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► **To cite this version:**

Swapnil Dhamal, Walid Ben-Ameur, Tijani Chahed, Eitan Altman. Optimal Investment Strategies for Competing Camps in a Social Network: A Broad Framework. 2018. hal-01712288v1

**HAL Id: hal-01712288**

**<https://inria.hal.science/hal-01712288v1>**

Preprint submitted on 19 Feb 2018 (v1), last revised 20 Nov 2018 (v3)

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# Optimal Investment Strategies for Competing Camps in a Social Network: A Broad Framework

Swapnil Dhamal, Walid Ben-Ameur, Tijani Chahed, and Eitan Altman

**Abstract**—We study the problem of optimally investing in nodes of a social network in a competitive setting, wherein two camps attempt to drive the average opinion of the population in their favor. Using DeGroot-Friedkin model of opinion dynamics, we formulate the problem as a zero-sum game with its players being the two camps. We derive optimal investment strategies for both camps, and show that a random investment strategy is optimal when the underlying network follows a popular class of weight distributions. We study a broad framework, where we consider several settings of the problem, namely, when the influence of a camp on a node is a concave function of its investment on that node, when a camp aims at maximizing competitor’s investment or deviation from its desired investment, and when one of the camps has uncertain information about the values of the model parameters. We also study the problem under common coupled constraints on the combined investments by the camps and derive optimal strategies of the two camps, and hence quantify the first-mover advantage. For a quantitative and illustrative study, we conduct simulations on real-world datasets and provide results and insights.

**Index Terms**—Social networks, opinion dynamics, elections, zero-sum games, common coupled constraints, decision under uncertainty, Stackelberg game, linear programming, convex optimization.

## I. INTRODUCTION

Opinion dynamics is a natural phenomenon in a system of cognitive agents, and is a well-studied topic across several disciplines. It is highly relevant to applications such as elections, viral marketing, propagation of ideas or behaviors, etc. In this paper, we consider two competing camps who aim to maximize the adoption of their respective opinions in a social network. In particular, we consider a strict competition setting where the opinion value of one camp is denoted by  $+1$  and that of the other camp by  $-1$ ; we refer to these camps as good and bad camps respectively. Opinion adoption by a population can be quantified in a variety of ways; here we consider a well-accepted way, namely, the average or equivalently, the sum of the opinion values of the nodes in the network [1], [2]. Hence the good camp’s objective would be to drive this sum to be as high as possible, while bad camp would aim to minimize it.

The average or sum of the opinion values is of relevance in several applications. In a fund collection scenario, for instance, the magnitude of the opinion value of an individual can be viewed as the amount of funds and its sign as the camp towards which he or she is willing to contribute. Consider another example where a group of sensors or reporting agents are

assigned the job of reporting their individual measurements of a particular parameter or event, and the resulting measurement would be obtained by averaging the individual values. In this case, two competitors may aim to manipulate the resulting average (one perhaps for a good cause of avoiding panic, and another for elevating it).

While the opinion values can be unbounded in the above examples, there are scenarios which can be modeled aptly by bounded opinion values. In elections, for instance, a voter can vote at most once. Here one could view bounded opinion value of an individual as a proxy for the probability with which the individual would vote for a camp. For instance, an opinion value of  $v \in [-1, +1]$  could imply that the probability of voting for the good camp is  $(1 + v)/2$  and that of voting for the bad camp is  $(1 - v)/2$ . Hence the good (or respectively bad) camp would want to maximize (or respectively minimize) the sum of opinion values, since this sum would indicate the expected number of votes in favor of the good camp. Product adoption is another example where bounded opinion values could be well justified.

It is well known that social networks play a prime role in determining the opinions, preferences, behaviors, etc. of the constituent individuals [3]. There have been efforts to develop models which could determine how the individuals update their opinions based on the opinions of their connections, and hence study the dynamics of opinions in the network [4]. With such an underlying model of opinion dynamics, a camp would aim at maximizing the adoption of its opinion in a social network, in presence of a competitor. A camp could act on achieving this objective by strategically investing on selected individuals in a social network who could adopt its opinion, who could in turn influence the opinions of their connections, and so on. Based on the underlying application, this investment could be in the form of money, free products or discounts, attention, convincing discussions, etc. Thus given that both camps have certain budget constraints, the strategy of the good camp comprises of how much to invest on each node in a social network, so as to maximize the sum of opinion values of the nodes, while that of the bad camp comprises of how much to invest on each node, so as to minimize this sum.

This setup results in a game, and since we consider a strict competition setting with constraints such as budget (and other constraints as we shall encounter), the setup falls into the framework of constrained zero-sum games [5].

### A. Motivation

There have been studies to identify influential nodes and the amounts to be invested on them, specific to analytically

S. Dhamal, W. Ben-Ameur, and T. Chahed are with Samovar, Telecom SudParis, CNRS, Universite Paris-Saclay, France.

E. Altman is with Institut National de Recherche en Informatique et en Automatique (INRIA) Sophia Antipolis-Mediterranee, France.

tractable models of opinion dynamics (such as DeGroot) [6], [7], [2]. Such studies are important to complement the empirical and experimental studies, since they provide more concrete results and rigorous reasoning behind them. However, most of the studies are based in a very preliminary setting and a limited framework. This paper aims to consider a broader framework by motivating and analyzing a variety of settings, which could open interesting future directions for analytical study of opinion dynamics.

We study settings wherein the investment per node by a camp could be unbounded or bounded. Bounded investments could be viewed as discounts which cannot exceed 100%, attention capacity or time constraint of a voter to receive convincing arguments, company policy to limit the number of free samples that can be given to a customer, government policy of limiting the monetary investment by a camp on a voter, etc. As we will see, bounded investments in our model would result in bounded opinion values, which as explained earlier, could be transformed into probability of voting for a party or adopting a product.

While the linear influence function as per the DeGroot-Friedkin model is well studied, the influence of a camp on a node might not increase linearly with the corresponding investment. In fact, several social and economic settings follow a diminishing marginal returns property, which says that for higher investments, the marginal returns (influence in this context) are lower for a marginal increase in investment. A concave influence function naturally captures this property. We study such an influence function in both unbounded and bounded investment per node settings, and relate it to the skewness of investment in optimal strategies.

There are scenarios where a camp may want to maximize the total investment of the competing camp, so as to upset the latter's broad budget allocation, which might lead to reduction in its available budget for future investments or for other channels such as mass media advertisement. The latter may also have to implement unappealing actions such as increasing the product cost or seek further monetary sources in order to compensate for its investments. Alternatively, the camps may have been advised an investment strategy that might be optimal for a mediator or central authority, and deviating from this strategy would incur a penalty. For similar reasons as above, a camp may want to maximize the penalty incurred by the competing camp. We study these settings which capture the adversarial behavior of a camp towards another camp.

In the above settings, it did not matter whether the camps strategized simultaneously or sequentially. The equilibrium concept that we use to analyze the game, is Nash equilibrium. However, there could be settings where a sequential play would be more natural than a simultaneous one, which would result in a Stackelberg game. The sequence may be determined by a mediator or central authority which, for example, may be responsible for giving permissions for campaigning or scheduling product advertisements to be presented to an individual. The equilibrium concept we use for the game in these sequential play settings would be subgame perfect Nash equilibrium; also since we are concerned with zero-sum game, we will be wording the equilibrium in terms of maxmin or

minmax. Assuming the good camp plays first (without loss of generality), the bad camp would choose a strategy that minimizes the sum of opinion values as a best response to the good camp's strategy. Knowing this, the good camp would want to maximize this minimum value. We motivate two such settings that we study in this paper.

It would often be the case that the total attention capacity of a node or the time it could allot for receiving campaigning from both camps combined, is bounded. This leads us to study the game under common coupled constraints (CCC) that the sum of investments by the camps on any node is bounded. These are called common coupled constraints since the constraints of one camp are satisfied if and only if the constraints of the other camp are satisfied for every strategy profile.

Another sequential setting could lead to uncertainty, where the good camp (which plays first) may not have exact information regarding the network parameters. However, the bad camp (which plays second) would have perfect information regarding these parameters, which are either revealed over time or deduced based on the effect of the good camp's investment. Forecasting the optimal strategy of bad camp, we derive a robust strategy of the good camp which would give it a good payoff even in the worst case.

It can be noted that the common coupled constraints setting captures the first mover advantage, while the uncertainty setting captures the first mover disadvantage.

## B. Related Work

A principal part of opinion dynamics in a population is how agents update their opinions over time. One of the most well accepted and well studied approaches of updating an agent's opinion is based on imitation where each agent adopts the opinion of some of its neighbors with a certain probability. One such example is the DeGroot model [8] where each agent updates its opinion using a weighted convex combination of its neighbors' opinions. The model developed by Friedkin and Johnsen [9], [10] considers that, in addition to its neighbors' opinions, an agent also gives certain weightage to its initial opinion.

Acemoglu and Ozdaglar [4] review several other models of opinion dynamics. Lorenz [11] surveys modeling frameworks concerning continuous opinion dynamics under bounded confidence, wherein agents pay more attention to beliefs that do not differ too much from their own. Xia, Wang, and Xuan [12] give a multidisciplinary review of the field of opinion dynamics as a combination of the social processes which are conventionally studied in social sciences, and the analytical and computational tools developed in mathematics, physics and complex system studies. Das, Gollapudi, and Munagala [13] show that the widely studied theoretical models of opinion dynamics do not explain their experimental observations, and hence propose a new model as a combination of the DeGroot model and the Voter model [14], [15]. Parsegov et al. [16] develop a multidimensional extension of Friedkin-Johnsen model, describing the evolution of the agents' opinions on several interdependent topics, and analyze its convergence.

Ghaderi and Srikant [17] consider a setting where an agent iteratively updates its opinion as a myopic best response to the opinions of its own and its neighbors, and hence study how the equilibrium and convergence to it depend on the network structure, initial opinions of the agents, the location of stubborn agents (forceful agents with unchanging opinions) and the extent of their stubbornness. Ben-Ameur, Bianchi, and Jakubowicz [18] analyze the convergence of some widespread gossip algorithms in the presence of stubborn agents and show that the network is driven to a state which exclusively depends on the stubborn agents. Jia et al. [19] propose an empirical model combining the DeGroot and Friedkin models, and hence study the evolution of self-appraisal, social power, and interpersonal influences for a group of individuals who discuss and form opinions. Halu et al. [20] consider the case of two interacting social networks, and hence study the case of political elections using simulations.

Yildiz, Ozdaglar, and Acemoglu [21] study the problem of optimal placement of stubborn agents in the discrete binary opinions setting, so as to have maximum influence on the society, assuming the location of the competing stubborn agents to be given. Gionis, Terzi, and Tsaparas [1] study the problem of identifying a set of target individuals whose positive opinions about an information item would maximize the overall positive opinion for the item in the social network, from an algorithmic and experimental perspective. Ballester, Calvó-Armengol, and Zenou [22] study optimal targeting by analyzing a noncooperative network game with local payoff complementarities. Sobehy et al. [23] propose strategies to win an election using a Mixed Integer Linear Programming approach.

The basic model we study is similar to that studied by Grabisch et al. [2], that is, a zero-sum game with two camps holding distinct binary opinion values, aiming to select a set of nodes to invest on, so as to influence the average opinion that eventually emerges in the network. Their study, however, considers non-negative matrices and focuses on the existence and the characterization of equilibria in a preliminary setting, where the influence and cost functions are linear, camps have network information with certainty, and there is no bound on combined investment by the camps per node. Dubey, Garg, and De Meyer [6] study existence and uniqueness of Nash equilibrium, while also considering convex cost functions. The study, however, does not consider the possibility of bounded investment on a node and the implications on the extent of skewness of investment owing to the convexity of cost functions. Bimpikis, Ozdaglar, and Yildiz [7] provide a sharp characterization of the optimal targeted advertising strategies and highlight their dependence on the underlying social network structure, in a preliminary setting. Their study also emphasizes the effect of absorption centrality, which forms an important part of our study.

The problem of maximizing information diffusion in social networks under popular models such as Independent Cascade and Linear Threshold, has been extensively studied [3], [24], [25]. The competitive setting has resulted in several game theoretic studies of this problem [26], [27], [28]. There have been preliminary studies addressing interaction among

different informations, where the spread of one information influences the spread of the others [29], [30].

