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## On singularities of minimum time control-affine systems

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39 the underlying normal hyperbolicity of the system, we provide a stratification whose  
 40 strata carry a smooth flow; the extremal flow is thus piecewise smooth, and contin-  
 41 uous. This regular behavior is obtained as a consequence of a straightening normal  
 42 form lemma, around the singular locus, proved at the end of the section. In Section 4,  
 43 we investigate the kind of singularity of the flow encountered when crossing strata.  
 44 Thanks to a suitable normal form, we prove that the associated regular-singular tran-  
 45 sition results into a logarithmic term, implying that the flow belongs to the log – exp  
 46 category [24]. This behaviour of the flow strongly contrasts with the situation studied  
 47 in [11] where singularities of the flow were stable; here, singularities are destroyed  
 48 by small perturbations of the initial conditions, unless these perturbations belong to  
 49 a codimension one stratum. We then apply these results to the controlled circular  
 50 restricted three body problem, in Section 5. We finally investigate global properties  
 51 of the flow and give upper bounds on the number of switches of the control for this  
 52 particular nonlinear system. Note that such bounds for time minimization are given  
 53 in the linear case in [5]. In contrast to [10], where a subset of the switching set was  
 54 studied, we treat here the general case using a comparison principle.

55 **2. Setting.** Let  $M$  be a smooth (that is  $\mathcal{C}^\infty$ -smooth) 4-dimensional manifold,  
 56 and let us consider the following control-affine system:

$$57 \quad (2.1) \quad \dot{x}(t) = F_0(x(t)) + u_1(t)F_1(x(t)) + u_2(t)F_2(x(t)), \quad |u(t)| = \sqrt{u_1^2(t) + u_2^2(t)} \leq 1.$$

58 Given endpoint conditions  $x(0) = x_0$ ,  $x(t_f) = x_f$ , one can consider the minimization  
 59 of the final time,  $t_f$ . The corresponding Hamiltonian writes

$$60 \quad H(x, p, u) = H_0 + u_1 H_1 + u_2 H_2, \quad H_i := \langle p, F_i(x) \rangle, \quad i = 0, 1, 2,$$

61 and the classical Pontrjagin maximum principle [4] provides a necessary condition for  
 62 optimality, allowing us to work with a true—that is independent of the control—Ha-  
 63 miltonian system, yet at the expense of introducing singularities.

64 **THEOREM 2.1** (Pontrjagin maximum principle). *Let  $u : [0, t_f] \rightarrow \mathbf{R}^2$  be an essen-*  
 65 *tially bounded time minimizing control of (2.1), and let  $x$  be the associated trajectory.*  
 66 *There exists a Lipschitz curve  $p(t) \in T_{x(t)}^*M$  such that, almost everywhere on  $[0, t_f]$ ,*

$$67 \quad (2.2) \quad \dot{x}(t) = \frac{\partial H}{\partial p}(x(t), p(t), u(t)), \quad \dot{p}(t) = -\frac{\partial H}{\partial x}(x(t), p(t), u(t)),$$

68

$$69 \quad (2.3) \quad H(x(t), p(t), u(t)) = \max_{|v| \leq 1} H(x(t), p(t), v),$$

70 and  $H(x(t), p(t), u(t)) \geq 0$ . Moreover,  $p$  does not vanish on  $[0, t_f]$ .

71 Triples  $(x, p, u)$  solutions of (2.2)-(2.3) are called *extremals*, and their projections on  
 72  $M$  *extremal trajectories*. We denote by  $z = (x, p)$  elements of the cotangent bundle  
 73  $T^*M$ , where  $p$  belongs to the fiber  $T_x^*M$ . We define the *switching surface*,

$$74 \quad \Sigma := \{z = (x, p) \in T^*M \mid H_1(z) = H_2(z) = 0\}.$$

75 Extremals along which  $H_1$  and  $H_2$  do not vanish simultaneously are called *bang arcs*.  
 76 An extremal is said to be *bang-bang* if it is a concatenation of bang arcs. The following  
 77 proposition is clear.

78 PROPOSITION 2.2. *An extremal lying out of  $\Sigma$  is an integral curve of the maxi-*  
 79 *mized Hamiltonian*

$$80 \quad H_0(z) + \sqrt{H_1^2(z) + H_2^2(z)}.$$

81 *The associated control belongs to  $\mathbf{S}^1$  and is equal to*

$$82 \quad (2.4) \quad u = \frac{1}{\sqrt{H_1^2 + H_2^2}}(H_1, H_2).$$

83 Let now  $\bar{z} = (\bar{x}, \bar{p})$  belong to  $\Sigma$ . We are interested in the local behavior of the extremal  
 84 flow in a neighbourhood of this singular point. We make the following transversality  
 85 assumption (remember that the ambient manifold is 4-dimensional):

$$86 \quad (\text{A}) \quad \text{Span}_{\bar{x}}\{F_1, F_2, F_{01}, F_{02}\} = T_{\bar{x}}M,$$

87 where  $F_{ij} := [F_i, F_j]$  denotes the Lie bracket of vector fields. As a derivation on  
 88 functions,  $[F_i, F_j]$  is the commutator  $F_i F_j - F_j F_i$ , while in coordinates  $[F_i, F_j](x) =$   
 89  $F_j'(x)F_i(x) - F_i'(x)F_j(x)$ . Property (A) is generic among vector fields and points of  
 90  $\Sigma$ , and holds in particular for control systems arising from mechanical systems (see  
 91 Section 5). Since the adjoint vector cannot be zero, assumption (A) implies that, for  
 92  $z$  in a neighbourhood of  $\bar{z}$ ,

$$93 \quad H_1^2(z) + H_2^2(z) + H_{01}^2(z) + H_{02}^2(z) \neq 0,$$

94 where  $H_{ij} := \{H_i, H_j\}$  now denotes the Poisson bracket of functions on the cotangent  
 95 bundle. In accordance with the definition of Lie brackets, in coordinates

$$96 \quad \{H_i, H_j\} = \sum_{k=1}^n \frac{\partial H_i}{\partial p_k} \frac{\partial H_j}{\partial x_k} - \frac{\partial H_i}{\partial x_k} \frac{\partial H_j}{\partial p_k}.$$

97 **3. Stratification of the extremal flow.** Following [9], we partition  $\Sigma$  accord-  
 98 ing to

$$99 \quad \Sigma_- := \{z \in \Sigma \mid H_{12}^2(z) < H_{02}^2(z) + H_{01}^2(z)\},$$

$$100 \quad \Sigma_+ := \{z \in \Sigma \mid H_{12}^2(z) > H_{02}^2(z) + H_{01}^2(z)\},$$

$$101 \quad \Sigma_0 := \{z \in \Sigma \mid H_{12}^2(z) = H_{02}^2(z) + H_{01}^2(z)\}.$$

103 In this paper, we focus on the case  $\Sigma_-$  which is the relevant one for mechanical  
 104 systems, as explained Section 5. The other situations (in particular  $\Sigma_0$ ) will be tackled  
 105 in a forthcoming work [17]. The main results of this section refine the analysis in [1]  
 106 by using normal hyperbolicity of the system to provide a suitable stratification under  
 107 our standing assumption (A).

108 PROPOSITION 3.1. *Let  $\bar{z}$  belong to  $\Sigma_-$ ; there exists a neighbourhood  $O_{\bar{z}}$  of  $\bar{z}$  in*  
 109  *$T^*M$  such that (i) for every  $z \in O_{\bar{z}}$  there exists a unique extremal passing through  $z$ ;*  
 110 *(ii) every such extremal intersects  $\Sigma$  at most once in  $O_{\bar{z}}$ .*

111 *Remark 3.2.* In the terminology of [6],  $\Sigma_-$  points along the extremal are called  
 112 *Fuller times of order zero*. In that paper, *Fuller times of higher order* are defined  
 113 inductively, and it is shown that, generically, there are only finite order Fuller times.  
 114 An upper bound on the order exists that only depends on the dimension of the ambient  
 115 manifold. We focus here on the structure of the extremal flow in the neighbourhood of  
 116 a single  $\Sigma_-$  point. As we shall see in Section 5, for usual mechanical systems  $\Sigma = \Sigma_-$ ;  
 117 this precludes accumulation of switching points (Fuller phenomenon).

118 Let us consider an extremal  $z : [0, t_f] \rightarrow T^*M$  as in Proposition 3.1, initializing  
119 at  $z(0) = \bar{z}_0$  and crossing  $\Sigma$  once at  $\bar{z} \in \Sigma_-$  and time  $\bar{t} \in (0, t_f)$ . The following holds.

120 **THEOREM 3.3.** *There exists an open neighbourhood  $O_{\bar{z}_0} \subset T^*M \setminus \Sigma$  of  $\bar{z}_0$  such*  
121 *that the extremal flow  $(t, z_0) \mapsto z(t, z_0)$  is well defined and continuous on  $[0, t_f] \times$*   
122  *$O_{\bar{z}_0}$ . Moreover,  $O_{\bar{z}_0} = S^0 \cup S^s$  where  $S^s$  is a codimension one submanifold of initial*  
123 *conditions leading to  $\Sigma_-$ , and where  $S^0 = O_{\bar{z}_0} \setminus S^s$ . The time  $t_\Sigma(z_0)$  to reach  $\Sigma$  from*  
124 *an initial condition  $z_0$  is well defined and smooth for  $z_0$  in  $S^s$ . Both  $S^0$  and  $S^s$*   
125 *are stable by the flow which is smooth on  $[0, t_f] \times S^0$  and on  $([0, t_f] \times S^s) \setminus \Delta$ , where*  
126  $\Delta = \{(t_\Sigma(z_0), z_0), z_0 \in S^s\}$ .

