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► **To cite this version:**

Jean-Baptiste Caillau, Jacques Fejoz, Michaël Orioux, Robert Roussarie. Singularities of min time affine control systems. 2018. hal-01718345v1

**HAL Id: hal-01718345**

**<https://inria.hal.science/hal-01718345v1>**

Preprint submitted on 27 Feb 2018 (v1), last revised 4 Nov 2021 (v2)

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# Singularities of min time affine control systems

J.-B. Caillau\*    J. Féjóz†    M. Orioux‡    R. Roussarie§

February 2018

## Abstract

Affine control problems arise naturally from controlled mechanical systems. Building on previous results [2, 8], we go one step further and prove that the extremal flow given by Pontrjagin maximum principle can be stratified. We also study the regularity of this flow in terms of regular-singular transition and prove that, as in the nilpotent approximation, the singularity of time minimizing extremals is logarithmic. We finally give global bounds on the number of such singularities and apply our results to the control of two and three bodies in space mechanics.

## Introduction

In this paper we study minimum-time affine control systems, of the form

$$\dot{x} = F_0(x) + u_1 F_1(x) + u_2 F_2(x),$$

where  $u$  is contained in some euclidean ball and all vector fields are smooth. We aim at developing a general theory and applying it to space mechanics, and more precisely to the controlled Kepler problem and the controlled restricted planar three-body problems. In our configuration, the controlled spacecraft (a satellite for instance) is under the influence of two primaries in circular motion. The mass of the satellite is supposed negligible with respect to the mass of the two primaries; see [16] for more details on the restricted 3-body problem. The dynamics is

$$\ddot{q} + \nabla V_\mu(q) - 2J\dot{q} = u$$

where  $V_\mu$  is the potential described in section 4, and fits our generic hypothesis (A) given in section 1, and  $u$ , the control, being the thrust of the engine. In [8], the nilpotent case

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has been extensively treated and the average problem is considered. See also [6], where geodesic convexity is proved for the averaged system in the case  $\mu = 0$  (Kepler). Here we will be interested in the original (as opposed to averaged) system. In this matter, the recent work [7] proved the non integrability of the extremal system, and [10] where the  $L^1$  minimization is studied, necessary and sufficient conditions for optimality are given. Recently, sufficient conditions for optimality have been also proved in [1] for a minimum time affine control system in a slightly different context.

In section 1, we start by recalling some classical results of geometric optimal control, with a particular emphasis on the Pontrjagin Maximum Principle, which reduces the problem to the study of a singular Hamiltonian system. For the sake of simplicity, we carry out the arguments in a 4-dimensional manifold, but the method can be adapted to a  $2m$ -dimensional manifold with an  $m$ -dimensional affine control. We study the local structure of the Hamiltonian flow under generic assumptions in section 2. The beginning of our study is builds upon the analysis in [8] and goes one step further than the recent paper [2] where the flow is proved to be well-defined and continuous: Using the underlying normal hyperbolicity of the system, we provide a stratification such that the flow is smooth on each stratum. We investigate in section 3 the kind of singularity of the flow encountered when crossing strata. Thanks to a suitable normal form, we prove that the associated regular-singular transition results in a logarithmic term, implying it belongs to the log-exp category, [11]. We apply these results to the control of the circular restricted three-body problem in section 4. We finally investigate global properties of the flow and give upper bounds on the number of switchings of the control for this nonlinear system. In contrast with [9] where a subset of the switching set was studied, we treat here the general case using a comparison *à la Sturm*. (We also note that such bounds for time minimization are given in the linear case in [4].)

## Notations.

- Smooth will mean of class  $\mathcal{C}^\infty$ .
- $M$  is a 4-dimensional smooth manifold.
- Let  $F_0, F_1$  and  $F_2$  be smooth vector fields on  $M$ .
- Let  $TM$  and  $T^*M$  be respectively the tangent and cotangent bundles of  $M$ , and  $\pi : T^*M \rightarrow M$  be the canonical projection.
- Let  $|v|$  be the standard Euclidean norm of a vector  $v \in \mathbf{R}^n$ .
- Let  $[\cdot, \cdot]$  be the standard Lie bracket of vector fields on manifolds. For two vector fields  $F_i, F_j$  we denote their Lie bracket by  $F_{ij} = [F_i, F_j]$ .
- Similarly, let  $\{\cdot, \cdot\}$  be the standard Poisson bracket on  $T^*M$ : In Darboux coordinates  $(x, p)$ ,  $\{f, g\} = \frac{\partial f}{\partial x} \frac{\partial g}{\partial p} - \frac{\partial g}{\partial x} \frac{\partial f}{\partial p}$ . For two smooth Hamiltonians  $H_i$  on  $T^*M$ , we denote their Poisson bracket by  $H_{ij} := \{H_i, H_j\}$ , so that if  $z$  is a Hamiltonian trajectory of  $H$ ,  $\frac{d}{dt} f(z(t)) = \{H, f\}(z(t))$ . More generally,  $H_{i,i_1,\dots,i_n} := \{H_i, H_{i_1\dots i_n}\}$ .

- For  $f \in \mathcal{C}^\infty(T^*M)$  (smooth real-valued function on  $T^*M$ ), let  $\text{ad } f : \mathcal{C}^\infty(M) \curvearrowright$ ,  $\text{ad } f(g) = \{f, g\}$ .
- For an normally hyperbolic equilibrium point  $z \in M$ , let  $W^s(z)$  (respectively  $W^u(z)$ ) be its stable (respectively unstable) manifold.
- Let  $J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  be the standard two-dimensional symplectic matrix.

## 1 Setting

We first briefly introduce the general minimum-time affine control system

$$\dot{x} = F_0(x) + u_1 F_1(x) + u_2 F_2(x), \quad x \in M, \quad u \in U \quad (1)$$

where  $U$  is the set of controls  $u \in L^\infty([0, t_f], \mathbf{R}^2)$  constrained in the unit Euclidean ball  $B$  of  $\mathbf{R}^2$ . (One can equally let the admissible controls be in  $L^1_{loc}([0, t_f], \mathbf{R}^2)$  rather than  $L^\infty([0, t_f], \mathbf{R}^2)$ , without much difference.)

The associated minimum-time control problem is:

$$\begin{cases} \dot{x}(t) = F_0(x(t)) + u_1(t)F_1(x(t)) + u_2(t)F_2(x(t)), & t \in [0, t_f], \quad u \in U \\ x(0) = x_0 \\ x(t_f) = x_f \\ t_f \rightarrow \min. \end{cases} \quad (2)$$

By definition, the associated *pseudo-Hamiltonian* is

$$H(x, p, u) = H_0 + u_1 H_1 + u_2 H_2, \quad H_i = \langle p, F_i(x) \rangle, \quad i = 0, 1, 2.$$

The first main theorem of the theory, known as the Pontrjagin Maximum Principle, provides a necessary condition for optimality, allowing us to work with a true—that is independent of the control—Hamiltonian system, possibly with singularities. Since the extremals lie in the cotangent bundle of the phase space, the dimension is doubled. See for instance [3].

**Theorem 1** (Pontrjagin Maximum Principle (PMP)). *Let  $(x, u)$  be a minimum-time trajectory of (2). There exists a Lipschitz curve  $p(t) \in T_{x(t)}^*M^*$ ,  $t \in [0, t_f]$ , such that, almost everywhere on  $[0, t_f]$*

(i)  $(x, p)$  is a solution of the Hamiltonian system associated with  $H$ :

$$\dot{x} = \frac{\partial H}{\partial p}(x, p, u), \quad \dot{p} = -\frac{\partial H}{\partial x}(x, p, u), \quad (3)$$

(ii)  $H(x(t), p(t), u(t)) = \max_{v \in B} H(x(t), p(t), v)$ ,

(iii)  $H(x(t), p(t), u(t)) \geq 0$ .

Moreover,  $p(\cdot)$  does not vanish on  $[0, t_f]$ . Solutions of (3) are called *extremals*, and their projections on  $M$ , *extremal trajectories*. We define the *singular locus*, or *switching surface*, as

$$\Sigma = \{z \in T^*M, H_1(z) = H_2(z) = 0\}.$$

Extremals along which  $(H_1, H_2)$  does not vanish are called *bang arcs* ( $u$  takes values in  $\partial B$ ). An extremal is said to be *bang-bang* if it is a concatenation of bang arcs. The following proposition is immediate.

**Proposition 1.** *An extremal lying out of  $\Sigma$  is an integral curve of the maximized Hamiltonian*

$$H^*(z) = H_0(z) + \sqrt{H_1^2(z) + H_2^2(z)},$$

and the associated control is

$$u = \frac{1}{\sqrt{H_1^2 + H_2^2}}(H_1, H_2)$$

(thus it lies in  $\mathbb{S}^1$ ).

Extremals lying outside  $\Sigma$  are bang arcs and we will see that those crossing  $\Sigma$  are bang-bang extremals. In what follows, we make the following assumption:

$$\det(F_1(x), F_2(x), F_{01}(x), F_{02}(x)) \neq 0, \text{ for almost all } x \in M. \quad (\text{A})$$

This property is generic among vector fields, and holds for most systems coming from mechanical systems. It is a sufficiency condition for global controllability, provided the drift  $F_0$  is recurrent, thus, for instance, controllability holds for the controlled Kepler and on some Hill's region of the restricted circular three body problem (see [8]). Also, since the adjoint vector cannot be zero, assumption (A) implies  $H_1^2(z) + H_2^2(z) + H_{01}^2(z) + H_{02}^2(z) \neq 0$  along an extremal  $z$ .