There have been studies considering games with constraints. A notable study by Rosen [31] shows existence of equilibrium in a constrained game, and its uniqueness in a strictly concave game. Altman and Solan [32] study constrained games, where the strategy set available to a player depends on the choice of strategies made by other players. The authors show that, in constrained zero-sum games, the value of the game need not exist (that is, maxmin and minmax values need not be the same) and contrary to general functions, maxmin value can be larger than minmax.

The topic of decision under uncertainty has been of interest to the game theory and optimization communities. One possible way of analyzing decision under uncertainty is using robust optimization tools. Ben-Tal, El Ghaoui, and Nemirovski [33] present a thorough review of these tools.

### C. Contributions of the Paper

A primary goal of this work is to provide a broad framework for optimal investment strategies for competing camps in a social network, and propose and explore several aspects of the problem. In particular, we study several well-motivated variants of a constrained zero-sum game where two competing camps aim to maximize the adoption of their respective opinions, under the DeGroot-Friedkin model of opinion dynamics. Our specific contributions are as follows:

- We show that a random investment strategy is optimal when the underlying network follows a particular popular class of weight distributions. (Section III-C)
- We further investigate when the influence of a camp on a node is a concave function of its investment on that node, for the cases of both unbounded and bounded investment per node, and hence provide implications for extent of skewness of the investment strategies. (Section IV)
- We look at the complementary problem where a camp acts as an adversary to the competing camp by aiming to maximize the latter's investment. We also look at the problem where a camp aims to maximize the deviation from the desired investment of the competing camp. (Section V)
- We study the problem under common coupled constraints that the combined investment by the good and bad camps on any given node cannot exceed a certain limit. We study the maxmin and minmax values in this setting and present some interesting implications. (Section VI)
- We look at the setting where one of the camps would need to make decision under uncertainty. (Section VII)
- Using simulations, we illustrate our analytically derived results on real-world social networks and present more insights based on the observations. (Section VIII)

## II. MODEL

Consider a social network with  $N$  as its set of nodes and  $E$  as its set of weighted and directed edges. Two competing camps (good and bad) aim at maximizing the adoption of their respective opinions in the social network. We consider a strict

TABLE I  
NOTATION TABLE

$v_i^0$	the initial opinion of node $i$
$w_{ii}^0$	weightage given to the initial opinion by node $i$
$w_{ig}$	weightage given by node $i$ to the good camp's opinion
$w_{ib}$	weightage given by node $i$ to the bad camp's opinion
$w_{ij}$	weightage given by node $i$ to the opinion of node $j$
$x_i$	investment made by good camp to directly influence node $i$
$y_i$	investment made by bad camp to directly influence node $i$
$k_g$	budget of the good camp
$k_b$	budget of the bad camp
$v_i$	the resulting opinion of node $i$

competition setting where the opinion value of the good camp is denoted by +1 and that of the bad camp by -1. We now present the parameters of the considered model of opinion dynamics and the update rule, followed by the underlying game and the various settings we study.

#### A. Parameters

Prior to the process of opinion dynamics, every node holds a bias in opinion which could have been formed owing to various factors such as the node's fundamental views, its experiences, past information from news and other sources, opinion dynamics in the past, etc. We denote this opinion bias of a node  $i$  by  $v_i^0$  and the weightage that the node attributes to it by  $w_{ii}^0$ .

The good and bad camps attempt to directly influence the nodes so that their opinions are driven towards being positive and negative, respectively. This direct influence depends on the investment or effort made by the camps, and also on how much a node weighs the camps' opinions. A given amount of investment may have different influence on different nodes based on how much these nodes weigh the camps' recommendations. We denote the investment made by the good and bad camps on node  $i$  by  $x_i$  and  $y_i$  respectively, and the weightage that node  $i$  attributes to them by  $w_{ig}$  and  $w_{ib}$  respectively. Since the influence of good camp (opinion of +1) on node  $i$  should be an increasing function of both  $x_i$  and  $w_{ig}$ , we assume the influence to be  $w_{ig}x_i$  for simplicity (maintaining the linearity of DeGroot-Friedkin model). Similarly,  $w_{ib}y_i$  is the influence of bad camp on node  $i$ . Also note that since the good and bad camps hold the opinions +1 and -1 respectively, the net influence owing to the direct recommendations from these camps is  $(w_{ig}x_i - w_{ib}y_i)$ .

Both the camps have budget constraints stating that the good camp can invest a total amount of  $k_g$  across all the nodes, while the bad camp can invest a total amount of  $k_b$ .

The network effect is captured by how much a node is influenced by each of its friends or connections, that is, how much weightage is attributed by a node to the opinion of each of its connections. Let  $v_j$  be the opinion held by node  $j$  and  $w_{ij}$  be the weightage attributed by node  $i$  to the opinion

of node  $j$ . Consistent with the DeGroot-Friedkin model, we assume the influence on node  $i$  owing to node  $j$ , to be  $w_{ij}v_j$ , thus the net influence owing to all of its connections being  $\sum_{j \neq i} w_{ij}v_j$ . We denote the weightage attributed by node  $i$  to its own opinion  $v_i$ , by  $w_{ii}$ . It is to be noted that we do not make any assumption regarding the sign of the edge weights, that is, they could be negative as well (as justified in [34], [35]). A negative edge weight  $w_{ij}$  can be interpreted as some form of distrust that node  $i$  holds on node  $j$ , that is,  $i$  would be driven towards adopting an opinion that is opposite to that held or suggested by  $j$ .

Table I presents the required notation. Consistent with the standard opinion dynamics models, we have the condition on the influence weights on any node  $i$  that they sum to at maximum 1 (since a node updates its opinions using a weighted 'convex' combination of the influencing factors).

$$\forall i, \quad |w_{ii}^0| + |w_{ii}| + \sum_{j \neq i} |w_{ij}| + |w_{ig}| + |w_{ib}| \leq 1$$

A standard assumption for guaranteeing convergence of the dynamics is that

$$|w_{ii}| + \sum_{j \neq i} |w_{ij}| < 1$$

This natural assumption is actually well suited for our model where we generally would have non-zero weights allotted to the influence outside of the network, namely, the influence due to bias ( $w_{ii}^0$ ) and campaigning ( $w_{ig}, w_{ib}$ ).

#### B. Opinion Update Rule

Let the nodes update their opinions in discrete time steps starting with time step 0. Let  $v_i^{(\tau)}$  be the opinion of node  $i$  at time step  $\tau$ , and  $v_i^{(0)} = v_i^0$ . With the aforementioned factors into consideration, each node  $i$  updates its opinion at each step, using the following update rule (which can be viewed as an extension of the DeGroot-Friedkin update rule):

$$v_i^{(\tau)} = w_{ii}^0 v_i^0 + w_{ii} v_i^{(\tau-1)} + \sum_{j \neq i} w_{ij} v_j^{(\tau-1)} + w_{ig} x_i - w_{ib} y_i \quad (1)$$

For any given node  $i$ , the static components are  $x_i, y_i, v_i^0$  (which are weighted by  $w_{ig}, w_{ib}, w_{ii}^0$ ), while the dynamic components are  $v_j$ 's (weighted by  $w_{ij}$ 's). The static components remain unchanged while the dynamic ones get updated in every time step.

Let  $\mathbf{w}$  be the matrix consisting of the elements  $w_{ij}$  for each pair  $(i, j)$  (note that  $\mathbf{w}$  contains only the network weights and not  $w_{ig}, w_{ib}, w_{ii}^0$ ). Let  $\mathbf{v}$  be the vector consisting of the opinions  $v_i$ ,  $\mathbf{v}^0$  and  $\mathbf{w}^0$  be the vectors consisting of the elements  $v_i^0$  and  $w_{ii}^0$  respectively,  $\mathbf{x}$  and  $\mathbf{y}$  be the vectors consisting of the investments  $x_i$  and  $y_i$  respectively,  $\mathbf{w}_g$  and  $\mathbf{w}_b$  be the vectors consisting of the weights  $w_{ig}$  and  $w_{ib}$  respectively. Let the operation  $\circ$  denote Hadamard product (elementwise product) of vectors, that is,  $(\mathbf{a} \circ \mathbf{b})_i = a_i b_i$ . Let Hadamard power be expressed as  $(\mathbf{a}^{\circ p})_i = a_i^p$ .

Assuming  $\mathbf{v}^{(\tau)}$  to be the vector consisting of the opinions  $v_i^{(\tau)}$ , the update rule (1) can be written in matrix form as

$$\mathbf{v}^{(\tau)} = \mathbf{w} \mathbf{v}^{(\tau-1)} + \mathbf{w}^0 \circ \mathbf{v}^0 + \mathbf{w}_g \circ \mathbf{x} - \mathbf{w}_b \circ \mathbf{y} \quad (2)$$

**Proposition 1.** *The dynamics defined by the update rule in (2) converges to  $\mathbf{v} = (\mathbf{I} - \mathbf{w})^{-1}(\mathbf{w}^0 \circ \mathbf{v}^0 + \mathbf{w}_g \circ \mathbf{x} - \mathbf{w}_b \circ \mathbf{y})$ .*

*Proof.* The recursion in (2) can be simplified as

$$\begin{aligned} \mathbf{v}^{(\tau)} &= \mathbf{w}\mathbf{v}^{(\tau-1)} + \mathbf{w}^0 \circ \mathbf{v}^0 + \mathbf{w}_g \circ \mathbf{x} - \mathbf{w}_b \circ \mathbf{y} \\ &= \mathbf{w} \left( \mathbf{w}\mathbf{v}^{(\tau-2)} + \mathbf{w}^0 \circ \mathbf{v}^0 + \mathbf{w}_g \circ \mathbf{x} - \mathbf{w}_b \circ \mathbf{y} \right) \\ &\quad + \mathbf{w}^0 \circ \mathbf{v}^0 + \mathbf{w}_g \circ \mathbf{x} - \mathbf{w}_b \circ \mathbf{y} \\ &= \mathbf{w}^2 \mathbf{v}^{(\tau-2)} + (\mathbf{I} + \mathbf{w})(\mathbf{w}^0 \circ \mathbf{v}^0 + \mathbf{w}_g \circ \mathbf{x} - \mathbf{w}_b \circ \mathbf{y}) \\ &= \mathbf{w}^3 \mathbf{v}^{(\tau-3)} + (\mathbf{I} + \mathbf{w} + \mathbf{w}^2)(\mathbf{w}^0 \circ \mathbf{v}^0 + \mathbf{w}_g \circ \mathbf{x} - \mathbf{w}_b \circ \mathbf{y}) \end{aligned}$$

When  $\tau \rightarrow \infty$ , the procedure can be continued to arrive at

$$\mathbf{v}^{(\tau)} = \lim_{\eta \rightarrow \infty} \mathbf{w}^\eta \mathbf{v}^0 + \left( \sum_{\eta=0}^{\infty} \mathbf{w}^\eta \right) (\mathbf{w}^0 \circ \mathbf{v}^0 + \mathbf{w}_g \circ \mathbf{x} - \mathbf{w}_b \circ \mathbf{y}) \quad (3)$$

Note that  $\mathbf{w}$  is a strictly substochastic matrix, with the sum of each of its rows being strictly less than 1. Hence the spectral radius of  $\mathbf{w}$  is less than 1, implying that

$$\begin{aligned} \{\mathbf{w}^\eta\}_{\eta \rightarrow \infty} &\rightarrow 0 \\ \Rightarrow (\mathbf{I} - \mathbf{w})(\mathbf{I} + \mathbf{w} + \mathbf{w}^2 + \dots) &= \mathbf{I} \\ \Rightarrow (\mathbf{I} - \mathbf{w})^{-1} &= \mathbf{I} + \mathbf{w} + \mathbf{w}^2 + \dots = \sum_{\eta=0}^{\infty} \mathbf{w}^\eta \end{aligned}$$

This further implicitly means that  $(\mathbf{I} - \mathbf{w})$  is invertible. Also, since  $\mathbf{v}^0$  is a constant, we have  $\lim_{\eta \rightarrow \infty} \mathbf{w}^\eta \mathbf{v}^0 = 0$ . Hence as  $\tau \rightarrow \infty$ , Equation (3) can be written as

$$\mathbf{v}^{(\tau)} = (\mathbf{I} - \mathbf{w})^{-1}(\mathbf{w}^0 \circ \mathbf{v}^0 + \mathbf{w}_g \circ \mathbf{x} - \mathbf{w}_b \circ \mathbf{y})$$

which is a constant row vector, that is, the dynamics converges to this steady state of opinion values.  $\square$

### III. THE FUNDAMENTAL PROBLEM

We now present the fundamental problem of competitive opinion dynamics under the DeGroot-Friedkin model.

