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# Surrogate-Assisted Bounding-Box Approach for Optimization Problems with Approximated Objectives

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Project-Teams CARDAMOM

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**Abstract:** In this work, we present a novel framework to perform multi-objective optimization, when considering an error on the objective functions. In many engineering optimization problems, the computation of the objective functions are affected by an error arising from the model employed for the computation. For example, in the case of uncertainty-based optimization the objective functions are statistics of a performance of interest which is uncertain due to the variability of the system input variables. These estimated objectives are affected by an error, which can be modeled with a confidence interval. The framework proposed here is general and aims at dealing with any error affecting a given objective function. The strategy is based on the extension of the Bounding-Box concept to the Pareto optima, where the error can be regarded with the abstraction of an interval (in one-dimensional problems) or a Bounding-Box (in multi-dimensional problems) around the estimated value. This allows the computation of an approximated Pareto front, whose accuracy is strongly dependent on the acceptable computational cost. This approach is then supplemented by the construction of an evolutive surrogate model on the objective functions, iteratively refined during the optimization process. This allows ultimately to further reduce the computation cost of the Pareto front with approximations of the objective functions at a negligible cost.

The validation of the proposed method is accomplished first by proving the mathematical convergence toward the true continuous Pareto front under some hypothesis. Secondly, a numerical algorithm is proposed and its performances are assessed on several numerical test-cases. Results are systematically compared to a simple Double-loop approach and to the classical Bounding-Box method.

**Key-words:** Multi-objective optimization, Error Bounding-Boxes, Cost reduction, Surrogate-assisting strategy, Uncertainty-based optimization

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## Le framework SABBa pour les problèmes d'optimisation avec des objectifs approximatés

**Résumé :** Dans ce papier, nous présentons un nouveau cadre permettant la résolution d'optimisation multi-objectifs tout en considérant un bruit sur les fonctions objectifs. Dans la majorité des problèmes d'optimisation en ingénierie, le calcul des fonctions objectifs s'accompagne intrinsèquement d'une erreur provenant du modèle employé. Par exemple, les fonctions objectifs d'un problème d'optimisation sous incertitudes sont des moments statistiques estimés sur les fonctions de sortie, incertaines de par les variabilités d'entrée. Ces objectifs approximatés sont donc affectés par une erreur, qui peut être représentée par un intervalle de confiance. Le cadre proposé ici reste général et aspire à pouvoir traiter avec des erreurs de n'importe quelle origine sur les fonctions objectifs. La stratégie repose sur l'extension du concept de Boîtes d'erreur conservatives aux optima de Pareto, avec une erreur interprétée comme un intervalle (dans les problèmes mono-objectif) ou une Boîte conservative (dans les problèmes multi-dimensions) autour de la valeur approximatée. Cela permet d'approximer le front de Pareto, avec une précision fortement dépendante du coût de calcul acceptable. Cette approche est de plus couplée à la construction d'un modèle de substitution évolutif sur les fonctions objectifs, raffiné itérativement pendant le processus d'optimisation. Cela permet de réduire à nouveau le coût global de l'optimisation grâce à des approximations lues directement sur le modèle de substitution, pour un coût négligeable.

La validation de la méthode proposée se fera tout d'abord en prouvant formellement la convergence vers le vrai front de Pareto continu sous quelques suppositions. Ensuite, un algorithme est proposé et appliqué à plusieurs cas-test. Les résultats sont méthodiquement comparés à l'approche Double-boucle directe et à la méthode des Boîtes d'erreur conservatives classique.

**Mots-clés :** Optimisation multi-objectifs, Boîtes d'erreur conservative, Réduction du coût, Assistance par modèle de substitution, Optimisation dans l'incertain

# 1 Introduction

A key ingredient in any optimization method is the evaluation of the objective function, which provides the metric to discriminate a good and feasible design, according to a set of objectives and constraints. In many real applications, the evaluation of the objective function can be affected by a source of uncertainty, noise or error. A clear distinction has to be made among these elements. A possible classification is presented in [1], where four classes are identified: (i) noise, caused for example by sensor measurements or random simulations, (ii) uncertainty for robustness (or reliability) analysis, where the model/numerical method used to compute the objective function includes stochastic variables, (iii) approximation error due to an approximate model of the exact objective function (e.g. a response surface) and (iv) time-varying functions.

Here, this paper aims at dealing with objective functions subject to approximation errors. This kind of errors on the objective functions could notably arise when performing optimization under uncertainties, requiring the computation of statistical moments, tail probabilities or other probabilistic measures. Such errors can also be linked to the variability of the measure of the objective values, in the context of noisy optimization.

In the context of noise- and uncertainty-based optimization, many ideas have been developed in the last decades to cope with uncertain parameters, assess the reliability of a system or improve the convergence of optimization algorithms despite noisy evaluations.

Basing on Taguchi's robust design concept, several studies have been performed (see [2] for a review). A strong effort has been provided on the reformulation of optimization problems with probabilistic or worst-case constraints ([3, 4, 5, 6]) or robustness measures. Jin et al [7] introduced a trade-off between performance and robustness by adding a robustness measure, computed by means of First-Order Second-Moment (analyzed in [8]) or Monte-Carlo sampling. Similar ideas are exploited with a Decision Support Problem in [9], with a weighted single-objective problem in [10], with a probabilistic indicator in [11] and coupled with percentile constraints and inverse reliability in [3].

An other class of techniques to perform optimization in noisy or uncertain context are the stochastic search algorithm, such as Evolutionary Algorithm. Review papers [1, 12] depicted the main methods to deal with uncertainties in an EA environment, namely averaging, robust index selection or robustness measures. In this context, robustness measure can be seen as a new fitness assessment, as in [13], or as an additional robustness index objective in [14]. Deb et al [15] introduced two concepts of robustness, by revising the objective function or applying an additional constraint on the problem, both computed by averaging. However, in literature, the main strategy to account for uncertainty on the objective functions is to use probabilistic ranking of the individuals. More precisely, Stochastic Pareto Genetic Algorithm, developed in [16], computes probabilistic dominations between individual, assuming normal distributions, and uses them for ranking. Hughes [17] assumes normal distributions and computes the probability of wrong decision when comparing two individuals to propose a probabilistic extension of the Non-dominated Sorting Genetic Algorithm ranking. This work was then supplemented by [18] with a bayesian learning of the variances of input noises. Teich [19] followed an idea similar to [17] but with a uniform distribution of the objective functions, yielding a ranking which is computed by summing probabilistic dominance. Robust evolutionary optimization has also been performed with coevolutionary algorithms such as in [20], to perform worst-case optimization. Other heuristics like Simulated Annealing and Pure Random Search have been developed in the noisy optimization context in [21] and [22]. Model-based approaches have also been compared and extended to noisy optimization problems in [23]. Finally, some works [24, 25] introduced new dominance criteria for non-stochastic interval uncertainties, while [26] introduced a worst-case methodology. Epistemic uncertainties have also been treated by means of evidence theory in

[27, 28].

Other techniques used for robust optimization are based on Stochastic programming (see [29] for a detailed review). These works are mostly focused on choosing optimal robustness measures. However, the practical applications are based on Sample Average Approximations and require expensive black-box evaluations to be approximated by surrogate models. In scenario-based optimization, alternative robustness concepts have been introduced to enrich the decision maker and introduce human-like reasoning. Some of these concepts are discussed and compared in [30, 31, 32, 33].

In the review of Mlakar et al [34], several techniques of optimization under uncertainty have been compared, and the concept of Bounding-Boxes and the associated Pareto dominance rules have been introduced. In Fusi et al [35], some concepts related to deterministic robustness measures (stochastic programming) and probabilistic boxes in the original objective space (robust evolutionary algorithms) have been reformulated in order to associate Bounding-Boxes approximations to robustness measures, so that approximated measures can be taken into account before full convergence. In practice, this means that the boxes associated to non-efficient and dominated designs would not be refined while Pareto optimal boxes would be converged up to a given threshold.

The first contribution of this paper relies on providing some mathematical evidences about the convergence of the Bounding-Box approach in the measures and design space for a general problem of multi-objective optimization with approximated objective functions. One drawback of the approach presented in [35] consists in the presence of substantial gains only during the first iterations of the optimization, with a less significant impact when converging toward the Pareto front. This can be easily explained since the convergence toward Pareto optimal designs leads to more and more potential optima, with very refined but costly evaluations of the Bounding-Box.

For curing this issue, the second main contribution of this paper is the coupling of the Bounding-Box approach with a surrogate-assisted method permitting to further reduce the computational cost mostly during the last iterations of the optimization algorithm. Accelerative strategy using surrogate models have been studied, notably within the optimization process, like in [36], and extensively in Evolutionary Strategy context ([37, 38, 39, 40, 41]). The strategy chosen here is to iteratively refine an evolutive metamodel each time an evaluation is performed, in order to bypass costly evaluations where convergence is detected. This coupling has several a priori advantages over classical approaches. In fact, the Bounding-Box approach quickly drives the optimization process in the Pareto optimal area, meaning that the evolutive metamodel would only need to be converged over this area.

The aim of this paper is to introduce a framework called SABBa (Surrogate-Assisted Bounding-Box approach), and to provide mathematical and numerical evidence about its convergence properties. After having further explained the strategy in Section 2, convergence of the Pareto front will be mathematically analyzed under several assumptions in Section 3. Then, the general pseudo-algorithm of the framework is presented in Section 4 and three analytical test-cases are fully analyzed in Section 5 in order to highlight the main behaviors of the coupling in terms of computational cost and accuracy. Finally, conclusive remarks and future works are drawn in Section 6.

## 2 Overview of the SABBa framework

A multi-objective optimization problem can be stated as

$$\begin{aligned} & \text{minimize/maximize: } \mathbf{f}(\mathbf{x}) \\ & \text{by changing: } \mathbf{x} \in \mathcal{X} \end{aligned} \quad (1)$$

where the  $m$  objective functions are collected in a vector  $\mathbf{f} \in \mathbb{R}^m$  and  $\mathbf{x} \in \mathcal{X}$  are the  $n$  design variables included in the design domain  $\mathcal{X} \subset \mathbb{R}^n$ .

In Eq. 1,  $\mathbf{f}$  represent a user-defined measure. The main hypothesis of the proposed framework consists in assuming that these measures can not be computed exactly, but only be approximated. They yield a specific error, which can potentially be reduced through additional evaluations. The objective function is approximated with a quantity  $\tilde{\mathbf{f}}^k$ , which corresponds to a level of refinement of the error  $\boldsymbol{\varepsilon}^k$ , so that  $\tilde{\mathbf{f}}^k = \mathbf{f} - \boldsymbol{\varepsilon}^k$ . Of course,  $\boldsymbol{\varepsilon}^k$  has to be estimated as it depends on  $\mathbf{f}$ . This quantity is supposed to be estimated by  $\tilde{\boldsymbol{\varepsilon}}^k$ , with  $|\boldsymbol{\varepsilon}^k| \leq \tilde{\boldsymbol{\varepsilon}}^k$ , *i.e.*  $\mathbf{f} \in [\tilde{\mathbf{f}}^k - \tilde{\boldsymbol{\varepsilon}}^k, \tilde{\mathbf{f}}^k + \tilde{\boldsymbol{\varepsilon}}^k]$ . This  $m$ -dimensional interval (represented in Figure 1) is named as a box, *i.e.*  $\mathbf{f} \in \mathcal{B}(\tilde{\mathbf{f}}^k, \tilde{\boldsymbol{\varepsilon}}^k)$ , using a notation introduced in the following definition:

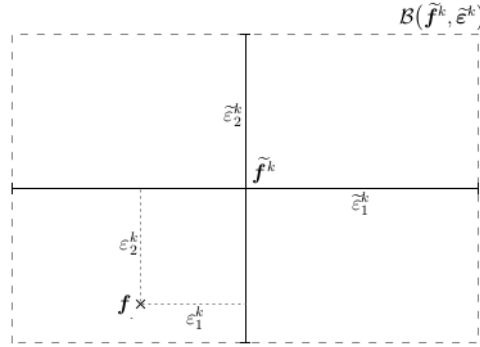


Figure 1: Bounding-Box approximation

**Definition 1.** A  $m$ -dimensional box is defined by its center and half-width vectors as follows

$$\mathcal{B}(\mathbf{a}, \mathbf{r}) = \{\mathbf{b} \in \mathbb{R}^m \mid \mathbf{b} \in [\mathbf{a} - \mathbf{r}, \mathbf{a} + \mathbf{r}]\} \in \wp(\mathbb{R}^m)$$

$\forall(\mathbf{a}, \mathbf{r}) \in \mathbb{R}^m \times \mathbb{R}_+^m$ , and with  $\wp(\mathbb{R}^m)$  the power set of  $\mathbb{R}^m$ .

### Remarks

- $\mathcal{B}(\mathbf{a}, \mathbf{0}) \equiv \mathbf{a}$
- To lighten the notations, with  $\mathbf{f} : \mathcal{X} \rightarrow \mathbb{R}^m$ , we set:

$$\mathcal{B}_{\mathbf{f}}(\mathbf{x}, \mathbf{r}) \equiv \mathcal{B}(\mathbf{f}(\mathbf{x}), \mathbf{r}), \quad \forall(\mathbf{x}, \mathbf{r}) \in \mathcal{X} \times \mathbb{R}^m$$



## 2.1 Pareto dominance

In multi-objective optimization, objective functions are naturally conflicting, since there is no one optimal design anymore but rather a set of trade-off optima, which are not dominated by any other design for all the objectives at the same time. These designs are named as Pareto-optimal and the ensemble is called a Pareto front. The designs of the Pareto front are non-dominated according to the following Pareto dominance rule:

**Definition 2.** For two vectors  $\mathbf{a}, \mathbf{b} \in \mathbb{R}^m$ , and with  $\mathcal{I}_1^m = \llbracket 1, m \rrbracket$ :

$$\begin{aligned} \mathbf{a} \succ \mathbf{b} \text{ (a dominates b)} &\iff \forall j \in \mathcal{I}_1^m, \pm a_j \leq \pm b_j \quad \text{and} \\ &\quad \exists j \in \mathcal{I}_1^m, \pm a_j < \pm b_j \\ \mathbf{a} \succ\!\succ \mathbf{b} \text{ (a strictly dominates b)} &\iff \forall j \in \mathcal{I}_1^m, \pm a_j < \pm b_j \\ \mathbf{a} \sim \mathbf{b} \text{ (a is indifferent to b)} &\iff \mathbf{a} \not\succeq \mathbf{b} \text{ and } \mathbf{b} \not\succeq \mathbf{a} \end{aligned}$$

$\forall j \in \mathcal{I}_1^m$ , the symbol  $\pm$  (implicitly  $\pm_j$ ) indicates the goal in the  $j^{\text{th}}$  dimension:

$$\pm = \begin{cases} + & \text{for minimization} \\ - & \text{for maximization} \end{cases}$$

**Definition 3.** For two designs  $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ , where  $\mathcal{X}$  is the design variables domain, and  $\mathbf{f}$  the objective function, we introduce the following notations:

$$\begin{aligned} \mathbf{x} \overset{\mathbf{f}}{\succ} \mathbf{y} &\iff \mathbf{f}(\mathbf{x}) \succ \mathbf{f}(\mathbf{y}) \\ \mathbf{x} \overset{\mathbf{f}}{\succ\!\succ} \mathbf{y} &\iff \mathbf{f}(\mathbf{x}) \succ\!\succ \mathbf{f}(\mathbf{y}) \\ \mathbf{x} \overset{\mathbf{f}}{\sim} \mathbf{y} &\iff \mathbf{f}(\mathbf{x}) \sim \mathbf{f}(\mathbf{y}) \end{aligned}$$

**Remark** Note that the objective functions appears explicitly in the notation of Pareto dominance.

**Definition 4.** A design  $\mathbf{x} \in \mathcal{X}$  is said to be non-dominated (or efficient) regarding a set  $\mathcal{A} \subseteq \mathcal{X}$  if and only if:

$$\nexists \mathbf{y} \in \mathcal{A}, \mathbf{y} \overset{\mathbf{f}}{\succ} \mathbf{x}$$

Then,  $\mathbf{x}$  is said to be Pareto-optimal iff  $\mathbf{x}$  is non-dominated regarding  $\mathcal{X}$ .

However, this definition of Pareto dominance can not be retained in the following since  $\mathbf{f}$  is assumed to be unknown. Only the approximate value  $\tilde{\mathbf{f}}^k$  and its conservative error  $\tilde{\boldsymbol{\varepsilon}}^k$  are used, considering that  $\mathbf{f} \in \mathcal{B}(\tilde{\mathbf{f}}^k, \tilde{\boldsymbol{\varepsilon}}^k)$ . A new dominance criterion is then introduced.