127 *Remark 3.4.* Although the analysis is drawn on a 4-dimensional manifold with  
128 control on the 2-disk, it will be clear from the proofs that the same results hold in  
129 dimension  $2m$  with control on the  $m$ -dimensional unit ball. Besides, while we assume  
130 that the reference extremal departing at  $\bar{z}_0$  crosses only once  $\Sigma$  for  $t \in [0, t_f]$ , the  
131 analysis can be readily extended to an extremal with a finite number of contacts with  
132  $\Sigma$  at  $\Sigma_-$  points.

133 Let us provide a simple example to illustrate the situation described by Proposition 3.1  
134 and Theorem 3.3. Consider the control system

$$135 \quad (3.1) \quad \begin{cases} \dot{x}_1(t) = 1 + x_3(t), & \dot{x}_3(t) = u_1(t), \\ \dot{x}_2(t) = x_4(t), & \dot{x}_4(t) = u_2(t), \end{cases}$$

136 with control in the 2-disk,  $u_1^2 + u_2^2 \leq 1$ . The maximized Hamiltonian (from Pontrjagin  
137 maximum principle) is

$$138 \quad H(x, p) = p_1(1 + x_3) + p_2x_4 + \sqrt{p_3^2 + p_4^2}$$

139 ( $p_i$  being the adjoint variable of  $x_i$ ), and the codimension two submanifold

$$140 \quad \Sigma = \{p_3 = 0\} \cap \{p_4 = 0\} = \Sigma_-$$

141 is the singular locus. (Note that the distribution  $\{F_1, F_2\}$  is involutive, so the bracket  
142  $H_{12}$  vanishes.) The adjoint states  $p_1$  and  $p_2$  are constant, let  $a = -p_1(0)$  and  $c =$   
143  $-p_2(0)$ . We get  $p_3(t) = at + b$ ,  $p_4(t) = ct + d$ , with  $b = p_3(0)$  and  $d = p_4(0)$ . Then

$$144 \quad (3.2) \quad \dot{x}_3(t) = \frac{at + b}{\sqrt{(at + b)^2 + (ct + d)^2}},$$

145 so that singularities occur when  $(at + b, ct + d)$  vanishes for some  $t$ , that is when  
146  $ad - bc = 0$ , which defines the codimension one submanifold  $S^s = \{p_1p_4 - p_2p_3 =$   
147  $0\} \setminus \{p = 0\}$  (remember that the adjoint state  $p$  cannot be zero for a minimum-time  
148 extremal by virtue of the maximum principle). We get a symmetric dynamics for  $x_4$   
149 and end up with the same submanifold. One verifies that this stratum is stable by  
150 the flow of the maximized Hamiltonian. Outside  $S^s$ , we can explicitly solve (3.2) and  
151 obtain

$$152 \quad x_3(t, z_0) = x_3(0) + \frac{a}{a^2 + c^2} (\sqrt{(at + b)^2 + (ct + d)^2} - \sqrt{b^2 + d^2})$$

$$153 \quad -c \frac{ad - bc}{(a^2 + c^2)^{3/2}} \left[ \operatorname{argsh} \left( \frac{(a^2 + c^2)t + ab + cd}{ad - bc} \right) - \operatorname{argsh} \left( \frac{ab + cd}{ad - bc} \right) \right].$$

154 It is clear that the flow is smooth outside  $S^s$ . If  $a$  and  $c$  are zero,  $p_3$  and  $p_4$  become  
 155 constant, and since  $p$  cannot vanish there are no contacts with  $\Sigma$ . Now observe that  
 156 the flow is defined on  $S^s$  by

$$157 \quad x_3(t, z_0) = x_3(0) + \frac{a}{a^2 + c^2} (\sqrt{(at + b)^2 + (ct + d)^2} - \sqrt{b^2 + d^2})$$

158 for all  $z_0 \in S^s \setminus \{p = 0\}$ . Restricted to  $S^s$ , the flow is smooth outside switches. We  
 159 also have global continuity on  $T^*M$ , but not Lipschitz continuity. Furthermore, on  
 160 this simple model a singularity of  $y \ln y$  type appears when crossing  $S^s$ , that is when  
 161 the determinant  $ad - bc$  goes to zero. We prove in Section 4 that the singularities in  
 162 the general case are not worse than in this example.

163 **3.1. Proof of Proposition 3.1.** According to assumption (A), the mapping

$$164 \quad (x, p_1, p_2, p_3, p_4) \mapsto (x, H_1, H_2, H_{01}, H_{02})$$

165 defines a smooth change of coordinates in a small enough neighbourhood  $O_{\bar{z}}$  of  $\bar{z}$ . A  
 166 polar blow-up is used to study the dynamics near the singularity, adding an  $\mathbf{S}^1$ -fiber  
 167 above  $\rho = 0$ :

$$168 \quad (H_1, H_2) = (\rho \cos \theta, \rho \sin \theta), \quad (\rho, \theta) \in \mathbf{R} \times \mathbf{S}^1.$$

169 In polar coordinates, (2.4) reads  $u = (\cos \theta, \sin \theta)$ , and  $\Sigma = \{\rho = 0\}$ . Computing, the  
 170 dynamics is

$$171 \quad (3.3) \quad X : \begin{cases} x' = \rho(F_0(x) + \cos \theta \cdot F_1(x) + \sin \theta \cdot F_2(x)), \\ \rho' = \rho(\cos \theta \cdot H_{01} + \sin \theta \cdot H_{02}), \\ \theta' = H_{12} + \cos \theta \cdot H_{02} - \sin \theta \cdot H_{01}, \\ H'_{01} = \rho(H_{001} + \cos \theta \cdot H_{101} + \sin \theta \cdot H_{201}), \\ H'_{02} = \rho(H_{002} + \cos \theta \cdot H_{102} + \sin \theta \cdot H_{202}), \end{cases}$$

172 where we have changed time from  $t$  to  $s$  ( $' = d/ds$ ) with  $dt = \rho ds$  to get rid of  
 173 the  $1/\rho$  singularity on  $\theta$ , and denoted  $X$  the corresponding vector field. In this  
 174 new time, the autonomous vector field in the right hand side of (3.3) is smooth,  
 175 which implies existence and uniqueness of maximal solutions through a point, as  
 176 well as smoothness of the flow. Note that when  $\rho$  vanishes, only  $\theta$  is not constant;  
 177 in particular  $\Sigma = \{\rho = 0\}$  is invariant by the flow. In the following, we denote  
 178  $\bar{H}_{ij} = H_{ij}(\bar{z})$ ,  $i, j = 0, 1, 2$ . The next lemma establishes that in each part of  $\Sigma$ , the  
 179 derivative of  $\theta$  has a different number of equilibria.

180 **LEMMA 3.5.** *Let  $z$  belong to  $\Sigma$ ; the mapping  $\theta \mapsto H_{12} + \cos \theta \cdot H_{02} - \sin \theta \cdot H_{01}$  has*  
 181 *(i) two zeros, denoted by  $\theta_-$  and  $\theta_+$ , for  $z$  in  $\Sigma_-$ ; (ii) exactly one zero for  $z$  in  $\Sigma_0$ ; (iii)*  
 182 *no zero for  $z$  in  $\Sigma_+$ . In the  $z \in \Sigma_-$  case, the two mappings  $(x, H_{01}, H_{02}) \mapsto \theta_{\pm}$*   
 183 *are well defined and smooth in a neighbourhood of  $(\bar{x}, \bar{H}_{01}, \bar{H}_{02})$ .*

184 *Proof.* Setting  $(H_{01}, H_{02}) = (r \cos \phi, r \sin \phi)$  where  $r \neq 0$  under assumption (A),

$$185 \quad H_{12} + \cos \theta \cdot H_{02} - \sin \theta \cdot H_{01} = H_{12} - r \sin(\theta - \phi),$$

186 so  $H_{12}/r = \sin(\theta - \phi)$  has two solutions,  $\theta_-$  and  $\theta_+$ , if  $\bar{z} \in \Sigma_-$ , no solution if  $z \in \Sigma_+$   
 187 and exactly one if  $z \in \Sigma_0$  (since  $H_{12}(z)/r = \pm 1$ ). The variables  $(x, \theta, H_{01}, H_{02})$  define  
 188 a local chart on  $\Sigma = \{\rho = 0\}$ , and the mapping

$$189 \quad g : (x, \theta, H_{01}, H_{02}) \mapsto H_{12} + \cos \theta \cdot H_{02} - \sin \theta \cdot H_{01}$$

190 verifies

$$191 \quad \frac{\partial g}{\partial \theta}(x, \theta_{\pm}, H_{01}, H_{02}) = -r \cos(\theta_{\pm} - \phi) = \pm \sqrt{r^2 - H_{12}^2} \neq 0,$$