#### A. Introduction of the Fundamental Problem

The problem of maximizing opinion adoption can be modeled as an optimization problem. In particular, owing to the perfect competition assumption, this problem can be modeled as the following maxmin problem. Here our objective is to determine our strategy (the values of  $x_i$  such that they satisfy certain constraints) so that we maximize the average opinion value, knowing that the competitor would play its best response strategy (the values of  $y_i$  which also need to satisfy certain constraints). Assuming linear constraints for setting the problem in the framework of linear program, we represent these constraints by  $\mathbf{Ax} \leq \mathbf{b}$  and  $\mathbf{Cy} \leq \mathbf{d}$ , respectively, where  $\mathbf{A}, \mathbf{C}$  are matrices and  $\mathbf{b}, \mathbf{d}$  are vectors.

Owing to  $x_i$  and  $y_i$  being investments, we have the natural constraints that  $x_i, y_i \geq 0, \forall i$ . We can hence write the maxmin optimization problem in its general form as

$$\begin{aligned} \max_{\mathbf{x} \geq 0} \min_{\mathbf{y} \geq 0} \sum_i v_i \\ \text{s.t. } \forall i: v_i - \sum_{j \neq i} w_{ij} v_j - w_{ii} v_i + w_{ib} y_i = w_{ig} x_i + w_{ii}^0 v_i^0 \end{aligned}$$

From Proposition 1, we have

$$\begin{aligned} \mathbf{v} &= (\mathbf{I} - \mathbf{w})^{-1}(\mathbf{w}^0 \circ \mathbf{v}^0 + \mathbf{w}_g \circ \mathbf{x} - \mathbf{w}_b \circ \mathbf{y}) \\ \Rightarrow \mathbf{1}^T \mathbf{v} &= \mathbf{1}^T (\mathbf{I} - \mathbf{w})^{-1}(\mathbf{w}^0 \circ \mathbf{v}^0 + \mathbf{w}_g \circ \mathbf{x} - \mathbf{w}_b \circ \mathbf{y}) \end{aligned}$$

Let the constant  $\mathbf{1}^T (\mathbf{I} - \mathbf{w})^{-1} = (((\mathbf{I} - \mathbf{w})^{-1})^T \mathbf{1})^T = ((\mathbf{I} - \mathbf{w}^T)^{-1} \mathbf{1})^T = \mathbf{r}^T$ . So the above is equivalent to

$$\sum_i v_i = \sum_i r_i (w_{ig} x_i + w_{ii}^0 v_i^0) - \sum_i r_i w_{ib} y_i \quad (4)$$

So the optimization problem becomes

$$\max_{\substack{\mathbf{Ax} \leq \mathbf{b} \\ \mathbf{x} \geq 0}} \min_{\substack{\mathbf{Cy} \leq \mathbf{d} \\ \mathbf{y} \geq 0}} \sum_i r_i (w_{ig} x_i + w_{ii}^0 v_i^0) - \sum_i r_i w_{ib} y_i$$

This can be achieved by solving two independent optimization problems, namely,

$$\max_{\substack{\mathbf{Ax} \leq \mathbf{b} \\ \mathbf{x} \geq 0}} \sum_i r_i w_{ig} x_i \quad \text{and} \quad \min_{\substack{\mathbf{Cy} \leq \mathbf{d} \\ \mathbf{y} \geq 0}} \sum_i r_i w_{ib} y_i$$

which can be easily solved.

For studying the problem in a broader framework, we consider the special case that we introduced in Section II.

1) *The Special Case of Overall Budget Constraints:* The special case in Section II that discusses overall budget constraints  $k_g$  and  $k_b$  for the good and bad camps respectively, corresponds to  $\sum_i x_i \leq k_g$  and  $\sum_i y_i \leq k_b$ . That is, we have  $\mathbf{A} = \mathbf{C} = \mathbf{1}^T, \mathbf{b} = k_g, \mathbf{d} = k_b$ .

The solution to the optimization problem is

$$\begin{aligned} x_i^* &= 0, \quad \forall i \notin \arg \max_i r_i w_{ig} \\ \sum_{i \in \arg \max_i r_i w_{ig}} x_i^* &= k_g, \quad \text{if } \max_i r_i w_{ig} > 0 \\ x_i^* &\geq 0 \end{aligned} \quad (5)$$

$$\begin{aligned} \text{and} \quad y_i^* &= 0, \quad \forall i \notin \arg \max_i r_i w_{ib} \\ \sum_{i \in \arg \max_i r_i w_{ib}} y_i^* &= k_b, \quad \text{if } \max_i r_i w_{ib} > 0 \\ y_i^* &\geq 0 \end{aligned} \quad (6)$$

Note that if  $\max_i r_i w_{ig} \leq 0$ , then  $x_i^* = 0, \forall i$  and if  $\max_i r_i w_{ib} \leq 0$ , then  $y_i^* = 0, \forall i$ .

Equations (6) and (5) lead to the following result.

**Proposition 2.** *In Setting III-A1, it is optimal for the good and bad camps to invest their budgets in node  $i$  with maximum value of  $r_i w_{ig}$  and  $r_i w_{ib}$  respectively, subject to the value being positive.*

**Insight 1.** *The parameter  $r_i$  could be viewed as the influence power of node  $i$  on the network, while  $w_{ig}$  and  $w_{ib}$  are respectively the influence powers of the good and bad camps on node  $i$ . So it is clear why these parameters factor into the result. Furthermore, the strategies of the players are independent which arises from the sum of steady state values of nodes as derived in (4). The linearity of the model and unconstrained investment on nodes allow both the camps to exhaust their budgets by concentrating their entire investments on one node possessing the highest value of  $r_i w_{ig}$  or  $r_i w_{ib}$  respectively. Also, the camps' strategies are independent of the initial opinions, since we aim to optimize the sum of the opinion values without considering their relative values.*

It can be seen that  $r_i$  acts as the absorption centrality of node  $i$  [7]. It can also be viewed as a variant of Katz centrality [36]. Katz centrality of node  $i$  is defined as the  $i^{\text{th}}$  element of vector  $\left( (\mathbf{I} - \alpha \mathcal{A}^T)^{-1} - \mathbf{I} \right) \mathbf{1}$  where  $\mathcal{A}$  is the adjacency matrix and  $0 < \alpha < \frac{1}{|\rho|}$  where  $\rho$  is the largest eigenvalue of  $\mathcal{A}$ . In our case,  $\mathcal{A}$  is replaced by the weighted adjacency matrix  $\mathbf{w}$ , for which  $|\rho| < 1$ , and we have  $\alpha = 1$ . The subtraction of the vector  $\mathbf{1}$  is common for all nodes, thus its relative effect can be ignored. Also, recall that

$$\mathbf{r}^T = \mathbf{1}^T (\mathbf{I} - \mathbf{w})^{-1} = \mathbf{1}^T \left( \mathbf{I} + \sum_{\eta=1}^{\infty} \mathbf{w}^\eta \right)$$

So if we have  $w_{ij} \geq 0$  for all pairs of nodes  $(i, j)$ , we will have that all elements of vector  $\mathbf{r}$  are at least 1. That is,  $w_{ij} \geq 0, \forall (i, j) \implies r_i \geq 1, \forall i$ .

2) *The Case of Bounded Investment Per Node*: This setting, as motivated earlier, includes additional constraints on the amount of investment per node by a camp:  $x_i, y_i \leq 1, \forall i$ . With respect to generic constraints, this is a special case with  $A = C = \begin{pmatrix} \mathbf{1}^T \\ \mathbf{I} \end{pmatrix}$ ,  $b = \begin{pmatrix} k_g \\ \mathbf{1} \end{pmatrix}$ ,  $d = \begin{pmatrix} k_b \\ \mathbf{1} \end{pmatrix}$ .

From Equation (4), an optimal  $\mathbf{x}$  can be obtained as follows. Let  $\mathbb{I}_{r_i w_{ig} > 0} = 1$  if  $r_i w_{ig} > 0$ , and 0 otherwise. Let  $\omega_1, \omega_2, \dots, \omega_n$  be the ordering of nodes in decreasing values of  $r_i w_{ig}$  with any tie-breaking rule. So (4) is maximized with respect to  $\mathbf{x}$  when

$$\begin{aligned} x_i &= 1 \cdot \mathbb{I}_{r_i w_{ig} > 0}, \text{ for } i = \omega_1, \dots, \omega_{\lfloor k_g \rfloor} \\ x_i &= (k_g - \lfloor k_g \rfloor) \cdot \mathbb{I}_{r_i w_{ig} > 0}, \text{ for } i = \omega_{\lfloor k_g \rfloor + 1} \\ x_i &= 0, \text{ for } i = \omega_{\lfloor k_g \rfloor + 2}, \dots, \omega_n \end{aligned}$$

An optimal  $\mathbf{y}$  is analogous, hence the following result.

**Proposition 3.** *In Setting III-A2, it is optimal for the good camp to invest in nodes one at a time, subject to a maximum investment of 1 per node, in decreasing order of values of  $r_i w_{ig}$  until either the budget  $k_g$  is exhausted or we reach a node with a non-positive value of  $r_i w_{ig}$ . The optimal strategy of the bad camp is analogous.*

Also, from Proposition 1, if a camp's investment per node is bounded by 1 unit, the opinion value of every node would be bounded between  $-1$  and  $+1$ . As stated earlier, such bounded opinion value is relevant to elections and product adoption examples, where the bounded opinion value of a node could be translated into the probability of the node voting for a camp or adopting a particular product.

### B. Maxmin versus Minmax Values

With no bounds on investment per node, it is clear that the maxmin and minmax values are the same, since the strategies of the camps are mutually independent, that is,

$$\max_{\mathbf{x} \geq 0} \min_{\mathbf{y} \geq 0} \sum_i v_i = \min_{\mathbf{y} \geq 0} \max_{\mathbf{x} \geq 0} \sum_i v_i \quad (7)$$

The equality would hold even with mutually independent constraints on the camps' investment on a node, that is,

$$\max_{0 \leq x \leq 1} \min_{0 \leq y \leq 1} \sum_i v_i = \min_{0 \leq y \leq 1} \max_{0 \leq x \leq 1} \sum_i v_i \quad (8)$$

We cannot compare the values in (7) and (8), in general. For instance, if all  $i$ 's have equal values of  $r_i w_{ib}$  and only one  $i$  has good value of  $r_i w_{ig}$ , then for  $k_g > 1$ , the value in (7) would be greater than that in (8). This can be seen using Equation (4); the value of  $\sum_i r_i (w_{ii}^0 v_i^0 - w_{ib} y_i)$  would stay the same while the value of  $\sum_i r_i w_{ig} x_i$  would be higher in (7) than in (8). On the other hand, if all  $i$ 's have equal values of  $r_i w_{ig}$  and only one  $i$  has good value of  $r_i w_{ib}$ , then for  $k_b > 1$ , the value in (8) would be greater than that in (7).

### C. Result for a Popular Class of Weight Distributions

We now present a result concerning a class of distribution of edge weights in a network, which includes the popular weighted cascade (WC) model.