## 2 Stratification of the extremal flow

We let  $\bar{z} \in \Sigma$ , and we are interested in the local behavior around  $\bar{z}$ . We know after [8] that there are essentially three cases, which lead to three stratas:  $\bar{z}$  is one of the three sets

$$\begin{aligned} \Sigma_- &= \{H_{12}(z)^2 < H_{02}(z)^2 + H_{01}(z)^2\} \\ \Sigma_+ &= \{H_{12}(z)^2 > H_{02}(z)^2 + H_{01}(z)^2\} \\ \Sigma_0 &= \{H_{12}(z)^2 = H_{02}(z)^2 + H_{01}(z)^2\}. \end{aligned}$$

The theorem below tackles the case of extremals that do not lie entirely in  $\Sigma$ , i.e., the ones initializing outside  $\Sigma$ .

**Theorem 2.** *Assume (A) holds. Then there exists a neighborhood  $O_{\bar{z}}$  of  $\bar{z}$  such that*

- (i) there exists a unique solution for system (2) in  $O_{\bar{z}}$ ;
- (ii) there is at most one switch on  $O_{\bar{z}}$ ;
- (iii) and extremals crossing  $\Sigma$  are bang-bang.

Furthermore, if  $\bar{z} \in \Sigma_-$ , the extremal flow  $z : (t, z_0) \in [0, t_f] \times O_{\bar{z}} \mapsto z(t, z_0) \in M$  is piecewise smooth. More precisely, it can be stratified as follows:

$$O_{\bar{z}} = S_0 \cup S_1 \cup \Sigma$$

where  $S_1$  is the codimension-one submanifold of initial conditions leading to the switching surface,  $S_0 = O_{\bar{z}} \setminus (S_1 \cup \Sigma)$ . Both are stable by the flow, which is smooth on  $S_0$ , and on  $[0, t_f] \times S_1 \setminus \Delta$  where  $\Delta = \{(\bar{t}(z_0), z_0), z_0 \in S_1\}$ , and  $\bar{t}(z_0)$  is the switching time of the extremal initializing at  $z_0$ , and continuous on  $O_{\bar{z}}$ . If  $\bar{z} \in \Sigma_+$ , no extremal intersects the singular locus, and therefore, the flow is smooth on  $O_{\bar{z}}$ .

In a slow enough time, so that we have sufficient regularity to define stable and unstable manifolds to  $\Sigma$ ,  $S_1$  is actually the global stable manifold.

*Example 1.* Before going into the proof, let us see a simple example given by a nilpotent approximation (only brackets of length smaller than 2 are non zero) of the two body controlled problem (see section 4):

$$\ddot{q} + \frac{q}{|q|^3} = u.$$

The nilpotent approximation is [8]

$$\begin{cases} \dot{x}_1 = 1 + x_3 & \dot{x}_3 = u_1 \\ \dot{x}_2 = x_4 & \dot{x}_4 = u_2 \end{cases} \quad (4)$$

with the same bound on the control  $|u| \leq 1$ . The maximum principle leads to the maximized Hamiltonian:

$$H(x, p) = p_1(1 + x_3) + p_2x_4 + \sqrt{p_3^2 + p_4^2}$$

( $p_i$  being the adjoint variable of  $x_i$ ), and the co-dimension two sub-manifold  $\Sigma = \{p_3 = 0\} \cap \{p_4 = 0\}$  is the singular locus. Notice that  $p_1$  and  $p_2$  are constant, note  $a = -p_1(0)$  et  $c = -p_2(0)$ . We then get  $p_3(t) = at + b$ ,  $p_4(t) = ct + d$ , with  $b = p_3(0)$  et  $d = p_4(0)$ . From (4), the regularity of the flow is given by the  $x_3$  and  $x_4$  components. We have

$$\dot{x}_3 = \frac{at + b}{\sqrt{(at + b)^2 + (ct + d)^2}}, \quad (5)$$

thus we see that singularity arises when  $t \mapsto (at + b, ct + d)$  vanishes, ie,  $ad - bc = 0$  which defines our codimension one submanifold  $S_1 = \{p_1p_4 - p_2p_3 = 0\} \setminus \{p = 0\}$  (remember that the adjoint state cannot be zero for a minimum-time extremal thanks to the Maximum

Principle). Naturally, we get a symmetric dynamics for  $x_4$  and end up with the same sub-manifold. One can notice that this strata is stable by the flow of  $H$ : If  $z_0 \in S_1$ ,  $z(t, z_0) = (x(t), p(t)) \in S_1$ . Outside  $S_1$ , we can explicitly solve (5), to obtain:

$$x_3(t, z_0) = \frac{a}{a^2 + c^2} (\sqrt{(at + b)^2 + (ct + d)^2} - \sqrt{b^2 + d^2}) - c \frac{ad - bc}{(a^2 + c^2)^{3/2}} \left[ \operatorname{argsh} \left( \frac{(a^2 + c^2)t + ab + cd}{ad - bc} \right) - \operatorname{argsh} \left( \frac{ab + cd}{ad - bc} \right) \right] + x_3(0). \quad (6)$$

It becomes then obvious that the flow of the nilpotent approximation is smooth outside  $S_1$ . If  $a$  and  $c$  are null,  $p_3$  and  $p_4$  become constant, and since  $p$  cannot vanish, switching never occurs. Now observe that the flow can be continuously extended to  $S_1$  by

$$x_3(t, z_0) = \frac{a}{a^2 + c^2} (\sqrt{(at + b)^2 + (ct + d)^2} - \sqrt{b^2 + d^2}) + x_3(0)$$

for all  $z_0 \in S_1 \setminus \{p = 0\}$ . Restricted to  $S_1$ , the flow is locally smooth, except at switches. We also have global continuity, even-though the flow is not Lipschitz continuous. Furthermore, on this simple model a singularity of the type " $z \ln z$ " appears when crossing  $S_1$ . This assumption is the subject of section 3.

Let us now prove theorem 2. Consider a local chart in  $O_{\bar{z}}$ ,

$$z \in T^*M \mapsto (x, p_1, p_1, p_3, p_4) \in \mathbb{R}^4 \times \mathbb{R}^4 \mapsto (x, H_1, H_2, H_{01}, H_{02}).$$

This map is a smooth change of coordinates according to assumption (A). Then, a polar blow-up is used to study the dynamics near the singularity by setting

$$(H_1, H_2) = (\rho \cos \theta, \rho \sin \theta), \quad (\rho, \theta) \in \mathbb{R} \times \mathbb{S}^1.$$

In polar coordinates we have an expression for the control  $u = (\cos \theta, \sin \theta)$ , and  $\Sigma = \{\rho = 0\}$ . At this point, the dynamics is the following:

$$\begin{cases} \dot{x} = F_0(x) + \cos \theta F_1(x) + \sin \theta F_2(x) \\ \dot{\rho} = \cos \theta H_{01} + \sin \theta H_{02} \\ \dot{\theta} = \frac{1}{\rho} (H_{12} + \cos \theta H_{02} - \sin \theta H_{01}) \\ \dot{H}_{01} = H_{001} + \cos \theta H_{101} + \sin \theta H_{201} \\ \dot{H}_{02} = H_{002} + \cos \theta H_{102} + \sin \theta H_{202} \end{cases} \quad (7)$$

The vector fields  $F_i$  are smooth, and then, so are the  $H_i$  and all their Poisson brackets. Setting a new time to desingularize  $dt = \rho ds$  we get:

$$\begin{cases} x' = \rho(F_0(x) + \cos(\theta)F_1(x) + \sin(\theta)F_2(x)) \\ \rho' = \rho(\cos \theta H_{01} + \sin \theta H_{02}) \\ \theta' = H_{12} + \cos(\theta)H_{02} - \sin(\theta)H_{01} \\ H'_{01} = \rho(H_{001} + \cos \theta H_{101} + \sin \theta H_{201}) \\ H'_{02} = \rho(H_{002} + \cos \theta H_{102} + \sin \theta H_{202}). \end{cases} \quad (8)$$

In this new time, the autonomous vector field is smooth, which imply existence and uniqueness, as well as smoothness of the flow. We will now note  $X$  for the vector field of the dynamical system (8). Note that when  $\rho$  is null, only  $\theta$  is non constant, in particular  $\Sigma = \{\rho = 0\}$  is invariant by the flow (in time  $s$ ). Also note the following formula:

$$\text{ad } \rho = \text{ad } H_0 + \cos \theta \text{ad } H_1 + \sin \theta \text{ad } H_2. \quad (9)$$

In the following we will denote  $\pi(\bar{z}) = \bar{x}$ ,  $H_{ij}(\bar{z}) = \bar{H}_{ij}$ ,  $i, j = 0, 1, 2$ . The next lemma establishes the the crucial following fact: In each part of  $\Sigma$ , the derivative on  $\theta$  has a different number of equilibria:

**Lemma 1.** (i)  $\forall z \in \Sigma_-, \theta \mapsto H_{12}(z) + \cos(\theta)H_{02}(z) - \sin(\theta)H_{01}(z)$  has two zeros, noted  $\theta_-$  and  $\theta_+$ . Furthermore, they can be defined by the implicit function theorem: There is a map  $y = (x, H_{01}, H_{02}) \in Y \cap \Sigma \mapsto \theta_{\pm}(y)$  where  $Y$  is small enough neighborhood of  $(\bar{x}, \bar{H}_{01}, \bar{H}_{02})$ .

