathbb{P}(X > Q(k/n)) + |t| \mathbb{E}(|X - M_p(k/n)|^{p-2} \mathbb{1}_{\{X > Q(k/n)\}})) & \text{if } p \geq 2 \end{cases} \end{aligned}$$

for some (generic) constant  $C > 0$  whose value may change throughout the proof. Recall that when  $p \geq 2$ ,

$$\begin{aligned} & \mathbb{E}(|X - M_p(k/n)|^{p-2} \mathbb{1}_{\{X > Q(k/n)\}}) \\ &= \mathbb{E}([X - M_p(k/n)]^{p-2} \mathbb{1}_{\{X > M_p(k/n)\}}) + \mathbb{E}([M_p(k/n) - X]^{p-2} \mathbb{1}_{\{Q(k/n) < X < M_p(k/n)\}}) \end{aligned}$$

and use Lemma 1 in this case to obtain, for  $n$  large enough:

$$\begin{aligned}
& |Z_n(M_p(k/n) + t; p) - Z_n(M_p(k/n); p)| \\
& \leq \begin{cases} C|t|^{p-1} \left( \mathbb{1}_{\{X > Q(k/n)\}} + \frac{k}{n} \right) & \text{if } 1 < p < 2 \\ C \left[ |t|^{p-1} \left[ \mathbb{1}_{\{X > Q(k/n)\}} + \frac{k}{n} \right] + |t| \left[ |X - M_p(k/n)|^{p-2} \mathbb{1}_{\{X > Q(k/n)\}} + [M_p(k/n)]^{p-2} \frac{k}{n} \right] \right] & \text{if } p \geq 2. \end{cases}
\end{aligned}$$

Squaring, integrating, using Lemma 1 again when  $p \geq 2$ , and recalling that  $|t| \leq |u|M_p(k/n)/\sqrt{k}$ , we find, for  $n$  large enough,

$$\begin{aligned}
& \mathbb{E}(|Z_n(M_p(k/n) + t; p) - Z_n(M_p(k/n); p)|^2) \\
& \leq \begin{cases} C|t|^{2(p-1)} \frac{k}{n} & \text{if } 1 < p < 2 \\ C \left( |t|^{2(p-1)} \frac{k}{n} + |t|^2 [M_p(k/n)]^{2(p-2)} \frac{k}{n} \right) & \text{if } p \geq 2 \end{cases} \\
& \leq \begin{cases} C|t|^{2(p-1)} \frac{k}{n} & \text{if } 1 < p < 2 \\ C|t|^2 [M_p(k/n)]^{2(p-2)} \frac{k}{n} & \text{if } p \geq 2. \end{cases} \tag{38}
\end{aligned}$$

Finally, for  $1 < p < 2$ ,

$$\begin{aligned}
\frac{n}{[M_p(k/n)]^{2p}} \left( \int_0^{uM_p(k/n)/\sqrt{k}} \sqrt{\frac{k}{n}} |t|^{2(p-1)} dt \right)^2 &= \frac{k}{[M_p(k/n)]^{2p}} \left( \int_0^{uM_p(k/n)/\sqrt{k}} |t|^{p-1} dt \right)^2 \\
&= O(k^{1-p}) = o(1) \tag{39}
\end{aligned}$$

while for  $p \geq 2$ ,

$$\begin{aligned}
\frac{n}{[M_p(k/n)]^{2p}} \left( \int_0^{uM_p(k/n)/\sqrt{k}} \sqrt{\frac{k}{n}} [M_p(k/n)]^{2(p-2)} |t|^2 dt \right)^2 &= \frac{k}{[M_p(k/n)]^4} \left( \int_0^{uM_p(k/n)/\sqrt{k}} |t| dt \right)^2 \\
&= O(k^{-1}) = o(1). \tag{40}
\end{aligned}$$