**Proposition 4.** *Let  $N_i = \{j : w_{ij} \neq 0\}$ ,  $d_i = |N_i|$ , and  $j \in N_i \iff i \in N_j$ . If  $\forall i, w_{ig} = w_{ib} = w_{ii}^0 = \frac{1}{\alpha + d_i} = w_{ij}, \forall j \in N_i$ , where  $\alpha > 0$ , then  $r_i w_{ig} = r_i w_{ib} = \frac{1}{\alpha}, \forall i$ .*

*Proof.* We know that

$$\begin{aligned} (\mathbf{I} - \mathbf{w}^T) \mathbf{r} &= \mathbf{1} \\ \iff \mathbf{r} &= \mathbf{1} + \mathbf{w}^T \mathbf{r} \\ \iff \forall i : r_i &= 1 + \sum_{j \in N_i} w_{ji} r_j = 1 + \sum_{j \in N_i} \left( \frac{1}{\alpha + d_j} \right) r_j \end{aligned}$$

Let us assume  $r_i = \gamma(\alpha + d_i)$ , where  $\gamma$  is some constant. If this satisfies the above equation, the uniqueness of  $r_i$  ensures that it is the only solution. Hence we have

$$\begin{aligned} \forall i : \gamma(\alpha + d_i) &= 1 + \sum_{j \in N_i} \gamma = 1 + \gamma d_i \\ \iff \gamma &= \frac{1}{\alpha} \end{aligned}$$

$$\therefore \forall i : r_i w_{ig} = r_i w_{ib} = \frac{\alpha + d_i}{\alpha} \cdot \frac{1}{\alpha + d_i} = \frac{1}{\alpha}$$

□

The above result implies that models which assign weights for all  $i$  such that  $w_{ig} = w_{ib} = w_{ii}^0 = \frac{1}{\alpha + d_i} = w_{ij}, \forall j \in N_i$ , are suitable for the use of a random strategy, since the decision parameters for the two camps ( $r_i w_{ig}$  and  $r_i w_{ib}$ ) are the same for each node  $i$ . That is, in these models, a random strategy that exhausts the entire budget is optimal. These models include the popular weighted cascade model, which would assign the weights this way with  $\alpha = 3$ .

## IV. EFFECT OF CONCAVE INFLUENCE FUNCTION

The linear influence function (1) without any bounds on investment per node leads to a solution which suggests concentrating the investment on a single node (Proposition 2). A concave influence function would account for the diminishing influence of a camp with increasing investment on a node, which would advise against such concentrated investment. For the purpose of our analysis so as to arrive at more specific results, we consider functions of the form  $x^{1/t}$  as the influence when the investment is  $x$ . It is to be noted, however, that it can be extended to other concave functions since we use a common framework of convex optimization.

### A. The Case of Unbounded Investment per Node

$$\begin{aligned} & \max_{\substack{x_i \leq k_g \\ x_i \geq 0}} \min_{\substack{y_i \leq k_b \\ y_i \geq 0}} \sum_i v_i \\ \text{s.t. } \forall i: & v_i - \sum_{j \neq i} w_{ij} v_j - w_{ii} v_i + w_{ib} y_i^{1/t} = w_{ig} x_i^{1/t} + w_{ii}^0 v_i^0 \end{aligned}$$

**Proposition 5.** *In Setting IV-A, for  $t > 1$ , it is optimal for the good and bad camps to invest in node  $i$  proportional to  $(r_i w_{ig})^{\frac{t}{t-1}}$  and  $(r_i w_{ib})^{\frac{t}{t-1}}$ , subject to positivity of  $r_i w_{ig}$  and  $r_i w_{ib}$  respectively.*

A proof of Proposition 5 is provided in Appendix A.

**Remark 1** (A Note on Skewness of Investment). *When we compare the results for lower and higher values of  $t$ , the investment made by the good camp has an exaggerated correlation with the value of  $r_i w_{ig}$  for lower values of  $t$ . In particular, the investment made is very skewed towards nodes with high values of  $r_i w_{ig}$  when  $t$  is very low, while it is proportional to  $r_i w_{ig}$  when  $t$  is very high. Note that  $t = 1$  corresponds to the linear case in Setting III-A1 where the investment is extremely skewed with each camp investing its entire budget on only one node.*

The implication of skewness can be linked to user perception of fairness [37]. Suppose a node  $p$  is such that  $r_p w_{pg} = \max_i r_i w_{ig}$ , and it is the unique node with this maximum value, and suppose a node  $q$  is such that  $r_q w_{qg} = r_p w_{pg} - \epsilon$ , where  $\epsilon$  is positive and infinitesimal. As per node  $q$ , the investment strategy would be fair if the investment in  $q$  is not much less than that in  $p$ , since they are almost equally valuable. However,  $t = 1$  leads to a highly skewed investment where  $p$  receives  $k_g$  and  $q$  receives 0, which can be perceived as unfair by node  $q$ . As  $t$  increases, the investment becomes less skewed; in particular,  $t \rightarrow \infty$  leads to investment on a node  $i$  to be proportional to  $r_i w_{ig}$ , which could be perceived as fair by the nodes.

### B. The Case of Bounded Investment Per Node

With the additional constraints  $x_i \leq 1$  and  $y_i \leq 1, \forall i$ , the optimal investment strategies are given by Proposition 6. We provide its proof in Appendix B.

**Proposition 6.** *Let  $\hat{\gamma} > 0$  be the solution of*

$$\sum_{i: r_i w_{ig} \in (0, t\hat{\gamma})} \left( \frac{r_i w_{ig}}{t\hat{\gamma}} \right)^{\frac{t}{t-1}} + \sum_{i: r_i w_{ig} > t\hat{\gamma}} 1 = k_g$$

*(It can be shown that  $\hat{\gamma}$  exists and is unique). Then in Setting IV-B, it is optimal for the good camp to follow the following investment strategy*

$$\begin{aligned} x_i^* &= 0, \text{ if } r_i w_{ig} \leq 0 \\ x_i^* &= 1, \text{ if } r_i w_{ig} > t\hat{\gamma} \\ x_i^* &= \left( k_g - \sum_{i: r_i w_{ig} > t\hat{\gamma}} 1 \right) \left( \frac{(r_i w_{ig})^{\frac{t}{t-1}}}{\sum_{i: r_i w_{ig} \in (0, t\hat{\gamma})} (r_i w_{ig})^{\frac{t}{t-1}}} \right), \\ & \text{if } r_i w_{ig} \in (0, t\hat{\gamma}) \end{aligned}$$

*If there does not exist a  $\hat{\gamma} > 0$  (because the number of nodes with  $r_i w_{ig} > 0$  is less than  $k_g$ ), we invest 1 on all nodes with  $r_i w_{ig} > 0$  and 0 on all other nodes. The optimal strategy of the bad camp is analogous.*

Note that for  $r_i w_{ig} \in (0, t\hat{\gamma}]$ , we can alternatively write  $x_i^* = \left( \frac{r_i w_{ig}}{t\hat{\gamma}} \right)^{\frac{t}{t-1}}$ , which would be between 0 and 1. So the nodes with positive values of  $r_i w_{ig}$  should be classified into two sets, that containing nodes with  $r_i w_{ig} \in (0, t\hat{\gamma}]$  (for which  $x_i^* \in (0, 1]$ ) and that containing nodes with  $r_i w_{ig} > t\hat{\gamma}$  ( $x_i^*$  forcefully limited to 1). So we can effectively start with all nodes in the former set (meaning  $t\hat{\gamma} \geq \max_i r_i w_{ig}$ ) and then transfer nodes to the latter set as per descending values of  $r_i w_{ig}$  (as we reduce  $t\hat{\gamma}$ ), until we have two sets, one with  $x_i^* = \left( \frac{r_i w_{ig}}{t\hat{\gamma}} \right)^{\frac{t}{t-1}} = \left( \frac{(r_i w_{ig})^{\frac{t}{t-1}}}{\sum_{i: r_i w_{ig} \in (0, t\hat{\gamma})} (r_i w_{ig})^{\frac{t}{t-1}}} \right) \leq 1$  and the other with  $x_i^*$  forcefully limited to 1.

**Insight 2.** *The solution suggests that the optimal solution can be obtained using a trial-and-error iterative process. A camp could use the optimal strategy for the unbounded case suggested in Proposition 5. If for any  $i$ , we get  $x_i^* > 1$ , we assign  $x_i^* = 1$  to node  $i$  with the highest value of  $r_i w_{ig}$ , and use Proposition 5 again by excluding node  $i$  and decrementing the available budget by 1. This process would be repeated until  $x_i^* \leq 1, \forall i$ .*

## V. ACTING AS COMPETITOR'S ADVERSARY

In this setting, a camp explicitly acts to maximize the competitor's investment or deviation from its desired investment required to drive the sign of the average opinion value of the population in its favor. Without loss of generality, we consider that the good camp acts as the adversary.

### A. The Case of Unbounded Investment per Node

$$\begin{aligned} & \max_{\substack{x_i \leq k_g \\ x_i \geq 0}} \min_{y_i \geq 0} \sum_i y_i \\ \text{s.t. } & \sum_i v_i \leq 0 \end{aligned}$$

$$\sum_i v_i = \sum_i r_i (w_{ig} x_i + w_{ii}^0 v_i^0) - \sum_i r_i w_{ib} y_i$$

**Proposition 7.** *In Setting V-A, it is optimal for the good camp to invest its budget in node  $i$  with maximum value of  $r_i w_{ig}$ , subject to it being positive. For the bad camp, it is optimal to invest the minimum amount in node  $i$  with maximum value of  $r_i w_{ib}$ , subject to its positivity, required to draw the average opinion value in its favor. (If there does not exist any node  $i$  with positive value of  $r_i w_{ib}$ , it is optimal for the bad camp to not invest at all). The investment made by the bad camp is*

$$\max \left\{ \frac{1}{\max_j w_{jb} r_j} \left( k_g \max\{\max_i r_i w_{ig}, 0\} + \sum_i r_i w_{ii}^0 v_i^0 \right), 0 \right\}$$

A proof of Proposition 7 is provided in Appendix C.



**Remark 2** (Maximizing Competitor's Deviation). *Let the desired investments for the good and bad camps be  $(\bar{x}_i)$  and  $(\bar{y}_i)$ , respectively. Thus the optimization problem is*

$$\max_{\substack{\sum_i (x_i - \bar{x}_i)^2 \leq k_g \\ x_i \geq 0}} \min_{y_i \geq 0} \sum_i (y_i - \bar{y}_i)^2$$

$$\text{s.t. } \sum_i v_i \leq 0$$

$$\forall i: v_i - \sum_{j \neq i} w_{ij} v_j - w_{ii} v_i + w_{ib} y_i = w_{ig} x_i + w_{ii}^0 v_i^0$$

Let  $\hat{\gamma} > 0$  be the solution of

$$\sum_{\substack{i: r_i w_{ig} \\ \geq -2\gamma \bar{x}_i}} \left( \frac{r_i w_{ig}}{2\gamma} \right)^2 + \sum_{\substack{i: r_i w_{ig} \\ < -2\gamma \bar{x}_i}} (\bar{x}_i)^2 = k_g$$

Then it is optimal for the good camp to follow the following investment strategy

$$x_i^* = 0, \text{ if } r_i w_{ig} < -2\gamma \bar{x}_i$$

$$= \bar{x}_i + \text{sgn}(r_i w_{ig}) \left( k_g - \sum_{\substack{i: r_i w_{ig} \\ < -2\gamma \bar{x}_i}} (\bar{x}_i)^2 \right)^{\frac{1}{2}} \left( \frac{(r_i w_{ig})^2}{\sum_{\substack{i: r_i w_{ig} \\ \geq -2\gamma \bar{x}_i}} (r_i w_{ig})^2} \right)^{\frac{1}{2}}$$

$$\text{if } r_i w_{ig} \geq -2\gamma \bar{x}_i$$

If there does not exist a  $\hat{\gamma} > 0$  (because  $\sum_{i: r_i w_{ig} < 0} (\bar{x}_i)^2 < k_g$  and no node with  $r_i w_{ig} > 0$ ), we invest 0 on any node with  $r_i w_{ig} < 0$  and  $\bar{x}_i$  on any node with  $r_i w_{ig} = 0$ .

This can be proved on similar lines as Proposition 6; here we obtain

$$(x_i^* - \bar{x}_i)^2 = \left( k_g - \sum_{\substack{i: r_i w_{ig} \\ < -2\gamma \bar{x}_i}} (\bar{x}_i)^2 \right) \left( \frac{(r_i w_{ig})^2}{\sum_{\substack{i: r_i w_{ig} \\ \geq -2\gamma \bar{x}_i}} (r_i w_{ig})^2} \right)$$

and the optimal square root is determined by  $\text{sgn}(r_i w_{ig})$  (since a positive  $r_i w_{ig}$  would mean a higher optimal investment as opposed to a negative  $r_i w_{ig}$ ). Here, it is possible that a node  $i$  is invested on even if it has negative value of  $r_i w_{ig}$ , in order to have the investment close to  $\bar{x}_i$ .

### B. The Case of Bounded Investment per Node

The optimal strategies of the camps can be easily obtained for this setting on similar lines as Proposition 3.