(ii)  $\forall z \in \Sigma_+, \theta \mapsto H_{12}(z) + \cos(\theta)H_{02}(z) - \sin(\theta)H_{01}(z)$  has no zero.

(iii)  $\forall z \in \Sigma_0, \theta \mapsto H_{12}(z) + \cos(\theta)H_{02}(z) - \sin(\theta)H_{01}(z)$  has exactly one zero.

**Proof of Lemma 1.** Indeed, with polar coordinates on the brackets  $(H_{01}, H_{02}) = r(\cos \psi, \sin \psi)$ , we get  $H_{12}(z) + \cos(\theta)H_{02}(z) - \sin(\theta)H_{01}(z) = H_{12}(z) - r \sin(\theta - \psi)$  and conclude by noting that  $H_{12}(z)/r = \sin(\theta - \psi)$  has two solutions,  $\theta_-$  and  $\theta_+$ , if  $\bar{z} \in \Sigma_-$ , no solution if  $\bar{z} \in \Sigma_+$  and one if  $\bar{z} \in \Sigma_0$  (since  $H_{12}(z)/r = \pm 1$ ). To check that we can use the implicit function theorem and define

$$g : (x, \theta, H_{01}, H_{02}) \in \Sigma \mapsto H_{12}(z) + \cos(\theta)H_{02}(z) - \sin(\theta)H_{01}(z)$$

and notice that

$$\frac{\partial g}{\partial \theta}(y, \theta_{\pm}) = -(\cos \theta_{\pm} H_{01}(z) + \sin \theta_{\pm} H_{02}(z)) = -r \cos(\theta_{\pm} - \psi) = \pm \sqrt{r(z)^2 - H_{12}(z)^2} \neq 0$$

for  $y \in Y$ . □

**Case  $\bar{z} \in \Sigma_-$ .** Let us briefly recall some notions about normal hyperbolicity.

*Definition 1.* A diffeomorphism  $f : M \hookrightarrow M$  is said to be normally hyperbolic along a compact submanifold  $N$  if  $N$  is invariant by  $f$  and the tangent bundle of  $M$  along  $N$  has a splitting  $T_z M = E(z)^u \oplus T_z N \oplus E(z)^s$ ,  $z \in N$ , such that  $df|_{E^{u,s}}(x) = E^{u,s}(f(x))$  ( $f$  preserve the splitting), and there exists  $\lambda_1 \leq \mu_1 < \lambda_2 \leq \mu_2 < \lambda_3 \leq \mu_3$ , with  $\mu_1 < 1 < \lambda_3$ , such that

$$\lambda_1 \leq \|df|_{E^s}\| \leq \mu_1, \quad \lambda_2 \leq \|df|_{TN}\| \leq \mu_2, \quad \lambda_3 \leq \|df|_{E^u}\| \leq \mu_3. \quad (10)$$



This property can be described by saying that the contraction (resp. expansion) in the the stable (resp. unstable) direction is stronger than tangentially to  $N$ . The distributions  $E^s$  and  $E^u$  turn out to be locally integrable and one can construct the local stable and unstable manifolds,  $W(z)^s$  and  $W(z)^u$  respectively tangent to  $E^s(z)$  and  $E^u(z)$  at each point  $z \in N$ . Also, define  $W^{so} = \bigcup_{z \in N} W^s(z)$  and  $W^{uo} = \bigcup_{z \in N} W^u(z)$  the local stable (resp. unstable) manifold of  $N$ . Define also  $l_s, l_u$  as the biggest integer such that  $\mu_1 \leq \lambda_2^{l_u}$  and  $\mu_2^{l_s} \leq \lambda_3$ .

We now recall a theorem of Hirsch, Pugh and Shub (theorem 3.5 in [12], see also [15]) giving the regularity of the manifold in terms of the ratio of the contraction and expansion rates.

**Theorem 3** (Hirsch, Pugh, Shub). *Any  $f$ -invariant submanifold which is close enough to  $N$  is included in  $W^{so} \cup W^{uo}$ . Furthermore,  $W^{so}$  and  $W^{uo}$  are smooth submanifolds of class  $C^{l_s}$  and  $C^{l_u}$  respectively.*

In our case, we have two codimension-two submanifolds of equilibrium points, namely  $\bar{z}_+ = (x, 0, \theta_+(y), H_{01}, H_{02})$  and  $\bar{z}_- = (x, 0, \theta_-(y), H_{01}, H_{02})$ . We set  $\cos \theta_- H_{01} + \sin \theta_- H_{02} = -\sqrt{r^2 - H_{12}^2} < 0$  (so the value for  $\theta_+$  will be the opposite). The Jacobian of the system (5) has two non-zero eigenvalues at those points:

$$\cos \theta_{\pm} H_{01} + \sin \theta_{\pm} H_{02}$$

and its opposite, and a 6-dimensional kernel. Given the spectrum of the Jacobian in  $\bar{z}_{\pm}$  we have a unidimensional stable submanifold  $W^s(\bar{z}_{\pm})$ , and a unidimensional unstable submanifold  $W^u(\bar{z}_{\pm})$  in each equilibria  $\bar{z}_{\pm}$ . The flow is thus normally hyperbolic to the manifold  $N = \{z_-(x, 0, \theta_-(y), H_{01}, H_{02}), z_- \in O_{\bar{z}}\}$ : The tangent space is splitted as in Definition 1, with  $\lambda_2$  and  $\mu_2$  being 1. On  $N$  the dynamics is trivial: Every point is an equilibrium. Hence there exists a unique trajectory converging to  $\bar{z}_-$  (in infinite time  $s$ ) in the stable manifold  $W^s(\bar{z}_-)$ . On  $\Sigma$  however, everything is constant except  $\theta$ , which makes a heteroclinic connexion from  $\theta_-$  to  $\theta_+$ . Then, an extremal will converge to  $\bar{z}_+$  when  $s \rightarrow -\infty$ . Since  $\rho' = \rho(\cos \theta H_{01} + \sin \theta H_{02})$ , in  $\bar{z}_-$  there could be no extremal going out of  $\Sigma$ , if we keep  $\rho \geq 0$ . The situation is symmetric in  $\bar{z}_+$ . The system (8) is autonomous, so there is a unique extremal passing through  $\bar{z}$ , besides, this is happening in finite time  $t$ . Indeed, notice that  $z \mapsto \cos \theta H_{01}(z) + \sin \theta H_{02}(z) = -\sqrt{r^2(z) - H_{12}^2(z)}$  is smooth, and as such, bounded on  $O_{\bar{z}}$ . It is also negative, and note  $C < 0$  a negative upper bound. Now by considering the dynamics of  $\rho$ , see that  $\dot{\rho}(s) \leq \rho(s)C$ . Hence, by Gronwall's lemma  $\rho(s) < \rho(0)e^{Cs}$ . Now the time interval before reaching  $\Sigma$  satisfies

$$\Delta t < \int_0^{\infty} \rho(s) ds < \rho(0)/C < \infty.$$

From now on we will investigate the flow regularity.

**Lemma 2.** *There exists a codimension one submanifold  $S_1 = \bigcup_{z \in N} W^s(z)$ —the set of initial conditions leading to  $\Sigma$ —preserved by the flow, and on which it is locally smooth. More precisely, the map  $z : [0, t_f] \times S_1 \setminus \Delta \rightarrow O_{\bar{z}}$  is smooth,*

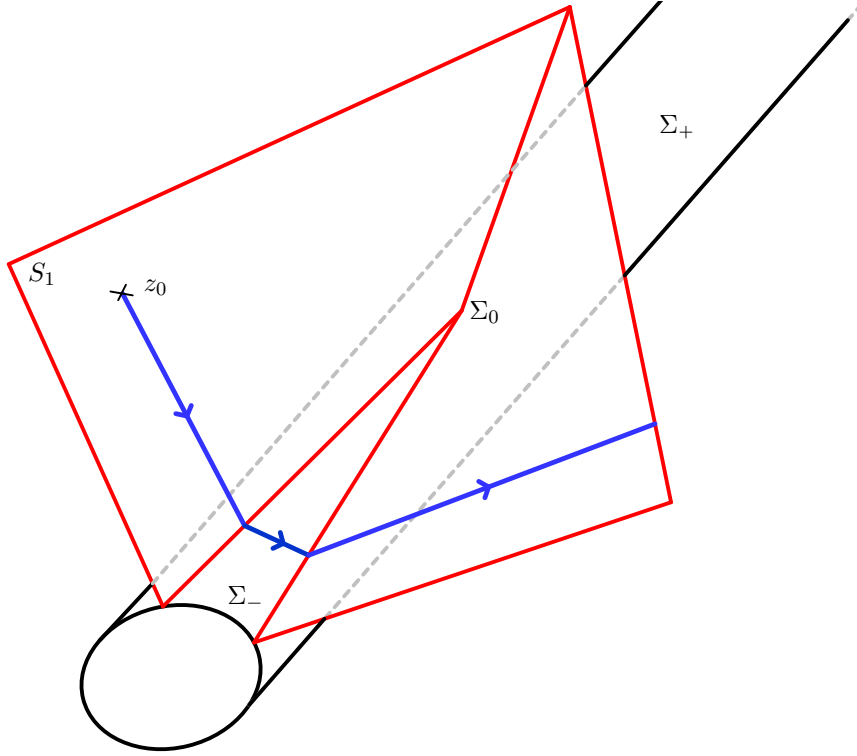


Figure 1: Stratification of the flow into regular submanifolds.

where  $\Delta = \{(\bar{t}(z_0), z_0), z_0 \in S_1\}$ ,  $\bar{t}(z_0)$  being the switching time of the extremal passing through  $z_0$ .