192 so the implicit function theorem allows to conclude.  $\square$

193 To complete the proof of Proposition 3.1, let us recall that a diffeomorphism  $f$  of  
 194 manifold  $M$  onto itself is said to be normally hyperbolic along a compact submanifold  
 195  $N$  if the submanifold  $N$  is invariant by  $f$  and if (i) every fibre of the tangent bundle  
 196 of  $M$  along  $N$  admits a splitting  $T_x M = E^u(x) \oplus T_x N \oplus E^s(x)$  for all  $x \in N$  such that  
 197  $f'(x) \cdot E^s(x) = E^s(f(x))$  and  $f'(x) \cdot E^u(x) = E^u(f(x))$  ( $f$  preserves the splitting);  
 198 (ii) there exists  $\lambda_1 \leq \mu_1 < \lambda_2 \leq \mu_2 < \lambda_3 \leq \mu_3$ , with  $\mu_1 < 1 < \lambda_3$ , such that (the  
 199 endomorphism norm below being induced by a given Riemannian structure on  $M$ )

$$200 \quad (3.4) \quad \lambda_1 \leq |f'|_{E^s} \leq \mu_1, \quad \lambda_2 \leq |f'|_{TN} \leq \mu_2, \quad \lambda_3 \leq |f'|_{E^u} \leq \mu_3.$$

201

202 The distributions  $E^s$  and  $E^u$  are locally integrable, and one can construct the local  
 203 stable and unstable manifolds,  $W(x)^s$  and  $W(x)^u$ , respectively tangent to  $E^s(x)$  and  
 204  $E^u(x)$  at each point  $x \in N$ . One also defines  $\mathcal{W}^s := \bigcup_{x \in N} W^s(x)$  and  $\mathcal{W}^u :=$   
 205  $\bigcup_{x \in N} W^u(x)$ , the local stable and unstable manifolds of  $N$ . Let also  $l_s$  and  $l_u$  be  
 206 the biggest integers such that  $\mu_1 \leq \lambda_2^{l_u}$  and  $\mu_2^{l_s} \leq \lambda_3$ . The following holds (see  
 207 (Theorem 3.5 in [14], or [18]), giving the regularity of the stable and unstable manifolds  
 208 in terms of the ratio of the contraction and expansion rates.

209 **THEOREM 3.6** (Hirsch, Pugh, Shub). *Any  $f$ -invariant submanifold which is close*  
 210 *enough to  $N$  is included in  $\mathcal{W}^s \cup \mathcal{W}^u$ . Furthermore,  $\mathcal{W}^s$  and  $\mathcal{W}^u$  are submanifolds of*  
 211 *class  $\mathcal{C}^{l_s}$  and  $\mathcal{C}^{l_u}$ , respectively.*

212 In our case, this result is applied on the cotangent bundle  $T^*M$  of the original state  
 213 manifold  $M$ , and we have two codimension two submanifolds of equilibrium points,  
 214 namely

$$215 \quad z_{\pm} = (x, \rho = 0, \theta = \theta_{\pm}(x, H_{01}, H_{02}), H_{01}, H_{02}),$$

216 parameterized locally by  $y := (x, H_{01}, H_{02})$  in a neighbourhood of  $(\bar{x}, \bar{H}_{01}, \bar{H}_{02})$ . We  
 217 set

$$218 \quad (3.5) \quad \cos \theta_- \cdot H_{01} + \sin \theta_- \cdot H_{02} = -\sqrt{r^2 - H_{12}^2} < 0,$$

219 and the opposite for  $\theta_+$ . The Jacobian of the system (3.3) has two non-zero eigenvalues  
 220 at those points:  $\cos \theta_{\pm} \cdot H_{01} + \sin \theta_{\pm} \cdot H_{02}$  and their opposite, and a 6-dimensional kernel.  
 221 So we have a one-dimensional stable submanifold  $W^s(z_{\pm})$ , and a one-dimensional  
 222 unstable submanifold  $W^u(z_{\pm})$  in every equilibrium  $z_{\pm}$ . The flow is thus normally  
 223 hyperbolic to the manifold

$$224 \quad N_- = \{z = z_-(y) = (\rho = 0, \theta = \theta_-(y), y)\},$$

225 and symmetrically to

$$226 \quad N_+ = \{z = z_+(y) = (\rho = 0, \theta = \theta_+(y), y)\}.$$

227 We now focus on the case of  $N_-$ , the analysis being the same for  $N_+$ . On  $N_-$   
 228 the dynamics is trivial: every point is an equilibrium. Hence there exists a unique  
 229 trajectory converging to  $z_-$  when  $s \rightarrow \infty$  in the stable manifold  $W^s(z_-)$ . On  $\Sigma$ ,





252  $7 = \dim N + 1$ , every fiber  $W^s(z)$  being of dimension one. The stratum we look for is

$$253 \quad S^s := \mathcal{W}_-^s \cap \{\rho > 0\}.$$

254 One can similarly define  $\mathcal{W}_+^u := \bigcup_{z \in N_+} W^u(z)$ , also  $\mathcal{C}^\infty$ -smooth of codimension one,  
255 and  $S^u := \mathcal{W}_+^u \cap \{\rho > 0\}$ .

256 To understand the regularity of the flow on this strata, we use a normal form to  
257 rewrite the system in the neighbourhood of the equilibria  $\theta_-$ . (The same approach  
258 also works near  $\theta_+$ .) Using again polar coordinates  $(H_{01}, H_{02}) = (r \cos \phi, r \sin \phi)$ , the  
259 dynamics (3.3) writes

$$260 \quad (3.6) \quad \begin{cases} \rho' = r\rho \cos(\theta - \phi), \\ \theta' = H_{12} - r \sin(\theta - \phi), \\ \xi' = \rho h(\rho, \theta, \xi), \end{cases}$$

261 where  $\xi = (x, r, \phi)$  and  $h$  is a smooth function. We set  $\psi = \theta - \phi$ , rescale the time  
262 according to  $dv = r ds$  (as (A) implies  $r > 0$  in the neighbourhood of  $\bar{z}$ ), and study a  
263 system with the following structure (the derivation wrt.  $v$  still being noted  $'$ ):

$$264 \quad (3.7) \quad \begin{cases} \rho' = \rho \cos \psi, \\ \psi' = g(\rho, \psi, \xi) - \sin \psi =: G(\rho, \psi, \xi), \\ \xi' = \rho h(\rho, \psi, \xi), \end{cases}$$

265 where  $g$  is a smooth function (so is  $G$ ) defined on an open set  $O$  of  $\mathbf{R}^2 \times D$ ,  $D$  being  
266 a compact domain of  $\mathbf{R}^6$ . As  $H_{12}$  is a smooth function in  $(H_1, H_2) = (\rho \cos \theta, \rho \sin \theta)$ ,  
267 because it is smooth wrt. the change of coordinates given by assumption (A),

$$268 \quad g(\rho, \psi, \xi) = a(\xi) + \rho b(\xi, \psi) + O(\rho^2)$$

269 since when  $\rho = 0$ ,  $g$  does not depend anymore on  $\theta$ . Besides,  $|g| < 1$  on  $O$  since it  
270 is a small neighbourhood of  $\Sigma_-$ . Equilibria occur when  $\rho = G = 0$ . They are semi-  
271 hyperbolic, since they are outside  $\{\psi = \pm\pi/2\}$  for  $\bar{z}$  in  $\Sigma_-$ . More precisely, it was  
272 shown in the previous subsection that the flow of this system is normally hyperbolic  
273 to the manifold  $\{\rho = 0\} \cap \{G = 0\}$ . For each  $\xi$ , we retrieve the two equilibria  $z_\pm$ ,  
274 and the previously defined  $\theta_\pm$  are mapped to  $\psi_\pm$  in the new set of coordinates for the  
275 blow-up. Indeed, thanks to the structure of  $g$ , we get  $\partial g / \partial \psi(0, \psi, \xi) = 0$ , so

$$276 \quad \frac{\partial G}{\partial \psi}(0, \psi, \xi) = -\cos \psi \neq 0,$$

277 and there exist two smooth functions  $\psi_\pm(\xi)$  that allow to parameterize anew the two  
278 pieces of  $\{G = 0\} \cap \{\rho = 0\}$  according to

$$279 \quad N_\pm = \{z = z_\pm(\xi) = (\rho = 0, \psi = \psi_\pm(\xi), \xi)\}.$$

280 The following result is a preparation lemma for the normal form computation.

281 **LEMMA 3.7.** *In a neighbourhood of  $N_-$ , the vector field  $X$  giving system (3.3) is*  
282 *smoothly conjugated to a vector field  $Y$  (that is there exists a smooth diffeomorphism*  
283  *$\Psi$  s.t.  $\Psi_* X = Y$ ) such that*

$$284 \quad (3.8) \quad Y : \begin{cases} \rho' = -\rho(1 + O(|\rho| + |\omega|)), \\ \omega' = \omega + O((|\rho| + |\omega|)^2), \\ \xi' = \rho O(|\rho| + |\omega|), \end{cases}$$