**Proposition 8.** *In Setting V-B, it is optimal for the good camp to invest in nodes one at a time, subject to a maximum investment of 1 per node, in decreasing order of values of  $r_i w_{ig}$  until either the budget  $k_g$  is exhausted or we reach a node with a non-positive value of  $r_i w_{ig}$ . Say the so derived optimal investment on node  $i$  is  $x_i^*$ . The optimal strategy of the bad camp is to invest in nodes one at a time, subject to a maximum investment of 1 per node, in decreasing order of values of  $r_i w_{ig}$  until  $\sum_i r_i w_{ib} y_i \geq \sum_j r_j (w_{jg} x_j^* + w_{jj}^0 v_j^0)$ .*

## VI. COMMON COUPLED CONSTRAINTS RELATING BOUNDS ON COMBINED INVESTMENT PER NODE

We now consider a setting in which the combined investment on a node by both camps is bounded by a certain limit. Without loss of generality, assume this limit to be 1. This leads

to the introduction of common coupled constraints (CCC),  $x_i + y_i \leq 1, \forall i \in N$ . The problem now becomes

$$\max_{\substack{\sum_i x_i \leq k_g \\ x_i \geq 0}} \min_{\substack{\sum_i y_i \leq k_b \\ 0 \leq y_i \leq (1-x_i)}} \sum_i v_i$$

$$\text{s.t. } \forall i: v_i - \sum_{j \neq i} w_{ij} v_j - w_{ii} v_i + w_{ib} y_i = w_{ig} x_i + w_{ii}^0 v_i^0$$

The inner term is

$$\min \sum_i v_i$$

$$\text{s.t. } \mathbf{y} \geq \mathbf{0}$$

$$\sum_i y_i \leq k_b \text{ or } -\sum_i y_i \geq -k_b \leftarrow \alpha$$

$$\forall i: v_i - \sum_{j \neq i} w_{ij} v_j - w_{ii} v_i + w_{ib} y_i = w_{ig} x_i + w_{ii}^0 v_i^0 \leftarrow z_i$$

$$\forall i: x_i + y_i \leq 1 \text{ or } -y_i \geq -(1-x_i) \leftarrow \gamma_i$$

Its dual problem can be written as

$$\max -\alpha k_b + \sum_i (z_i (w_{ig} x_i + w_{ii}^0 v_i^0) - \gamma_i (1-x_i)) \quad (9)$$

$$\text{s.t. } \forall i: (1-w_{ii}) z_i - \sum_{j \neq i} w_{ji} z_j = 1 \leftarrow v_i \quad (10)$$

$$\forall i: w_{ib} z_i - \gamma_i - \alpha \leq 0 \leftarrow y_i \quad (11)$$

$$\alpha \geq 0$$

$$\forall i: z_i \in \mathbb{R}, \gamma_i \geq 0$$

As earlier, from (10), we have  $z_i = ((\mathbf{I} - \mathbf{w}^T)^{-1} \mathbf{1})_i = r_i$ . For satisfying Constraint (11), it is required that

$$\forall i: r_i w_{ib} - \gamma_i - \alpha \leq 0 \text{ or } \gamma_i \geq r_i w_{ib} - \alpha \quad (12)$$

To maximize objective function (9), it is required that  $\gamma_i$  should be as low as possible (knowing that  $1-x_i \geq 0$ ). So the above condition  $\gamma_i \geq r_i w_{ib} - \alpha$  along with  $\gamma_i \geq 0$  gives

$$\forall i: \gamma_i = \max\{r_i w_{ib} - \alpha, 0\}$$

So we need to maximize the objective function with respect to  $\gamma_i, x_i, \forall i$  and  $\alpha$ . For this purpose, let us define a set with respect to  $\alpha$ , namely,

$$J_\alpha = \{j: r_j w_{jb} - \alpha \geq 0\}$$

So the objective function to be maximized is

$$-\alpha k_b - \sum_{j \in J_\alpha} (r_j w_{jb} - \alpha)(1-x_j) + \sum_i r_i (w_{ig} x_i + w_{ii}^0 v_i^0) \quad (13)$$

which is equal to

$$\alpha \left( \sum_{j \in J_\alpha} (1-x_j) - k_b \right) - \sum_{j \in J_\alpha} (1-x_j) r_j w_{jb} + \sum_i r_i (w_{ig} x_i + w_{ii}^0 v_i^0) \quad (14)$$

**Claim 1.** *It is sufficient to search the values of  $\alpha \in \{r_j w_{jb}\}_{j: r_j w_{jb} > 0} \cup \{0\}$  to find an optimal solution.*

*Proof.* Since  $\alpha \geq 0$ , we have  $\alpha \neq r_j w_{jb}$  for any  $r_j w_{jb} < 0$ . Consider a range  $[r_l w_{lb}, r_u w_{ub}]$  for a consecutive pair of distinct values of  $r_j w_{jb}$ . If a range has both these values negative, we do not search for  $\alpha$  in that range, since  $\alpha \geq 0$ . If a range has  $r_l w_{lb} \leq 0$  and  $r_u w_{ub} > 0$ , we search for  $\alpha$  in

$[0, r_u w_{ub}]$ . We will now determine an optimal value of  $\alpha$  in the valid searchable subset of  $[r_l w_{lb}, r_u w_{ub}]$ , for a given  $\mathbf{x}$ .

Case 1: If  $\alpha = r_l w_{lb}$  (where  $r_l w_{lb} \geq 0$ ):

We have  $J_\alpha = \{j : r_j w_{jb} \geq r_l w_{lb}\}$ . The value of the objective function (13) becomes

$$-r_l w_{lb} k_b - \sum_{j \in J_\alpha} (r_j w_{jb} - r_l w_{lb})(1 - x_j) + \sum_i r_i (w_{ig} x_i + w_{ii}^0 v_i^0)$$

Case 2: If  $\alpha \in (r_l w_{lb}, r_u w_{ub}]$ :

We have  $J_\alpha = \{j : r_j w_{jb} \geq r_u w_{ub}\}$ .

Case 2a: If  $\sum_{j \in J_\alpha} (1 - x_j) - k_b \geq 0$ , we have an optimal  $\alpha = r_u w_{ub}$  (from (14)).

Case 2b: Instead, if  $\sum_{j \in J_\alpha} (1 - x_j) - k_b < 0$ , we have optimal  $\alpha \rightarrow r_l w_{lb}$  if  $r_l w_{lb} \geq 0$ , and the optimal value is the same as that for  $\alpha = r_l w_{lb}$  (Case 1). Note here that if  $r_l w_{lb} < 0$  and  $r_u w_{ub} \geq 0$ , we would have optimal  $\alpha = 0$ .

Case 3: If  $r_l w_{lb} = \max_i r_i w_{ib}$ , that is, when we are looking for  $\alpha \geq \max_i r_i w_{ib}$ . For  $\alpha = \max_i r_i w_{ib}$ , we have  $J_\alpha = \{\arg \max_i r_i w_{ib}\}$  and so the term  $\sum_{j \in J_\alpha} (r_j w_{jb} - \alpha)(1 - x_j)$  in (13) vanishes. For  $\alpha > \max_i r_i w_{ib}$ , we have  $J_\alpha = \{\}$  and so the term  $\sum_{j \in J_\alpha} (r_j w_{jb} - \alpha)(1 - x_j)$  in (13) vanishes in this case too. So the objective function to be maximized becomes

$$-\alpha k_b + \sum_i r_i (w_{ig} x_i + w_{ii}^0 v_i^0)$$

for which the optimal  $\alpha = \max_i r_i w_{ib}$  (the lowest value of  $\alpha$  such that  $\alpha \geq \max_i r_i w_{ib}$ ).

The above cases show that it is sufficient to search the values of  $\alpha \in \{r_j w_{jb}\}_{j:r_j w_{jb} > 0} \cup \{0\}$  to determine an optimal value of the objective function.  $\square$

Now that we have established that the only possible values of optimal  $\alpha$  are  $\{r_j w_{jb}\}_{j:r_j w_{jb} > 0} \cup \{0\}$ , we can assume optimal  $\alpha = r_{\hat{j}} w_{\hat{j}b}$  for  $\hat{j} \in \{j : r_j w_{jb} > 0\} \cup \{d\}$ , where the dummy node  $d$  is such that  $r_d w_{db} = 0$ .

Recalling the objective function in (14),

$$\begin{aligned} & \sum_i r_i (w_{ig} x_i + w_{ii}^0 v_i^0) + \alpha \left( \sum_{j \in J_\alpha} (1 - x_j) - k_b \right) - \sum_{j \in J_\alpha} (1 - x_j) r_j w_{jb} \\ &= \sum_i r_i (w_{ig} x_i + w_{ii}^0 v_i^0) - \left[ \sum_{j \in J_\alpha} (1 - x_j) r_j w_{jb} + \alpha \left( k_b - \sum_{j \in J_\alpha} (1 - x_j) \right) \right] \\ &= \sum_i r_i (w_{ig} x_i + w_{ii}^0 v_i^0) \\ & \quad - \left[ \sum_{j \in J_\alpha} (1 - x_j) r_j w_{jb} + \left( k_b - \sum_{j \in J_\alpha} (1 - x_j) \right) r_{\hat{j}} w_{\hat{j}b} \right] \end{aligned}$$

Let  $I_\alpha = \{j : r_j w_{jb} > \alpha\}$ ,  $P_\alpha = \{j : r_j w_{jb} = \alpha = r_{\hat{j}} w_{\hat{j}b}\}$ . So the objective function is

$$\begin{aligned} & \sum_i r_i (w_{ig} x_i + w_{ii}^0 v_i^0) - \left[ \sum_{j \in P_\alpha} (1 - x_j) r_j w_{jb} - \sum_{j \in P_\alpha} (1 - x_j) r_{\hat{j}} w_{\hat{j}b} \right. \\ & \quad \left. + \sum_{j \in I_\alpha} (1 - x_j) r_j w_{jb} + \left( k_b - \sum_{j \in I_\alpha} (1 - x_j) \right) r_{\hat{j}} w_{\hat{j}b} \right] \\ &= \sum_i r_i (w_{ig} x_i + w_{ii}^0 v_i^0) \\ & \quad - \left[ \sum_{j \in I_\alpha} (1 - x_j) r_j w_{jb} + \left( k_b - \sum_{j \in I_\alpha} (1 - x_j) \right) r_{\hat{j}} w_{\hat{j}b} \right] \end{aligned}$$

Comparing this with generic objective function (4) and since it should hold for any  $r_i, w_{ig}, w_{ib}, w_{ii}^0, v_i^0$ , it is necessary that the coefficients of non-zero values of  $r_i w_{ib}$  are the same in both forms of the objective function. This along with the fact that  $\forall j \in I_\alpha : r_j w_{jb} > 0$  (since  $\alpha \geq 0$ ), gives  $\sum_{j \in I_\alpha} y_j = \sum_{j \in I_\alpha} (1 - x_j)$ . Also if  $r_{\hat{j}} w_{\hat{j}b} > 0$ , then  $\sum_{j \in P_\alpha} y_j = k_b - \sum_{j \in I_\alpha} (1 - x_j)$ . And for all other terms, we have  $\sum_{j \notin J_\alpha} y_j = 0$ . Since  $\forall j : 0 \leq y_j \leq 1 - x_j$ , these are equivalent to

$$\forall j \in I_\alpha : y_j = 1 - x_j, \forall j \notin J_\alpha : y_j = 0, \sum_{j \in P_\alpha} y_j = k_b - \sum_{j \in I_\alpha} (1 - x_j) \quad (15)$$

To check for the consistency of budget of the bad camp, it is necessary that  $\sum_{j \in I_\alpha} y_j \leq k_b$ . This gives the constraint  $\sum_{j \in I_\alpha} (1 - x_j) \leq k_b$  or equivalently,

$$\sum_{j \in I_\alpha} x_j \geq |I_\alpha| - k_b \quad (16)$$

Also if  $r_{\hat{j}} w_{\hat{j}b} > 0$ , for the consistency of investment on the nodes in  $P_\alpha$  (that is,  $\forall j \in P_\alpha : x_j + y_j \leq 1$ ), it is necessary that  $\sum_{j \in P_\alpha} y_j \leq \sum_{j \in P_\alpha} (1 - x_j)$  or equivalently,  $k_b - \sum_{j \in I_\alpha} (1 - x_j) \leq \sum_{j \in P_\alpha} (1 - x_j)$  or equivalently,

$$\sum_{j \in I_\alpha} x_j + \sum_{j \in P_\alpha} x_j \leq |I_\alpha| + |P_\alpha| - k_b \quad (17)$$