Combine finally (37), (38), (39) and (40) to complete the proof. ■

## C Additional figures

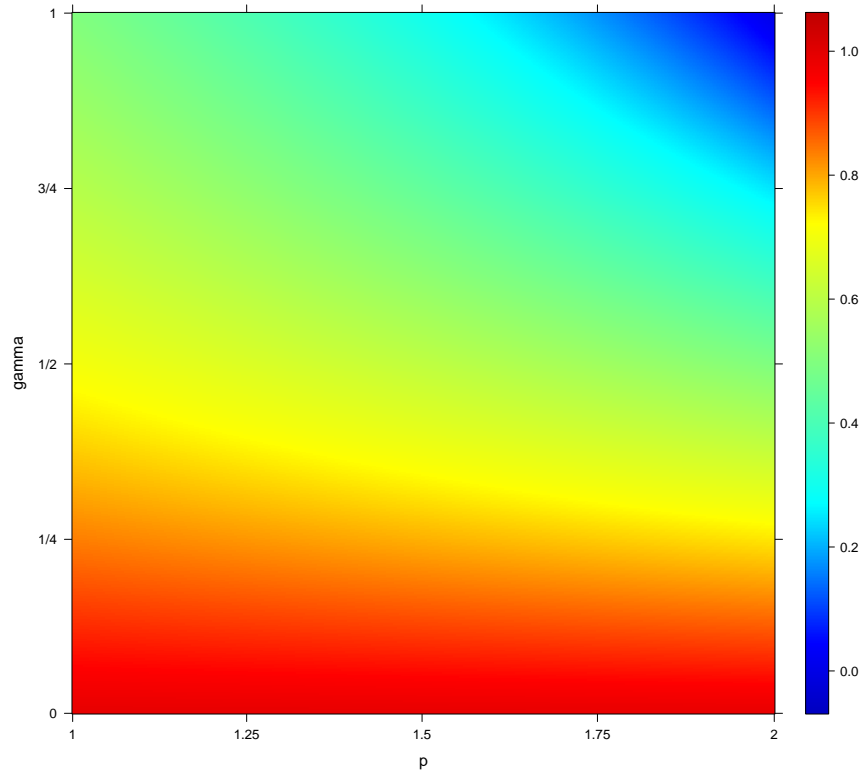


Figure C.1: Behaviour of  $(p, \gamma) \mapsto \kappa(p, \gamma)$  for  $p \in [1, 2]$  and  $\gamma \in (0, 1)$ .

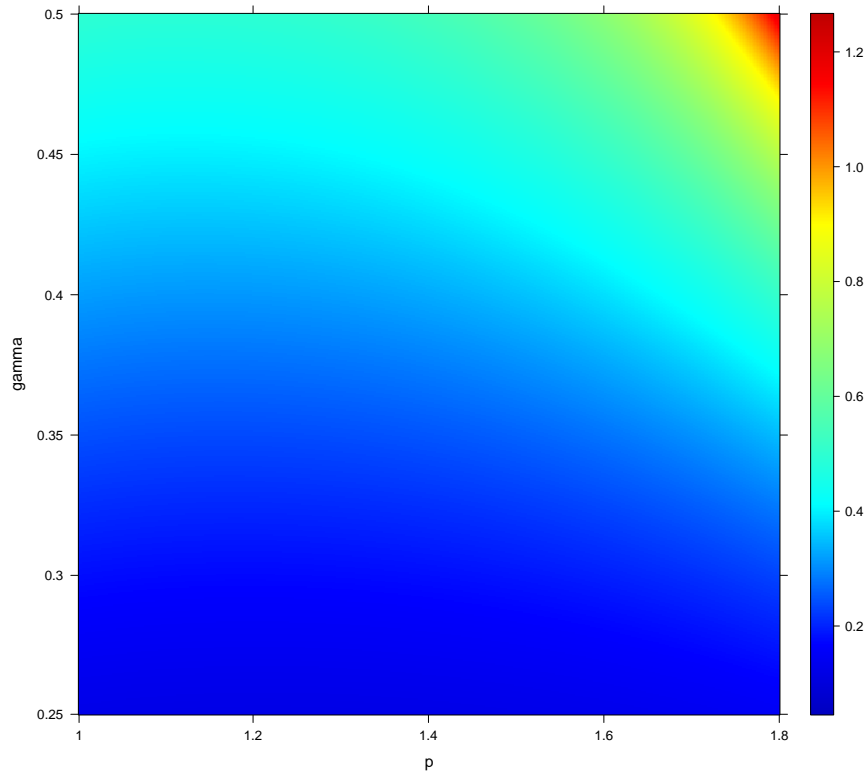
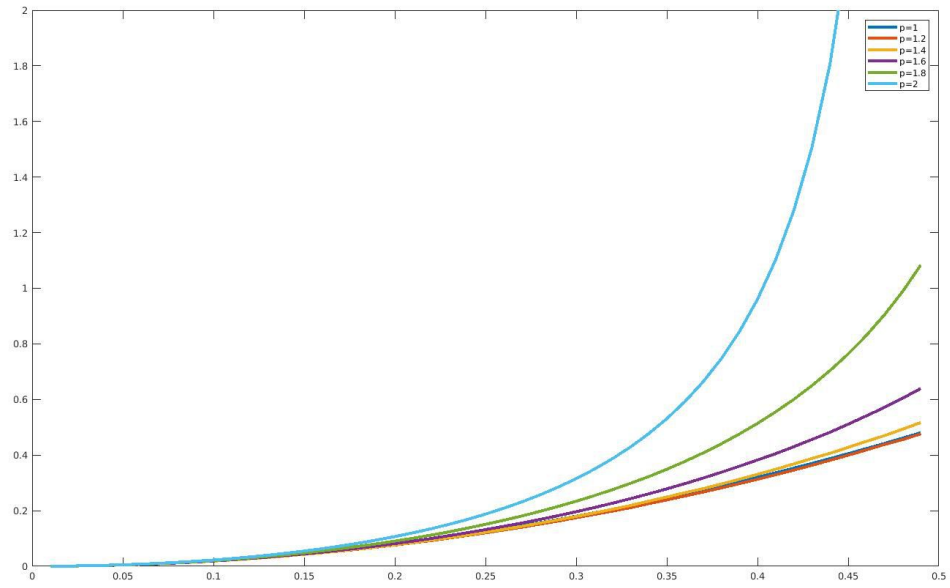