285 *in suitable coordinates  $(\rho, \omega, \xi)$ .*

286 *Proof.* Let us set  $\omega = \psi - \psi_-(\xi)$  along  $\{G = 0\}$ , and study the system near  $\omega = 0$ .  
 287 In these new coordinates,

$$288 \quad \begin{cases} \rho' = \rho \cos(\omega + \psi_-(\xi)), \\ \omega' = g(\rho, \omega + \psi_-(\xi), \xi) - \sin(\omega + \psi_-(\xi)) - \rho \frac{\partial \psi}{\partial \xi}(\xi) \cdot h(\rho, \omega + \psi_-(\xi), \xi), \\ \xi' = \rho h(\rho, \omega + \psi_-(\xi), \xi). \end{cases}$$

289 Then

$$290 \quad g(\rho, \omega + \psi_-(\xi), \xi) = a(\xi) + \rho b(\omega + \psi_-(\xi), \xi) + O(\rho^2),$$

291 so  $g(0, \psi_-(\xi), \xi) = \sin(\psi_-(\xi)) = a(\xi)$ . So (3.3) is equivalent to

$$292 \quad (3.9) \quad \begin{cases} \rho' = \lambda(\xi)\rho(1 + O(|\rho| + |\omega|)), \\ \omega' = \beta(\xi)\rho - \lambda(\xi)\omega + O((|\rho| + |\omega|)^2), \\ \xi' = \rho(\gamma(\xi) + O(|\rho| + |\omega|)), \end{cases}$$

293 with  $\lambda(\xi) = \cos(\psi_-(\xi))$  and  $\beta, \gamma$  smooth functions. The Jacobian matrix of the right  
 294 hand side in (3.9) is

$$295 \quad \begin{pmatrix} \lambda(\xi) & 0 & 0 \\ \beta(\xi) & -\lambda(\xi) & 0 \\ \gamma(\xi) & 0 & 0 \end{pmatrix}.$$

296 Let us change coordinates further in order to diagonalize this Jacobian. Consider  
 297  $\tilde{\omega} = \omega + g_1(\xi)\rho$  and  $\tilde{\xi} = \xi + g_2(\xi)\rho$ , with  $g_1$  and  $g_2$  to be chosen. One has

$$298 \quad \tilde{\omega}' = \omega' + \frac{\partial g_1}{\partial \xi}(\xi)\xi'\rho + g_1(\xi)\rho',$$

299 so  $\tilde{\omega}' = (\beta(\xi) + 2g_1(\xi)\lambda(\xi))\rho - \lambda(\xi)\tilde{\omega} + O((|\rho| + |\omega|)^2)$ , and by picking  $g_1 = -\beta/(2\lambda)$   
 300 we obtain what we look for. Indeed, with this change of variables,  $O((|\rho| + |\omega|)^k) =$   
 301  $O((|\rho| + |\tilde{\omega}|)^k)$  for all  $k$ ; moreover

$$302 \quad \tilde{\xi}' = \xi' + \frac{\partial g_2}{\partial \xi}(\xi)\xi'\rho + g_2(\xi)\rho' = \rho(\gamma(\xi) + g_2(\xi)\lambda(\xi)) + O((|\rho| + |\omega|)^2),$$

303 and we choose  $g_2 = -\gamma/\lambda$ . In these new variables,

$$304 \quad \begin{cases} \rho' = \lambda(\tilde{\xi})\rho(1 + O(|\rho| + |\tilde{\omega}|)), \\ \tilde{\omega}' = -\lambda(\tilde{\xi})\tilde{\omega} + O((|\rho| + |\tilde{\omega}|)^2), \\ \tilde{\xi}' = \rho O(|\rho| + |\tilde{\omega}|). \end{cases}$$

305 A smooth change of time finishes the proof.  $\square$

306 **PROPOSITION 3.8** ( $\mathcal{C}^\infty$ -normal form). *Set  $\Omega = \rho\omega$ ; there exist  $A, B, C$  smooth*  
 307 *functions on a neighbourhood of  $\{0\} \times D$  such that the vector field  $Y$  in (3.8) is*  
 308 *smoothly conjugated to*

$$309 \quad (3.10) \quad Y^\infty : \begin{cases} \rho' = -\rho(1 + \Omega A(\Omega, \xi)), \\ \omega' = \omega(1 + \Omega B(\Omega, \xi)), \\ \xi' = \Omega C(\Omega, \xi). \end{cases}$$

310 We postpone the proof of Proposition 3.8 to the end of the section and proceed to  
 311 prove Theorem 3.3.

312 LEMMA 3.9. *For  $z_0$  in  $S^s$ , the contact point  $z_\Sigma(z_0)$  and the contact time  $t_\Sigma(z_0)$*   
 313 *with  $\Sigma$  are well defined. These two mappings are smooth functions of  $z_0$  on  $S^s$ .*

314 *Proof.* Thanks to Proposition 3.8, the globally invariant manifold  $S^s$ , fibered by  
 315 stable manifolds, is straightened to  $\{\omega = 0, \rho \geq 0\}$ . But on  $\{\omega = 0\}$ , we have the  
 316 trivial dynamics

$$317 \quad \begin{cases} \rho' = -\rho, \\ \omega' = 0, \\ \xi' = 0. \end{cases}$$

318 In particular, associating to an initial condition  $z_0 = (\rho_0, 0, \xi_0)$  on the stable stratum  
 319 the contact point with  $\Sigma$  is simply projecting to  $(0, 0, \xi_0)$ , so the mapping  $z_0 \mapsto z_\Sigma(z_0)$   
 320 is smooth on  $S^s$ . Moreover,  $\rho(v) = e^{-v}\rho_0$ , where  $v$  is the new time  $dt = \rho dv/(r\lambda(\xi))$   
 321 (see the proof of Proposition 3.8). As a result,

$$322 \quad t_\Sigma(z_0) = \int_0^\infty \frac{\rho(v)}{r(\xi(v))\lambda(\xi(v))} dv = \frac{\rho_0}{r(\xi_0)\lambda(\xi_0)}$$

323 which is also smooth on  $S^s$ . □

324 Let  $t \in [0, t_f]$ , and let  $z_0$  be close enough to the initial condition  $\bar{z}_0$  of the reference  
 325 extremal. If  $z_0$  does not belong to  $S^s$ , the extremal curve  $z(t, z_0)$  does not meet  $\Sigma$   
 326 and is well defined. If  $(t, z_0)$  belongs to  $([0, t_f] \times S^s) \setminus \Delta$ , either  $t < t_\Sigma(z_0)$  and  $z(t, z_0)$   
 327 is again the value at  $t$  of the smooth extremal curve, or  $t > t_\Sigma(z_0)$  and our analysis  
 328 shows that  $z(t, z_0)$  is uniquely defined and such that

$$329 \quad z(t, z_0) = z(t - t_\Sigma(z_0), z_\Sigma(z_0)).$$

330 It is clear that this construction leads to a flow which is smooth on  $[0, t_f] \times S^0$ , and  
 331 also on  $([0, t_f] \times S^s) \setminus \Delta$  thanks to Lemma 3.9. To conclude the proof of Theorem 3.3,  
 332 it remains to prove that the flow is continuous on  $O_{\bar{z}_0}$ .

333 Let  $z_0$  and  $z_1$  belong to  $O_{\bar{z}_0}$ , with  $z_0 \in S^s$ , two close times  $t_0, t_1$  in  $[0, t_f]$ , and  $O_\delta$   
 334 be a small neighbourhood of  $\bar{z} := z_\Sigma(\bar{z}_0)$ . The extremal from  $z_0$  is passing through  
 335  $z_\Sigma(z_0) \in O_\delta$ . We want to control the quantity  $|z(t_1, z_1) - z(t_0, z_0)|$ . Suppose, without  
 336 loss of generality, that  $t_0, t_1 > t_\Sigma(z_0)$ . Let  $\varepsilon > 0$ , and denote  $t_\delta$  the contact time with  
 337  $O_\delta$  of the extremal starting from  $z_0$ , *resp.*  $t'_\delta$  the exit time from this neighbourhood.  
 338 We can choose  $z_0$  and  $z_1$  to be close enough, so that  $|z(t_\delta, z_0) - z(t_\delta, z_1)| < \varepsilon/3$ ; simply  
 339 because the flow is continuous when the singular locus is not crossed yet. We will  
 340 use the following Lemma which gives a uniform bound on the time interval spent by  
 341 extremals in a neighbourhood of  $\bar{z}$  to conclude (the result appears in [1], we give an  
 342 alternative proof):

343 LEMMA 3.10 ([1]). *For all  $\delta > 0$  there exists a neighbourhood  $O_\delta$  of  $\bar{z}$  in which*  
 344 *every extremal spends a time smaller than  $\delta$ .*

345 *Proof.* We will prove it in a neighbourhood of  $\bar{z}_-$ , the situation being symmetric  
 346 around  $\bar{z}_+$ . Let us define  $O_{\delta-} = \{z \mid \rho < \delta, |\theta - \theta_-| < \delta\}$ , on which  $z \mapsto \cos \theta \cdot$   
 347  $H_{01}(z) + \sin \theta \cdot H_{02}(z)$  is smooth and thus bounded. Now set

$$348 \quad M_\delta = \sup_{z \in O_{\delta-}} \cos \theta \cdot H_{01}(z) + \sin \theta \cdot H_{02}(z),$$