To check for the consistency of budget of the good camp, it is necessary that  $\sum_{j \in I_\alpha} x_j \leq k_g$  and  $\sum_{j \in I_\alpha} x_j + \sum_{j \in P_\alpha} x_j \geq 0$ . These along with Inequalities (16) and (17) give  $|I_\alpha| - k_b \leq k_g$  and  $|I_\alpha| + |P_\alpha| - k_b \geq 0$ , or equivalently,

$$|I_\alpha| \leq k_g + k_b \quad \text{and} \quad |I_\alpha| + |P_\alpha| \geq k_b \quad (18)$$

The sets  $I_\alpha$  and  $P_\alpha$  depend only on  $\hat{j}$ . So let the set of  $\hat{j}$ 's that satisfy the constraints in (18) be denoted by  $\tilde{J}$ , that is,

$$\tilde{J} = \{\hat{j} : |I_\alpha| \leq k_g + k_b \text{ and } |I_\alpha| + |P_\alpha| \geq k_b\}$$

The term  $\sum_i r_i w_{ii}^0 v_i^0$  being a constant, and substituting  $\alpha = r_{\hat{j}} w_{\hat{j}b}$ , objective function (13) becomes

$$\begin{aligned} & \max_{\mathbf{x}, \hat{j}} \sum_i r_i w_{ig} x_i + \sum_{j:r_j w_{jb} \geq r_{\hat{j}} w_{\hat{j}b}} x_j (r_j w_{jb} - r_{\hat{j}} w_{\hat{j}b}) \\ & \quad - \sum_{j:r_j w_{jb} \geq r_{\hat{j}} w_{\hat{j}b}} (r_j w_{jb} - r_{\hat{j}} w_{\hat{j}b}) - r_{\hat{j}} w_{\hat{j}b} k_b \\ & \iff \max_{\hat{j}} \left[ \max_{\mathbf{x}} \sum_i x_i (r_i w_{ig} + \max\{r_i w_{ib} - r_{\hat{j}} w_{\hat{j}b}, 0\}) \right. \\ & \quad \left. - \sum_i \max\{r_i w_{ib} - r_{\hat{j}} w_{\hat{j}b}, 0\} - r_{\hat{j}} w_{\hat{j}b} k_b \right] \quad (19) \end{aligned}$$

Hence the good camp's optimal strategy can be obtained by maximizing (19) with respect to  $\mathbf{x}$  and  $\hat{j} \in \tilde{J}$ , subject to Constraints (16) and (17), and  $x_i \in [0, 1], \forall i$ .

For a given  $\hat{j}$ , to obtain an optimal strategy of the good camp, it could start by investing a total of  $(|I_\alpha| - k_b)$  in nodes belonging to  $I_\alpha$  one at a time (subject to a maximum investment of 1 per node) in decreasing order of values of  $(r_i w_{ig} + \max\{r_i w_{ib} - r_{\hat{j}} w_{\hat{j}b}, 0\})$ ; this would ensure satisfaction of Constraint (16). Let  $\tilde{x}_i$  be the investment on node  $i$  after this step. It could then invest in nodes one at a time (maximum investment of  $1 - \tilde{x}_i$  in node  $i$ ) in decreasing order

of values of  $(r_i w_{ig} + \max\{r_i w_{ib} - r_{\hat{j}} w_{\hat{j}b}, 0\})$  until either the remaining budget  $(k_g - |I_\alpha| + k_b)$  is exhausted or a node with a negative value of  $(r_i w_{ig} + \max\{r_i w_{ib} - r_{\hat{j}} w_{\hat{j}b}, 0\})$  is reached, while respecting Constraint (17). Hence let  $\mathbf{x}_j^* = (x_{ji}^*)$  be the resulting optimal  $\mathbf{x}$  for the given  $\hat{j}$ , obtained using these steps.

The absolute optimal strategy of the good camp can now be computed by iterating over all  $\hat{j} \in \hat{J}$  and taking the one that maximizes (from Expression (19))

$$\begin{aligned} & \max_{\hat{j} \in \hat{J}} \sum_i x_{ji}^* (r_i w_{ig} + \max\{r_i w_{ib} - r_{\hat{j}} w_{\hat{j}b}, 0\}) \\ & - \sum_i \max\{r_i w_{ib} - r_{\hat{j}} w_{\hat{j}b}, 0\} - r_{\hat{j}} w_{\hat{j}b} k_b \end{aligned} \quad (20)$$

Recall that

$$\sum_i v_i = \sum_i r_i w_{ii}^0 v_i^0 + \sum_i r_i w_{ig} x_i - \sum_i r_i w_{ib} y_i$$

Since  $y_i \in [0, 1 - x_i], \forall i$ , the optimal strategy of the bad camp is to invest in nodes one at a time (subject to a maximum investment of  $1 - x_i$  per node) in decreasing order of values of  $r_i w_{ib}$  until either its budget  $k_b$  is exhausted or we reach a node with a negative value of  $r_i w_{ib}$ .

Note that if  $k_g$  and  $k_b$  are integers, it is an optimal investment strategy of the good and bad camps to invest one unit or not invest at all in a node.

**Insight 3.** Assuming a  $\hat{j}$ , the strategy of the good camp is to choose nodes with good values of  $(r_i w_{ig} + \max\{r_i w_{ib} - r_{\hat{j}} w_{\hat{j}b}, 0\})$ . That is, it chooses nodes with not only good values of  $r_i w_{ig}$ , but also good values of  $r_i w_{ib}$ . This is expected since the budget constraint per node allows the good camp (which plays first) to block those nodes on which the bad camp would have preferred to invest. Also, based on (15) and the definitions of  $J_\alpha, I_\alpha, P_\alpha$ , node  $\hat{j}$  can be viewed as a boundary for the bad camp's investment, that is, the bad camp would not invest in any node  $i$  such that  $r_i w_{ib} < r_{\hat{j}} w_{\hat{j}b}$ .

### Maxmin versus Minmax Values

Here, we compare the maxmin and minmax values of the game in the introductory setting with that in the common coupled constraints setting. The introduction of the total budget constraints per node disturbs the equality between maxmin and minmax, as we show now. Let  $(\mathbf{x}', \mathbf{y}')$  be an optimal maxmin strategy profile in (8). Adding the constraint  $\mathbf{0} \leq \mathbf{y} \leq \mathbf{1} - \mathbf{x}$  restricts the set of feasible strategies for the bad camp, and this set of feasible strategies and hence its optimal strategy now depends on  $\mathbf{x}$ . So we have

$$\max_{\mathbf{0} \leq \mathbf{x} \leq \mathbf{1}} \min_{\mathbf{0} \leq \mathbf{y} \leq \mathbf{1}} \sum_i v_i \leq \max_{\mathbf{0} \leq \mathbf{x} \leq \mathbf{1}} \min_{\mathbf{0} \leq \mathbf{y} \leq \mathbf{1} - \mathbf{x}} \sum_i v_i$$

Similarly,

$$\min_{\mathbf{0} \leq \mathbf{y} \leq \mathbf{1}} \max_{\mathbf{0} \leq \mathbf{x} \leq \mathbf{1}} \sum_i v_i \geq \min_{\mathbf{0} \leq \mathbf{y} \leq \mathbf{1}} \max_{\mathbf{0} \leq \mathbf{x} \leq \mathbf{1} - \mathbf{y}} \sum_i v_i$$

These two inequalities, along with (8), result in the following inequality,

$$\max_{\mathbf{0} \leq \mathbf{x} \leq \mathbf{1}} \min_{\mathbf{0} \leq \mathbf{y} \leq \mathbf{1} - \mathbf{x}} \sum_i v_i \geq \min_{\mathbf{0} \leq \mathbf{y} \leq \mathbf{1}} \max_{\mathbf{0} \leq \mathbf{x} \leq \mathbf{1} - \mathbf{y}} \sum_i v_i \quad (21)$$

This result, which is contrary to general functions (for which maxmin is less than or equal to minmax), has also been derived

in [32]. In our problem, this is a direct consequence of the first mover advantage, which restricts the strategy set of the second mover. In the maxmin case as analyzed earlier, the good camp invests in nodes with good values of  $(r_i w_{ig} + \max\{r_i w_{ib} - r_{\hat{j}} w_{\hat{j}b}, 0\})$  (assuming a  $\hat{j}$ ). That is, it is likely to invest in nodes with good values of  $r_i w_{ib}$  which are the preferred investees of the bad camp. Owing to total investment limit per node, the bad camp may not be able to make the desired investment. It can be shown on similar lines that, in the minmax case where the bad camp plays first, it would play symmetrically opposite, thus limiting the ability of good camp to invest in nodes with good values of  $r_i w_{ig}$ .

## VII. DECISION UNDER UNCERTAINTY

We look at the setting which considers the possibility that the good camp, which plays first, may not have complete or exact information regarding the extrinsic weights, namely,  $w_{ig}, w_{ib}, w_{ii}^0$ . The bad camp, however, which plays second, has perfect information regarding the values of these parameters, and hence it is known that it would act optimally. Forecasting the optimal strategy of the bad camp, the good camp aims at choosing a robust strategy which would give it a good payoff even in the worst case.

Let  $\mathbf{u} = \begin{pmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{pmatrix}$ , where  $\mathbf{u}_i = \begin{pmatrix} w_{ii}^0 \\ w_{ig} \\ w_{ib} \end{pmatrix}$ . That is,  $\mathbf{u} = (w_{11}^0 \ w_{1g} \ w_{1b} \ \cdots \ w_{nn}^0 \ w_{ng} \ w_{nb})^T$ .

Let  $U$  be a polytope defined by  $E\mathbf{u} \leq f$  (that is,  $\mathbf{u} \in U$ ). It can be viewed as the uncertainty set, which in this case, is a convex set. The polytope would be based on the application at hand and could be deduced from observations, predictions, etc. We use the framework of robust optimization [33] for solving this problem.

For the purpose of this specific topic of decision under uncertainty, let us assume that all the elements of  $\mathbf{u}$  are non-negative. This is to ensure bounded values of the parameters. For instance, if we have a constraint in the linear program such as  $w_{ig} + w_{ib} + w_{ii}^0 \leq \theta_i$ , the individual values  $w_{ig}, w_{ib}, w_{ii}^0$  can be unbounded. So for this specific setting of uncertain parameters, we will assume  $w_{ig}, w_{ib}, w_{ii}^0 \geq 0, \forall i$ .