Figure C.2: Behaviour of the asymptotic variance  $V$  of the direct empirical tail  $L^p$ -median estimator. Top panel: a plot of the function  $\gamma \mapsto V(p, \gamma)$ , on the interval  $(0, 1/2)$ , for  $p \in \{1, 1.2, 1.4, 1.6, 1.8, 2\}$ . Bottom panel: a plot of the function  $(p, \gamma) \mapsto V(p, \gamma)$  on the domain  $[1, 1.8] \times [0.25, 0.5]$ .

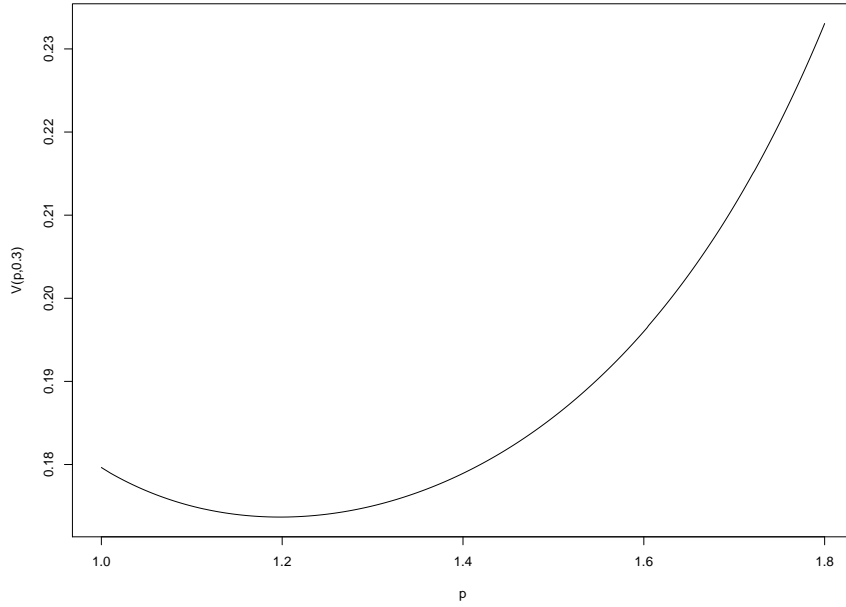


Figure C.3: Behaviour of the function  $p \mapsto V(p, 0.3)$  on the interval  $[1, 1.8]$ .