**Proof of Lemma 2.** We begin the proof by showing that  $S_1$  is a well defined codimension 1 submanifold. Recall that the Jacobian of the vector field has a convenient spectrum on  $N$ : A 6-dimensional kernel,

$$\cos \theta_- H_{01} + \sin \theta_- H_{02} = -\sqrt{r^2 - H_{12}^2},$$

and  $+\sqrt{r^2 - H_{12}^2}$ . As such, the flow is normally hyperbolic to  $N$ , then  $V = \bigcup_{z \in N} W^s(z)$  is the so-called stable manifold of  $N$ .  $V$  is smooth by theorem 3 since there is no dilatation nor contraction on  $TN$ . Then the submanifold we are looking for is  $S_1 = V \cap \{\rho > 0\}$ , and  $\dim S_1 = \dim N + \dim W^s(z) = 7$ . It remains to show that the map  $z : (t, z_0) \in [0, t_f] \times S_1 \setminus \Delta \mapsto z(t, z_0) \in O_{\bar{z}}$  is smooth. Let's set  $\bar{t}(z_0) = \int_0^\infty \rho(s, z_0) ds$  the contact time with the singularity  $\Sigma$ , we know that  $\bar{t}(z_0) < \infty$ . The flow is smooth in the time  $s$ , hence  $z_0 \mapsto \rho(s, z_0)$  is smooth and by the classical dominated convergence argument,  $\bar{t}(z_0)$  is smoothly depending on  $z_0 \in O_{\bar{z}}$ . Obviously  $z : (t, z_0) \in [0, \bar{t}(z_0)) \times S_1 \mapsto z(t, z_0) \in O_{\bar{z}}$  is smooth. By uniqueness  $z(t, z_0) = z(t - \bar{t}(z_0), z(\bar{t}(z_0), z_0))$ , and we use the fact that the

contact point with the singular locus  $\Sigma$  is depending smoothly on the initial condition on  $S_1$ , more precisely, the map  $z_0 \in S_1 \mapsto z(\bar{t}(z_0), z_0)$  is smooth. It is straightforward by writing  $z(\bar{t}(z_0), z_0) = \int_0^\infty X(z(s, z_0)) ds + z_0$ . The integrand is bounded uniformly with respect to  $z_0$ , indeed,  $s \mapsto z(s, z_0)$  is only parameterizing the stable manifold  $W^s(z(\bar{t}(z_0), z_0)) \subset O_{\bar{z}}$ , but this neighborhood is relatively compact (and independent of  $z_0$ ), and the vector field  $X$  is bounded on  $O_{\bar{z}}$ . Thus this integral is smooth with respect to the parameter  $z_0$ . Now notice that  $(t, z_0) \mapsto z(t - \bar{t}(z_0), z(\bar{t}(z_0), z_0)) = z(t, z_0)$  is an extremal initializing on  $\Sigma$  at time  $\bar{t}(z_0)$ . By the symmetry  $t \mapsto -t$ , this situation is analogous to the previous one, this map is smooth as long as  $t \neq \bar{t}(z_0)$ , which conclude the proof of Lemma 2.  $\square$

We now know that the flow is smooth outside of  $S_1$  and restricted to  $S_1$ , we also know that it is Lipschitz with respect to the time  $t$ . It remains to prove its global continuity on  $O_{\bar{z}}$ . Let  $z_0, z_1 \in O_{\bar{z}}$ ,  $t, t'$  in  $[0, t_f]$  and  $O_\delta \subset O_{\bar{z}}$  be a small neighborhood of  $\bar{z}$ . Thanks to the previous assumption, we can assume without loss of generality, that  $z_0 \in S_1$ ,  $z_1 \in O_{\bar{z}} \setminus S_1$  and that the extremal from  $z_0$  is passing through  $\bar{z}$ . We would like to control the quantity  $|z(t, z_0) - z(t', z_1)|$ . Let  $\varepsilon > 0$ , and note  $t_0$ , the contact time with  $O_\delta$ , and  $t'_0$  the exit time from this neighborhood: Namely  $t_0 = \inf\{t \in \mathbb{R}, \text{ s. t. } z(t, z_0) \in O_\delta\}$ . If  $z_0$  and  $z_1$  are close enough,  $|z(t_0, z_0) - z(t_0, z_1)| < \varepsilon/2$ ; simply because the flow is continuous when the singular locus is not crossed yet. We will use the following Lemma to conclude:

**Lemma 3** ([2]). *For all  $\delta > 0$  there exists a neighborhood  $O_\delta$  of  $\bar{z}$  in which every extremal spends a time interval uniformly (that is not depending on the extremal) smaller than  $\delta$ .*

**Proof of Lemma 3.** We will prove it in a neighborhood  $O_{\bar{z}_-}$  of  $\bar{z}_-$ , the situation being symmetric around  $\bar{z}_+$ , and an extremal in  $\Sigma$  spending 0 time  $t$ . Let's define  $O_\delta = \{z \in O_{\bar{z}_-}, \rho < \delta, |\theta - \theta_-| < \delta\}$ ,  $z \mapsto \cos(\theta)H_{01}(z) + \sin(\theta)H_{02}(z)$  is smooth and thus bounded on  $O_\delta$ . Now set

$$M_\delta = \sup_{z \in O_\delta} \cos(\theta)H_{01}(z) + \sin(\theta)H_{02}(z),$$

remember it is negative on  $O_{\bar{z}_-}$ . Then, for any extremal in  $O_\delta$ ,  $\frac{\dot{\rho}(s)}{\rho(s)} \leq M_\delta$ , which implies

$$\rho(s) < \rho(0)e^{M_\delta s}.$$

So that, we can control its time interval in  $O_\delta$  by

$$\Delta_{O_\delta} t < \int_0^\infty \rho(0)e^{M_\delta s} ds = -\frac{R}{M_\delta},$$

this quantity tends to 0 when  $R$  does, and the lemma is proved. Then, with a good choice of  $O_\delta$ ,  $|z(t'_0, z_0) - z(t_0, z_0)| \leq \varepsilon/2$ , and  $|z(t'_0, z_0) - z(t_0, z_1)| \leq |z(t'_0, z_0) - z(t_0, z_0)| + |z(t_0, z_0) - z(t_0, z_1)| \leq \varepsilon$ . Now notice that,  $z(t, z_0) = z(t - t'_0, z(t'_0, z_0))$ , and we use the regularity of the system when the singular locus is not crossed to conclude.  $\square$

*Remark 1.* In the case  $\bar{z} \in \Sigma_-$ , ie, the only case when switching occur, we can quantify the jump on the control at a switching time  $\bar{t}$  in terms of Poisson brackets:

$$u(\bar{t}_\pm) = (\cos \theta_\pm, \sin \theta_\pm) = \frac{1}{r^2}(-H_{02}H_{12} \pm H_{01}\sqrt{r^2 - H_{12}^2}, H_{01}H_{12} \pm H_{02}\sqrt{r^2 - H_{12}^2}).$$

**Case  $\bar{z} \in \Sigma_+$ .** The dynamics in the initial time is still given by:

$$\begin{cases} \dot{x} = F_0(x) + \cos(\theta)F_1(x) + \sin(\theta)F_2(x) \\ \dot{\rho} = \cos \theta H_{01} + \sin \theta H_{02} \\ \dot{\theta} = \frac{1}{\rho}(H_{12} + \cos(\theta)H_{02} - \sin(\theta)H_{01}) \\ \dot{H}_{01} = H_{001} + \cos \theta H_{101} + \sin \theta H_{201} \\ \dot{H}_{02} = H_{002} + \cos \theta H_{102} + \sin \theta H_{202}. \end{cases} \quad (11)$$

In particular, by lemma 1 we already know  $\dot{\theta}$  is everywhere nonzero. Without loss of generality let's assume  $\rho\dot{\theta} = H_{12} + \cos(\theta)H_{02} - \sin(\theta)H_{01} > 0$  on  $O_{\bar{z}}$ . The following lemma implies the results on this case:

**Lemma 4.** *In a neighborhood  $O_{\bar{z}}$  of  $\bar{z} \in \Sigma_+$  we have the estimate:  $\exists K, c > 0, \forall t$  s.t  $z(t) \in O_{\bar{z}}, \rho(t) \geq K\rho_0 e^{-ct}$ .*

**Proof of the Lemma.** Without loss of generality let's assume  $H_{12} + \cos(\theta)H_{02} - \sin(\theta)H_{01} > 0$  on  $O_{\bar{z}}$ .  $O_{\bar{z}}$  is relatively compact, and as such, the continuous map  $u : z \rightarrow H_{12}(z) + \cos(\theta(z))H_{02}(z) - \sin(\theta(z))H_{01}(z)$ , is bounded from below and above by two positive constant  $c_1$  and  $c_2$ :  $c_2 > u(z) > c_1 > 0$  for all  $z \in O_{\bar{z}}$ . One can notice that  $\frac{d}{dt}(\rho(H_{12} + \cos \theta H_{02} - \sin \theta H_{01})) = \rho(\dot{H}_{12} + \cos \theta \dot{H}_{02} - \sin \theta \dot{H}_{01})$ . By remark 9,  $\dot{H}_{ij} = H_{0ij} + \cos \theta H_{1ij} + \sin \theta H_{2ij}$  which also define a smooth map. As such  $\rho(\dot{H}_{12} + \cos \theta \dot{H}_{02} - \sin \theta \dot{H}_{01}) \geq a$  with  $a < 0$  some constant. Then we have:

$$\frac{d}{dt}(\rho u) \geq a\rho \Leftrightarrow \rho(t)u(t) \geq \rho_0 u(0) + a \int_0^t \rho(s) ds.$$