Since the good camp aims to optimize in the worst case of parameter values, while the bad camp has knowledge of these values with certainty, the optimization problem is

$$\max_{\substack{\sum_i x_i \leq k_g \\ x_i \geq 0}} \min_{E\mathbf{u} \leq f} \min_{\substack{\sum_i y_i \leq k_b \\ y_i \geq 0}} \sum_i r_i w_{ig} x_i + \sum_i r_i w_{ii}^0 v_i^0 - \sum_i r_i w_{ib} y_i$$

If  $\max_j r_j w_{jb} > 0$ , that is, the bad camp has at least one feasible node to invest on, then we have  $\sum_i r_i w_{ib} y_i = k_b \max_j r_j w_{jb}$ , else we have  $\sum_i r_i w_{ib} y_i = 0$ . For arriving at a concise solution, let  $d$  be a dummy node such that  $r_d w_{db} = 0$ . So now we have  $\sum_i r_i w_{ib} y_i = k_b \max_{j \in N \cup \{d\}} r_j w_{jb}$ . The optimization problem thus is

$$\begin{aligned} & \max_{\substack{\sum_i x_i \leq k_g \\ x_i \geq 0}} \min_{\mathbf{u}} \sum_i r_i w_{ig} x_i + \sum_i r_i w_{ii}^0 v_i^0 - k_b \max_{j \in N \cup \{d\}} r_j w_{jb} \\ & \text{s.t. } E\mathbf{u} \leq f \end{aligned}$$

Note that there are in general  $n + 1$  possibilities for  $\max_{j \in N \cup \{d\}} r_j w_{jb}$ . We could write a linear program for

each possibility of  $i_0 = \arg \max_{j \in N \cup \{d\}} r_j w_{jb}$ . For a fixed  $i_0 \in N \cup \{d\}$ , the inner term is

$$\begin{aligned} \min_{\mathbf{u}} \sum_i r_i w_{ig} x_i + \sum_i r_i w_{ii}^0 v_i^0 - k_b r_{i_0} w_{i_0 b} \\ \text{s.t. } E\mathbf{u} \leq f \quad \leftarrow \alpha_{i_0} \\ \forall i : r_i w_{ib} \leq r_{i_0} w_{i_0 b} \quad \leftarrow \beta_{i_0 i} \end{aligned}$$

For this problem to be feasible, the constraint set should be non-empty. Let  $N_f$  be the subset of  $N \cup \{d\}$  consisting of nodes  $i_0$  such that the constraint set satisfying  $E\mathbf{u} \leq f$  and  $\forall i : r_i w_{ib} \leq r_{i_0} w_{i_0 b}$  is non-empty. Its dual is the following

$$\begin{aligned} \max -\alpha_{i_0}^T f \\ \text{s.t. } \forall i \in N \setminus \{i_0\} : -\alpha_{i_0}^T E_{ib} - \beta_{i_0 i} r_i \leq 0 \quad \leftarrow w_{ib} \\ \text{if } i_0 \neq d : -\alpha_{i_0}^T E_{i_0 b} + \sum_{i \neq i_0} \beta_{i_0 i} r_{i_0} \leq -k_b r_{i_0} \quad \leftarrow w_{i_0 b} \\ \forall i : -\alpha_{i_0}^T E_{ig} \leq r_i x_i \quad \leftarrow w_{ig} \\ \forall i : -\alpha_{i_0}^T E_{ii} \leq r_i v_i^0 \quad \leftarrow w_{ii}^0 \\ \alpha_{i_0} \geq \mathbf{0} \\ \forall i : \beta_{i_0 i} \geq 0 \end{aligned}$$

We need to find a common  $\mathbf{x}$  for all possibilities of  $i_0 \in N_f$ . So we have a constraint on the value of the dual, say  $\rho$ , namely,  $\rho \leq -\alpha_{i_0}^T f, \forall i_0$ . We hence obtain a solution to the optimization problem by solving the following LP.

$$\begin{aligned} \max \rho \\ \text{s.t. } \sum_i x_i \leq k_g \\ \forall i : x_i \geq 0 \\ \forall i_0 \in N_f : \left\{ \begin{array}{l} \rho + \alpha_{i_0}^T f \leq 0 \\ \forall i \in N \setminus \{i_0\} : -\alpha_{i_0}^T E_{ib} - \beta_{i_0 i} r_i \leq 0 \\ \text{if } i_0 \neq d : -\alpha_{i_0}^T E_{i_0 b} + \sum_{i \neq i_0} \beta_{i_0 i} r_{i_0} \leq -k_b r_{i_0} \\ \forall i : -\alpha_{i_0}^T E_{ig} - r_i x_i \leq 0 \\ \forall i : -\alpha_{i_0}^T E_{ii} \leq r_i v_i^0 \\ \alpha_{i_0} \geq \mathbf{0} \\ \forall i : \beta_{i_0 i} \geq 0 \end{array} \right. \end{aligned}$$

## VIII. SIMULATIONS AND RESULTS

Throughout this paper, we analytically derived the optimal investment strategies of competing camps in a social network for driving the opinion of the population in their favor. We hence presented either closed-form expressions or algorithms with polynomial running time. With the aim of determining implications of the analytically derived results on real-world networks and obtaining further insights, we conducted a simulation study on two popular network datasets. We present the setup and observed results, and provide insights behind them.

### A. Simulation Setup

For conducting study on a real-world network dataset, we consider an academic collaboration network obtained from co-authorships in the ‘‘High Energy Physics - Theory’’ papers published on the e-print arXiv from 1991 to 2003. It contains

15,233 nodes and 31,376 links among them, and is popularly denoted as NetHEPT. This network exhibits many structural features of large-scale social networks and is widely used for experimental justifications, for example, in [24], [38], [39]. For the purpose of graphical illustration, we use the popular Zachary’s Karate club dataset consisting of 34 nodes and 78 links among them [40].

Proposition 4 showed that some popular models of distributing edge weight in a graph would result in random strategy being optimal, and so may not be suited for simulations. Hence in order to transform an undirected unweighted network dataset into a weighted directed one for our simulations, we consider that for any node  $i$ , the tuple  $(w_{ii}^0, w_{ig}, w_{ib})$  is randomly generated such that

$$\begin{aligned} \forall i : w_{ii}^0 + w_{ig} + w_{ib} = 0.5, \text{ and} \\ w_{ij} = \frac{0.5}{d_i}, \text{ if there is an edge between } i \text{ and } j. \end{aligned}$$

Figure 1 presents the values of the parameters used for experimentation on the Karate club dataset. The size and color saturation of a node  $i$  represent the value of the parameter mentioned in the corresponding caption (bigger size and higher saturation implies higher value). Unless otherwise specified, we consider  $k_g = k_b = 5$  for this dataset. Also, unless otherwise specified, we consider  $v_i^0 = 0, \forall i$ .

### B. Simulation Results

Table II presents the quantitative results of our simulations on Karate club and NetHEPT datasets. For both the datasets, the overall opinion value is positive in the fundamental unbounded setting (III-A1). The results of the good camp doubling its budget can also be seen. The effect of bounded investment (Setting III-A2) can be seen; for these particular datasets, the maxmin value decreases implying that the value of  $r_i w_{ig}$  is probably concentrated on one node, while that of  $r_i w_{ib}$  is well distributed, thus giving the bad camp an advantage in bounded (and hence distributed) investment. Figure 3 (a) illustrates the case with bounded investment per node ( $x_i, y_i \leq 1, \forall i$ ) for the Karate club dataset. Here, the label ‘c’ means that that node is invested on by both the camps with 1 unit, while ‘g’ and ‘b’ mean that the node is invested on by the good and bad camp, respectively.

Considering the complementary setting of maximizing the competitor’s (bad camp’s) investment, we could see how much budget the bad camp required to draw the overall opinion in its favor. As expected from the results of previous setting, the investment required in the unbounded setting (V-A) is more than the budget available in Setting III-A1. In the bounded setting (V-B), for Karate club dataset, we can see that the bad camp could have driven the overall opinion in its favor without expending its entire budget. For NetHEPT dataset, as expected from the previous setting, the investment required by the bad camp in the bounded setting (V-B) is less than that required in the unbounded setting (V-A).

Results under the concave influence function are presented for both unbounded (IV-A) and bounded (IV-B) cases for  $t = 2$  and  $t = 10$ . Figure 2 shows the effect of the value of  $t$  in the concave influence function unbounded setting, on

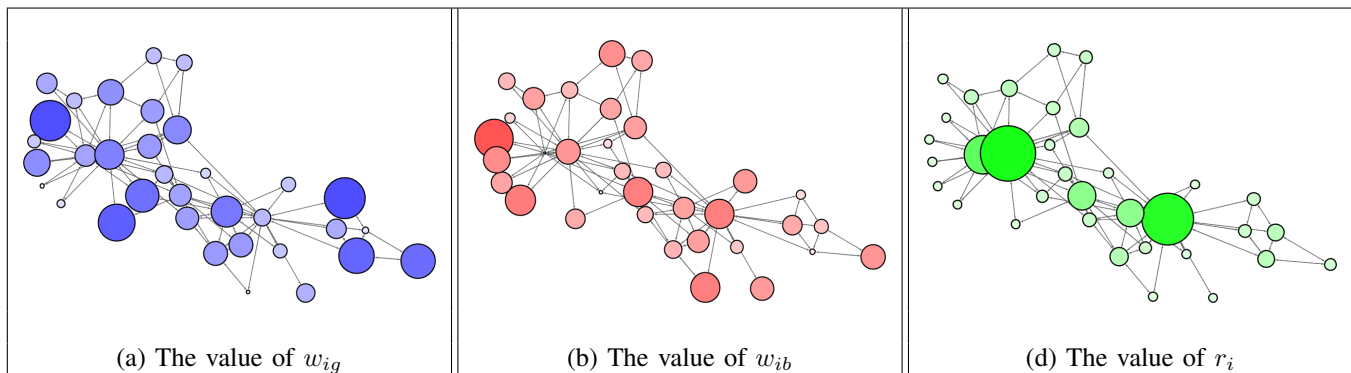


Fig. 1. Details about the Karate club dataset used in our simulations. The size and color saturation of a node  $i$  represent the value of the parameter.

TABLE II  
RESULTS FOR KARATE CLUB AND NETHEPT DATASETS

Setting		Section	Karate club			NetHEPT		
Aspect	Case		$k_g$	$k_b$	Results	$k_g$	$k_b$	Results
Fundamental	Unbounded	III-A1	5	5	$\max \min \sum_i v_i = 0.1564$	100	100	$\max \min \sum_i v_i = 73.2539$
			10	5	$\max \min \sum_i v_i = 5.8811$	200	100	$\max \min \sum_i v_i = 347.0770$
	Bounded	III-A2	5	5	$\max \min \sum_i v_i = -0.0538$	100	100	$\max \min \sum_i v_i = 2.8513$
Concave ( $t = 2$ )	Unbounded	IV-A	5	5	$\max \min \sum_i v_i = 0.4581$	100	100	$\max \min \sum_i v_i = -0.8446$
			20	20	$\max \min \sum_i v_i = 0.9163$	400	400	$\max \min \sum_i v_i = -1.6892$
	Bounded	IV-B	5	5	$\max \min \sum_i v_i = 0.4612$	100	100	$\max \min \sum_i v_i = -0.8446$
			20	20	$\max \min \sum_i v_i = 1.2653$	400	400	$\max \min \sum_i v_i = -1.7117$
Concave ( $t = 10$ )	Unbounded	IV-A	5	5	$\max \min \sum_i v_i = 1.1180$	100	100	$\max \min \sum_i v_i = -12.4212$
			20	20	$\max \min \sum_i v_i = 1.2842$	400	400	$\max \min \sum_i v_i = -14.2682$
	Bounded	IV-B	5	5	$\max \min \sum_i v_i = 1.1180$	100	100	$\max \min \sum_i v_i = -12.4212$
			20	20	$\max \min \sum_i v_i = 1.3104$	400	400	$\max \min \sum_i v_i = -14.2682$
Complementary	Unbounded	V-A	5	-	$\max \min \sum_i y_i = 5.1404$	100	-	$\max \min \sum_i y_i = 136.5231$
	Bounded	V-B	5	-	$\max \min \sum_i y_i = 4.8936$	100	-	$\max \min \sum_i y_i = 102.7266$
CCC	Bounded	VI	5	5	$\max \min \sum_i v_i = 1.5399$ $\min \max \sum_i v_i = -0.8900$	100	100	$\max \min \sum_i v_i = 7.4843$ $\min \max \sum_i v_i = -3.6795$

the distribution of investment and final opinion values for the Karate club dataset. In Figures 2 (a), (b), (d), (e), the size and color saturation of a node represent the amount of investment on it by the camp mentioned in the corresponding caption. With careful observation, it can be seen that for  $t = 2$ , the investments are more skewed, while for  $t = 10$ , the investments by the good and bad camps on a node  $i$  are close to being proportional to the values of  $r_i w_{ig}$  and  $r_i w_{ib}$ , respectively. In Figures 2 (c) and (f), the shape and color of a node represent its opinion sign (blue circle implies positive, red square implies negative), while its size and color saturation represent its opinion magnitude.

In the unbounded case, for some values of budgets, there exist nodes for which either  $x_i$  or  $y_i$  or both exceed value 1. So for the bounded case, the camps are directed to have different investment strategies than in the unbounded case; the effects can be seen in Table II where the values are different in Settings IV-A and IV-B for the same values of

budgets. In some scenarios, however, the investment strategies inadvertently assured  $x_i, y_i \leq 1, \forall i$  even for the unbounded case; so the investment strategies remain the same in both settings if the budgets remain unchanged. A careful analysis of the values would indicate that the constraints  $x_i, y_i \leq 1$  are likely to come into picture for some nodes, for lower values of  $t$  and higher values of budgets. Lower values of  $t$  lead to skewed investment and so a higher likelihood of some nodes having investment more than 1 in the unbounded case. Similarly, higher values of budgets scale up the investments on the nodes, resulting in a higher likelihood of some nodes having investment more than 1 in the unbounded case. This can also be inferred from the analytically derived investment strategies of the camps.