349 and remember it is negative on  $O_{\delta-}$ . Then, for any extremal in  $O_{\delta-}$ ,  $\dot{\rho}(s) \leq M_\delta \rho(s)$ ,  
 350 which implies

$$351 \quad \rho(s) \leq \rho(0)e^{M_\delta s}.$$

352 So we bound the time spent in  $O_{\delta-}$  by

$$353 \quad \Delta_{O_{\delta-}} t \leq \int_0^\infty \rho(0) e^{M_\delta s} ds = -\frac{\delta}{M_\delta}.$$

354 As  $M_\delta$  tends to a negative value when  $\delta$  tends to zero, this quantity tends to 0 when  $\delta$   
 355 does, so every extremal spends an arbitrarily small time in that set. The same holds  
 356 for the time interval  $\Delta_{O_{\delta+}} t$  in a similarly defined neighbourhood of  $\bar{z}_+$ , say  $O_{\delta+}$ . This  
 357 settles the case of extremals going through the singular locus. Let  $z$  be an extremal  
 358 without singularity. The time interval in a neighbourhood  $O_\delta$  can be expressed as  
 359 follows:

$$360 \quad \Delta_{O_\delta} t = \Delta_{O_{\delta-}} t + \int_{s_-}^{s_+} \rho ds + \Delta_{O_{\delta+}} t,$$

361 as  $z$  exits  $O_{\delta-}$  at time  $s_-$  and enters  $O_{\delta+}$  at time  $s_+$  (see Figure 2). To tackle the  
 362 central part of this sum, let us go back to system (3.10).

$$363 \quad Y^\infty : \begin{cases} \rho' = -\rho(1 + \Omega A(\Omega, \xi)), \\ \omega' = \omega(1 + B(\Omega, \xi)), \\ \xi' = \Omega C(\Omega, \xi), \end{cases}$$

364 which, by a time rescaling, is equivalent to

$$365 \quad Y^\infty : \begin{cases} \rho' = -\rho(1 + \Omega \tilde{A}(\Omega, \xi)), \\ \omega' = \omega, \\ \xi' = \Omega \tilde{C}(\Omega, \xi), \end{cases}$$

366 with  $\tilde{A}$  standing for  $(A - B)/(1 + \Omega B)$ , and  $\tilde{C}$  for  $C/(1 + \Omega B)$ . Because of the change  
 367 of coordinates  $\tilde{\omega} = \omega + g_1(\xi)\rho$ , when  $s = s_-$  one has  $\tilde{\omega}_\pm := \tilde{\omega}(s_\pm) = \delta + g_1(\xi_\pm)\rho(s_\pm)$ .  
 368 Let us denote  $s_2$  the new time, with  $ds_2 = ds/(r\lambda(1 + \Omega B))$ . The denominator is  
 369 bounded below by a positive constant in  $O_\delta$ , so  $s_+ - s_- \leq M(s_{2+} - s_{2-})$ . We also  
 370 have  $s_{2+} - s_{2-} = \ln(\tilde{\omega}_+/\tilde{\omega}_-)$ . So

$$371 \quad \int_{s_-}^{s_+} \rho ds \leq M\delta \ln \left( \frac{\delta + g_1(\xi_+)\rho_+}{\delta + g_1(\xi_-)\rho_-} \right)$$

372 which tends to 0 whenever  $\delta$  does. This concludes the proof of the lemma.  $\square$

373 Then, with a good choice of  $O_\delta$ ,  $|z(t'_\delta, z_0) - z(t_\delta, z_0)|$  and  $|z(t'_\delta, z_1) - z(t_\delta, z_1)|$  are  
 374 smaller than  $\varepsilon/3$ , so that  $|z(t'_\delta, z_0) - z(t'_\delta, z_1)| \leq |z(t'_\delta, z_0) - z(t_\delta, z_0)| + |z(t_\delta, z_0) -$   
 375  $z(t_\delta, z_1)| + |z(t_\delta, z_1) - z(t'_\delta, z_1)| \leq \varepsilon$ . Now notice that  $z(t_0, z_0) = z(t_0 - t'_\delta, z(t'_\delta, z_0))$ ,  
 376 and use the regularity of the system when the singular locus is not crossed to conclude  
 377 the proof of Theorem 3.3.

378 *Remark 3.11.* In the case  $\bar{z} \in \Sigma_-$ , we can compute the jump on the control at  
 379 the switching time  $\bar{t}$  in terms of Poisson brackets:

$$380 \quad (3.11) \quad u(\bar{t}_+) - u(\bar{t}_-) = \frac{2\sqrt{r^2(\bar{z}) - \bar{H}_{12}^2}}{r^2(\bar{z})} (\bar{H}_{01}, \bar{H}_{02}).$$

381 *Proof (of Proposition 3.8).* The argument is based on a generalization of the  
 382 Poincaré-Dulac theorem. Denote  $H^l$  the space of homogeneous polynomials of de-  
 383 gree  $l$  in  $\mathbf{R}^n$  with smooth coefficients in  $\xi \in \mathbf{R}^k$ . We recall that for a linear vec-  
 384 tor field  $X$  that does not depend on  $\xi$  (and has no component in the  $\xi$  direction),

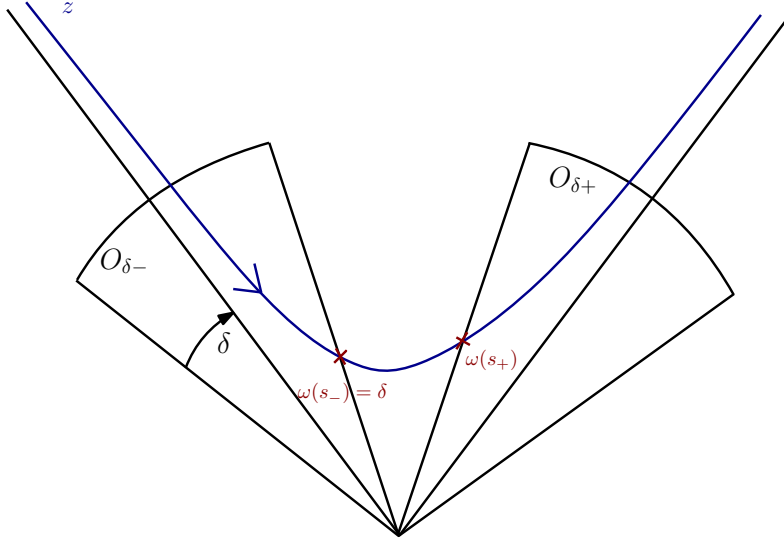


FIG. 2. Extremal entering  $O_\delta$  with  $\theta(s_-) = \theta_- + \delta$ , that is  $\omega(s_-) = \delta$ .

385  $H^l = \text{im}[X, \cdot]_{|H^l} + \ker[X, \cdot]_{|H^l}$ . A vector field  $Z$  is said to be resonant with  $X$  if  
 386  $Z \in \ker[X, \cdot]$

387 **LEMMA 3.12.** *Let  $X(x, \xi)$  be a smooth vector field in  $\mathbf{R}^n \times \mathbf{R}^k$ ,  $X(0, \xi) = 0$ .  
 388 Denote by  $X_1$  its linear part. Then, if  $X_1$  does not depend on  $\xi$ , there exist  $g_i \in$   
 389  $H^i \cap \ker[X_1, \cdot]$ ,  $i = 2, \dots, l$ , and a smooth vector field  $R_l$  with zero  $l$ -jet such that, in  
 390 a neighbourhood of zero,  $X$  is smoothly conjugate to*

$$391 \quad X_1 + g_2 + \dots + g_l + R_l, \quad l \in \mathbf{N}.$$

392 *Proof.* We will follow [23] and reason by induction on  $l$ , then treat the case  $l = \infty$ .  
 393 For  $l = 1$ , the result is trivial:  $X = X_1 + R_1$  where  $R_1$  has zero first jet at zero. Suppose  
 394 by induction that  $g_1, \dots, g_{l-1}$ , and  $R_{l-1}$  are as desired,  $l \geq 2$ ;  $R_{l-1}$  has a zero  $l-1$   
 395 jet (at zero) and can be written as

$$396 \quad R_{l-1} = [X_1, Z] + g_l + R_l,$$

397 where  $Z \in H^l$ ,  $g_l \in \ker[X_1, \cdot]$ , and  $R_l$  is a smooth vector field with zero  $l$ -jet. Now,

$$398 \quad [X, Z] = [X_1, Z] + \sum_{i=2}^l [g_i, Z] + [R_{l-1}, Z] = [X_1, Z] + R'_l,$$