Now using Gronwall's lemma:  $\rho(t)u(t) \geq \rho_0 u(0) e^{A \int_0^t 1/u(s) ds}$ . Finally

$$\rho(t) \geq \rho_0 \frac{c_1}{c_2} e^{a/c_2 t}.$$

We conclude the proof of theorem 2 by taking  $c = -a/c_2 > 0$  and  $K = c_1/c_2$ . For extremals lying in  $\Sigma_+$ , the following proposition holds:

**Proposition 2.** *There exists a singular flow inside  $\Sigma_+$  and  $\Sigma_0$ , on which we have the control feedback:  $u = \frac{1}{H_{12}}(-H_{02}, H_{01})$ , and the singular flow is smooth. It is solution of the Hamiltonian system  $H = H_0 - \frac{H_{02}}{H_{12}}H_1 + \frac{H_{01}}{H_{12}}H_2$ .*

**Proof.**  $H_1$  and  $H_2$  are identically vanishing along a singular extremal, and so are they derivative with respect to the time  $t$ . Then, for the time interval on which  $\frac{dH_i}{dt}(z(t)) = H_{0i}(z(t)) + u_j H_{ji}(z(t)) = 0$  for  $i = 1, 2$ . Those identities give the wanted feedback. Since  $|u| \leq 1$ , this cannot happen in  $\Sigma_+$ . So  $H_{12}$  cannot be zero, and the singular flow is smooth.  $\square$

*Remark 2.* It has been noticed in [8] that extremals lying in  $\Sigma_+$  cannot be optimal by the Goh condition.

### 3 Regular-singular transition

In the applications, whenever the distribution is involutive the interesting case is  $\Sigma_-$ , and we have the stratification defined in the previous section. This stratification of the flow raises the question of the transition: How does the flow behave when one is getting close to the stratum  $S_1$ ? We answer that question by considering the Poincaré map between two well chosen sections. Recall the dynamic in the time  $s$ :

$$\begin{cases} x' = \rho(F_0(x) + \cos(\theta)F_1(x) + \sin(\theta)F_2(x)) \\ \rho' = \rho(\cos \theta H_{01} + \sin \theta H_{02}) \\ \theta' = H_{12} + \cos(\theta)H_{02} - \sin(\theta)H_{01} \\ H'_{01} = \rho(H_{001} + \cos \theta H_{101} + \sin \theta H_{201}) \\ H'_{02} = \rho(H_{002} + \cos \theta H_{102} + \sin \theta H_{202}). \end{cases}$$

in polar coordinates on the Poisson brackets:  $(H_{01}, H_{02}) = r(\cos \phi, \sin \phi)$  one gets:

$$\begin{cases} \rho' = r\rho \cos(\theta - \phi) \\ \theta' = H_{12} - r \sin(\theta - \phi) \\ \xi' = \rho h(\rho, \theta, \xi) \end{cases} \quad (12)$$

where  $\xi = (x, r, \phi)$  and  $h$  is a smooth function. We can set  $\psi = \theta - \phi$ , rescale the time according to  $dv = rds$ , and study a system with the following structure (the derivation still being noted  $'$ ):

$$\begin{cases} \rho' = \rho \cos \psi \\ \psi' = g(\rho, \psi, \xi) - \sin \psi = G(\rho, \psi, \xi) \\ \xi' = \rho h(\rho, \psi, \xi) \end{cases} \quad (13)$$

where  $g, h$  are smooth functions defined on an open set  $O$  of  $\mathbb{R} \times \mathbb{R} \times D$ ,  $D$  being a compact domain of  $\mathbb{R}^k$ ;  $h$  has values in  $\mathbb{R}^k$  and  $g(\rho, \psi, \xi) = a(\xi) + \rho b(\xi, \psi) + O(\rho^2)$  and  $|g| < 1$  on  $O$ . (This comes from the fact that  $H_{12}$  is a smooth function in  $(\rho \cos \theta, \rho \sin \theta)$ .) Equilibria occur when  $\rho = 0$ ,  $G = 0$  and are semi-hyperbolic, since they're outside  $\{\psi = \pm\pi/2\}$ . More precisely, it was shown in the last section that the flow of this system is normally hyperbolic to the manifold  $\{\rho = 0\} \cap \{G = 0\}$ . For each  $\xi$ , there are two equilibria  $z_{\pm}$ . Thanks to the structure of  $g$ , we get  $\frac{\partial g}{\partial \psi}(0, \psi, \xi) = 0$ , thus  $\frac{\partial G}{\partial \psi}(0, \psi, \xi) = -\cos \psi \neq 0$ .  $\{G = 0\} \cap \{\rho = 0\}$  can then be parameterized by the map  $\xi \mapsto \psi(\xi)$  by the implicit

function theorem. Equilibria are then given by  $(0, \psi_{12}(\xi), \xi)$  coordinates, and define two codimension two submanifolds. In each equilibria, the stable and unstable manifolds are of dimension one. Their reunion form a codimension one submanifold,  $S_{12} = \bigcup_{\xi} W^s(z_{12})$ .  $S_1$  is thus the submanifold of initial condition leading to an equilibrium (in infinite time). The aim of the following work is to study the regular-singular transition, or more precisely, the type of singularity occurring when one crosses  $S_1$ .

Introducing  $\omega = \psi - \psi(\xi)$  along  $\{G = 0\}$  (the analysis is similar for  $\psi_2$  on the unstable manifold), we will study (13) near  $\omega \sim 0$ . The change of coordinates  $(\rho, \psi, \xi) \mapsto (\rho, \omega, \xi)$  gives

$$\begin{cases} \rho' = \rho \cos(\omega + \psi(\xi)) \\ \omega' = g(\rho, \omega + \psi(\xi), \xi) - \sin(\omega + \psi(\xi)) - \rho \frac{\partial \psi}{\partial \xi}(\xi) \cdot h(\rho, \omega + \psi(\xi), \xi) \\ \xi' = \rho h(\rho, \omega + \psi(\xi), \xi) \end{cases}$$

As  $g$  has the form given by (ii),  $g(\rho, \omega + \psi(\xi), \xi) = a(\xi) + \rho b(\omega + \psi(\xi), \xi) + O(\rho^2)$ , and  $g(0, \psi(\xi), \xi) = \sin(\psi(\xi)) = a(\xi)$ . So that system (1) is equivalent to

$$\begin{cases} \rho' = \lambda(\xi)\rho(1 + O(\omega^2)) \\ \omega' = \beta(\xi)\rho - \lambda(\xi)\omega + O((\rho + |\omega|)^2) \\ \xi' = \rho(\gamma(\xi) + O(\rho + |\omega|)) \end{cases} \quad (14)$$

with  $\lambda(\xi) = \cos(\psi(\xi))$  and  $\beta, \gamma$  smooth functions (depending on the derivatives of  $g, h$ ). The Jacobian matrix of (14) is

$$\begin{pmatrix} \lambda(\xi) & 0 & 0 \\ \beta(\xi) & -\lambda(\xi) & 0 \\ \gamma(\xi) & 0 & 0 \end{pmatrix}.$$

Let us now find a change of coordinates making this Jacobian diagonal:  $(\rho, \omega, \xi) \mapsto (\rho, \tilde{\omega}, \tilde{\xi}) = (\rho, \omega + A(\xi)\rho, \xi + B(\xi)\rho)$ . We get  $\tilde{\omega}' = \omega' + \frac{\partial A}{\partial \xi}(\xi) \cdot \xi' \rho + A(\xi)\rho' = \omega' + A(\xi)\rho' + O(\rho^2)$ . Thus  $\tilde{\omega}' = (\beta(\xi) + 2A(\xi)\lambda(\xi))\rho - \lambda(\xi)\tilde{\omega} + O((\rho + |\omega|)^2)$  and by picking  $A = -\frac{\beta}{\lambda}$  we obtain what we were looking for: Indeed, with this change of variables,  $O((\rho + |\omega|)^k) = O((\tilde{\rho} + |\tilde{\omega}|)^k)$  for all  $k$ . Now,  $\tilde{\xi}' = \xi' + \frac{\partial A}{\partial \xi}(\xi) \cdot \xi' \rho + B(\xi)\rho' = \rho(\gamma(\xi) + B(\xi)\lambda(\xi)) + O((\rho + |\omega|)^2)$  and we pick  $B(\xi) = -\frac{\gamma}{\lambda}$ . Slightly abusing of the notations (except for  $\rho$ ), we still note the new variables  $(\rho, \omega, \xi)$  and obtain the new vector field

$$\begin{cases} \rho' = \lambda(\xi)\rho(1 + O(\rho)) \\ \omega' = -\lambda(\xi)\omega + O((\rho + |\omega|)^2) \\ \xi' = \rho O(\rho + |\omega|) \end{cases} \quad (15)$$

System (15) is smoothly equivalent to

$$Y : \begin{cases} \rho' = -\rho(1 + O(\rho)) \\ \omega' = \omega + O((\rho + |\omega|)^2) \\ \xi' = \rho O(\rho + |\omega|) \end{cases} \quad (16)$$

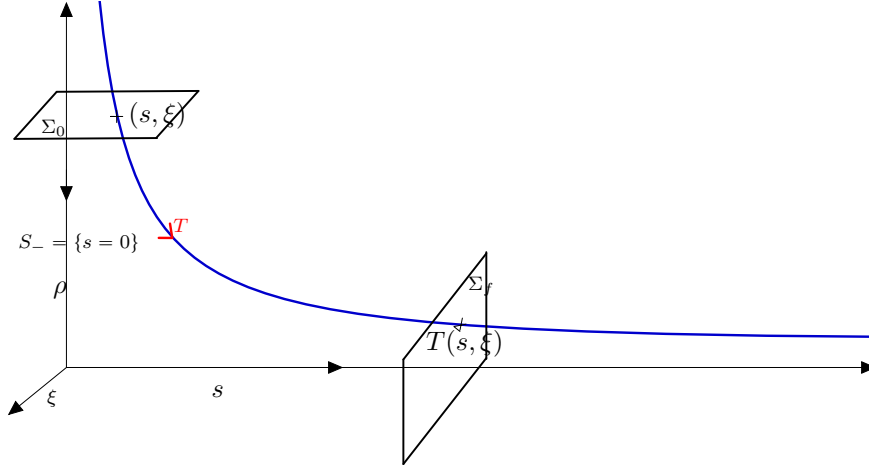


Figure 2: Poincaré map between the two sections.