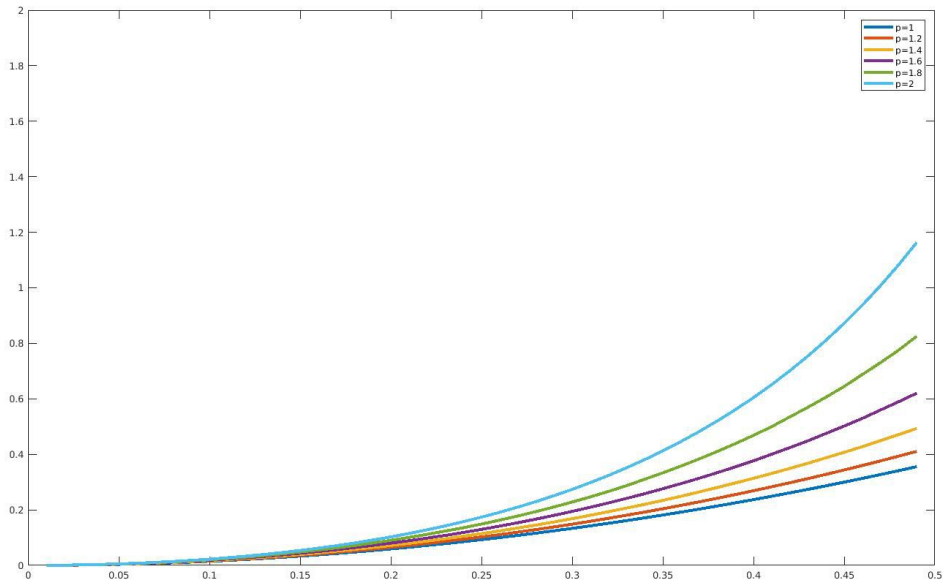


Figure C.4: Behaviour of the asymptotic variance  $\mathfrak{v}$  of the indirect tail  $L^p$ -median estimator: plot of the function  $\gamma \mapsto \mathfrak{v}(p, \gamma)$ , on the interval  $(0, 1/2)$ , for  $p \in \{1, 1.2, 1.4, 1.6, 1.8, 2\}$ .

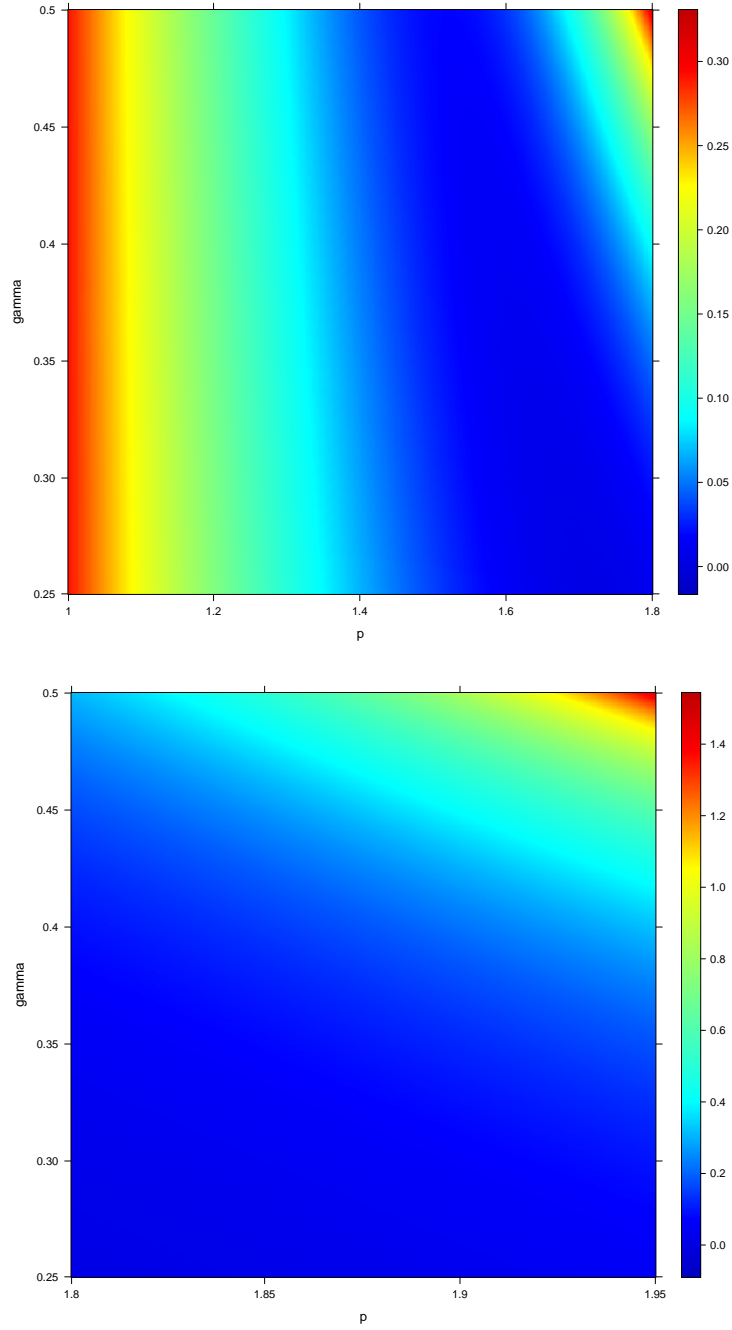


Figure C.5: A comparison between the asymptotic variances  $V$  and  $\mathbf{v}$  with plots of the function  $(p, \gamma) \mapsto \log(V(p, \gamma)/\mathbf{v}(p, \gamma))$ . Top: on the domain  $(p, \gamma) \in [1, 1.8] \times [0.25, 0.5]$ , bottom: on the domain  $(p, \gamma) \in [1.8, 1.95] \times [0.25, 0.5]$ .

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