399 where  $R'_l$  has zero  $l$ -jet. Note  $\phi_Z$  the flow of  $Z$ , and consider  $X^t := (\phi_Z^t)_* X$ . One has

$$400 \quad \frac{dX^t}{dt} = [X, Z] = [X_1, Z] + R'_l,$$

401 so that  $X^t = X^0 + t[X_1, Z] + R_{l,t}$  with  $j^l(R_{l,t})(0) = 0$ . Since  $Z$  is a homogeneous  
 402 polynomial of degree  $l$ , it has zero  $l-1$  jet, so  $X$  and  $X^t$  have the same  $l-1$  jet,  
 403 which means that

$$404 \quad X^0 = X_1 + g_2 + \dots + g_l + [X_1, Z].$$

405 For  $t = -1$ ,  $X^{-1} = X_1 + g_2 + \dots + g_l + R_{l,-1}$ , and  $\phi_Z^{-1}$  conjugates the two vector fields,  
 406 which ends the proof by induction. The above construction provides a sequence of  
 407 formal diffeomorphisms  $\varphi_l = \phi_Z^{-1}$  ( $Z \in H^l$ ) such that  $(\varphi_l)_*X = X_1 + g_2 + \dots + g_l + R_l$ .  
 408 Also notice that  $\varphi_l$  and  $\varphi_{l+1}$  have the same  $l$ -jet. This defines a sequence of coefficients  
 409  $g_l(\xi)$  for all  $l$ .  $\square$

410 By a generalization of Borel theorem proved by Malgrange in [15], we know that there  
 411 exists a smooth function  $\varphi$  such that the  $l$ -jet of  $\varphi_l$  and  $\varphi$  are identical for all  $l \in \mathbf{N}$ .  
 412 We can also realize, using the same theorem, the formal series given by the resonant  
 413 monomials by a smooth vector field  $X^\infty$ . Thus we have  $\varphi_*(X) = X^\infty + R^\infty$ , where  
 414  $R^\infty$  has zero infinite jet. We begin by looking for monomials that are resonant with  
 415 the linearized vector field of  $Y$ ,

$$416 \quad Y_1 = -\rho \frac{\partial}{\partial \rho} + \omega \frac{\partial}{\partial \omega}$$

417 (monomials  $X$  for which  $[Y_1, X] = 0$ ). The Lie bracket with  $Y_1$  treats  $\xi$  as a constant:  
 418 the map  $X \mapsto [Y_1, X]$  is linear in  $\xi$ . There are three different cases for such monomials  
 419 and

$$420 \quad [Y_1, a(\xi)\rho^i\omega^j \frac{\partial}{\partial \rho}] = (i - j - 1)a(\xi)\rho^i\omega^j \frac{\partial}{\partial \rho},$$

$$421 \quad [Y_1, b(\xi)\rho^i\omega^j \frac{\partial}{\partial \omega}] = (i + 1 - j)b(\xi)\rho^i\omega^j \frac{\partial}{\partial \omega},$$

$$422 \quad [Y_1, c(\xi)\rho^i\omega^j \frac{\partial}{\partial \xi}] = (i - j)c(\xi)\rho^i\omega^j \frac{\partial}{\partial \xi}.$$

423  
 424  
 425 Setting  $\Omega := \rho\omega$ , the monomials we are looking for are

$$426 \quad a(\xi)\rho\Omega^k \frac{\partial}{\partial \rho}, \quad b(\xi)\omega\Omega^k \frac{\partial}{\partial \omega}, \quad c(\xi)\Omega^k \frac{\partial}{\partial \xi}, \quad k \in \mathbf{N}.$$

427 The lemma allows us to state that the infinite jet of  $Y$  can be formally developed on  
 428 the resonant monomials, so  $Y$  is formally conjugate to

$$429 \quad W : \begin{cases} \rho' = -\rho(1 + \sum_{i \geq 1} a_i(\xi)\Omega^i), \\ \omega' = \omega(1 + \sum_{i \geq 1} b_i(\xi)\Omega^i), \\ \xi' = \rho \sum_{i \geq 1} c_i(\xi)\Omega^i. \end{cases}$$

430 Notice that, by putting  $\Omega$  in factor in the formal series above, we get a formal field  
 431 of the required form (3.10). There exists  $Y^\infty$ , a smooth vector field on  $O$ , such that  
 432  $W = Y^\infty + R^\infty$  where  $R^\infty$  is a smooth function with zero infinite jet along  $D$ . At this  
 433 stage, we have that  $Y$  is smoothly equivalent to  $Y^\infty + R^\infty$ . The last step consists in  
 434 killing the flat perturbation  $R^\infty$ . This can be achieved by the path's method: instead  
 435 of looking for a diffeomorphism sending  $Y_0 := Y$  on  $Y_1 := Y^\infty + R^\infty$ , we search for a  
 436 one parameter family (path) of diffeomorphism  $(g_t)_t$  such that

$$437 \quad (3.12) \quad g_t^* Y_0 = Y_t,$$

438  $Y_t$  being a path of vector fields joining  $Y_0$  and  $Y_1$ . Consider the linear path  $Y_t =$   
 439  $(1 - t)Y_0 + tY_1$ ,  $t \in [0, 1]$ . Differentiating (3.12) with respect to  $t$  we get

$$440 \quad (3.13) \quad \frac{\partial}{\partial t}(g_t^* Y_0) = \dot{Y}_t = Y_1 - Y_0 = R^\infty.$$

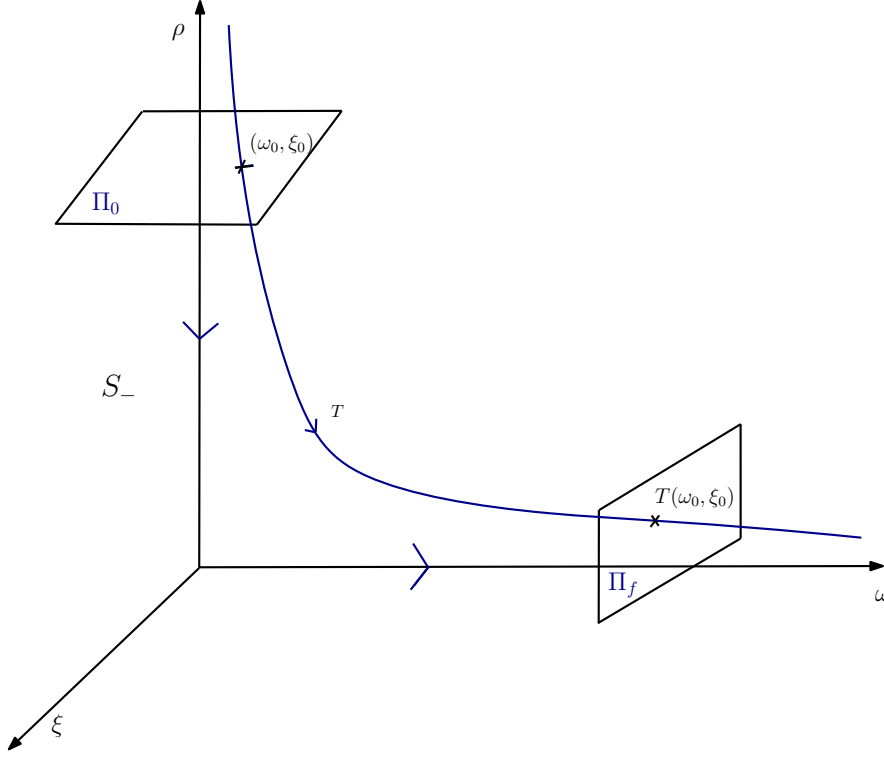


FIG. 3. Transition map between the two sections.

441 The family  $g_t$  defines a family of vector fields  $Z_t$  by

$$442 \quad Z_t(g_t(x)) = \frac{\partial}{\partial t} g_t(x),$$

443 and conversely we obtain the desired path of diffeomorphisms by integrating these  
444 fields. As a consequence, (3.13) can be rewritten

$$445 \quad (3.14) \quad [Y_t, Z_t] = R^\infty.$$

446 We just showed that getting rid of the flat perturbation  $R^\infty$  boils down to finding a  
447 solution to (3.14). It has been proved in [21] (see Theorem 10), that this equation has  
448 a solution. This ends the proof of Proposition 3.8.  $\square$

449 **4. Regular-singular transition.** The existence of a stratification of the flow  
450 in the  $\Sigma_-$  case raises the question of the transition: how does the flow behave when  
451 one is getting close to the stratum  $S^s$ ? We answer this question by considering the  
452 Poincaré map between two well chosen sections. Using the normal form given by  
453 Proposition 3.8, it is possible to make a precise statement: for given  $\rho_0$  and  $\omega_f$ , both  
454 positive, consider the two sections  $\Pi_0 \subset \{\rho = \rho_0\}$  and  $\Pi_f \subset \{\omega = \omega_f\}$ . As  $\Pi_0$  is  
455 transverse to  $\{\omega = 0\}$ , it can be parameterized by  $(\omega, \xi)$  coordinates. Similarly,  $\Pi_f$  is  
456 transverse to  $\{\rho = 0\}$  and can be parameterized by  $(\rho, \xi)$  coordinates (see Figure 3).

457 **THEOREM 4.1.** *Let  $T : \Pi_0 \rightarrow \Pi_f$  be the Poincaré mapping between the two sec-*  
458 *tions,  $T(\omega_0, \xi_0) = (\rho(\omega_0, \xi_0), \xi(\omega_0, \xi_0))$ . There exist smooth functions  $P$  and  $X$  defined*

459 on a neighbourhood of  $\{(0,0)\} \times D$  such that

$$460 \quad T(\omega_0, \xi_0) = (P(\omega_0 \ln \omega_0, \omega_0, \xi_0), X(\omega_0 \ln \omega_0, \omega_0, \xi_0)).$$

461 *Remark 4.2.* The mapping  $T$  belongs to the log-exp category [24]. (See [8] for the  
462 role of this category in sub-Riemannian geometry.)