Following [34] and [35], this dominance is denoted as *Boxed Pareto dominance*. The basic criterion yields that a box  $\mathcal{B}(\mathbf{a}, \mathbf{r})$  dominates a box  $\mathcal{B}(\mathbf{b}, \mathbf{r}')$  if and only if any point of  $\mathcal{B}(\mathbf{a}, \mathbf{r})$  dominates (in the classical way) every points of  $\mathcal{B}(\mathbf{b}, \mathbf{r}')$ . The boxed dominance rule can be interpreted as a classical dominance rule between the least performing point of  $\mathcal{B}(\mathbf{a}, \mathbf{r})$  and the most performing one of  $\mathcal{B}(\mathbf{b}, \mathbf{r}')$ , as seen in the following definition.

**Definition 5.** For two boxes  $(\mathcal{B}(\mathbf{a}, \mathbf{r}), \mathcal{B}(\mathbf{b}, \mathbf{r}')) \in \wp(\mathbb{R}^m)^2$ ,

$$\begin{aligned} \mathcal{B}(\mathbf{a}, \mathbf{r}) \underset{\mathcal{B}}{\succ} \mathcal{B}(\mathbf{b}, \mathbf{r}') &\iff \forall j \in \mathcal{I}_1^m, \pm a_j + r_j \leq \pm b_j - r'_j \quad \text{and} \\ &\quad \exists j \in \mathcal{I}_1^m, \pm a_j + r_j < \pm b_j - r'_j \\ \mathcal{B}(\mathbf{a}, \mathbf{r}) \underset{\mathcal{B}}{\succ\sim} \mathcal{B}(\mathbf{b}, \mathbf{r}') &\iff \forall j \in \mathcal{I}_1^m, \pm a_j + r_j < \pm b_j - r'_j \\ \mathcal{B}(\mathbf{a}, \mathbf{r}) \underset{\mathcal{B}}{\sim} \mathcal{B}(\mathbf{b}, \mathbf{r}') &\iff \mathcal{B}(\mathbf{a}, \mathbf{r}) \not\underset{\mathcal{B}}{\succ} \mathcal{B}(\mathbf{b}, \mathbf{r}') \text{ and } \mathcal{B}(\mathbf{b}, \mathbf{r}') \not\underset{\mathcal{B}}{\succ} \mathcal{B}(\mathbf{a}, \mathbf{r}) \end{aligned}$$

Of course, the following equivalence follows:

$$\forall \mathbf{a}, \mathbf{b} \in \mathbb{R}^m, \mathcal{B}(\mathbf{a}, 0) \underset{\mathcal{B}}{\succ} \mathcal{B}(\mathbf{b}, 0) \iff \mathbf{a} \succ \mathbf{b} \quad (2)$$

which traduces the consistency between the two domination rules when the size of the box tends to zero.

**Remark** It can be noted that the relation between  $\mathcal{B}(\mathbf{a}, \mathbf{r})$  and  $\mathcal{B}(\mathbf{b}, \mathbf{r}')$  is the same as between  $\mathcal{B}(\mathbf{a}, \mathbf{r}')$  and  $\mathcal{B}(\mathbf{b}, \mathbf{r})$ .

## 2.2 Structure of the framework

The main objective of the proposed framework is to obtain a converged Pareto front with a low computational cost, by exploiting at best the information of the error associated to the computation of the objective function. Basic ingredients for defining the framework are the definition of the Bounding-Box (provided in Section 2), and two criteria for deciding if using (or not) the surrogate model instead of the true evaluation of the objective functions, and if refining (or not) a BB, respectively.

Note that the population at a given iteration is represented by two sets of boxes: i) the first is defined basing on true approximated objectives with the size given by their associated conservative error; ii) the second on objective values computed by means of the surrogate model with the box size given by the surrogate model error. Here, we report the main steps during one iteration of the optimization process:

- For each new design, check if the surrogate error approximation is inferior to a user-defined threshold  $\mathbf{s}_1$  (denoted as Criterion 1 in the following). This is done for assessing whether the current surrogate model can be used for bypassing the computation of the objective functions.
- For the designs where criterion 1 is not respected, compute a first approximation  $\mathcal{B}(\tilde{\mathbf{f}}^0, \tilde{\boldsymbol{\varepsilon}}^0)$  (cf. Fig. 1).
- For all these computed boxes, apply the boxed-dominance concept. Then, check if the conservative error is inferior to a user-defined threshold  $\mathbf{s}_2$  (denoted as Criterion 2 in the following). When criterion 2 is not respected, then refine all non-dominated boxes. Any dominated box is assumed to be inefficient and is frozen without further refinement (cf. Fig. 2).
- When all computed boxes are either dominated or fully converged to the threshold  $\mathbf{s}_2$ , refine the surrogate model in order to improve its approximation error and produce new designs by means of the chosen optimization method.

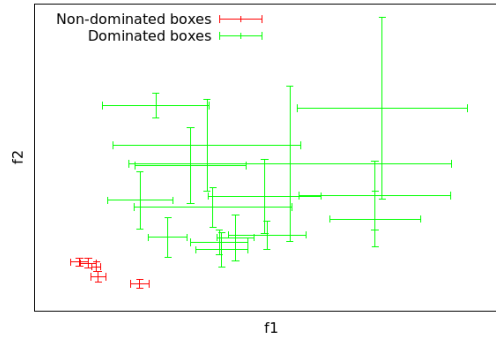


Figure 2: Bounding-Boxes refinement strategy

Note that this framework is conceived for any multi-objective optimization problem with approximated information, and is then independent from the optimization method, the method to estimate the error associated to the Bounding-Box, the metrics, etc.

Intuitively, boxed dominance is harder to achieve than classical dominance, since the number of non-dominated boxes should be always higher or equal to the number of non-dominated designs. Note that the wider boxes are, the higher the number of non-dominated designs is. However, a consistent cost reduction is realized by means of the Bounding-Box approach as some designs feature such poor performances that even wide boxes are dominated by the efficient ones. Thus, the dominated and non-efficient boxes are not further refined. In this case, much less evaluations are required in order to approximate the objective functions. Globally, the Bounding-Box approach allows a quick convergence toward the Pareto front, where the boxes have to be refined, and the surrogate-assisted strategy should be effective in order to reduce further computational cost.

The following section deals with mathematical aspects of the surrogate assisted Bounding-Box approach. From a theoretical point of view, it aims at proving that the framework permits the convergence toward the real Pareto-optimal set when the number of designs and the refinement of the boxes tend to infinity. From a more practical point of view, we aim to show also that the strategy is robust, meaning that among a discrete set of designs, all the truly non-dominated ones stay within the boxed Pareto-optimal set of the framework at any refinement level. Moreover, the strategy should systematically find an efficient and robust box size to assign to interpolated designs, thus building a deep coupling between the surrogate and the Bounding-Boxes.

### 3 Mathematical framework of the proposed approach

The goal of this section is twofold. First, we want to prove the convergence of a set of interest toward the exact Pareto-optimal area when the number  $N$  of optimization iterations and the refinement  $k$  of the objective functions tend to infinity.

Secondly, the convergence of the method is demonstrated when the surrogate-assisted strategy is used.

### 3.1 Bounding-Box definition and convergence analysis

Based on the Pareto dominance defined in the preceding section, some sets are introduced that represent Pareto-optimal sets. The set  $\mathcal{P}$  represents the true continuous Pareto front of the problem. His discrete counterpart is labeled as  $\tilde{\mathcal{P}}$ . Finally, we indicate with  $\tilde{\mathcal{P}}_{\mathcal{B}}$  the set of Pareto-optimal boxes.

**Definition 6.** Let us introduce the following sets, with  $\mathcal{I}_1^N = \llbracket 1, N \rrbracket$ ,  $\mathcal{A} \subset \mathbb{R}^m$ , designs  $\mathbf{a}_i \in \mathbb{R}^m$  and boxes  $\mathcal{B}_i \in \wp(\mathbb{R}^m)$ :

$$\begin{aligned}\mathcal{P}(\mathcal{A}) &= \{\mathbf{a} \in \mathcal{A} \mid \nexists \mathbf{b} \in \mathcal{A}, \mathbf{b} \succ \mathbf{a}\} \\ \tilde{\mathcal{P}}\left(\{\mathbf{a}_i\}_{i=1}^N\right) &= \{\mathbf{a}_i, i \in \mathcal{I}_1^N \mid \nexists j \in \mathcal{I}_1^N, \mathbf{a}_j \succ \mathbf{a}_i\} \\ \tilde{\mathcal{P}}_{\mathcal{B}}\left(\{\mathcal{B}_i\}_{i=1}^N\right) &= \{\mathcal{B}_i, i \in \mathcal{I}_1^N \mid \nexists j \in \mathcal{I}_1^N, \mathcal{B}_j \succ_{\mathcal{B}} \mathcal{B}_i\}\end{aligned}$$

In the case of multi-objective Pareto analysis (considering  $\mathbf{f}$  as objective function and  $\mathcal{X}$  as design space),  $\mathcal{A}$  represents  $\mathbf{f}(\mathcal{X})$  and designs  $\mathbf{a}_i$  as well as the center of boxes  $\mathcal{B}_i$  are elements of  $\mathbf{f}(\mathcal{X})$ . Note that  $\mathcal{P}(\mathbf{f}(\mathcal{X}))$  is simplified to  $\mathcal{P}$ , *i.e.* the Pareto front of the problem, while  $\tilde{\mathcal{P}}$  is always linked to a given set of designs.

**Definition 7.** With  $\mathbf{f}$  the objective functions,  $\mathcal{X}$  the design space,  $\mathbf{x}_i \in \mathcal{X}$  and  $\mathbf{r}_i \in \mathbb{R}^m$ , let us define

$$\begin{aligned}\mathcal{X}_{\mathcal{P}}^{\mathbf{f}} &= \{\mathbf{x} \in \mathcal{X} \mid \mathbf{f}(\mathbf{x}) \in \mathcal{P}\} \\ \mathcal{X}_{\tilde{\mathcal{P}}}^{\mathbf{f}}\left(\{\mathbf{x}_i\}_{i=1}^N\right) &= \left\{\mathbf{x}_i, i \in \mathcal{I}_1^N \mid \mathbf{f}(\mathbf{x}_i) \in \tilde{\mathcal{P}}\left(\{\mathbf{f}(\mathbf{x}_i)\}_{i=1}^N\right)\right\} \\ \mathcal{X}_{\tilde{\mathcal{P}}_{\mathcal{B}}}^{\mathbf{f}}\left(\{(\mathbf{x}_i, \mathbf{r}_i)\}_{i=1}^N\right) &= \left\{\mathbf{x}_i, i \in \mathcal{I}_1^N \mid \mathcal{B}_{\mathbf{f}}(\mathbf{x}_i, \mathbf{r}_i) \in \tilde{\mathcal{P}}_{\mathcal{B}}\left(\{\mathcal{B}_{\mathbf{f}}(\mathbf{x}_i, \mathbf{r}_i)\}_{i=1}^N\right)\right\}\end{aligned}$$

Practically,  $\mathcal{X}_{\mathcal{P}}^{\mathbf{f}}$ ,  $\mathcal{X}_{\tilde{\mathcal{P}}}^{\mathbf{f}}\left(\{\mathbf{x}_i\}_{i=1}^N\right)$  and  $\mathcal{X}_{\tilde{\mathcal{P}}_{\mathcal{B}}}^{\mathbf{f}}\left(\{(\mathbf{x}_i, \mathbf{r}_i)\}_{i=1}^N\right)$  represent the ensemble of design variables associated to the Pareto Front,  $\mathcal{P}$ ,  $\tilde{\mathcal{P}}$  and  $\tilde{\mathcal{P}}_{\mathcal{B}}$ , respectively.

**Remark**  $\mathcal{X}_{\tilde{\mathcal{P}}_{\mathcal{B}}}^{\mathbf{f}}\left(\{(\mathbf{x}_i, \mathbf{r}_i)\}_{i=1}^N\right)$  can be explicitly defined as follows:

$$\mathbf{x}_i \in \mathcal{X}_{\tilde{\mathcal{P}}_{\mathcal{B}}}^{\mathbf{f}}\left(\{(\mathbf{x}_i, \mathbf{r}_i)\}_{i=1}^N\right) \iff \forall k \in \mathcal{I}_1^N, \exists j \in \mathcal{I}_1^m, \pm f_j(\mathbf{x}_i) - r_{ij} < \pm f_j(\mathbf{x}_k) + r_{kj}. \quad (3)$$

**Proposition 1.** Note that  $\forall \{(\mathbf{x}_i, \mathbf{r}_i)\}_{i=1}^N \in (\mathcal{X} \times \mathbb{R}^m)^N$ ,

$$\mathcal{X}_{\tilde{\mathcal{P}}}^{\mathbf{f}}\left(\{\mathbf{x}_i\}_{i=1}^N\right) \subseteq \{\mathbf{x}_i\}_{i=1}^N \text{ and } \mathcal{X}_{\tilde{\mathcal{P}}_{\mathcal{B}}}^{\mathbf{f}}\left(\{(\mathbf{x}_i, \mathbf{r}_i)\}_{i=1}^N\right) \subseteq \{\mathbf{x}_i\}_{i=1}^N$$

The proof is provided in A.

Note that the inclusion becomes strict whenever a design does not satisfy the non-domination condition.

Finally, two last sets are introduced,  $\mathcal{P}_c$  and  $\tilde{\mathcal{P}}_c$ , which represent the continuous extension of  $\mathcal{P}$  and  $\tilde{\mathcal{P}}$ , respectively. These sets contains all the dominated but not strictly dominated points of  $\mathbb{R}^m$  with respect to  $\mathcal{P}$  and  $\tilde{\mathcal{P}}$ .

**Definition 8.** With  $\mathcal{A} \subset \mathbb{R}^m$  and  $\forall i, \mathbf{a}_i \in \mathbb{R}^m$ , let us define  $\mathcal{P}_c$  and  $\tilde{\mathcal{P}}_c$ , as follows

$$\begin{aligned}\mathcal{P}_c(\mathcal{A}) &= \mathcal{P}(\mathcal{A}) \cup \{\mathbf{b} \in \mathbb{R}^m \mid \exists \mathbf{z} \in \mathcal{P}(\mathcal{A}), \mathbf{z} \succ \mathbf{b} \ \& \ \nexists \mathbf{z} \in \mathcal{P}(\mathcal{A}), \mathbf{z} \succ \mathbf{b}\} \\ \tilde{\mathcal{P}}_c\left(\{\mathbf{a}_i\}_{i=1}^N\right) &= \tilde{\mathcal{P}}\left(\{\mathbf{a}_i\}_{i=1}^N\right) \cup \left\{\mathbf{b} \in \mathbb{R}^m \mid \exists \mathbf{z} \in \tilde{\mathcal{P}}\left(\{\mathbf{a}_i\}_{i=1}^N\right), \mathbf{z} \overset{\mathcal{P}}{\succ} \mathbf{b} \ \& \ \nexists \mathbf{z} \in \tilde{\mathcal{P}}\left(\{\mathbf{a}_i\}_{i=1}^N\right), \mathbf{z} \overset{\mathcal{P}}{\succ} \mathbf{b}\right\}\end{aligned}$$

Usually, as for Def. 6,  $\mathcal{A} \equiv \mathbf{f}(\mathcal{X})$  and  $\mathcal{P}_c(\mathcal{A})$  is denoted as  $\mathcal{P}_c$ .

### 3.1.1 Mathematical formulation for error-based boxes

As stated in Section 2, the proposed framework is developed in the context of approximated objective functions, assuming that the error can be estimated and potentially reduced. The goal of this section is to provide some definitions and to introduce the so-called Bounding-Box recursive strategy.

**Definition 9.** An error is assumed on the computation of each objective function  $f_j$ . The approximated result  $\tilde{f}_j$  differs from the function  $f_j$  by an additive error, as follows:

$$\forall \mathbf{x} \in \mathcal{X}, \forall j \in \mathcal{I}_1^m, f_j(\mathbf{x}) = \tilde{f}_j(\mathbf{x}) + \varepsilon_j(\mathbf{x})$$

**Assumption 1.** Here, let us assume that a conservative error  $\tilde{\varepsilon}$  can be computed on  $\tilde{\mathbf{f}}$ , meaning that:

$$\forall \mathbf{x} \in \mathcal{X}, \forall j \in \mathcal{I}_1^m, |\varepsilon_j(\mathbf{x})| \leq \tilde{\varepsilon}_j(\mathbf{x}).$$

Then,  $\forall \mathbf{x} \in \mathcal{X}, \mathbf{f}(\mathbf{x}) \in \mathcal{B}_{\tilde{\mathbf{f}}}(\mathbf{x}, |\boldsymbol{\varepsilon}(\mathbf{x})|) \subseteq \mathcal{B}_{\tilde{\mathbf{f}}}(\mathbf{x}, \tilde{\boldsymbol{\varepsilon}}(\mathbf{x}))$ .

As shown in the following, assuming that this error is conservative is necessary for demonstrating the convergence of the proposed approach.

It is assumed that  $\mathbf{f}$  can be potentially improved by reducing the error through a refinement procedure. This conservative error is supposed to converge to zero when the refinement tends to infinity.