The Jacobian of this system is diagonal and even constant. Let us now state a smooth normal form theorem:

**Proposition 3** ( $\mathcal{C}^\infty$ -normal form). *Let  $u = \rho\omega$ , then there exist  $A, B, C$  smooth functions on a neighborhood of  $D \times 0_u$  such that  $Y$  is equivalent to*

$$Y^\infty : \begin{cases} \rho' = -\rho(1 + uA(u, \xi)) \\ \omega' = \omega(1 + uB(u, \xi)) \\ \xi' = uC(u, \xi) \end{cases} \quad (17)$$

Under this change of coordinates, the globally invariant manifold  $S_-$ , fibered by stable manifolds, becomes  $\{\omega = 0\}$ , and the equilibria are  $\{(0, 0, \xi)\}$ . We can now make a precise statement: For given  $\rho_0$  et  $\omega_f$ , both positive, consider the two sections  $\Pi_0 \subset \{\rho = \rho_0\}$  and  $\Pi_f \subset \{\omega = \omega_f\}$ . As  $\Pi_0$  is transverse to  $\{\omega = 0\}$ , it can be parameterized by  $(\omega, \xi)$  coordinates. Similarly,  $\Pi_f$  is transverse to  $\{\rho = 0\}$  and can be parameterized by  $(\rho, \xi)$  coordinates.

**Theorem 4.** *Let  $T : \Pi_0 \rightarrow \Pi_f$  be the Poincaré mapping between the two sections,  $T(\omega_0, \xi_0) = (\rho(\omega_0, \xi_0), \xi(\omega_0, \xi_0))$ . Then,  $T$  is a smooth function in  $(\omega_0 \ln \omega_0, \omega_0, \xi_0)$  as there exist smooth functions  $R$  and  $X$  defined on a neighborhood of  $\{0\} \times \{0\} \times D$  such that*

$$T(\omega_0, \xi_0) = (R(\omega_0 \ln \omega_0, \omega_0, \xi_0), X(\omega_0 \ln \omega_0, \omega_0, \xi_0)).$$

Thus,  $T$  belongs to the log-exp category.

One can look once again at the example of section 2.2, the nilpotent approximation

for the minimum time Kepler problem. Recall we had

$$x_3(t, z_0) = \frac{a}{a^2 + c^2} (\sqrt{(at + b)^2 + (ct + d)^2} - \sqrt{b^2 + d^2}) - c \frac{ad - bc}{(a^2 + c^2)^{3/2}} \left[ \operatorname{argsh} \left( \frac{(a^2 + c^2)t + ab + cd}{ad - bc} \right) - \operatorname{argsh} \left( \frac{ab + cd}{ad - bc} \right) \right] + x_3(0) \quad (18)$$

giving the regularity. When one crosses  $\Sigma$  the determinant  $ad - bc$  becomes 0, and one indeed get singularities of the form " $z \ln z$ ".

**Proof of Theorem 4.** Before proving the normal form result, let's demonstrate how it implies Theorem 4. First, note that system (17) is equivalent to

$$\begin{cases} \omega' = \omega \\ \rho' = -\rho(1 + uA(u, \xi)) \\ \xi' = uC(u, \xi) \end{cases} \quad (19)$$

with  $A$  standing now for  $\frac{A-B}{1+uB}$ , and  $C$  for  $\frac{C}{1+uB}$ . It has the same trajectories, and thus the same Poincaré mapping between the two sections. The transition time is given by the first equation:  $t(\omega_0) = \ln(\omega_f/\omega_0)$ . (The singular-regular transition occurs when  $\omega_0 \rightarrow 0$ , and the transition time tends to infinity.) Still noting  $u = \rho\omega$ , (19) implies

$$\begin{cases} u' = -u^2 A(u, \xi), \\ \xi' = uC(u, \xi), \end{cases} \quad (20)$$

that we want to integrate from an initial condition on  $\Pi_0$  in time  $t(\omega_0)$ . We extend this system by the trivial equation  $\omega'_0 = 0$ , and denote  $\varphi$  its associated flow. Then,  $T(\omega_0, \xi_0) = \varphi(\ln(\omega_f/\omega_0), \omega_0, \rho_0\omega_0, \xi_0)$  (on  $\Pi_0$ ,  $u_0 = \rho_0\omega_0$ ). It is not the form we are looking for since  $\ln(\omega/\omega_0)$  is not regular at  $\omega_0 = 0$ , but we have the following estimate on the  $u$  coordinate of the flow.

**Lemma 5.** *There exists a constant  $M$  such that, for small enough  $\omega_0 > 0$ ,  $\xi \in D$ , and integration time  $t \leq \ln(\omega_f/\omega_0)$ ,*

$$0 \leq u(t, \omega_0, \rho_0\omega_0, \xi_0) \leq M\omega_0.$$

**Proof of the Lemma.** Compare the dynamics of  $u$  in (20) with  $v' = -v^2$ , which integrates according to

$$v(t, v_0) = \frac{v_0}{1 + v_0 t}$$

for  $v_0 > 0$ . So

$$v(\ln(\omega_f/\omega_0), \rho_0\omega_0) = \frac{\rho_0\omega_0}{1 + \rho_0\omega_0 \ln(\omega_f/\omega_0)},$$

hence the estimate as  $s_0 \ln(\omega_f/\omega_0)$  is small enough for small enough  $\omega_0 > 0$ .  $\square$



Let us now make a change of time and consider the following rescaled system:

$$\begin{cases} s'_0 = 0, \\ u' = -(u^2/\omega_0)A(u, \xi), \\ \xi' = (u/\omega_0)C(u, \xi). \end{cases} \quad (21)$$

For  $\omega_0 > 0$ , its flow  $\tilde{\varphi}$  is well defined and the Poincaré mapping is obtained by evaluating it in time  $\omega_0 \ln(\omega_f/\omega_0)$ ,

$$T(\omega_0, \xi_0) = \tilde{\varphi}(\omega_0 \ln(\omega_f/\omega_0), \omega_0, \rho_0 \omega_0, \xi_0).$$

We use the blow up on  $\{u = \omega = 0\}$  to prove that  $T$  has the required form. Set  $f(u, \omega, \xi) = (\eta, \omega, \xi)$  with  $\eta = u/\omega$ : In coordinates  $(\omega, \eta, \xi)$ , the pulled back system writes

$$Z : \begin{cases} \omega'_0 = 0, \\ \eta' = -\eta^2 A(\eta \omega_0, \xi), \\ \xi' = \eta C(\eta \omega_0, \xi), \end{cases} \quad (22)$$

The vector field  $Z$  is actually smooth. The blow up map  $f$  send a cone  $-\eta_0 \omega \leq u \leq \eta_0 \omega$  on a rectangle  $-\eta_0 \leq \eta \leq \eta_0$ ,  $-\omega_0 \leq \omega \leq \omega_0$ . According to the previous Lemma, we only need to evaluate its flow  $\hat{\varphi}(t, \omega_0, \eta_0, \xi_0)$  on a band  $\omega_0 \in [-\omega_1, \omega_1]$ ,  $\eta_0 \in [-M, M]$ ,  $\xi \in D$ , to compute  $\tilde{\varphi}$  is time  $\omega_0 \ln(\omega_f/s_0)$ . As  $\hat{\varphi} = (\hat{\eta}, \hat{\xi})$  is smooth on such a band, we eventually get

$$T(\omega_0, \xi_0) = (\hat{\eta}(\omega_0 \ln(\omega_f/\omega_0), \omega_0, \rho_0, \xi_0), \hat{\xi}(\omega_0 \ln(\omega_f/\omega_0), \omega_0, \rho_0, \xi_0),$$

which has the desired regularity. *Box*

*Remark 3.* Notice that when  $\omega \rightarrow 0$ , the Poincaré map goes in the invariant submanifold  $\{\rho = 0\}$ , although in infinite time.

Let us finally prove Proposition 3.

**Proof of Proposition 3.** Let us first state a generalization of the Poincaré-Dulac theorem. First we introduce some notation. Note  $H^l$  the space of homogeneous polynomials of degree  $l$  in  $\mathbb{R}^n$  with smooth coefficients in  $\xi \in \mathbb{R}^k$ . We recall that for a linear vector field  $X$  which does not depends on  $\xi$  (and has no component in the  $\xi$  direction),  $H^l = \text{Im}[X, \cdot]_{|H^l} + \ker[X, \cdot]_{|H^l}$ . A vector field  $Z$  is said to be resonant with  $X$  if  $Z \in \ker[X, \cdot]$ .