463 *Proof.* The system (3.10) is equivalent to

$$464 \quad (4.1) \quad \begin{cases} \omega' = \omega, \\ \rho' = -\rho(1 + \Omega \tilde{A}(\Omega, \xi)), \\ \xi' = \Omega \tilde{C}(\Omega, \xi). \end{cases}$$

465 It has the same trajectories, and thus the same Poincaré mapping between the two  
466 sections. The transition time is given by the first equation:  $s(\omega_0) = \ln(\omega_f/\omega_0)$ . (The  
467 singular-regular transition occurs when  $\omega_0 \rightarrow 0$ , and the transition time tends to  
468 infinity.) Still noting  $u = \rho\omega$ , (4.1) implies

$$469 \quad (4.2) \quad \begin{cases} \Omega' = -\Omega^2 \tilde{A}(\Omega, \xi), \\ \xi' = \Omega \tilde{C}(\Omega, \xi), \end{cases}$$

470 that we want to integrate from an initial condition on  $\Pi_0$  in time  $s(\omega_0)$ . We extend  
471 this system by the trivial equation  $\omega'_0 = 0$ , and denote  $\varphi$  its associated flow. Then,  
472  $T(\omega_0, \xi_0) = \varphi(\ln(\omega_f/\omega_0), \omega_0, \rho_0\omega_0, \xi_0)$  (remember that on  $\Pi_0$ ,  $\Omega_0 = \rho_0\omega_0$ ). It is not  
473 the form we are looking for since  $\ln(\omega/\omega_0)$  is not regular at  $\omega_0 = 0$ , but we have the  
474 following estimate on the  $u$  coordinate of the flow.

475 **LEMMA 4.3.** *There exists a constant  $M$  such that, for small enough  $\omega_0 > 0$ ,  
476  $\xi \in D$ , and integration time  $t \leq \ln(\omega_f/\omega_0)$ ,*

$$477 \quad 0 \leq \Omega(t, \omega_0, \rho_0\omega_0, \xi_0) \leq M\omega_0.$$

478 *Proof.* Compare the dynamics of  $\Omega$  in (4.2) with  $v' = -v^2$ , which integrates  
479 according to

$$480 \quad v(t, v_0) = \frac{v_0}{1 + v_0 t}$$

481 for  $v_0 > 0$ . So

$$482 \quad v(\ln(\omega_f/\omega_0), \rho_0\omega_0) = \frac{\rho_0\omega_0}{1 + \rho_0\omega_0 \ln(\omega_f/\omega_0)},$$

483 hence the estimate as  $\omega_0 \ln(\omega_f/\omega_0)$  is small enough for small enough  $\omega_0 > 0$ .  $\square$

484 Let us make a change of time, and consider the following rescaled system:

$$485 \quad (4.3) \quad \begin{cases} \omega'_0 = 0, \\ \Omega' = -(\Omega^2/\omega_0)\tilde{A}(\Omega, \xi), \\ \xi' = (\Omega/\omega_0)\tilde{C}(\Omega, \xi). \end{cases}$$

486 For  $\omega_0 > 0$ , its flow  $\tilde{\varphi}$  is well defined and the Poincaré mapping is obtained by  
487 evaluating it in time  $\omega_0 \ln(\omega_f/\omega_0)$ :

$$488 \quad T(\omega_0, \xi_0) = \tilde{\varphi}(\omega_0 \ln(\omega_f/\omega_0), \omega_0, \rho_0\omega_0, \xi_0).$$



489 We make a blow-up on  $\{\Omega = \omega = 0\}$  to prove that  $T$  has the required regularity. Set  
 490  $f(\Omega, \omega, \xi) = (\eta, \omega, \xi)$  with  $\eta = \Omega/\omega$ : in coordinates  $(\omega, \eta, \xi)$ , the pulled back system  
 491 writes

$$492 \quad (4.4) \quad Z : \begin{cases} \omega'_0 = 0, \\ \eta' = -\eta^2 \tilde{A}(\eta\omega_0, \xi), \\ \xi' = \eta \tilde{C}(\eta\omega_0, \xi). \end{cases}$$

493 The vector field  $Z$  is actually smooth. The blow-up map  $f$  sends the cone  $-\eta_0\omega \leq$   
 494  $\Omega \leq \eta_0\omega$  onto the rectangle  $-\eta_0 \leq \eta \leq \eta_0$ ,  $-\omega_0 \leq \omega \leq \omega_0$ . According to the previous  
 495 lemma, we only need to evaluate its flow  $\hat{\varphi}(t, \omega_0, \eta_0, \xi_0)$  on a band  $\omega_0 \in [-\omega_1, \omega_1]$ ,  
 496  $\eta_0 \in [-M, M]$ ,  $\xi \in D$ , to compute  $\tilde{\varphi}$  in time  $\omega_0 \ln(\omega_f/\omega_0)$ . As  $\hat{\varphi} = (\hat{\eta}, \hat{\xi})$  is smooth  
 497 on such a band, we eventually get

$$498 \quad T(\omega_0, \xi_0) = (\hat{\eta}(\omega_0 \ln(\omega_f/\omega_0), \omega_0, \rho_0, \xi_0), \hat{\xi}(\omega_0 \ln(\omega_f/\omega_0), \omega_0, \rho_0, \xi_0)),$$

499 which has the desired regularity.  $\square$

500 **5. Application to mechanical systems.** In this section, we particularize our  
 501 study to mechanical systems associated with a potential. We go further in the specific  
 502 case of the potential of the restricted three body problem. Let  $Q$  be an open subset  
 503 of the plane  $\mathbf{R}^2$ , let  $g : TQ = Q \times \mathbf{R}^2 \rightarrow \mathbf{R}^2$  be a smooth function on the (trivial)  
 504 tangent bundle, and consider the following controlled mechanical system on  $M = TQ$ :

$$505 \quad (5.1) \quad \ddot{q}(t) + g(q(t), \dot{q}(t)) = u(t), \quad u_1^2(t) + u_2^2(t) \leq 1.$$

506 A simple rescaling allows to take into account a more general constraint on the control,  
 507  $|u(t)| \leq \varepsilon$ , for any positive  $\varepsilon$ . Given a smooth potential  $V : Q \rightarrow \mathbf{R}$ , an important  
 508 particular case of such a mechanical system is obtained for  $g(q, v) = \nabla V(q)$ , in which  
 509 case  $g$  is independent of the velocity coordinate  $v$  (this corresponds to a system with  
 510 kinetic energy  $\frac{v^2}{2}$ ). System (5.1) is control-affine of the form (2.1) studied in the  
 511 previous sections with

$$512 \quad F_0(q, v) = v \frac{\partial}{\partial q} - g(q, v) \frac{\partial}{\partial v}, \quad F_1(q, v) = \frac{\partial}{\partial v_1}, \quad F_2(q, v) = \frac{\partial}{\partial v_2}.$$

513 **PROPOSITION 5.1.** *Minimum time controls of (5.1) are piecewise smooth with a*  
 514 *finite number of singularities. At such singularities, the control rotates instantaneously*  
 515 *of an angle  $\pi$  (generating so-called " $\pi$ -singularities" [10]).*

516 *Proof.* One checks that  $\{F_1, F_2, F_{01}, F_{02}\}$  have rank four at any point of  $M$ , so  
 517 assumption (A) holds. As  $H_{12} = 0$  everywhere,  $\Sigma = \Sigma_-$ : contacts of minimum time  
 518 extremals with  $\Sigma$  are isolated according to Proposition 3.1, so there are finitely many  
 519 of them, and at such points  $u(\bar{t}+) = -u(\bar{t}-)$  by virtue of (3.11).  $\square$

520 Theorems 3.3 and 4.1 also apply. Every initial condition leading to a  $\pi$ -singularity  
 521 has a neighbourhood that can be stratified, with a codimension one stratum leading  
 522 to neighbouring  $\pi$ -singularities. The extremal flow is continuous on this neighbour-  
 523 hood, and crossing this stratum generates logarithmic (in the sense of Theorem 4.1)  
 524 singularities on the flow.