**Assumption 2.** The sequence  $(\tilde{\boldsymbol{\varepsilon}}^k)_{k \in \mathbb{N}_+}$  is considered in the following, where  $k$  represent the "degree of refinement". No constraint is assigned on its monotony but it is assumed that:

$$\forall j \in \mathcal{I}_1^m, \lim_{k \rightarrow +\infty} \tilde{\varepsilon}_j^k = 0.$$

Then, this states that an infinite refinement would lead to an error equal to zero. Note that in engineering applications, this refinement is often the biggest source of computational cost.

The goal of this error-driven optimization is to obtain a refined discrete Pareto-front, where the output of interest is  $\mathcal{X}_{\tilde{\mathcal{P}}}^{\mathbf{f}}$ . Note that the direct computation of this set is impossible without the knowledge of  $\mathbf{f}$ , then this set is approximated by means of sets which can be refined, basing only on  $\tilde{\mathbf{f}}^k$  and  $\tilde{\boldsymbol{\varepsilon}}^k$ .

Let us now define the classical and recursive (BB) approaches, which are introduced to obtain an approximation of the refined discrete front. Note that the following development is written in the design space, but can be easily extended to the objective space.

**Definition 10.** For  $\{\mathbf{x}_i\}_{i=1}^N \in \mathcal{X}^N$ ,  $\mathbf{f}$  the objective functions and  $\tilde{\mathbf{f}}^k$  refinable approximations,  $\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}, \text{classical}}^{\mathbf{f}, 0} = \tilde{\mathcal{X}}_{\tilde{\mathcal{P}}, \text{BB}}^{\mathbf{f}, 0} = \{\mathbf{x}_i\}_{i=1}^N$  and the associated initial conservative errors  $\{\tilde{\boldsymbol{\varepsilon}}^0(\mathbf{x}_i)\}_{i=1}^N$ , let us introduce a decreasing sequence  $(N^k)_{k \in \mathbb{N}_+}$ , with  $N^0 = N$ , so that we define  $\forall k \in \mathbb{N}_+$ :

$$\begin{aligned} \tilde{\mathcal{X}}_{\tilde{\mathcal{P}}, \text{classical}}^{\mathbf{f}, k+1} &= \mathcal{X}_{\tilde{\mathcal{P}}_{\mathcal{B}}}^{\tilde{\mathbf{f}}^k} \left( \left\{ \left( \tilde{\mathcal{X}}_{\tilde{\mathcal{P}}, \text{classical}_i}^{\mathbf{f}, 0}, \tilde{\boldsymbol{\varepsilon}}^k \left( \tilde{\mathcal{X}}_{\tilde{\mathcal{P}}, \text{classical}_i}^{\mathbf{f}, 0} \right) \right) \right\}_{i=1}^{N^k} \right) \\ \tilde{\mathcal{X}}_{\tilde{\mathcal{P}}, \text{BB}}^{\mathbf{f}, k+1} &= \mathcal{X}_{\tilde{\mathcal{P}}_{\mathcal{B}}}^{\tilde{\mathbf{f}}^k} \left( \left\{ \left( \tilde{\mathcal{X}}_{\tilde{\mathcal{P}}, \text{BB}_i}^{\mathbf{f}, k}, \tilde{\boldsymbol{\varepsilon}}^k \left( \tilde{\mathcal{X}}_{\tilde{\mathcal{P}}, \text{BB}_i}^{\mathbf{f}, k} \right) \right) \right\}_{i=1}^{N^k} \right) \end{aligned}$$

As the Bounding-Box approach is developed in the following, the notations are slightly lightened as follows:

$$\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f}, k+1} = \mathcal{X}_{\tilde{\mathcal{P}}_{\mathcal{B}}}^{\tilde{\mathbf{f}}^k} \left( \left\{ \left( \tilde{\mathcal{X}}_{\tilde{\mathcal{P}}_i}^{\mathbf{f}, k}, \tilde{\boldsymbol{\varepsilon}}^k \left( \tilde{\mathcal{X}}_{\tilde{\mathcal{P}}_i}^{\mathbf{f}, k} \right) \right) \right\}_{i=1}^{N^k} \right) \quad (4)$$

where, as it might be expected,  $\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k}$  represents the approximation of  $\mathcal{X}_{\tilde{\mathcal{P}}}^{\mathbf{f}}$  at the  $k^{\text{th}}$  refinement of the boxes, by using  $\tilde{\mathbf{f}}^k$  instead of  $\mathbf{f}$ . Note that this set remains a boxed Pareto-optimal set.

The so-called classical approach is then the double-loop or nested optimization, *i.e.* all boxes are fully refined before checking the Pareto-optimality. The difference with the Bounding-Box approach lies in the recursive aspect of the BB sequence, which is of major importance. As seen in Assumption 2, the most time-consuming step consists in decreasing the conservative error  $\tilde{\varepsilon}^k$ . In the Bounding-Box approach, Proposition 1 states that  $\forall k \in \mathbb{N}_+$ ,  $\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k} \subseteq \tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k-1} \subseteq \dots \subseteq \tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},0}$ . Then, as seen in the proof of this proposition, any non-satisfied non-domination condition implies a strict inclusion between two sets of the sequence. Hence, if any box is dominated, the size  $N^k$  ( $= \text{Card}(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k})$ ) of the set of interest starts decreasing, meaning that the refinement of  $\tilde{\varepsilon}^k$  is required only on  $N^k < N$  designs. On the contrary, with the classical approach, the  $N$  designs are refined at each iteration.

The next section is devoted to the convergence of the BB set sequence.

### 3.1.2 Convergence analysis

This section aims to demonstrate that the sequence  $(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k})_{k \in \mathbb{N}_+}$  theoretically converge to the real Pareto-optimal set. More precisely, the goal is to demonstrate the convergence of the continuous Pareto front  $\tilde{\mathcal{P}}_c$  built on the approximated measures  $\tilde{\mathbf{f}}^k(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k})$  toward the real continuous front  $\mathcal{P}_c$ .

To this extent, one more assumption is needed: it is assumed that there is no singular value in the Pareto-optimal set associated to the objective functions  $\mathbf{f}$ . Then, the designs  $\mathbf{x}_i$  are assumed to be spread enough around the optimal set when  $N$  increases.

**Assumption 3.** For a given multi-objective optimization, it is assumed that:

$$\forall \mathbf{y} \in \mathcal{X}_{\tilde{\mathcal{P}}}^{\mathbf{f}}, \exists \mathcal{D} \subseteq \mathcal{X} \text{ so that } \mathbf{y} \in \mathcal{D}, \tilde{\mathcal{D}} \neq \emptyset \text{ and } \forall j \in \mathcal{I}_1^m, f_j \in \mathcal{C}^0(\mathcal{D}).$$

Let  $\mathcal{D}(\mathbf{y})$  be a set satisfying the previous condition for a given  $\mathbf{y} \in \mathcal{X}$ .

$$\forall \epsilon \in \mathbb{R}_+^*, \exists M \in \mathbb{N}_+, \forall \mathbf{y} \in \mathcal{X}_{\tilde{\mathcal{P}}}^{\mathbf{f}}, \exists k \in \mathcal{I}_1^M, \mathbf{x}_k \in \mathcal{D}(\mathbf{y}), \|\mathbf{x}_k - \mathbf{y}\| \leq \epsilon.$$

This notably traduces the fact that the optimizer converges on the whole Pareto front to refine the discrete front on the integrity of  $\mathcal{X}_{\tilde{\mathcal{P}}}^{\mathbf{f}}$  and that there exists a non-empty part of  $\mathcal{X}$  around each efficient design where the restriction of all  $f_j$  to it are continuous.

**Remarks** The following relation is trivial:

$$\mathcal{X}_{\tilde{\mathcal{P}}}^{\mathbf{f}}(\mathcal{X}_{\tilde{\mathcal{P}}}^{\mathbf{f}}(\{\mathbf{x}_i\}_{i=1}^N)) = \mathcal{X}_{\tilde{\mathcal{P}}}^{\mathbf{f}}(\{\mathbf{x}_i\}_{i=1}^N).$$

More generally,  $\forall \mathcal{S}$  s.t.  $\mathcal{X}_{\tilde{\mathcal{P}}}^{\mathbf{f}}(\{\mathbf{x}_i\}_{i=1}^N) \subseteq \mathcal{S} \subseteq \{\mathbf{x}_i\}_{i=1}^N$ ,

$$\mathcal{X}_{\tilde{\mathcal{P}}}^{\mathbf{f}}(\mathcal{S}) = \mathcal{X}_{\tilde{\mathcal{P}}}^{\mathbf{f}}(\{\mathbf{x}_i\}_{i=1}^N). \quad (5)$$

Note that the same can be said for the Pareto fronts  $\tilde{\mathcal{P}}$ .

**Proposition 2.**  $\forall \mathbf{y} \in \{\mathbf{x}_i\}_{i=1}^N, \mathbf{y} \notin \mathcal{X}_{\tilde{\mathcal{P}}}^{\mathbf{f}}(\{\mathbf{x}_i\}_{i=1}^N) \iff \exists \mathbf{y}' \in \mathcal{X}_{\tilde{\mathcal{P}}}^{\mathbf{f}}(\{\mathbf{x}_i\}_{i=1}^N), \mathbf{y}' \succ \mathbf{y}$ .

The proof is given in B.

**Lemma 1.**  $\forall \{(\mathbf{x}_i, \tilde{\varepsilon}(\mathbf{x}_i))\}_{i=1}^N \in (\mathcal{X} \times \mathbb{R}^m)^N$ ,  $\mathcal{X}_{\tilde{\mathcal{P}}}^f(\{\mathbf{x}_i\}_{i=1}^N) \subseteq \mathcal{X}_{\tilde{\mathcal{P}}_B}^f(\{(\mathbf{x}_i, \tilde{\varepsilon}(\mathbf{x}_i))\}_{i=1}^N)$

The proof is given in C.

The Proposition 2 states that any dominated design is dominated by at least one non-dominated design, which seems reasonable, as there has to be at least one dominant design in a given set. Lemma 1 introduces one of the main advantages of the Bounding-Box strategy by assessing the robustness of the boxed dominance compared to the classical one. Following a boxed approach, each non-boxed efficient design is retained as boxed efficient.

A novelty of this paper compared to Fusi's Bounding-Box approach in [35] is the possibility to impose a precision threshold for the convergence of the objective functions. In [35], *exact* estimations for non-dominated designs are computed (*i.e.* the associated error is considered equal to zero), while here, non-dominated boxes are only converged to a given user-defined threshold. Finally, this boxed Pareto front features a robust behavior, since it retains all the truly non-dominated designs, by adding only some almost efficient ones.

**Theorem 1.** For any initial set  $\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},0} = \{\mathbf{x}_i\}_{i=1}^N \in \mathcal{X}^N$ , approximation sequence  $(\tilde{\mathbf{f}}^k)_{k \in \mathbb{N}_+}$  and conservative error set sequence  $(\tilde{\varepsilon}^k)_{k \in \mathbb{N}_+}$ , the robustness is verified  $\forall k \in \mathbb{N}_+$ , *i.e.*:

$$\mathcal{X}_{\tilde{\mathcal{P}}}^f(\{\mathbf{x}_i\}_{i=1}^N) \subseteq \tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k} \quad (6)$$

**Proof.** For proving this, mathematical induction can be used.

Let us assume that  $\exists k \in \mathbb{N}_+$ , so that  $\mathcal{X}_{\tilde{\mathcal{P}}}^f(\{\mathbf{x}_i\}_{i=1}^N) \subseteq \tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k}$ . Then  $\forall \mathbf{x} \in \tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k}$ :

- if  $\mathbf{x} \in \mathcal{X}_{\tilde{\mathcal{P}}}^f(\{\mathbf{x}_i\}_{i=1}^N)$ ,  $\nexists \mathbf{y} \in \mathcal{X}$ , so that  $\mathbf{y} \succ \mathbf{x}$ . Hence,  $\nexists \mathbf{y} \in \tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k} \subseteq \mathcal{X}$ , so that  $\mathbf{y} \succ \mathbf{x}$ , and therefore  $\mathbf{x} \in \mathcal{X}_{\tilde{\mathcal{P}}}^f(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k})$ ;
- if  $\mathbf{x} \notin \mathcal{X}_{\tilde{\mathcal{P}}}^f(\{\mathbf{x}_i\}_{i=1}^N)$ , with Proposition 2,  $\exists \mathbf{y} \in \mathcal{X}_{\tilde{\mathcal{P}}}^f(\{\mathbf{x}_i\}_{i=1}^N) \subseteq \tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k}$ , so that  $\mathbf{y} \succ \mathbf{x}$ , therefore,  $\mathbf{x} \notin \mathcal{X}_{\tilde{\mathcal{P}}}^f(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k})$ .

which means that:

$$\mathcal{X}_{\tilde{\mathcal{P}}}^f(\{\mathbf{x}_i\}_{i=1}^N) = \mathcal{X}_{\tilde{\mathcal{P}}}^f(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k}) \quad (7)$$

Finally, Lemma 1 gives  $\mathcal{X}_{\tilde{\mathcal{P}}}^f(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k}) \subseteq \mathcal{X}_{\tilde{\mathcal{P}}_B}^f(\{(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k}, \tilde{\varepsilon}^k(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k}))\}_{i=1}^N) = \tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k+1}$ , which ends the inductive step of the proof, yielding  $\mathcal{X}_{\tilde{\mathcal{P}}}^f(\{\mathbf{x}_i\}_{i=1}^N) \subseteq \tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k+1}$ .

Of course,  $\mathcal{X}_{\tilde{\mathcal{P}}}^f(\{\mathbf{x}_i\}_{i=1}^N) \subseteq \{\mathbf{x}_i\}_{i=1}^N = \tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},0}$ , therefore, the mathematical induction proves that the robustness inclusion is verified.  $\square$

This first theorem illustrates how the framework may be used in practice. To further reduce computational cost, non-dominated boxes are converged to a non-zero width, while still retaining every efficient design. This low-cost Pareto-front approximation allows a fast analysis and can be supplemented by refined approximations in post-processing if needed.

Note that the convergence of the discrete approximated front extension  $\tilde{\mathcal{P}}_c(\tilde{\mathbf{f}}^k(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k}))$  toward the real continuous front  $\mathcal{P}_c$  has still to be proven. This would imply the convergence of the preimage, because of the hypothesis of continuity related to the Assumption 3.

**Theorem 2.** For any initial set  $\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},0} \in \mathcal{X}^N$ , approximation sequence  $(\tilde{\mathbf{f}}^k)_{k \in \mathbb{N}_+}$  and conservative error set sequence  $(\tilde{\boldsymbol{\varepsilon}}^k)_{k \in \mathbb{N}_+}$ , the following convergence is verified:

$$\lim_{(N,k) \rightarrow (+\infty, +\infty)} d_H\left(\tilde{\mathcal{P}}_c(\tilde{\mathbf{f}}^k(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k})), \mathcal{P}_c\right) = 0 \quad (8)$$

with the classical Hausdorff distance denoted as  $d_H$ .

**Proof.** The triangle inequality gives :

$$d_H\left(\tilde{\mathcal{P}}_c(\tilde{\mathbf{f}}^k(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k})), \mathcal{P}_c\right) \leq d_H\left(\tilde{\mathcal{P}}_c(\tilde{\mathbf{f}}^k(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k})), \tilde{\mathcal{P}}_c(\mathbf{f}(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k}))\right) + d_H\left(\tilde{\mathcal{P}}_c(\mathbf{f}(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k})), \mathcal{P}_c\right). \quad (9)$$

Assumption 1 implies that  $\forall \mathbf{x} \in \tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k}$ ,  $d_\infty(\mathbf{f}(\mathbf{x}), \tilde{\mathbf{f}}^k(\mathbf{x})) \leq \max_j \tilde{\boldsymbol{\varepsilon}}_j^k(\mathbf{x})$ .

Now, let us suppose that  $\exists \mathbf{a} \in \tilde{\mathcal{P}}_c(\mathbf{f}(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k}))$ , so that  $d_\infty(\mathbf{a}, \tilde{\mathcal{P}}_c(\tilde{\mathbf{f}}^k(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k}))) > \max_{(i,j)} \tilde{\boldsymbol{\varepsilon}}_j^k(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}_i}^{\mathbf{f},k})$ .

This means that  $\tilde{\mathcal{P}}_c(\tilde{\mathbf{f}}^k(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k})) \cap (\mathbf{a}, \tilde{\boldsymbol{\varepsilon}}_{max}) = \emptyset$  with  $\tilde{\boldsymbol{\varepsilon}}_{max}$  being the  $m$ -dimensional vector where each component is equal to  $\max_{(i,j)} \tilde{\boldsymbol{\varepsilon}}_j^k(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}_i}^{\mathbf{f},k})$ .