**Lemma 6.** *Let  $X(x, \xi)$  be a smooth vector field in  $\mathbb{R}^n \times \mathbb{R}^k$ ,  $X(0, \xi) = 0$ . Note  $X_1$  its linear part. Then, if  $X_1$  does not depend on  $\xi$ , there exist  $g_i \in H^i \cap \ker[X_1, \cdot]$ ,  $i = 2, \dots, l$  and a smooth vector field  $R_l$  with zero  $l$ -jet such that  $X$  is smoothly conjugate to in a neighborhood of zero to:*

$$X_1 + g_2 + \dots + g_l + R_l, \quad l \in \mathbb{N}$$

**Proof of Lemma 6.** We will follow [18] and reason by induction on  $l$ , then treat the case  $l = \infty$ . For  $l = 1$ , the result is trivial:  $X = X_1 + R_1$  where  $R_1$  has zero first jet in zero. Suppose now that  $g_1, \dots, g_{l-1}$ , and  $R_{l-1}$  are as in lemma 6,  $l \geq 2$ .  $R_{l-1}$  has a zero  $l-1$  jet (in zero), and then can be written as  $R_{l-1} = [X_1, Z] + g_l + R_l$ , where  $Z \in H^l$ ,  $g_l \in \ker[X_1, \cdot]$  and  $R_l$  is a smooth vector field with zero  $l$ -jet.  $[X, Z] = [X_1, Z] + \sum_{i=2}^l [g_i, Z] + [R_{l-1}, Z] = [X_1, Z] + R'_l$ , where  $R'_l$  has zero  $l$ -jet. Now, note  $\phi_Z$  the flow of  $Z$ , and consider  $(\phi_Z^t)_* X := X^t$ . We have  $\frac{d}{dt} X^t = [X, Z] = [X_1, Z] + R'_l$ , so that  $X^t = X^0 + t[X_1, Z] + R_{l,t}$ , with  $j^l(R_{l,t})(0) = 0$ . Since  $Z$  is a homogeneous polynomial of degree  $l$ , it has zero  $l-1$  jet, and  $X$  and  $X^t$  have the same  $l-1$  jet, that means  $X^0 = X_1 + g_2 + \dots + g_l + [X_1, Z]$ . Finally, we chose  $t = -1$  to get  $X^{-1} = X_1 + g_2 + \dots + g_l + R_{l,-1}$ , and  $\phi_Z^{-1}$  conjugates the two vector field, which ends the proof of the finite case. The above construction provide a sequence of formal diffeomorphisms  $\varphi_l = \phi_Z^{-1}$  ( $Z \in H^l$ ) with  $(\varphi_l)_* X = X_1 + g_2 + \dots + g_l + R_l$ . Also notice that  $\varphi_l$  and  $\varphi_{l+1}$  have the same  $l$ -jet. This define a sequence of coefficient  $g_l(\xi)$  for all  $l$ .  $\square$  Now remark that a generalization of Borel theorem, proved by Malgrange in [13] show that there exists a smooth function  $\psi$  such that the  $l$  jet of  $\varphi_l$  and  $\psi$  are identical for all  $l \in \mathbb{N}$ . We can also realize, using the same theorem, the formal series given by the resonant monomials by a smooth vector field  $X^\infty$ . Thus we have  $\psi_*(X) = X^\infty + R_\infty$ , where  $R_\infty$  has zero infinite jet. Thus, we begin by looking for monomials that are resonant with the linearized vector field of  $Y$ ,  $Y_1 = -\rho \frac{\partial}{\partial \rho} + s \frac{\partial}{\partial \omega}$  (monomials  $X$  for whom  $[Y_1, X] = 0$ ). First notice that the Lie bracket with  $Y_1$  treat  $\xi$  as a constant: The map  $X \mapsto [Y_1, X]$  is linear in  $\xi$ . Such monomials can be written  $a(\xi)\rho^i\omega^j \frac{\partial}{\partial \rho}$ ,  $b(\xi)\rho^i\omega^j \frac{\partial}{\partial \omega}$ , and  $c(\xi)\rho^i\omega^j \frac{\partial}{\partial \xi}$ . We get:

$$\begin{aligned} [Y_1, a(\xi)\rho^i\omega^j \frac{\partial}{\partial \rho}] &= (i-j-1)a(\xi)\rho^i\omega^j \frac{\partial}{\partial \rho} \\ [Y_1, b(\xi)\rho^i\omega^j \frac{\partial}{\partial \omega}] &= (i+1-j)b(\xi)\rho^i\omega^j \frac{\partial}{\partial \omega} \end{aligned}$$

and

$$[Y_1, c(\xi)\rho^i\omega^j \frac{\partial}{\partial \xi}] = (i-j)c(\xi)\rho^i\omega^j \frac{\partial}{\partial \xi}.$$

The monomials we are looking for are thus:  $a(\xi)\rho u^k \frac{\partial}{\partial \rho}$ ,  $b(\xi)s u^k \frac{\partial}{\partial \omega}$  and  $c(\xi)u^k \frac{\partial}{\partial \xi}$ ,  $k \in \mathbb{N}$ . The lemma allow us to state that the infinite jet of  $Y$  can be formally developed on the resonant monomials, ie,  $Y$  is formally conjugate to

$$W : \begin{cases} \rho' = -\rho(1 + \sum_{i \geq 1} a_i(\xi)u^i) \\ \omega' = \omega(1 + \sum_{i \geq 1} b_i(\xi)u^i) \\ \xi' = \rho \sum_{i \geq 1} c_i(\xi)u^i \end{cases} .$$

It remains to realize those series by smooth functions. By the remark above, there exist  $Y^\infty$  a smooth vector field on  $O$  such that  $W = Y^\infty + R_\infty$  where  $R_\infty$  is a smooth function with zero infinite jet along  $D$ . At this stage we have  $Y$  is smoothly equivalent to  $Y^\infty + R_\infty$ .

The last step consists in killing the flat perturbation  $R_\infty$ . This can be achieved by the path's method: Instead of looking for a diffeomorphism sending  $Y_0 := Y$  on

$Y_1 := Y^\infty + R_\infty$ , we search for a one parameter family (path) of diffeomorphism ( $g_t$ ) such that

$$g_t^* Y_0 = Y_t, \quad (23)$$

$Y_t$  being a path of vector fields joining  $Y_0$  and  $Y_1$ . Consider the linear path  $Y_t = (1-t)Y_0 + tY_1$ ,  $t \in [0, 1]$ . Differentiating (23) with respect to  $t$  we get

$$\frac{\partial}{\partial t}(g_t^* Y_0) = \dot{Y}_t = Y_1 - Y_0 = R_\infty. \quad (24)$$

Now, the family  $g_t$  define a family of vector fields  $Z_t$  by  $Z_t(g_t(x)) = \frac{\partial}{\partial t} g_t(x)|_t$ , reciprocally by integrating  $Z_t$  we obtain the desired path of diffeomorphism. Thus (24) can be rewritten

$$[Y_t, Z_t] = R_\infty. \quad (25)$$

We just showed that getting rid of the flat perturbation  $R_\infty$  boils down to find a solution to the equation (25). It has been proved in [17], theorem 10, that this equation has a solution.  $\square$

## 4 Application to mechanical systems

In this section, we specify our study to mechanical systems coming from a potential, and go further in the particular case of orbit transfer problem with gravitational coplanar two or three body potential. Consider a system of the form  $\ddot{q} + \nabla V(q) = 0$ . Let  $q \in M = \mathbb{R}^2 \setminus \{-\mu, 1 - \mu\}$  be the position vector, and note  $\mu$  the mass ratio of the two primaries. The dynamics of the controlled restricted elliptic three-body problem (CETBP) is the following -  $q^1$  and  $q^2$  are the positions of the two primaries, which by definition, are in elliptic motion around their center of mass:

$$\ddot{q} + \nabla V_\mu(t, q) = u,$$

with  $V_\mu(t, q) = \frac{1-\mu}{|q-q^1(t)|} + \frac{\mu}{|q-q^2(t)|}$ . One can notice that when  $\mu = 0$ , we are in the Keplerian case. The non-autonomous Maximized Hamiltonian given by a non-autonomous version of the Maximum Principle is

$$H(t, z) = p_q \cdot v - \nabla V_\mu(t, q) \cdot p_v + |p_v|, \text{ with } z = (x, p).$$

with of course  $p_q$ , (resp.  $p_v$ ) being the adjoint coordinate of  $q$ , (resp.  $v$ ).

The controlled circular restricted three body problem (CRTBP) is a reduction of the (CETBP) where the two primaries are in circular motion around their center of mass. Its dynamics can be express as an autonomous system, in the rotating frame. Written on the convenient affine control system form:  $x = (q, v)$ ,

$$\dot{x} = F_0(x) + u_1 F_1(x) + u_2 F_2(x)$$

with

$$F_0(x) = F_0(x) = v \cdot \partial_q + (-q + (1 - \mu) \frac{q + \mu}{|q + \mu|^3} + \mu \frac{q - 1 + \mu}{|q - 1 + \mu|^3} + 2Jv) \cdot \partial_v,$$

$F_1(x) = \partial_{v_1}$ ,  $F_2(x) = \partial_{v_2}$  in Cartesian coordinates. The maximized Hamiltonian of the (CRTBP) given by the maximum principle is:

$$H(z) = p_q \cdot v - J_\mu(q, u) \cdot p_v + |p_v|,$$

with

$$J_\mu(q, v) = -q + (1 - \mu) \frac{q + \mu}{|q + \mu|^3} + \mu \frac{q - 1 + \mu}{|q - 1 + \mu|^3} + 2Jv.$$

Before applying the result of section 2, let us make the following important remark: In both cases, the distribution generated by  $F_1$  and  $F_2$  is involutive as the two vector fields actually commute:

$$[F_1, F_2] = 0. \quad (26)$$

**Proposition 4.** *Assumption (A) is check for the Keplerian and circular restricted three-body problem.*

This is actually the consequence of a more general statement (see [8] for the proof).