525 An important application is the minimum time control of the elliptic restricted  
 526 three body problem [22]. Let  $\mu \in (0, 1)$  be the ratio of the two primary masses, and  
 527 let  $e \in [0, 1)$  be the eccentricity; the elliptic restricted problem seeks the trajectory

528 of a third body whose motion is influenced by the two primary masses, while these  
 529 masses are not influenced by the third negligible one. The two primaries are assumed  
 530 to be on elliptic orbits of common eccentricity  $e \in [0, 1)$  and common focus equal to  
 531 their center of mass set at the origin (see Figure 4). Let us denote  $q^1$  and  $q^2$  the  
 532 positions, depending on time, of the two primaries. Let  $\mathcal{Q}$  denote the open subset of  
 533 the time-position space

$$534 \quad \mathcal{Q} = \{(t, q) \in \mathbf{R} \times \mathbf{R}^2 \mid q \neq q^1(t) \text{ and } q \neq q^2(t)\}$$

535 outside collisions. On  $\mathcal{Q} \times \mathbf{R}^2$ , the dynamics describing the controlled motion of the  
 536 third body is

$$537 \quad (5.2) \quad \ddot{q}(t) + \nabla_q V_{\mu, e}(t, q(t)) = u(t),$$

538 where the time-dependent potential is

$$539 \quad V_{\mu, e}(t, q) = \frac{1 - \mu}{|q - q^1(t)|} + \frac{\mu}{|q - q^2(t)|}.$$

540 Note that  $q^1$  and  $q^2$  depend on the eccentricity  $e$  prescribing the motion of the two  
 541 primaries, hence the dependence of the potential both on  $\mu$  and  $e$ . A remarkable  
 542 situation occurs when  $e = 0$ . The two primaries are in circular motion around their  
 543 center of mass, and the system can be made autonomous by going into the moving  
 544 frame attached to the rotating bodies. In this frame, still denoting  $q$  the position  
 545 (now related to moving axes), the dynamics is exactly of the form (5.1) with

$$546 \quad g(q, v) = (1 - \mu) \frac{(q_1 + \mu, q_2)}{[(q_1 + \mu)^2 + q_2^2]^{3/2}} + \mu \frac{(q_1 - 1 + \mu, q_2)}{[(q_1 - 1 + \mu)^2 + q_2^2]^{3/2}} - q + 2(-v_2, v_1),$$

547 and  $Q = \mathbf{R}^2 \setminus \{(-\mu, 0), (1 - \mu, 0)\}$ . We refer to [9] for further details on the controlled  
 548 circular restricted three body problem. As stated at the beginning of the section,  
 549 Proposition 3.1 as well as Theorems 3.3 and 4.1 apply to the minimum time control  
 550 of the circular restricted problem. Besides, it turns that Proposition 5.1 remains  
 551 true for the more general elliptic restricted problem. The proof is based on a direct  
 552 estimation *à la Sturm* of the time between two singularities, and provides a global  
 553 bound on the number of these singularities. Let  $q : [0, t_f] \rightarrow \mathbf{R}^2$  be a minimum time  
 554 trajectory of the elliptic restricted three body problem (5.2). Let us denote

$$555 \quad \delta_1 = \min_{[0, t_f]} |q(t) - q^1(t)|, \quad \delta_2 = \min_{[0, t_f]} |q(t) - q^2(t)|,$$

556 and

$$557 \quad \delta_{12} = \frac{\delta_1 \delta_2}{[(1 - \mu)\delta_2^3 + \mu\delta_1^3]^{1/3}}.$$

558 **PROPOSITION 5.2.** *Singularities of minimum time trajectories of the elliptic re-*  
 559 *stricted three body problem are  $\pi$ -singularities. The number of  $\pi$ -singularities for a*  
 560 *minimum time trajectory is bounded above by  $\lfloor t_f / (\pi \delta_{12}^{3/2}) \rfloor$ , where  $t_f$  is the minimum*  
 561 *time.*

562 In the circular restricted case ( $e = 0$ ), this proposition provides a global bound on  
 563 the number of heteroclinic connections of the system after blow-up and time change  
 564 defined in Section 3; each  $\pi$ -singularity is associated with a pair of hyperbolic equilibria

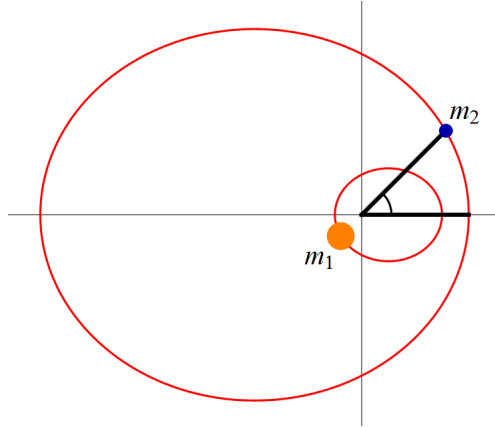


FIG. 4. *Elliptic restricted three body problem. The two primary masses,  $m_1$  and  $m_2$ , are on elliptic orbits of common eccentricity and common focus (located at their center of mass).*

565  $(\bar{z}_-, \bar{z}_+)$ , and going from one  $\pi$ -singularity to another generates a connection between  
 566  $\bar{z}_+^i$  and  $\bar{z}_-^{i+1}$ . Not surprisingly, this bound is expressed in terms of the distance to the  
 567 primaries, that is to the singularities of the original dynamics. An interesting open  
 568 question is to provide an estimate of the distance to the collisions—and thus a more  
 569 explicit bound on the number of  $\pi$ -singularities—in terms of the boundary conditions  
 570 in the position-velocity phase space of the restricted problem. Note also that when  
 571  $\mu = 0$ , we go back to the Kepler problem and  $\delta_1 = \delta_{12}$  is the distance to the collision.  
 572 The proof the proposition uses the following lemma.

573 LEMMA 5.3. *Let us consider the minimum-time control of*

$$574 \quad (5.3) \quad \ddot{q}(t) + \nabla_q V(t, q(t)) = u(t),$$

575 *where  $V$  is a smooth potential defined on a open subset  $O \subset \mathbf{R}^3$ . Let  $A(t, q)$  be a*  
 576 *symmetric matrix of order  $n$  whose entries depend continuously on  $(t, q) \in O$  such*  
 577 *that*

$$578 \quad A(t, q) \geq \nabla_{qq}^2 V(t, q), \quad (t, q) \in O.$$

579 *The following statement holds. Singularities of minimum time trajectories of (5.3)*  
 580 *are  $\pi$ -singularities. If  $\bar{t}_1 < \bar{t}_2$  are two such singularities, if  $A(t, q(t)) > \nabla_{qq}^2 V(t, q(t))$*   
 581 *for some  $t \in [\bar{t}_1, \bar{t}_2]$ , there exists a non-trivial solution of  $\ddot{y}(t) + A(t, q(t))y(t) = 0$  that*  
 582 *vanishes both at  $\bar{t}_1$  and  $\bar{t}'_2 < \bar{t}_2$ .*

583 *Proof.* Applying Pontrjagin maximum principle to (5.3), one gets the maximized  
 584 (time dependent) Hamiltonian

$$585 \quad H(t, q, v, p_q, p_v) = p_q \cdot v - p_v \cdot \nabla_q V(t, q) + |p_v|,$$

586 and  $u = p_v/|p_v|$  whenever  $p_v$  is not zero. As the equation on  $p_v$  is a second order  
 587 linear one,

$$588 \quad \ddot{p}_v(t) + \nabla_{qq}^2 V(t, q(t))p_v(t) = 0,$$

589 if  $p_v$  and  $\dot{p}_v$  vanish simultaneously then  $p_v$  is identically zero: this is impossible since  
 590  $p_q = -\dot{p}_v$  would also vanish, leading to  $p = (p_q, p_v)$  identically zero, a proscribed case  
 591 when minimizing time. So  $\dot{p}_v \neq 0$  when  $p_v = 0$ , and the zeros of  $p_v$  are isolated.  
 592 At such a point, the ratio  $p_v/|p_v|$  has opposite left and right limits, resulting in a  
 593  $\pi$ -singularity. Sturm's comparison Theorem [16] allows to conclude.  $\square$

594 *Proof (of Proposition 5.2).* Let

$$595 \quad A(t, q) = \begin{pmatrix} 1 + \frac{1-\mu}{|q-q^1(t)|^3} + \frac{\mu}{|q-q^2(t)|^3} & 0 \\ 0 & \frac{1-\mu}{|q-q^1(t)|^3} + \frac{\mu}{|q-q^2(t)|^3} \end{pmatrix}.$$

596 A straightforward calculation shows that

$$597 \quad \det(A(t, q) - \nabla_{qq}^2 V_\mu(t, q)) = 3 \left[ (1-\mu) \frac{(q_2 - q_2^1(t))^2}{|q - q^1(t)|^5} + \mu \frac{(q_2 - q_2^2(t))^2}{|q - q^2(t)|^5} \right] > 0$$

598 for  $(t, q)$  in  $\mathcal{Q}$ . By the previous lemma, the interval between two singularities is greater  
599 than the time interval between two zeros of the scalar equation

$$600 \quad (5.4) \quad \ddot{y}(t) + \left( \frac{1-\mu}{|q(t) - q^1(t)|^3} + \frac{\mu}{|q(t) - q^2(t)|^3} \right) y(t) = 0.$$

601 As

$$602 \quad \frac{1-\mu}{|q(t) - q^1(t)|^3} + \frac{\mu}{|q(t) - q^2(t)|^3} \leq \frac{1-\mu}{\delta_1^3} + \frac{\mu}{\delta_2^3},$$

603 a solution of (5.4) cannot have consecutive zeros in an interval of length smaller than

$$604 \quad \frac{\pi}{\sqrt{\frac{1-\mu}{\delta_1^3} + \frac{\mu}{\delta_2^3}}} = \pi \delta_{12}^{3/2}$$

605 by Sturm comparison again.  $\square$

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