Therefore, from Def. 8, i) whether  $\exists i \in \mathcal{I}_1^N$ , then  $(\tilde{\mathbf{f}}^k(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}_i}^{\mathbf{f},k}), 0) \succ_{\mathcal{B}} (\mathbf{a}, \tilde{\boldsymbol{\varepsilon}}_{max})$  or ii)  $\forall \mathbf{a}' \in (\mathbf{a}, \tilde{\boldsymbol{\varepsilon}}_{max})$ ,  $\nexists i \in \mathcal{I}_1^N$ , so that  $\tilde{\mathbf{f}}^k(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}_i}^{\mathbf{f},k}) \succ \mathbf{a}'$ .

- In the first case i), the dominance can be formulated as follows:  $\exists j \in \mathcal{I}_1^N$ , so that  $(\tilde{\mathbf{f}}^k(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}_j}^{\mathbf{f},k}), \tilde{\boldsymbol{\varepsilon}}_{max}) \succ_{\mathcal{B}} (\mathbf{a}, 0)$  and as  $\mathbf{a} \in \tilde{\mathcal{P}}_c(\mathbf{f}(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k}))$ ,  $\nexists i \in \mathcal{I}_1^N$ ,  $\mathbf{f}(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}_i}^{\mathbf{f},k}) \succ \mathbf{a}$ . Therefore, it can be inferred that  $\nexists i \in \mathcal{I}_1^N$ ,  $\mathbf{f}(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}_i}^{\mathbf{f},k}) \in (\tilde{\mathbf{f}}^k(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}_j}^{\mathbf{f},k}), \tilde{\boldsymbol{\varepsilon}}_{max})$ . However, this would mean  $\exists \mathbf{x} \in \tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k}$ ,  $d_\infty(\mathbf{f}(\mathbf{x}), \tilde{\mathbf{f}}^k(\mathbf{x})) > \tilde{\boldsymbol{\varepsilon}}_{max_i} > \max_j \tilde{\boldsymbol{\varepsilon}}_j^k(\mathbf{x})$ , which is contradictory with Assumption 1.
- The second case ii) implies that  $\exists \mathbf{b} \in \tilde{\mathcal{P}}_c(\tilde{\mathbf{f}}^k(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k}))$ , so that  $(\mathbf{a}, \tilde{\boldsymbol{\varepsilon}}_{max}) \succ_{\mathcal{B}} (\mathbf{b}, 0)$ . However,  $\exists j \in \mathcal{I}_1^N$ , so that  $\mathbf{f}(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}_j}^{\mathbf{f},k}) \succ \mathbf{a}$ . Therefore, it follows  $(\mathbf{f}(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}_j}^{\mathbf{f},k}), \tilde{\boldsymbol{\varepsilon}}_{max}) \succ_{\mathcal{B}} (\mathbf{b}, 0)$ . As  $\mathbf{b} \in \tilde{\mathcal{P}}_c(\tilde{\mathbf{f}}^k(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k}))$ , from Def. 8, it follows that  $\nexists i \in \mathcal{I}_1^N$ , so that  $\tilde{\mathbf{f}}^k(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}_i}^{\mathbf{f},k}) \succ \mathbf{b}$ , hence,  $\nexists i \in \mathcal{I}_1^N$ ,  $\tilde{\mathbf{f}}^k(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}_i}^{\mathbf{f},k}) \in (\mathbf{f}(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}_j}^{\mathbf{f},k}), \tilde{\boldsymbol{\varepsilon}}_{max})$ , which contradicts again Assumption 1.

Finally, it has been proven that  $\forall \mathbf{a} \in \tilde{\mathcal{P}}_c(\mathbf{f}(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k}))$ , it follows that  $d_\infty(\mathbf{a}, \tilde{\mathcal{P}}_c(\tilde{\mathbf{f}}^k(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k}))) \leq \max_{(i,j)} \tilde{\boldsymbol{\varepsilon}}_j^k(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}_i}^{\mathbf{f},k})$  by contradiction. This statement holds also when inverting the Pareto front continuous sets (real and approximated), and this can be proved in the same way. As a consequence, the Hausdorff distance can be written as follows:

$$d_H\left(\tilde{\mathcal{P}}_c(\tilde{\mathbf{f}}^k(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k})), \tilde{\mathcal{P}}_c(\mathbf{f}(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k}))\right) \leq \max_{(i,j)} \tilde{\boldsymbol{\varepsilon}}_j^k(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}_i}^{\mathbf{f},k}).$$

Hence, Assumption 2 implies that:

$$\lim_{k \rightarrow \infty} d_H\left(\tilde{\mathcal{P}}_c(\tilde{\mathbf{f}}^k(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k})), \tilde{\mathcal{P}}_c(\mathbf{f}(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k}))\right) = 0 \quad (10)$$

Let us focus now on the second part of the sum 9.



Of course,  $\forall \mathbf{a} \in \tilde{\mathcal{P}}_c(\mathbf{f}(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k}))$ ,  $\exists \mathbf{a}' \in \mathcal{P}_c$ , so that  $\mathbf{a}' \succ \mathbf{a}$  (or  $\mathbf{a}' = \mathbf{a}$ ). Moreover,  $\forall \mathbf{b} \in \mathbb{R}^m$  such that  $\exists \mathbf{a}' \in \mathcal{P}$ ,  $\mathbf{a}' \succ \mathbf{b}$ , Assumption 3 with Theorem 1 provides evidence that the recursive discrete efficient set converges toward the continuous real one and that this efficient set is included in  $\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k}$ . In other words,  $\forall k \in \mathbb{N}, \exists M \in \mathbb{N}^*, \exists l \in \mathcal{I}_1^M, s.t. \forall j \in \mathcal{I}_1^m, |a'_j - f_j(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}_l}^{\mathbf{f},k})| < |a'_j - b_j|$ . Thus,  $\forall k \in \mathbb{N}, \exists M \in \mathbb{N}^*, \exists l \in \mathcal{I}_1^M, \mathbf{f}(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}_l}^{\mathbf{f},k}) \succ \mathbf{b}$ . Hence,  $\tilde{\mathcal{P}}_c$  is always dominated by  $\mathcal{P}_c$  and any element dominated by  $\mathcal{P}_c$  is dominated by  $\tilde{\mathcal{P}}_c$  after a sufficient number of iterations. From Definition 8, it can be deduced that:

$$\lim_{N \rightarrow \infty} d_H(\tilde{\mathcal{P}}_c(\mathbf{f}(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k})), \mathcal{P}_c) = 0. \quad (11)$$

Finally, by combining Equations 9, 10 and 11, it comes:

$$\lim_{(N,k) \rightarrow (+\infty, +\infty)} d_H(\tilde{\mathcal{P}}_c(\tilde{\mathbf{f}}^k(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f},k})), \mathcal{P}_c) = 0,$$

which ends the proof.  $\square$

### 3.1.3 Some comments on the Bounding-Box strategy

From previous findings, it can be observed that:

- The convergence of the tightest boxes is not required here as the concept of boxes is robustly extended to Pareto optima.
- The convergence analysis has been carried out, by building continuous fronts on discrete sets of designs.

Note that the notations introduced in the previous paragraphs are well designed to be used in the effective algorithms.

One of the main conclusions of [35] was a significant drop in the performances of the Bounding-Box strategy throughout the execution, as the designs converge toward the Pareto optima. This observation has been the starting point of the additional feature of the framework, which is presented in the next section.

## 3.2 Surrogate-assisted model

### 3.2.1 Principles and mathematical proof

In [35], boxes are rarely dominated while being poorly refined, as for the first designs, because of the convergence toward  $\mathcal{X}_{\mathcal{P}}^{\mathbf{f}}$ . Fusi's strategy behaves at the last iterations of the algorithm as a classical optimization with tight refinement on every designs.

The targeted strategy here is to bypass the full computation of the objective functions by generating a surrogate model as a substitute for costly evaluations (when the error is low enough). Within the Bounding-Box approach, the evolutive surrogate model is more and more used throughout the execution as its accuracy improves each time some objective functions are computed. During the last iterations of the optimization, the surrogate model is massively used, since it should be very accurate in the optimal area near the Pareto front. Therefore, this approach should allow a faster estimation and densification of the Pareto front.

The evolutive surrogate model built on  $\mathbf{f}$  is noted as  $\tilde{\mathbf{f}}_d$ , with an associated conservative error  $\tilde{\boldsymbol{\varepsilon}}_d$ . Moreover, as these approximations depend on the amount of information known about  $\mathbf{f}$ , they are noted as  $\tilde{\mathbf{f}}_d^l$  and  $\tilde{\boldsymbol{\varepsilon}}_d^l$ , with  $l \in \mathbb{N}$  describing this amount of information, for example the number of training points used to build the metamodel. The following is then assumed:

**Assumption 4.** Given a function  $\mathbf{f}$ , both the surrogate model built on  $\mathbf{f}$  and its conservative error converge as follows:

$$\begin{aligned}\lim_{l \rightarrow \infty} \tilde{\mathbf{f}}_d^l &= \mathbf{f}, \\ \lim_{l \rightarrow \infty} \tilde{\boldsymbol{\varepsilon}}_d^l &= \mathbf{0}.\end{aligned}$$

For each level  $l$ , the surrogate model  $\tilde{\mathbf{f}}_d^l$  is used for the designs satisfying  $\forall j \in \mathcal{I}_1^m, \tilde{\boldsymbol{\varepsilon}}_{d_j}^l \leq s_{1_j}$  with  $s_{1_j} \in \mathbb{R}_+$  a threshold fixed by the user. This criterion is noted as  $\tilde{\boldsymbol{\varepsilon}}_d^l \leq \mathbf{s}_1$  in the following.

**Remark** It is important to note that the designs  $\mathbf{x}_i$  are added at a certain level  $l_{in} = l_{in}(\mathbf{x}_i)$  depending on the design. To lighten the notation, it is assumed that  $\tilde{\mathbf{f}}_d^{l_{in}}(\mathbf{x}_i) = \tilde{\mathbf{f}}_d^{l_{in}(\mathbf{x}_i)}(\mathbf{x}_i)$ ,  $\tilde{\boldsymbol{\varepsilon}}_d^{l_{in}}(\mathbf{x}_i) = \tilde{\boldsymbol{\varepsilon}}_d^{l_{in}(\mathbf{x}_i)}(\mathbf{x}_i)$ .

**Definition 11.** The objective function used in the following is a sum of a given  $\mathbf{f}_{opt}$  and the surrogate-based approximation:

$$\mathbf{f}_{OBJ} = \mathbf{f}_{opt} \cdot \mathbb{1}_{\tilde{\boldsymbol{\varepsilon}}_d^{l_{in}} > \mathbf{s}_1} + \tilde{\mathbf{f}}_d^{l_{in}} \cdot \mathbb{1}_{\tilde{\boldsymbol{\varepsilon}}_d^{l_{in}} \leq \mathbf{s}_1}$$

Note that  $\mathbf{f}_{opt}$  represents the original (not surrogate-assisted) objective function.

**Proposition 3.** Given the threshold  $\mathbf{s}_1$ , the Pareto optima of the non-boxed real problem are included in the following boxed Pareto optima:

$$\forall \{\mathbf{x}_i\}_{i=1}^N \in \mathcal{X}^N, \mathcal{X}_{\tilde{\mathcal{P}}}^{\mathbf{f}_{opt}}(\{\mathbf{x}_i\}_{i=1}^N) \subseteq \mathcal{X}_{\tilde{\mathcal{P}}_B}^{\mathbf{f}_{OBJ}}(\{(\mathbf{x}_i, \mathbf{r}(\mathbf{x}_i))\}_{i=1}^N)$$

with  $\mathbf{s}_1 = \mathbf{0}$  implying equality and  $\mathbf{r}$  defined from one of the following definitions:

$$\begin{aligned}\text{Rough} : \mathbf{r}(\mathbf{x}_i) &= \mathbf{s}_1; \\ \text{Refined} : \mathbf{r}(\mathbf{x}_i) &= \tilde{\boldsymbol{\varepsilon}}_d^{l_{in}}(\mathbf{x}_i); \\ \text{Accurate} : \mathbf{r}(\mathbf{x}_i) &= \tilde{\boldsymbol{\varepsilon}}_d^{l_{in}}(\mathbf{x}_i) \cdot \mathbb{1}_{\tilde{\boldsymbol{\varepsilon}}_d^{l_{in}}(\mathbf{x}_i) \leq \mathbf{s}_1}(\mathbf{x}_i).\end{aligned}$$

The proof is provided in D.

However, the surrogate model can not be directly built on the objective functions since they are affected by some errors. More precisely, given a threshold  $\mathbf{s}_2$ , an efficient design  $\mathbf{x} \in \mathcal{X}$  will converge up to this threshold, while a dominated one will only converge up to dominance. Let  $k_{min}$  (Note that  $k_{min}$  is a simplified notation of  $k_{min}(\mathbf{x})$ ) be the smallest integer so that the Bounding-Box conservative error  $\tilde{\boldsymbol{\varepsilon}}^{k_{min}}(\mathbf{x})$  is either converged to  $\mathbf{s}_2$  or dominated.

Then,  $\mathbf{f}_{OBJ}$  can be defined with  $\mathbf{f}_{opt} = \tilde{\mathbf{f}}^{k_{min}}$ , which provides the coupled objective function that will be used in the following sections:

$$\mathbf{f}_{OBJ} = \tilde{\mathbf{f}}^{k_{min}} \cdot \mathbb{1}_{\tilde{\boldsymbol{\varepsilon}}_d^{k_{min}} > \mathbf{s}_1} + \tilde{\mathbf{f}}_d^{l_{in}} \cdot \mathbb{1}_{\tilde{\boldsymbol{\varepsilon}}_d^{l_{in}} \leq \mathbf{s}_1} \quad (12)$$

**Remark** Note that  $\tilde{\mathbf{f}}^{k_{min}}$  and  $\tilde{\boldsymbol{\varepsilon}}^{k_{min}}$  are extended here to the entire domain  $\mathcal{X}$ , while their value have only been computed on some of the training points  $\mathbf{x}_i$ . However, there would always exist a refinement  $k_{min}$  which would satisfy one of the two criteria seen above (from Assumption

2). Therefore, there is a tricky choice to make on how to extend  $\tilde{\boldsymbol{\varepsilon}}^{k_{min}}(\mathbf{x})$ , or to compute a conservative approximation  $\tilde{\mathbf{s}}_2(\mathbf{x})$ , so that  $\forall \mathbf{x}$ ,

$$|\tilde{\mathbf{f}}^{k_{min}}(\mathbf{x}) - \mathbf{f}(\mathbf{x})| \leq \tilde{\boldsymbol{\varepsilon}}^{k_{min}}(\mathbf{x}) \leq \tilde{\mathbf{s}}_2(\mathbf{x}).$$

From a numerical point of view, because of the noise on the training points  $(\mathbf{x}_i, \tilde{\mathbf{f}}^{k_{min}}(\mathbf{x}_i))$  on which the evolutive surrogate model is built, a regression seems to be more accurate a-priori than an interpolation.