**Lemma 7.** *A second order controlled system on  $\mathbb{R}^n$*

$$\ddot{q} + g(q, \dot{q}) = u$$

*is a control-affine system on  $\mathbb{R}^{2m}$  with an involutive distribution  $\{F_1, \dots, F_m\}$  and a drift  $F_0$  such that  $\{F_1, \dots, F_m, F_{01}, \dots, F_{0m}\}$  has maximum rank.*

**Local properties.** The following proposition has been proved in [9] is a direct consequence of remark 1, and of (26).

**Proposition 5.** *The switching correspond to instant rotation of angle  $\pi$  of the control ( $\pi$ -singularity). If  $t$  is a switching time,  $u(t_-) = -u(t_+)$ .*

Now applying theorem 2, we obtain:

**Proposition 6.** *Let  $O_{\bar{z}}$  be a neighborhood of  $\bar{z} \in \Sigma$ : The local extremal flow of the minimum-time restricted circular three-body problem is piecewise smooth, and stratified into:  $O_{\bar{z}} = S_0 \cup S_1 \cup \Sigma$ ,  $S_1$  being the co-dimension one manifold of initial conditions leading to the singularity, and  $S_0$  its complementary. The flow is smooth on each strata and continuous on  $O_{\bar{z}}$ :*

- if  $z_0 \in S_1$  the extremal from  $z_0$  has exactly one  $\pi$ -singularity in  $O_{\bar{z}}$ ,
- Otherwise the extremal from  $z_0$  has no  $\pi$ -singularity.

*Besides, the regular-singular transition is Lipschitz continuous.*

**Proof.** Since (A) is checked, we just need to insure that  $\Sigma = \Sigma_-$  in this case. However  $H_{12} \equiv 0$  because of (26), implying  $H_{01}^2(z) + H_{02}^2(z) > H_{12}(z)^2$  for all  $z \in T^*M$ . The last statement is a direct consequence of theorem 4.  $\square$

**Global properties.** Those switchings are the so-called  $\pi$ -singularities. Since there is no accumulation of switching points, there is only a finite number of such singularities on a time interval  $[0, t_f]$ . We are now going to investigate global properties of such switching. Namely, the next proposition bounds the number of  $\pi$ -singularity along a time optimal trajectory.

*Definition 2.* We define  $\delta = \inf_{[0, t_f]} |q(t)|$ ,  $\delta_1 = \inf_{[0, t_f]} |q(t) - q^1(t)|$ ,  $\delta_2 = \inf_{[0, t_f]} |q(t) - q^2(t)|$ . This quantities represents the distance to the collisions in the two body, and restricted three-body problems respectively. Finally note  $\delta_{12}(\mu) = \frac{\delta_1 \delta_2}{((1-\mu)\delta_2^3 + \mu\delta_1^3)^{1/3}}$ .

Estimate on the global number of switching can be obtain by Sturm like theorems, we denote by  $[x]$  the integer part of a real number  $x$ :

**Proposition 7.** - *In the Keplerian case, there is at least a time interval of length  $\pi\delta^{3/2}$  between two  $\pi$ -singularities. On a time interval  $[0, t_f]$  the maximum amount of such singularities is  $N = \lfloor \frac{t_f}{\pi\delta^{3/2}} \rfloor$ .*

- *In the controlled restricted three-body problem with a mass ratio  $\mu$ , there is at least a time interval of length  $\pi\delta_{12}(\mu)^{3/2}$  between two  $\pi$ -singularities. On a time interval  $[0, t_f]$  the maximum amount of such singularities is  $N = \lfloor \frac{t_f}{\pi\delta_{12}(\mu)^{3/2}} \rfloor$ .*

One can notice that  $\delta_{12}(0) = \delta$ , which makes the proposition coherent. The proof is a immediate consequence of the more general lemma:

**Lemma 8.** *Let us consider the minimum-time control system coming from a  $\mathcal{C}^2$  potential  $V : \mathbf{R} \times O \rightarrow \mathbf{R}$ ,  $O \subset \mathbf{R}^n$ ,*

$$\ddot{q} + \nabla V_t(q) = u, \|u\| \leq 1. \quad (27)$$

*Let  $A_t(q) \in S_n(\mathbf{R})$  a continuous matrix, such that for all time  $t$ , and  $q \in O$ ,  $A_t(q) \geq \nabla^2 V_t(q)$ , then the following statement holds: If  $\bar{t}_1$  and  $\bar{t}_2$  are switching times for (27), there exists a non trivial solution of  $\ddot{y} + A_t(q(t))y = 0$  vanishing in  $\bar{t}_1$  and  $t' < \bar{t}_2$ .*

**Proof of Lemma 8.** Applying the P.M.P. to (27), one gets the maximized Hamiltonian  $H^*(q, v, p_q, p_v) = p_q \cdot v - p_v \cdot \nabla V_t(q) + |p_v|$ , and the feedback  $u^* = \frac{p_v}{|p_v|}$ . It remains to study the zeros of the adjoint state  $p_v$  to have access to the switching times. The equation on  $p_v$  is a second order linear equation:

$$\ddot{p}_v + \nabla_q^2 V_t(q) p_v = 0. \quad (28)$$

The following Sturm-like theorem is due to Morse and will allow us to conclude (see [14]):

**Theorem 5 (Morse).** *Let  $a, b \in \mathbb{R}$ , with  $b > a$ . Consider the two linear second order equations*

$$z'' + P(t)z = 0, \quad (29)$$

and

$$z'' + Q(t)z = 0, \quad (30)$$

*with  $P(t), Q(t)$  be two symmetric continuous  $n \times n$  matrices, such that  $Q(t) - P(t) \geq 0$ , and assume there exists a  $\bar{t}$  with  $Q(\bar{t}) - P(\bar{t}) > 0$ . If (29) has a non trivial solution  $y$ ,  $y(a) = y(b) = 0$ , then (30) has a non trivial solution which vanishes in  $a$  and  $c < b$ .*

If  $A_t(q)$  is a matrix as mentioned in Lemma 8, Morse's theorem gives the result by taking  $Q(t) = A_t(q(t))$  and the lemma is proved.

**Proof of Proposition 7.** We will only be interested the second statement, since it implies the first one. Recall the dynamics of the third mass is:

$$\ddot{q} = -\nabla_q V_\mu(t, q) + u,$$

with  $V_\mu(t, q) = -\frac{1-\mu}{|q-q^1(t)|} - \frac{\mu}{|q-q^2(t)|}$ , the non-autonomous potential. Let

$$A_t(q) = \begin{pmatrix} 1 + \frac{1-\mu}{|q-q^1(t)|^3} + \frac{\mu}{|q-q^2(t)|^3} & 0 \\ 0 & \frac{1-\mu}{|q-q^1(t)|^3} + \frac{\mu}{|q-q^2(t)|^3} \end{pmatrix} \in \mathcal{M}_2(\mathbb{R}).$$

A straightforward calculation shows that

$$\det(A_t(q) - \nabla_q^2 V_\mu(t, q)) = 3 \left[ (1-\mu) \frac{(q_2 - q_2^1)^2}{|q - q^1|^5} + \mu \frac{(q_2 - q_2^2)^2}{|q - q^2|^5} \right] > 0$$

as long as we don't have  $q_2(t) = q_2^1(t) = q_2^2(t)$ . But this cannot happen on the whole trajectory, and so Morse's theorem apply. By our lemma above, the minimum-time interval between two switching times is greater than the time interval between conjugate times of the solutions of:  $z_2'' + \left( \frac{1-\mu}{|q-q^1(t)|^3} + \frac{\mu}{|q-q^2(t)|^3} \right) z_2 = 0$ . But of course,  $\frac{1-\mu}{|q-q^1(t)|^3} + \frac{\mu}{|q-q^2(t)|^3} \leq \frac{1-\mu}{\delta_1^3} + \frac{\mu}{\delta_2^3}$ , and by Sturm's theorem in dimension one, solutions of this equation cannot have two conjugate times in a interval of length smaller than  $\frac{\pi}{\sqrt{\frac{1-\mu}{\delta_1^3} + \frac{\mu}{\delta_2^3}}}$ . Finally,

the distance between two switching times is in fact greater than

$$\frac{\pi \delta_1^{3/2} \delta_2^{3/2}}{\sqrt{(1-\mu)\delta_2^3 + \mu\delta_1^3}} = \pi \delta_{12}(\mu)^{3/2}.$$

□

One of the natural extensions for this work is to study the influence of averaging on the stratification. Indeed, in the controlled Kepler problem, one can average with respect to the fast angle, [8, 5] and reduce the dimension of the problem. Thus, the projection of the singular locus  $\Sigma$  becomes of codimension one and extremals will cross it generically. Fortunately, averaging regularizes the Hamiltonian and the study of the singularities still occurring has been initiated in the previously cited papers and should be pursued.

**Acknowledgements.** The authors thank Jean-Pierre Marco for many and fruitful exchanges on the work presented in this paper.

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