**Theorem 3.** *Given two user-defined thresholds  $\mathbf{s}_1$  and  $\mathbf{s}_2$ , the non-boxed discrete Pareto optima are included in the following boxed Pareto optima:*

$$\forall \{\mathbf{x}_i\}_{i=1}^N \in \mathcal{X}^N, \mathcal{X}_{\tilde{\mathcal{P}}}^{\mathbf{f}}(\{\mathbf{x}_i\}_{i=1}^N) \subseteq \mathcal{X}_{\tilde{\mathcal{P}}_{\mathbf{B}}}^{\mathbf{f}_{OBJ}}(\{(\mathbf{x}_i, \mathbf{r}(\mathbf{x}_i))\}_{i=1}^N),$$

where equality holds when  $\mathbf{s}_1 = \mathbf{s}_2 = \mathbf{0}$  and  $\tilde{\mathbf{s}}_2(\mathbf{x}) = \mathbf{s}_2$  for non-dominated designs, and where  $\mathbf{r}$  can represent a rough or accurate boxing, based on the threshold or the errors, as follows:

$$\begin{aligned} \text{Rough : } \mathbf{r}(\mathbf{x}_i) &= \mathbf{s}_1 + \tilde{\mathbf{s}}_2(\mathbf{x}_i); \\ \text{Accurate : } \mathbf{r}(\mathbf{x}_i) &= \tilde{\boldsymbol{\varepsilon}}^{k_{min}}(\mathbf{x}_i) + \tilde{\boldsymbol{\varepsilon}}_d^{l_{in}}(\mathbf{x}_i) \cdot \mathbb{1}_{\tilde{\boldsymbol{\varepsilon}}_d^{l_{in}}(\mathbf{x}_i) \leq \mathbf{s}_1}(\mathbf{x}_i). \end{aligned}$$

As observed before,  $\tilde{\boldsymbol{\varepsilon}}^{k_{min}}(\mathbf{x}_i)$  is only known when the whole computation is performed, therefore, a trade-off is defined using the conservative approximation, as follows:

$$\text{Trade-off : } \mathbf{r}(\mathbf{x}_i) = \tilde{\boldsymbol{\varepsilon}}^{k_{min}}(\mathbf{x}_i) \cdot \mathbb{1}_{\tilde{\boldsymbol{\varepsilon}}_d^{l_{in}}(\mathbf{x}_i) > \mathbf{s}_1}(\mathbf{x}_i) + (\tilde{\mathbf{s}}_2(\mathbf{x}_i) + \tilde{\boldsymbol{\varepsilon}}_d^{l_{in}}(\mathbf{x}_i)) \cdot \mathbb{1}_{\tilde{\boldsymbol{\varepsilon}}_d^{l_{in}}(\mathbf{x}_i) \leq \mathbf{s}_1}(\mathbf{x}_i)$$

**Proof.** The proof is quite straightforward and comes from the following inequalities:  
If  $\tilde{\boldsymbol{\varepsilon}}_d^{l_{in}}(\mathbf{x}_i) > \mathbf{s}_1$ ,

$$|\mathbf{f}_{OBJ}(\mathbf{x}_i) - \mathbf{f}(\mathbf{x}_i)| = |\tilde{\mathbf{f}}^{k_{min}}(\mathbf{x}_i) - \mathbf{f}(\mathbf{x}_i)| \leq \tilde{\boldsymbol{\varepsilon}}^{k_{min}}(\mathbf{x}_i) \leq \tilde{\mathbf{s}}_2(\mathbf{x}_i)$$

Else,

$$\begin{aligned} |\mathbf{f}_{OBJ}(\mathbf{x}_i) - \mathbf{f}(\mathbf{x}_i)| &\leq |\tilde{\mathbf{f}}_d^{l_{in}}(\mathbf{x}_i) - \tilde{\mathbf{f}}^{k_{min}}(\mathbf{x}_i)| + |\tilde{\mathbf{f}}^{k_{min}}(\mathbf{x}_i) - \mathbf{f}(\mathbf{x}_i)| \\ &\leq \tilde{\boldsymbol{\varepsilon}}_d^{l_{in}}(\mathbf{x}_i) + \tilde{\boldsymbol{\varepsilon}}^{k_{min}}(\mathbf{x}_i) \\ &\leq \tilde{\boldsymbol{\varepsilon}}_d^{l_{in}}(\mathbf{x}_i) + \tilde{\mathbf{s}}_2(\mathbf{x}_i) \\ &\leq \mathbf{s}_1 + \tilde{\mathbf{s}}_2(\mathbf{x}_i), \end{aligned}$$

which comes from Equation 12. □

### 3.2.2 Coupling with the Bounding-Box approach

While the Bounding-Box approach can be easily applied to any optimization with noisy outputs, the evolutive surrogate model requires to be iteratively built on the objective functions throughout the optimization process. Hence, using a parallel sampling method on the design space for the optimization would not benefit from the evolutive surrogate model construction. On the contrary, a genetic algorithm or any classical sequential optimization process would allow a refinement of the surrogate model at each generation or iteration of the optimization.

A sketch about the coupling is provided in Algorithm 1. Steps (a) to (c) are then detailed in the following sections.

---

**Algorithm 1** Algorithm overview

---

```

1: while Optimization running do
2:   Read new designs
3:   if Evolutive surrogate model precise enough for some designs then
4:     (a) Use surrogate to approximate objective functions
5:   else
6:     (b) Compute and (c) refine the boxed Pareto front to a given threshold
7:     Update the surrogate model
8:   end if
9: end while

```

---

**3.2.3 Some comments**

The evolutive surrogate model construction described in the previous section is well designed to supplement the Bounding-Box approach. We consider that the mathematical convergence of the coupled strategy can be easily demonstrated since the only difference is that the limit  $k \rightarrow +\infty$  becomes  $(\mathbf{s}_1, \mathbf{s}_2) \rightarrow (\mathbf{0}, \mathbf{0})$  by using the rough box size.

The drawback of the coupling with the evolutive surrogate is an increase of the box sizes associated to the designs whose values are computed with the surrogate. On the contrary, the main advantage is a significant decrease of the global computational cost, especially at the end of the execution, when the surrogate is considered converged toward the true function.

This strategy is well designed to be implemented as the variables  $k_{min}(\mathbf{x})$  and  $l_{in}(\mathbf{x})$  are fixed once computed, therefore  $\mathbf{f}_{OBJ}(\mathbf{x})$  and  $\mathbf{r}(\mathbf{x})$  have only to be stored after their computation.

In the following, a numerical algorithm is proposed and applied to some analytical test-cases in order to validate the framework, and get a first overview about the potential gains that could arise in industrial applications.

**4 Algorithm of the framework**

This section illustrates the numerical algorithm associated to the proposed framework.

As seen in the previous sections, two user-defined thresholds  $\mathbf{s}_1$  and  $\mathbf{s}_2$  have to be fixed. The value  $\mathbf{s}_1$  is compared to the evolutive surrogate error to determine whether the surrogate can be used. On the contrary,  $\mathbf{s}_2$  is compared to the error of approximation of the objective functions, providing the accuracy at which boxes are not refined anymore. Here, these thresholds are defined as a percentage of the objective functions' range, where this value is refined and updated throughout the optimization.

For illustrating the algorithm, three design sets  $\mathcal{D}$ ,  $\mathcal{D}_{new}$  and  $\mathcal{D}'$  are introduced. The set  $\mathcal{D}$  is initialized at the beginning of the algorithm, and contains all the non-dominated designs and is updated at each refinement of the objective functions. This set has been introduced in Definition 10, and is updated at the end of the execution, along with the associated boxes. The set  $\mathcal{D}_{new}$  represents the new designs at each optimization iteration. This set can contain several designs (e.g. a genetic algorithm's generation) or only one design (for sequential optimization). Finally, the set  $\mathcal{D}'$  contains the non-dominated designs of  $\mathcal{D}_{new}$ , whose values can be refined. In other words, at each optimization iteration, when a design is not approximated with the evolutive surrogate, it is assigned to  $\mathcal{D}'$  as long as it is non-dominated. As for  $\mathcal{D}$ , this set is updated every time an approximation is refined, but only  $\mathcal{D}'$  is reinitialized at each optimization iteration.

Then, the algorithm is straightforward and is provided in Alg. 2. The main steps are the following: i) in the subroutine `compareDesignMM`, if the error of the evolutive surrogate is low

enough to interpolate some designs, the associated  $\mathcal{D}'$  is filled and a first approximation for the other designs is computed; ii) the inner while loop refines each design to the  $s_2$  threshold one by one within the `iterateRefinement` subroutine and checks for new domination at each iteration; iii) finally, the evolutive surrogate is updated and new designs can be analyzed.

---

**Algorithm 2** Detailed pseudo-algorithm of the framework
 

---

```

1: Set  $s_1$  and  $s_2$ 
2: Initialize  $\mathcal{D}$  empty
3: while Optimization running do
4:   Read new designs  $\mathcal{D}_{new} = \{\mathbf{x}_i\}_{i=1}^N$ 
5:    $\mathcal{D} = \mathcal{D} \cup \mathcal{D}_{new}$ 
6:   Initialize  $\mathcal{D}'$  empty
7:   Call compareDesignMM( $\mathcal{D}_{new}, s_1, s_2, \tilde{\mathbf{f}}_d, \tilde{\boldsymbol{\varepsilon}}_d, \tilde{\mathbf{f}}^0, \tilde{\boldsymbol{\varepsilon}}^0, \mathbf{f}_{OBJ}, \mathbf{r}, \mathcal{D}'$ ) (Alg. 3)  $\triangleright (a,b)$ 
8:   Compute  $\mathcal{D} = \tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f}_{OBJ},0} = \mathcal{X}_{\tilde{\mathcal{P}}_B}^{\mathbf{f}_{OBJ}}((\mathcal{D}, \mathbf{r}(\mathcal{D})))$ 
9:    $\mathcal{D}' = \mathcal{D} \cap \mathcal{D}'$ 
10:   $k = 0$ 
11:  while  $\max(\tilde{\boldsymbol{\varepsilon}}^k(\mathbf{x})) > s_2, \mathbf{x} \in \mathcal{D}'$  do
12:     $k = k + 1$ 
13:    Select  $\mathbf{x}_r \equiv \mathbf{x}$  to refine in  $\mathcal{D}'$ 
14:    Call iterateRefinement( $\mathcal{D}', \mathbf{x}_r, \tilde{\mathbf{f}}^{k-1}, \tilde{\boldsymbol{\varepsilon}}^{k-1}, \tilde{\mathbf{f}}^k, \tilde{\boldsymbol{\varepsilon}}^k$ ) (Alg. 4)  $\triangleright (c)$ 
15:    for each  $\mathbf{x} \in \mathcal{D}'$  do
16:       $\mathbf{f}_{OBJ}(\mathbf{x}) = \tilde{\mathbf{f}}^k(\mathbf{x})$ 
17:       $\mathbf{r}(\mathbf{x}) = \tilde{\boldsymbol{\varepsilon}}^k(\mathbf{x})$ 
18:    end for
19:    Compute  $\mathcal{D} = \tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f}_{OBJ},k} = \mathcal{X}_{\tilde{\mathcal{P}}_B}^{\mathbf{f}_{OBJ}}\left(\left(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f}_{OBJ},k-1}, \mathbf{r}(\tilde{\mathcal{X}}_{\tilde{\mathcal{P}}}^{\mathbf{f}_{OBJ},k-1})\right)\right)$ 
20:     $\mathcal{D}' = \mathcal{D} \cap \mathcal{D}'$ 
21:  end while
22:  Update  $\tilde{\mathbf{f}}_d$  and  $\tilde{\boldsymbol{\varepsilon}}_d$ 
23: end while
24: Return  $\mathcal{D}$  and  $\mathcal{B}_{\mathbf{f}_{OBJ}}(\mathcal{D}, \mathbf{r}(\mathcal{D}))$ 

```

---

**Remark** In Algorithm 2,  $\{(\mathcal{D}_i, \mathbf{r}(\mathcal{D}_i))\}_{i=1}^{Card(\mathcal{D})}$  is lightened as  $(\mathcal{D}, \mathbf{r}(\mathcal{D}))$ , and  $\mathcal{B}_{\mathbf{f}_{OBJ}}(\mathcal{D}, \mathbf{r}(\mathcal{D}))$  is the set of boxes  $\mathcal{B}_{\mathbf{f}_{OBJ}}(\mathcal{D}_i, \mathbf{r}(\mathcal{D}_i))$ .

The two subroutines `compareDesignMM` and `iterateRefinement` are provided in the following, to complete Algorithm 2. First, `compareDesignMM` (in Algorithm 3) takes as inputs the set of new designs, the thresholds and the actual evolutive surrogate. It returns the first approximations of the objective functions, the coupled objective functions  $\mathbf{f}_{OBJ}$ , the boxes size  $\mathbf{r}$  and the non-dominated set of designs (which can be potentially refined). For each new design, the error of the evolutive surrogate is compared to the threshold  $s_1$ . If the error is small enough, the coupled objective functions are computed with the surrogate. Otherwise, first approximations are computed and the design is added to  $\mathcal{D}'$ .

**Algorithm 3** compareDesignMM( $\mathcal{D}_{new}, \mathbf{s}_1, \mathbf{s}_2, \tilde{\mathbf{f}}_d, \tilde{\boldsymbol{\varepsilon}}_d, \tilde{\mathbf{f}}^0, \tilde{\boldsymbol{\varepsilon}}^0, f_{OBJ}, \mathbf{r}, \mathcal{D}'$ )

---

```

1: for each  $\mathbf{x} \in \mathcal{D}_{new}$  do
2:   if  $\tilde{\boldsymbol{\varepsilon}}_d(\mathbf{x}) \leq \mathbf{s}_1$  then ▷ (a)
3:      $f_{OBJ}(\mathbf{x}) = \tilde{\mathbf{f}}_d(\mathbf{x})$ 
4:      $\mathbf{r}(\mathbf{x}) = \tilde{\mathbf{s}}_2(\mathbf{x}) + \tilde{\boldsymbol{\varepsilon}}_d(\mathbf{x})$ 
5:   else ▷ (b)
6:     Compute  $\tilde{\mathbf{f}}^0(\mathbf{x})$  and  $\tilde{\boldsymbol{\varepsilon}}^0(\mathbf{x})$ 
7:      $f_{OBJ}(\mathbf{x}) = \tilde{\mathbf{f}}^0(\mathbf{x})$ 
8:      $\mathbf{r}(\mathbf{x}) = \tilde{\boldsymbol{\varepsilon}}^0(\mathbf{x})$ 
9:      $\mathcal{D}' = \mathcal{D}' \cup \mathbf{x}$ 
10:  end if
11: end for
12: Return  $\tilde{\mathbf{f}}^0, \tilde{\boldsymbol{\varepsilon}}^0, f_{OBJ}, \mathbf{r}$  and  $\mathcal{D}'$ 

```

---

Then, `iterateRefinement` (in Algorithm 4) computes the approximation  $\tilde{\mathbf{f}}^k$  and its associated error for each design, which can be refined. The inputs are the non-dominated set of designs (which can be potentially refined) and the previous approximations. The outputs are the updated approximations. Note that the boxes are refined one at a time to sequentially check for domination.

**Algorithm 4** iterateRefinement( $\mathcal{D}', \mathbf{x}_r, \tilde{\mathbf{f}}^{k-1}, \tilde{\boldsymbol{\varepsilon}}^{k-1}, \tilde{\mathbf{f}}^k, \tilde{\boldsymbol{\varepsilon}}^k$ )

---

```

1: for each  $\mathbf{x} \in \mathcal{D}'$  do
2:   if  $\mathbf{x} = \mathbf{x}_r$  then
3:     Refine and compute  $\tilde{\mathbf{f}}^k(\mathbf{x})$  and  $\tilde{\boldsymbol{\varepsilon}}^k(\mathbf{x})$  from  $\tilde{\mathbf{f}}^{k-1}(\mathbf{x})$  and  $\tilde{\boldsymbol{\varepsilon}}^{k-1}(\mathbf{x})$  ▷ (c)
4:   else
5:      $\tilde{\mathbf{f}}^k(\mathbf{x}) = \tilde{\mathbf{f}}^{k-1}(\mathbf{x})$ 
6:      $\tilde{\boldsymbol{\varepsilon}}^k(\mathbf{x}) = \tilde{\boldsymbol{\varepsilon}}^{k-1}(\mathbf{x})$ 
7:   end if
8: end for
9: Return  $\tilde{\mathbf{f}}^k$  and  $\tilde{\boldsymbol{\varepsilon}}^k$ 

```

---

## 5 Test-cases and analysis

This section is devoted to the application of the framework on several test-cases and to the analysis of the performances. The following choices have been done:

- The optimization problems deal with analytical functions, and a noise is arbitrarily added to the objective functions depending on the number of evaluations computed, so that the approximation can be refined. The number of evaluations is used as an indicator to compare several strategies. In practice, the error  $\tilde{\boldsymbol{\varepsilon}}_i$  is computed in the following way for each objective function  $f_i$ :

$$\varepsilon_{noise_i} = \frac{\max(f_i) - \min(f_i)}{4 + 6X_1} \quad \text{with } X_1 \sim \mathcal{U}[0, 1]$$

$$\varepsilon_{bias_i} = \varepsilon_{noise_i} X_2 \quad \text{with } X_2 \sim \mathcal{U}[-1, 1]$$

$$\tilde{f}_i = f_i + \varepsilon_{bias_i}$$

$$\tilde{\boldsymbol{\varepsilon}}_i = 2 \cdot \varepsilon_{noise_i}$$

Then, this approximation is refined by choosing a refinement factor  $r_f > 1$ , as follows:

$$\begin{aligned}\tilde{f}_i &\leftarrow f_i + \frac{\tilde{f}_i - f_i}{r_f} \\ \tilde{\varepsilon}_i &\leftarrow \frac{\tilde{\varepsilon}_i}{r_f}\end{aligned}$$

- The optimization is performed by making a simple Monte-Carlo sampling, meaning that at each iteration, the set of new designs is constituted by 10 (ten) random designs. The effect of convergence to the optimal area will be studied in test-cases 2 and 3.
- The surrogate model is a noisy evaluation of the objective functions. In particular, this noise, linked to the metamodel error, is considered proportional to the distance, noted as  $d_{\min}$ , to the closest training point relatively to the parameter interval width. This distance is also proportional to the range covered by the objective functions. For each  $i$ , it holds that

$$\begin{aligned}\tilde{\varepsilon}_i &= \frac{d_{\min}}{\frac{1}{N_x} \sum_j (\max(x_j) - \min(x_j))} \cdot \frac{\max(f_i) - \min(f_i)}{2} \\ \varepsilon_{noise_i} &= \frac{\tilde{\varepsilon}_i}{2} X_1 \quad \text{with } X_1 \sim \mathcal{U}[0, 1] \\ \varepsilon_{bias_i} &= \varepsilon_{noise_i} X_2 \quad \text{with } X_2 \sim \mathcal{U}[-1, 1] \\ \tilde{f}_i &= f_i + \varepsilon_{bias_i}\end{aligned}$$

- Note also that the extension  $\tilde{\mathbf{s}}_2$  of the conservative error to interpolated boxes is simply taken equal to the threshold  $\mathbf{s}_2$ , which implies that only non-dominated interpolated boxes are ensured conservative.

In the following, the proposed framework (SABBa) is compared with three different approaches: i) the *classical approach* (denoted as *Class*), *i.e.* non-recursive Bounding-Box approach defined in Def. 10, where each box is refined to the threshold  $\mathbf{s}_2$ ; ii) the recursive approach, *i.e.* Fusi's Bounding-Box strategy [35], where only non-dominated boxes are refined to the threshold  $\mathbf{s}_2$ ; iii) the classical approach accelerated by means of the evolutive surrogate model (denoted as *Class-acc*), where the surrogate is built on the converged boxes.

## 5.1 Test-case 1: The BNH problem

The first test-case deals with a bi-objective constrained problem, proposed by Binh and Korn (1997):

$$\begin{aligned}\text{minimize: } \mathbf{f}(\mathbf{x}) &= \left( \begin{array}{c} 4x_1^2 + 4x_2^2 \\ (x_1 - 5)^2 + (x_2 - 5)^2 \end{array} \right) \\ \text{subject to: } & (x_1 - 5)^2 + x_2^2 \leq 25 \\ & (x_1 - 8)^2 + (x_2 + 3)^2 \geq 7.7 \\ \text{by changing: } & (x_1, x_2) \in [0, 5] \times [0, 3]\end{aligned}\tag{13}$$

The acceptable design area and its counterpart in the objective space are represented in Fig. 3. Note that the Pareto optimal set in the design space is represented by the thick black curve. The new designs are chosen randomly in  $[0, 5] \times [0, 3]$  under the two stated geometric constraints.

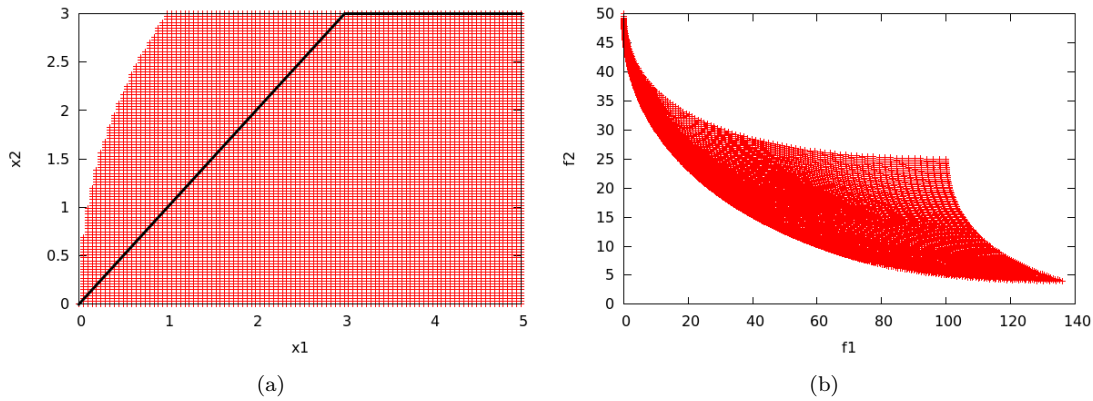


Figure 3: Test-case 1, a) acceptable design and Pareto optima (thick line), b) Pareto representation.

In Figures 4 and 5, the results in terms of optimal designs and objectives, for the classical and the recursive approaches are provided, respectively. This optimization is performed with 100 optimization iterations and  $s_2 = 1\%$ . Note that designs are represented as points in the design space and as boxes in the objective space.

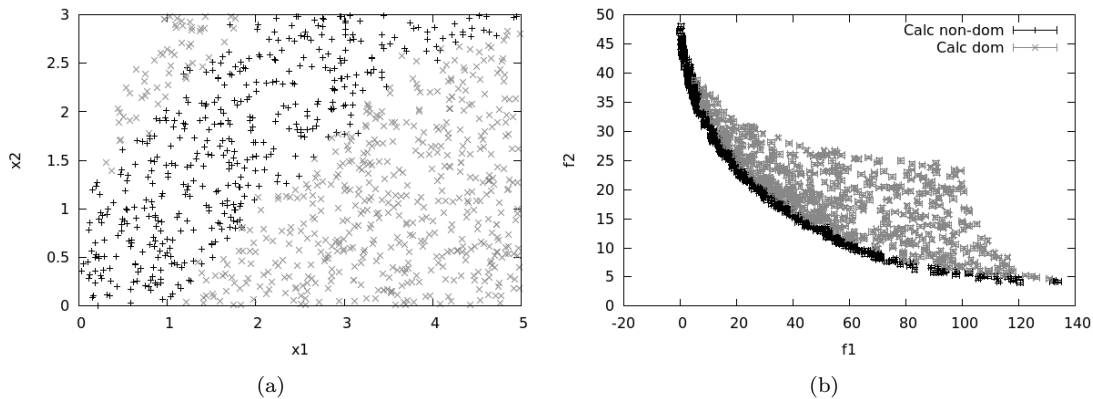


Figure 4: Test-case 1, *Class approach*, non-dominated designs in black and dominated one in grey: a) Design space, b) Objective space.

As expected, the Pareto front of these two strategies are very similar and both provided the same design area to be non-dominated. Of course, the difference lies in the objective space, where a significant portion of the boxes are not converged to  $s_2$  in the recursive approach, thus reducing the number of function evaluations, and then the computational cost of the optimization.

In Figures 6 and 7, the effect of the evolutive surrogate model is illustrated for the classical approach and the proposed framework, respectively. The thresholds  $s_2 = 1\%$  and  $s_1 = 1\%$  are used.

It can be observed in Figures 6 and 7 that enough information was gathered to allow the evolutive surrogate to converge and to bypass some function evaluations in order to further reduce the computational cost. It seems that the gain is more and more significant throughout



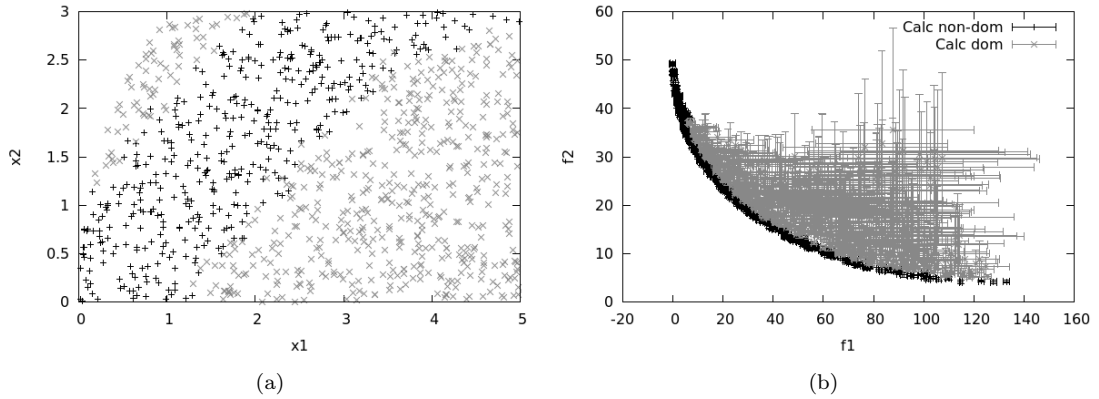


Figure 5: Test-case 1, *Fusi approach*, non-dominated designs in black and dominated one in grey: a) Design space, b) Objective space.

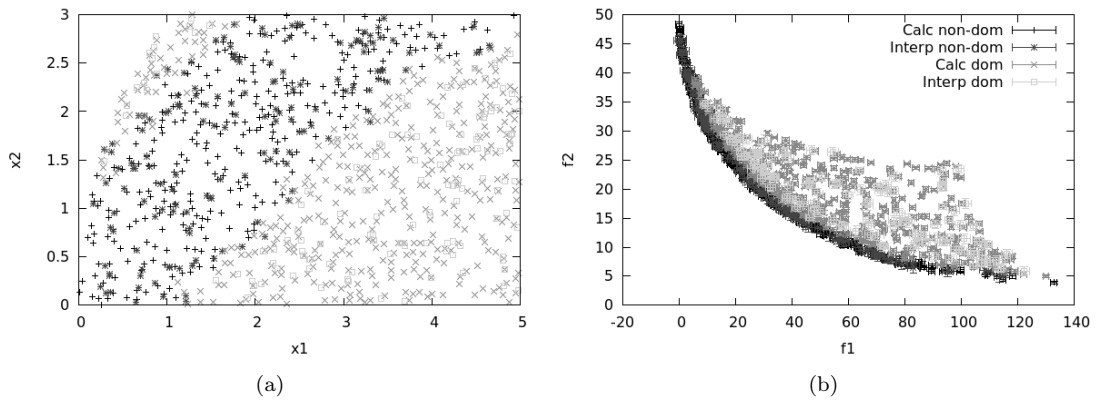


Figure 6: Test-case 1, *Class-acc approach*, non-dominated interpolated designs added in dark grey, dominated interpolated designs in light grey. a) design space, b) objective space.

the execution. More important, the non-dominated area is similar to the fully-computed classical approach shown in Figure 5, and so it is for the Pareto front.

The cost reduction is verified by comparing the number of function evaluations needed for each strategy. Note that this number is only a basic indicator to compare several strategies. Then, in Figure 8, the results obtained with different strategies in terms of number of function evaluations along the iterations of the optimization are reported:

- As expected, the recursive (denoted as Fusi [35]) strategy has a beneficial impact on the number of function evaluations. More precisely, it improves the slope of the cost curve by a ratio that can be interpreted as the mean computational cost in the recursive strategy compared to the full convergence cost. As nearly half of the designs are non-dominated in this test-case, the gain is quite limited. Indeed, a high number of boxes are fully converged.
- The coupling with the evolutive surrogate appears to be more and more effective throughout the optimization, as suggested previously. This is due to the fact that the surrogate attains

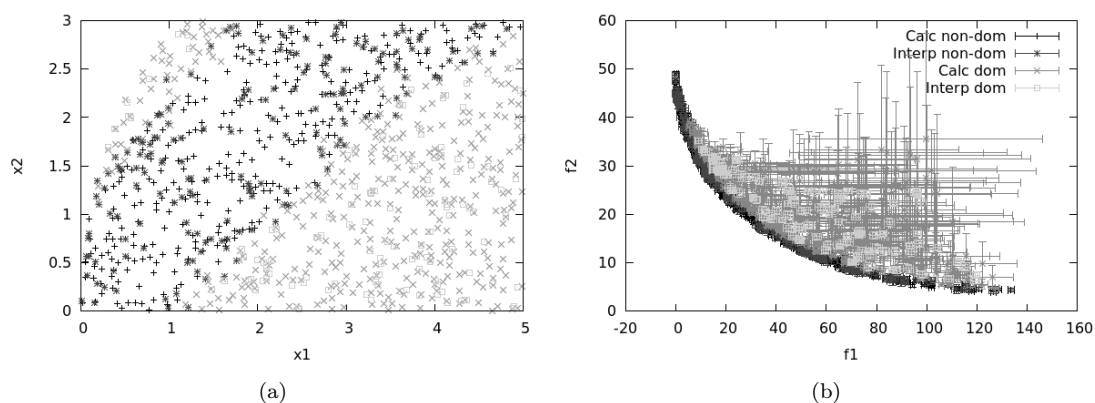


Figure 7: Test-case 1, *SABBa* approach, non-dominated interpolated designs added in dark grey, dominated interpolated designs in light grey: a) Design space, b) Objective space.

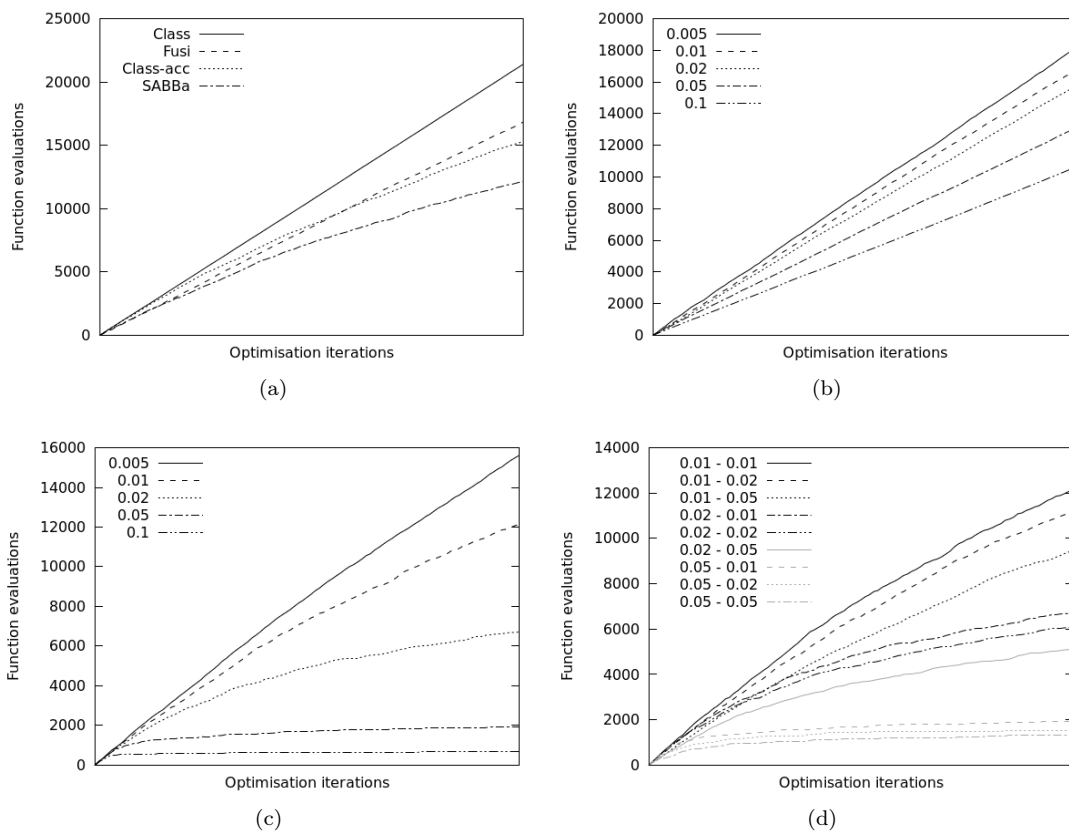


Figure 8: Test-case 1: a) number of function evaluations for each method of interest ( $s_1 = s_2 = 0.01$ ), b) Fusi's approach varying  $s_2$ , c) SABBa framework varying  $s_1$  with  $s_2 = 0.01$ ; d) varying both  $s_2$  and  $s_1$ .

the targeted accuracy after a certain number of iterations and does not accelerate the optimization process. Therefore, during the first iterations, the slope of the cost curve is the same as the non-assisted strategies.

In order to stress the influence of the tuning parameters, further comparisons are performed by varying  $s_2$  in Fusi's approach, and then  $s_1$  and both  $s_2$  and  $s_1$  with SABBa. Associated results in terms of the number of function evaluations are reported in Figure 8.

As it can be observed in Figure 8b,  $s_2$  has a significant impact on the computational cost. This can be explained as there are a lot of fully converged boxes. As a consequence, varying the convergence threshold influences all these boxes and has an important effect on the global cost. However, this threshold has a tremendous impact on the accuracy of the Pareto optimal set in the design space. This is qualitatively represented in Fig. 9 where the outputs with finest and coarsest threshold from Fig. 8b are compared.

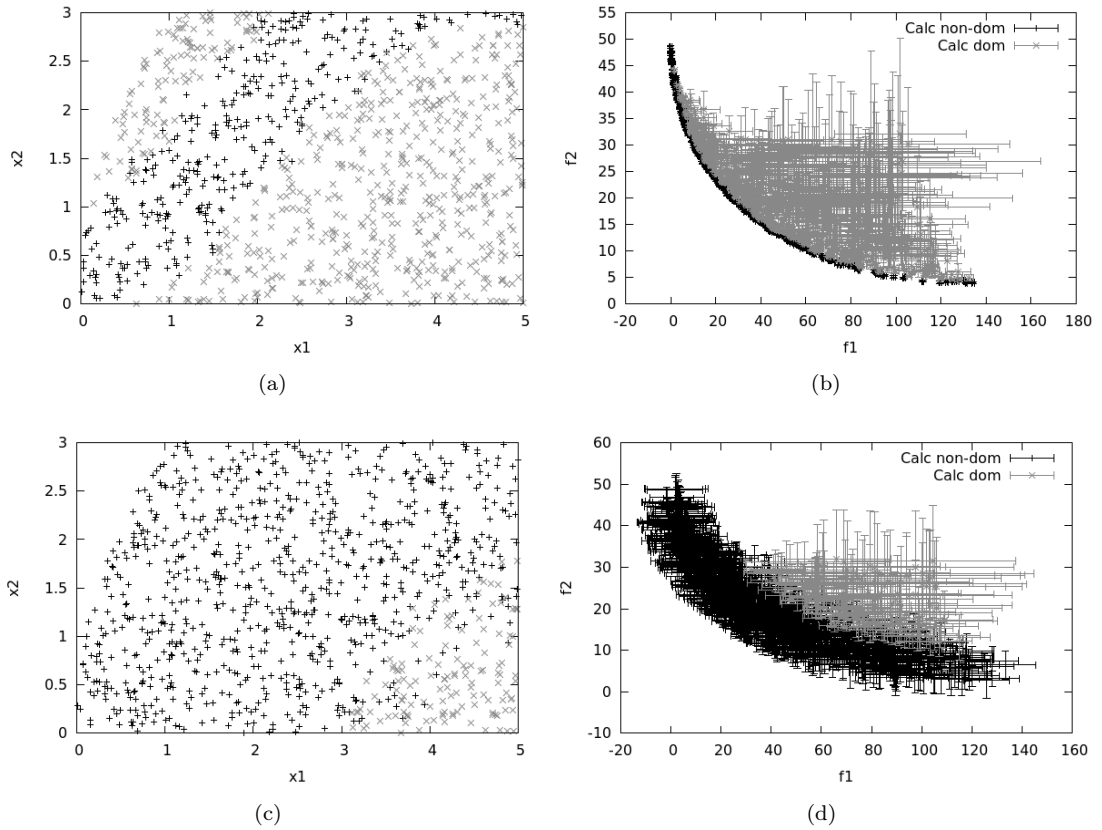


Figure 9: Impact of  $s_2$ : Outputs with  $s_2 = 0.005$  in a) design space and b) objective space ; Outputs with  $s_2 = 0.1$  in c) design space and d) objective space.

Finally, Figures 8c and 8d illustrate how it is possible to extremely reduce the computational cost by bypassing several evaluations by means of the surrogate model. This requires to trust the surrogate, where the probability of underestimating the error is the highest, when for example high local variations are not yet captured. Globally, the framework seems effective as a cost-reduction strategy but by requiring some tuning of the thresholds in order to provide the best

results.

The next test-case deals with a function with an optimal area, which is small with respect to the entire space of variation.

## 5.2 Test-case 2: The Triangle problem

This problem is a bi-objective unconstrained optimization:

$$\begin{aligned} \text{minimize: } \mathbf{f}(\mathbf{x}) &= \begin{pmatrix} \frac{x_1+x_2}{10} + |x_1 - x_2| \\ \frac{x_1}{5} + |x_2 - 2| \end{pmatrix} \\ \text{by changing: } (x_1, x_2) &\in [0, 10]^2 \end{aligned} \quad (14)$$

The design space with the optimal set (represented with a thick black curve) and the Pareto counterpart are represented in Fig. 10.

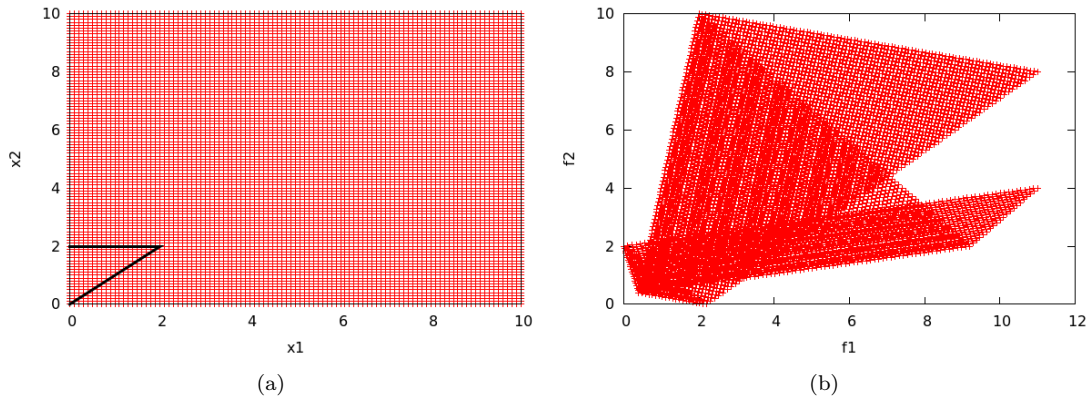


Figure 10: Test-case 2: a) acceptable design and Pareto optima (thick line), b) Pareto representation.

This small optimal area is well fit to study the effect of the convergence of the proposed framework. In particular, two different variants are assessed: in the first a), at each iteration, 10 random new designs in  $\mathcal{X} = [0, 10] \times [0, 10]$  are taken into account; in the second case b), 10 designs in a domain  $\mathcal{X}^k$  converging in  $[0, 2] \times [0, 2]$  throughout the iteration are considered, with  $\mathcal{X}^0 = \mathcal{X}$ .

These two variants are proposed here for assessing the behavior of the proposed framework over a drawback raised in [35], *i.e.* the decreasing rate of convergence of the Bounding-Box approach during the optimization.

### 5.2.1 Test-case 2 a): No convergence of the designs

In this variant, new designs are chosen completely randomly within the  $[0, 10]^2$  design space. Of course, this study gives poor Pareto front refinement and is highly inefficient, as it can be seen in Fig. 11.

In the following, the effect of the convergence of the new designs toward the optimal area is presented and discussed.

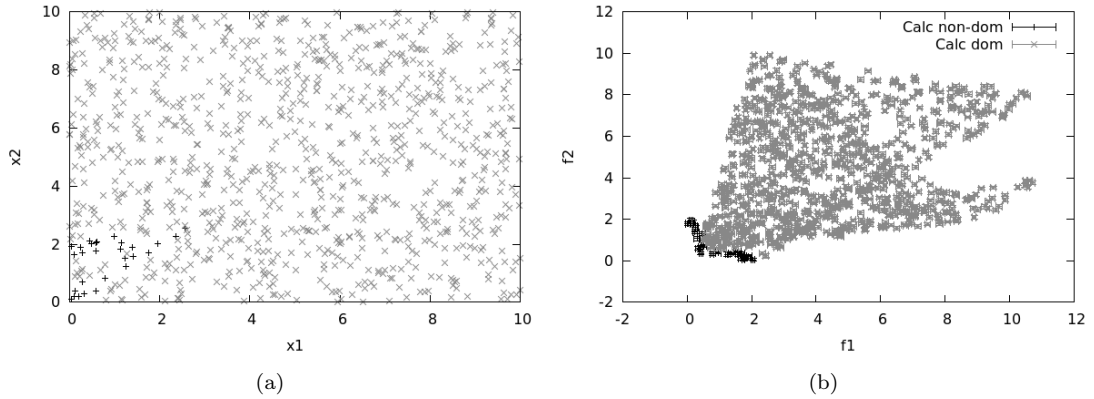


Figure 11: Test-case 2: *Classical approach*, non-converging domain for new designs: a) Design space, b) Objective space.

### 5.2.2 Test-case 2 b): Convergence of the designs

For this case, new designs are randomly chosen in a domain converging toward the optimal area. In practice, with  $k$  the optimization iteration, the new designs are chosen as follows, for  $i$  in  $\{1, 2\}$ ,

$$x_i = X \left( 2 + 8 \exp \left( -\frac{k}{45} \right) \right), \quad \text{with } X \sim \mathcal{U}[0, 1]$$

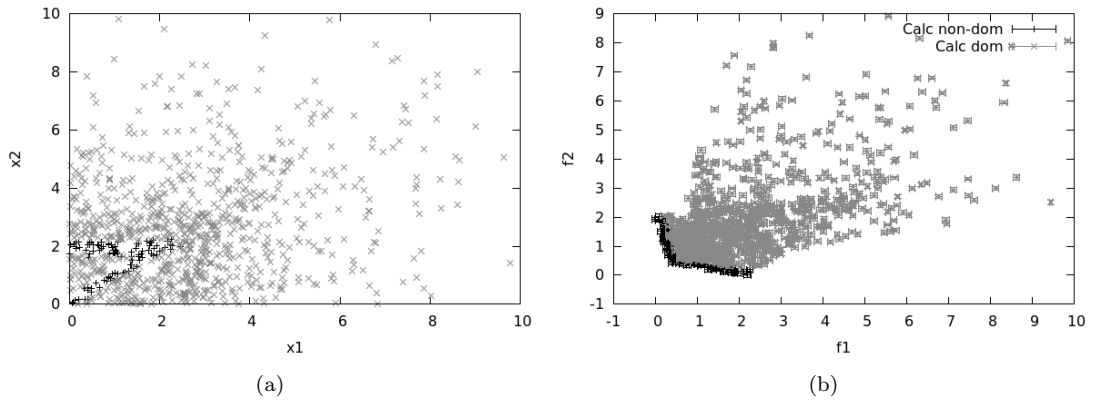


Figure 12: Test-case 2: *Classical approach*, converging domain for new designs: a) Design space, b) Objective space.

The main effect of the convergence toward the optimal area is a better refinement of the Pareto optima. This can be seen in Figs. 11 and 12, where the triangular optimal shape is more precisely captured in the design space, and so it is the Pareto front in the objective space.

A quantitative comparison is performed on the number of function evaluations in Fig. 13. As concluded in [35], it can be seen that the convergence toward the optimal area induces a decreasing impact of the recursive strategy. In practice, one can see the slope of the cost curve

slowly tending back to the one of the classical (solid) curve in Fig. 13b. This is due to the increasing ratio of non-dominated designs among the converging search domain. Hence, SABBa relies on the evolutive surrogate to become quickly accurate on the optimal area, which is highly sampled, in order to ultimately bypass all function evaluations in this area. Such a behavior can indeed be seen in Fig. 13b where both surrogate-assisted strategies (*Class-acc* and *SABBa*) benefit from the convergence to reach their plateau faster. Despite the cost increase of the *Fusi* strategy alone, SABBa allows fewer evaluations and a very reduced final slope.

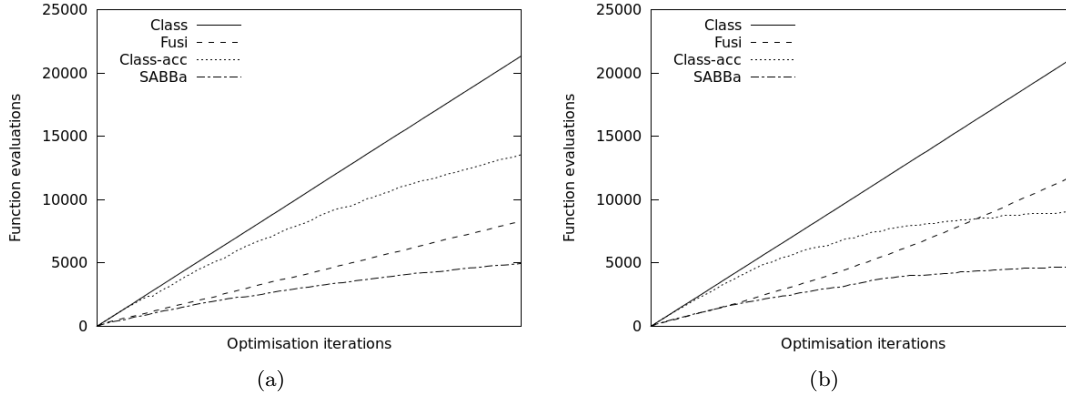


Figure 13: Test-case 2: number of function evaluations depending on the strategy for a) the non-converging case, b) the converging case.

**Remark** Fig. 13a can also be compared with Fig. 8a, where the gain induced by the recursive strategy was quite limited. Indeed, one can see in Fig. 11a that very few designs are non-dominated, which allows significant gains from the recursive strategy. By converging toward the optimal area, Figs. 12a has more and more non-dominated designs throughout the iterations, which explains the decreasing gains from the recursive approach alone. This can also be apprehended through the impact of  $s_2$  on the cost for both the converging and non-converging optimizations. As seen in Fig. 14a, the number of non-dominated designs is low enough for the  $s_2$  threshold to have a negligible impact on the computational cost. On the contrary, the convergence toward the optimal area increases the number of fully converged designs and therefore increases the impact of  $s_2$  throughout the iterations, which explains the separation of the curves in Fig. 14b.

### 5.3 Test-case 3: The Kursawe problem

The Kursawe bi-objective optimization is performed here, in order to study the framework behavior on a more complex Pareto front shape and a 3D design domain. The problem is the following:

$$\begin{aligned} \text{minimize: } \quad & \mathbf{f}(\mathbf{x}) = \left( \begin{array}{c} \sum_{i=1}^2 [-10 \exp(-0.2 \sqrt{x_i^2 + x_{i+1}^2})] \\ \sum_{i=1}^3 [|x_i|^{0.8} + 5 \sin(x_i^3)] \end{array} \right) \\ \text{by changing: } \quad & (x_1, x_2, x_3) \in [-5, 5]^3 \end{aligned} \quad (15)$$

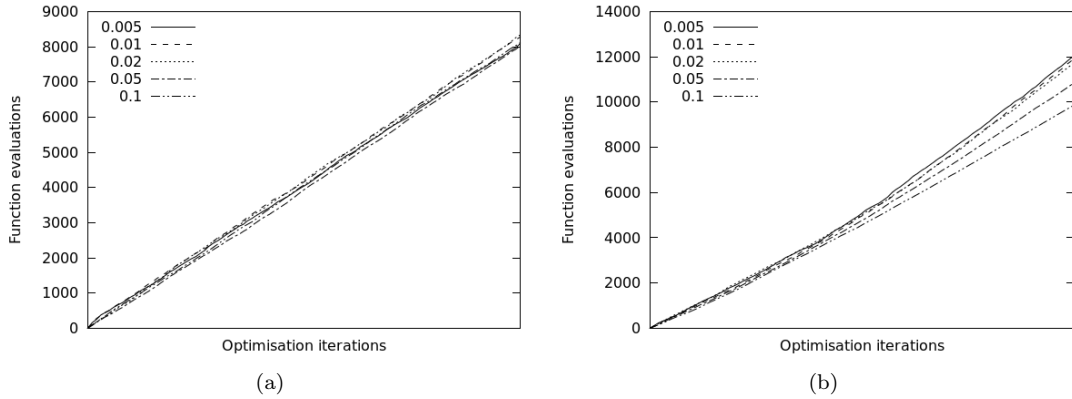


Figure 14: Test-case 2: number of function evaluations (*Fusi* strategy) depending on  $s_2$  for a) the non-converging case, b) the converging case.

This problem is known to have a quite complex Pareto front with several discontinuities, associated to a small area of the design space. In the following, the convergence toward the optimal zone is forced, similarly to test-case 2 b). The slope of the computational cost of the recursive strategies is expected to show a variable behavior, with a decreasing impact of these approaches throughout the optimization, as concluded from previous test-case.

The Pareto representation of the functions and the associated Pareto front are given in Figs. 15a and 15b. One can see that the front is highly discontinuous and that the range of the objective functions should allow for a high influence of the recursive strategy through poorly refined dominated boxes. The Pareto-optimal designs are represented in Fig. 15c in the  $[-5, 5]^3$  domain and in Fig. 15d in a close-up view. These designs form complex disjoint sets and the approximations will be qualitatively compared to Fig. 15c.

The SABBa framework is applied in the following with three levels of thresholds. First, one can see in Fig. 16 the outputs obtained with SABBa using a fine threshold, which gives results highly comparable to Fig. 15, but with only 28510 function evaluations over 48913 for the classical approach (gain = 42%).

The framework is also used with larger thresholds in order to further reduce the computational burden. It can be qualitatively seen that the optimal designs are less accurately captured in Fig. 17a and the associated Pareto front is depicted with a higher imprecision in Fig. 17b. However, this study is performed with only 8294 evaluations (gain = 83%).

Finally, very coarse user-defined thresholds are used. As it can be seen in Fig. 18a, non-dominated designs are still located in the right area, however local optimal shape is not captured. This can however be used to get a first approximation of the optimal set as the computational cost is here reduced to 3546 evaluations (gain = 93%).

The cost graphs are given in Fig. 19. It can be observed in Figs. 19a and 19c, as expected, that the convergence toward the optimal area is similar to that one of Figs. 13 and 14, with a reduction of the impact of the recursive strategy alone and an increasing sensitivity to  $s_2$  throughout the optimization.

The plateau is reached in this test-case before the end of the optimization process, and Fig. 19c illustrates the influence of the threshold  $s_1$  when the plateau is reached. The interaction between  $s_1$  and  $s_2$  can finally be observed in Fig. 19d. One can see that, because of the variable impact of  $s_2$  on the recursive approaches computational cost, the influence of this threshold on

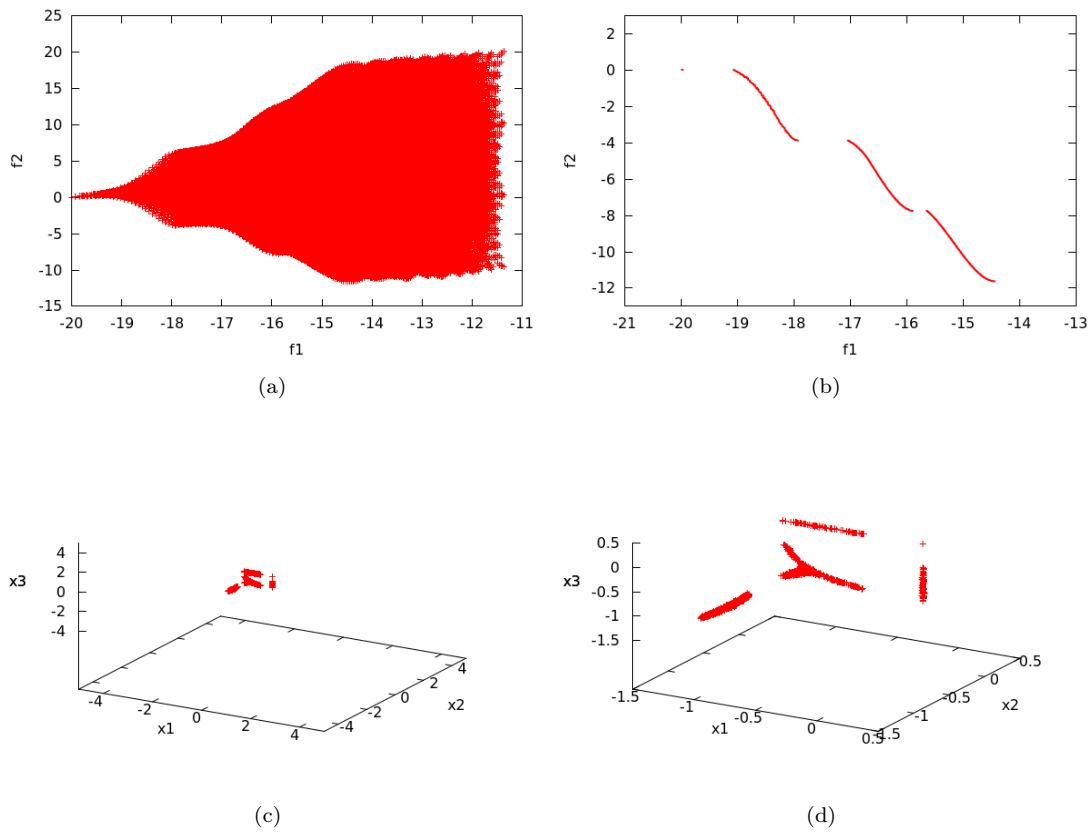


Figure 15: Test-case 3: a) image in the objective space, b) associated Pareto front, c) Pareto optima in the design space, d) in a close-up view.

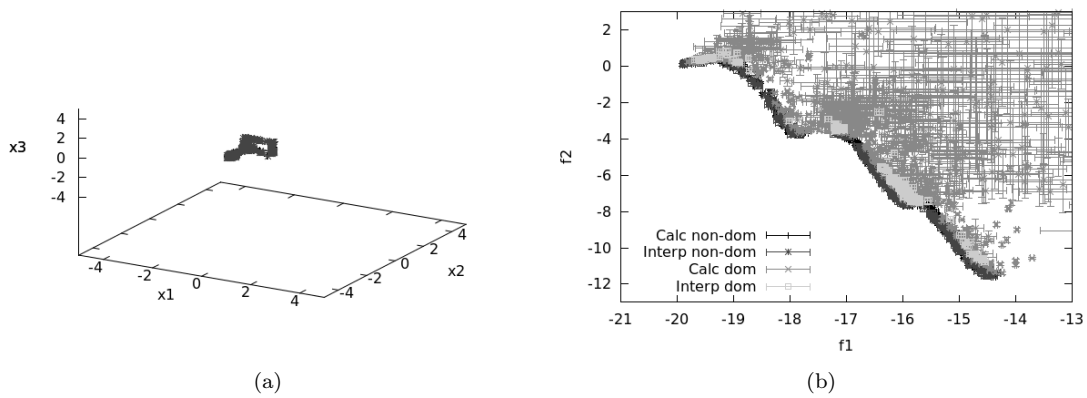


Figure 16: Test-case 3: SABBa outputs with fine thresholds in a) the design space, b) the objective space. Computational cost saving: 42%.



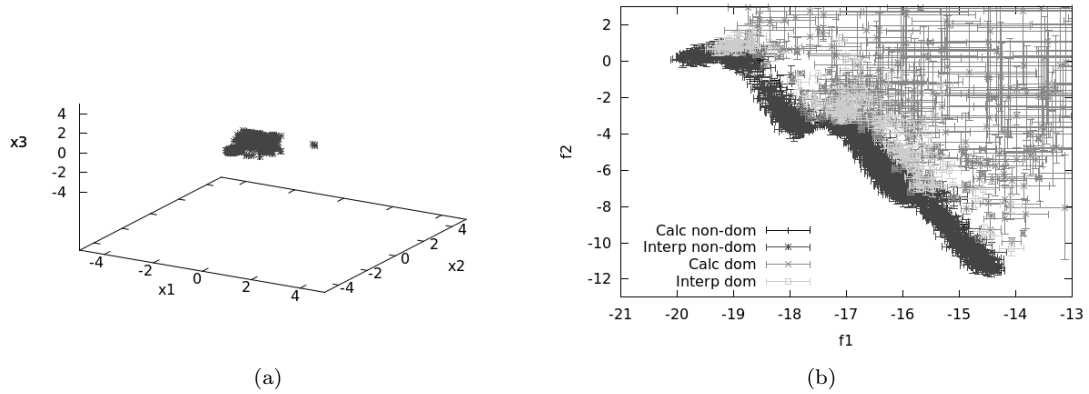


Figure 17: Test-case 3: SABBa outputs with moderate thresholds in a) the design space, b) the objective space. Computational cost saving: 83%.

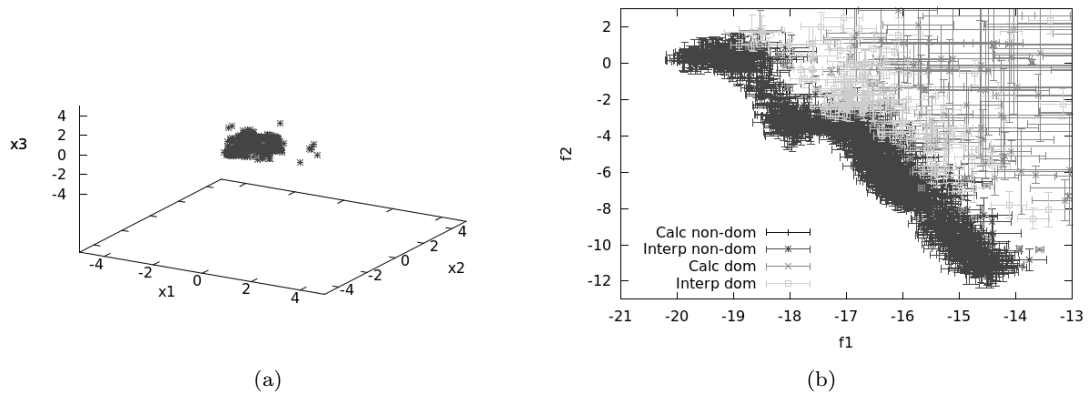


Figure 18: Test-case 3: SABBa outputs with coarse thresholds in a) the design space, b) the objective space. Computational cost saving: 93%.

the overall SABBa framework is highly dependent on  $s_1$ . In practice, with a coarse  $s_1$ , function evaluations will be bypassed before the convergence toward the optimal area, which explains why the last three curves of Fig. 19d are very similar, as  $s_2$  has a negligible impact. On the contrary, with a fine  $s_1$ , the surrogate-assisting strategy permits to avoid true computations after a significant number of iterations. In this case, many non-dominated boxes are computed and  $s_2$  has a non-negligible impact on the global cost, as it can be seen with the three first curves of the same figure.

It can be concluded that the SABBa framework yields here a decrease in the computational cost associated to the optimization process. The user-defined thresholds should however be carefully chosen as they impact both outputs accuracy and global computational cost.

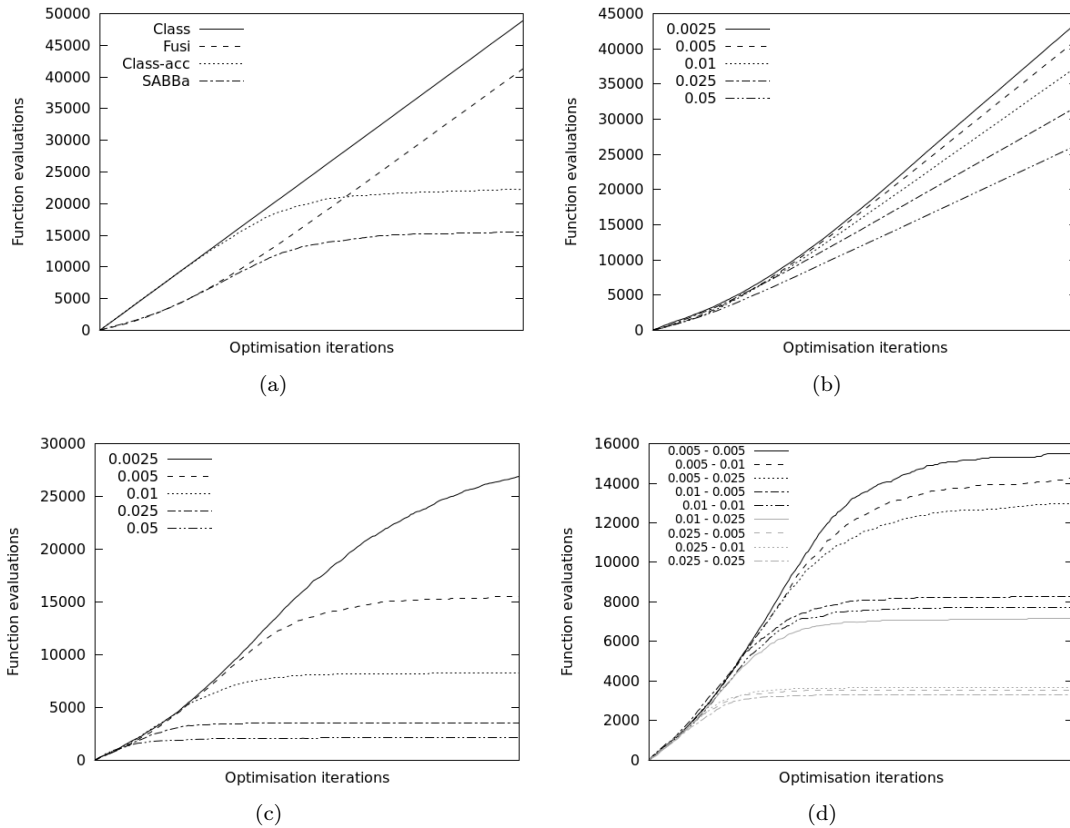


Figure 19: Test-case 3: a) number of function evaluations for each method of interest ( $s_1 = s_2 = 0.01$ ), b) Fusi's approach varying  $s_2$ , c) SABBa framework varying  $s_1$  with  $s_2 = 0.01$ ; d) varying both  $s_2$  and  $s_1$ .

## 6 Conclusive remarks

In this paper, a novel framework has been introduced to efficiently deal with multi-objective optimization problem with approximated objective functions. This kind of problem notably arise in uncertainty-based optimization, when robustness measures are chosen as objectives functions. The proposed strategy couples the recursive Bounding-Box approach for error-based objective functions developed in [35] with a surrogate-assisting strategy, based on a model refined on the flight, that aims at bypassing function evaluations. The intuitive idea is to use the Bounding-Box approach to quickly converge toward the optimal area through a poor refinement of non-efficient designs and to simultaneously build an evolutive surrogate, which refinement is driven by the optimization process toward the optimal area, to bypass the costly black-box evaluations when the surrogate error is estimated low enough. First, we have provided a mathematical proof of the robustness of the framework and of its convergence under some assumptions. Then, a numerical validation has been carried out, in order to validate the convergence, quantify the computational cost reduction and make explicit the main behaviors of the strategy.

Some innovative concepts, such as boxed Pareto dominance or the Pareto-optimal set sequence, have been formulated in the first sections. Some theorems have also been introduced

and proved in order to provide mathematical evidence about the convergence of the strategy toward the real Pareto-optimal set, the robustness of the imprecise approach (with non-zero thresholds) and the coupling with the surrogate-assisting strategy.

From a numerical point of view, the algorithm associated to the framework has been formulated and applied to several analytical test-cases.

The framework has been tested and analyzed on three different test-cases. It appears to yield a consistent cost-reduction compared to the Fusi [35] approach, which already performed better than the classical nested optimization problem. Accuracy of the outputs have been qualitatively analyzed with regards to the associated cost reduction. Finally, the influence of the user-defined thresholds have been investigated, in order to give the keys to tune them efficiently.

Future developments will be directed towards the application of the proposed framework to a problem of uncertainty-based multi-objective optimization in engineering applications.

## A Proof 1

**Proof.** The proof here is trivial. By definition,

$$\mathcal{X}_{\tilde{\mathcal{P}}}^f(\{\mathbf{x}_i\}_{i=1}^N) = \{\mathbf{x}_i, i \in \mathcal{I}_1^N \mid \text{non-domination condition}\} \subseteq \{\mathbf{x}_i, i \in \mathcal{I}_1^N\} \equiv \{\mathbf{x}_i\}_{i=1}^N$$

And the same can be said for the boxed Pareto optima.  $\square$

## B Proof 2

**Proof.** For  $\mathbf{y} \notin \mathcal{X}_{\tilde{\mathcal{P}}}^f(\{\mathbf{x}_i\}_{i=1}^N)$ , let us assume that  $\nexists \mathbf{y}' \in \mathcal{X}_{\tilde{\mathcal{P}}}^f(\{\mathbf{x}_i\}_{i=1}^N)$ ,  $\mathbf{y}' \succ \mathbf{y}$ , then  $\mathbf{y} \in \mathcal{X}_{\tilde{\mathcal{P}}}^f(\mathcal{X}_{\tilde{\mathcal{P}}}^f(\{\mathbf{x}_i\}_{i=1}^N) \cup \mathbf{y}) = \mathcal{X}_{\tilde{\mathcal{P}}}^f(\{\mathbf{x}_i\}_{i=1}^N)$  from Equation 5, which proves the first implication by contradiction. The second implication is immediate from the definition of  $\mathcal{X}_{\tilde{\mathcal{P}}}^f$ .  $\square$

## C Proof 3

**Proof.** By using the explicit definition of the Pareto front as in Equation 3, the proof is immediate as Assumption 1 gives  $\forall \mathbf{x} \in \mathcal{X}, \forall j \in \mathcal{I}_1^N, f_j(\mathbf{x}) \in [\tilde{f}_j(\mathbf{x}) - \tilde{\varepsilon}_j(\mathbf{x}), \tilde{f}_j(\mathbf{x}) + \tilde{\varepsilon}_j(\mathbf{x})]$ . Hence,  $\forall \mathbf{x}_i \in \mathcal{X}_{\tilde{\mathcal{P}}}^f(\{\mathbf{x}_i\}_{i=1}^N)$ :

$$\begin{aligned} \forall k \in \mathcal{I}_1^N, \exists j \in \mathcal{I}_1^m, \pm f_j(\mathbf{x}_i) &< \pm f_j(\mathbf{x}_k) \\ \pm \tilde{f}_j(\mathbf{x}_i) - \tilde{\varepsilon}_j(\mathbf{x}_i) &< \pm \tilde{f}_j(\mathbf{x}_k) + \tilde{\varepsilon}_j(\mathbf{x}_k) \end{aligned}$$

Therefore,  $\mathbf{x}_i \in \mathcal{X}_{\tilde{\mathcal{P}}_b}^f(\{(\mathbf{x}_i, \tilde{\varepsilon}(\mathbf{x}_i))\}_{i=1}^N)$ , which ends the proof.  $\square$

## D Proof 4

**Proof.** The proof is similar to the one of Lemma 1 and relies on the fact that  $\forall \mathbf{x}$ ,

$$|\mathbf{f}_{OBJ}(\mathbf{x}_i) - \mathbf{f}_{opt}(\mathbf{x}_i)| \leq \tilde{\varepsilon}_d^{lin}(\mathbf{x}_i)$$

for the refined, and that

$$\left(\tilde{\varepsilon}_d^{l_{in}}(\mathbf{x}_i) > \mathbf{s}_1\right) \Rightarrow \left(\mathbf{f}_{OBJ}(\mathbf{x}_i) = \mathbf{f}_{opt}(\mathbf{x}_i)\right)$$

for the accurate. Thus, the rough is immediate,

$$|\mathbf{f}_{OBJ}(\mathbf{x}_i) - \mathbf{f}_{opt}(\mathbf{x}_i)| \leq \mathbf{s}_1$$

All those inequalities come directly from Definition 11.  $\square$

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