Figures 3 (b-c) illustrate the effect of common coupled constraints  $x_i + y_i \leq 1, \forall i$ , for the Karate club dataset. The advantage of playing first is clearly visible from the overall opinion value as well as the distribution of opinion values in

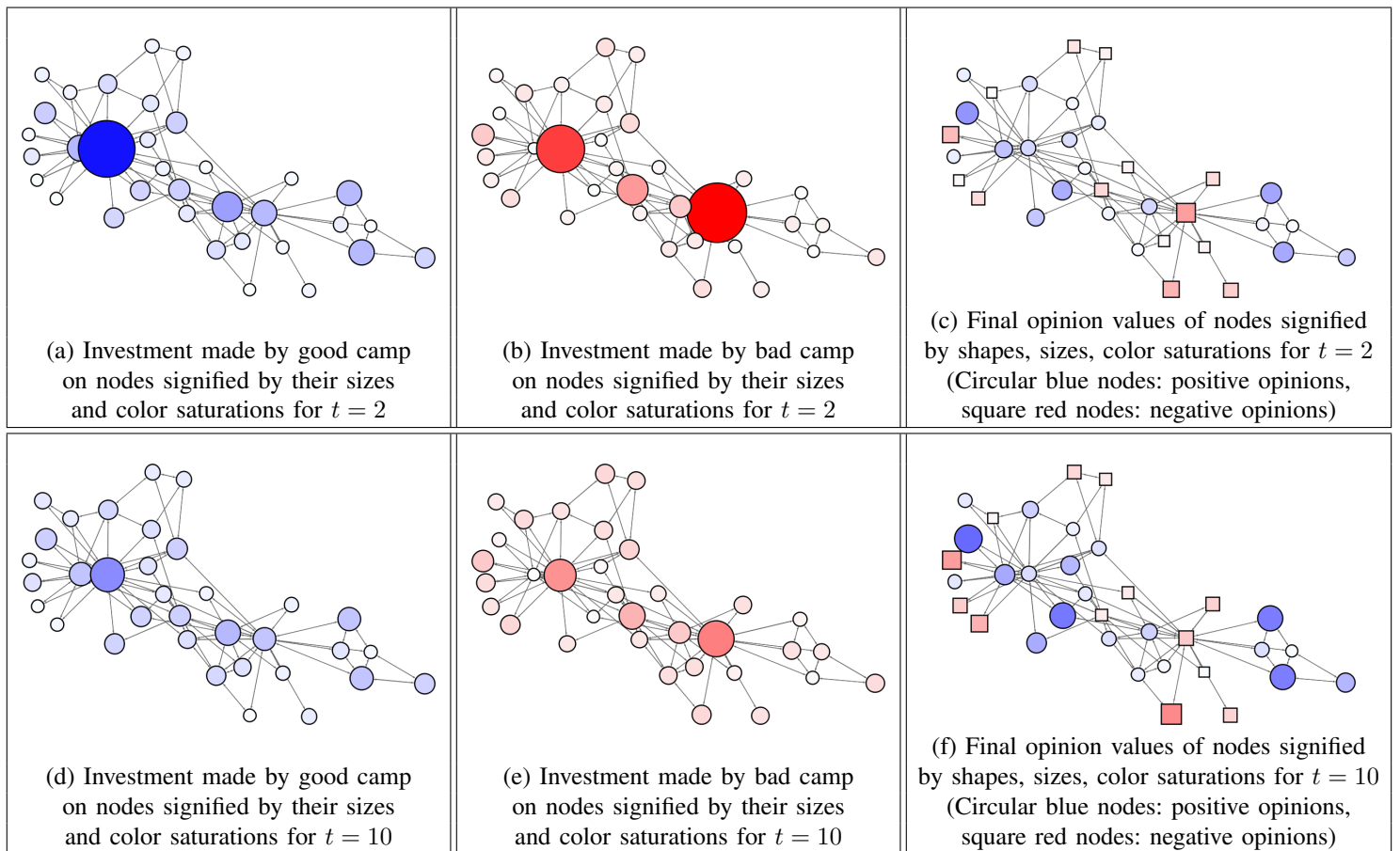


Fig. 2. Simulation results for the Karate club dataset with  $k_g = k_b = 5$  when the influence function is concave (Setting IV-A)

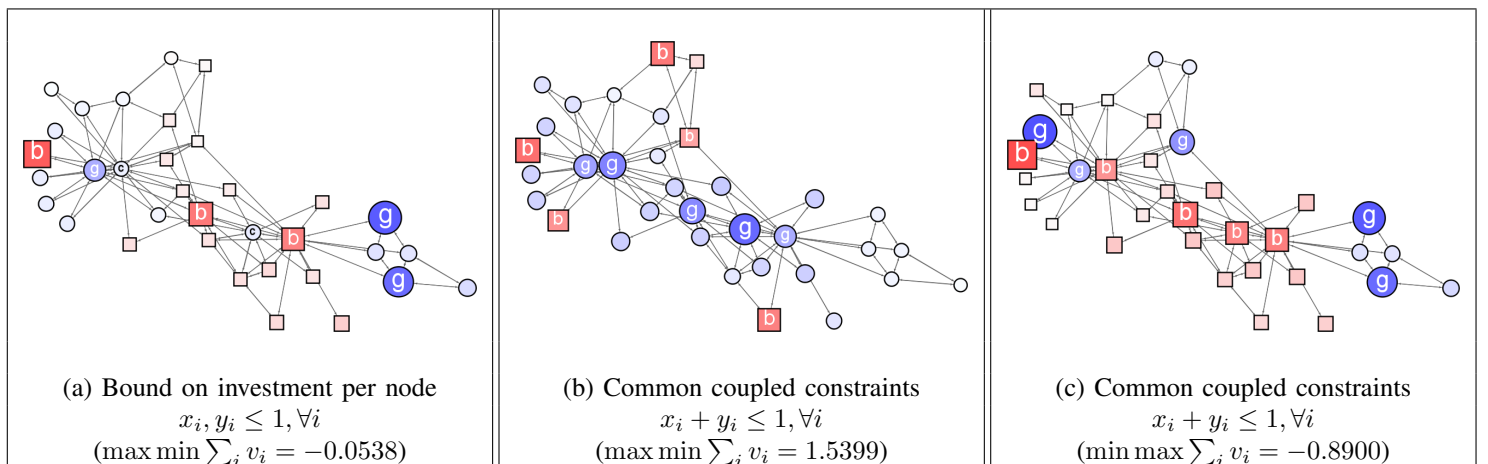


Fig. 3. Results in presence of additional constraints for the Karate club dataset with  $k_g = k_b = 5$ ; The nodes are labeled 'g/b/c' to signify if invested on by good/bad/both camps respectively. The sign of the opinion value of a node is signified by its shape and color (circle and blue for good, square and red for bad), while the absolute value of its opinion is signified by its size and color saturation.

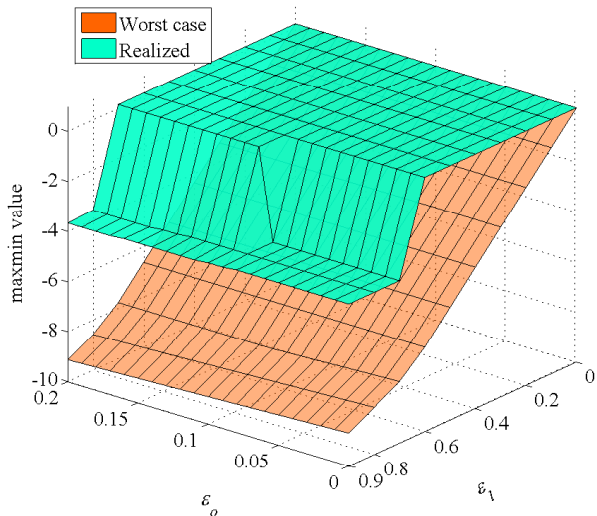


Fig. 4. The worst case and realized values of the maxmin value for different values of  $\epsilon_l$  and  $\epsilon_o$  for the Karate club dataset with  $k_g = k_b = 5$

the figures. Specifically, the value is the highest in Figure 3 (b) with a healthy distribution of positive opinion values, followed by the value in Figure 3 (a), followed by that in Figure 3 (c) which is dominated by negative opinion values.

For studying the effects of decision under uncertainty as analyzed in Section VII, we consider that the good camp (first mover) is uncertain about parameters  $w_{ig}$  and  $w_{ib}$  up to a certain limit. In particular, there is a fractional uncertainty of  $\epsilon_l$  regarding the values of these parameters, while there is a fractional uncertainty of  $\epsilon_o$  regarding the values of the sums of these parameters over the entire network.  $\epsilon_l$  can be hence viewed as local uncertainty and  $\epsilon_o$  as global uncertainty. Let  $\hat{w}_{ig}$  and  $\hat{w}_{ib}$  be the underlying ground truth values for a node  $i$  (the actual values destined to be realized).

$$\begin{aligned}
 (1 - \epsilon_l)\hat{w}_{ig} &\leq w_{ig} \leq (1 + \epsilon_l)\hat{w}_{ig} \\
 (1 - \epsilon_l)\hat{w}_{ib} &\leq w_{ib} \leq (1 + \epsilon_l)\hat{w}_{ib} \\
 (1 - \epsilon_o) \sum_i \hat{w}_{ig} &\leq \sum_i w_{ig} \leq (1 + \epsilon_o) \sum_i \hat{w}_{ig} \\
 (1 - \epsilon_o) \sum_i \hat{w}_{ib} &\leq \sum_i w_{ib} \leq (1 + \epsilon_o) \sum_i \hat{w}_{ib}
 \end{aligned}$$

It is clear that the latter two constraints would come into picture only if  $\epsilon_o < \epsilon_l$  (this would usually be the case since, though there may be significant relative deviation for the individual parameters, the relative deviation of their sum is usually low owing to significant balancing of positive and negative deviations of the individual parameters). For different values of  $\epsilon_l$  and  $\epsilon_o$ , Figure 4 presents the maxmin values: (a) as computed by the good camp (first mover) as its worst case value using our robust optimization approach and (b) as realized based on the ground truth. We assume the ground truth values as depicted in Figure 1. It can be seen that for a large enough range of values of  $\epsilon_l$  and  $\epsilon_o$ , though the good camp computes the worst case maxmin value to be very low, the corresponding realized value is the same as when the good camp is certain about the parameter values. The uncertainty factor starts affecting it only for very high values of  $\epsilon_l$  and  $\epsilon_o$ .

## IX. CONCLUSION

We used a variant of DeGroot-Friedkin model for opinion dynamics and studied a zero-sum game framework for optimal investment strategies for two competing camps in a social network. We arrived at closed form expressions and efficient algorithms for a number of well-motivated settings. Our analysis arrived at a decision parameter analogous to the Katz centrality measure. We also showed that for some of the popular models of weighing edges similar to weighted cascade, a random strategy is indeed optimal.

We derived a linear program for obtaining optimal strategy for a camp to whom the parameters' values are uncertain, while playing against a camp who has exact knowledge about the parameter values. In our simulations, we observed that a camp is likely to get affected due to uncertainty only for considerable amount of uncertainty. We studied the complementary problem where a camp aims to maximize its competitor's investment required to drive the average opinion value of the population in its favor, and saw that the optimal strategies fundamentally remain the same, albeit with different forms of the exact optimal strategies. We further looked at a setting where the influence of a camp on a node is a concave function of the amount of investment, and derived that a more concave function results in a less skewed investment strategy. We also studied the formulated game under common coupled constraints, where the constraints of one player are satisfied if and only if the constraints of the other player are satisfied for every strategy profile. We analytically derived the optimal strategies of both the camps, and hence quantified the first-mover advantage. We showed all the results quantitatively as well as illustratively using simulations on network datasets.

This work has several interesting directions for future work, of which we mention a few. It would be interesting to study other models of opinion dynamics in literature with respect to optimal investment strategies of competing camps and investigate if it is possible to arrive at closed form expressions under them. The two camps setting can be extended to multiple camps where each camp would attempt to drive the opinion of the population towards its own, which would act as a generalization of the two opinion classes as was in our case, positive and negative. It would be interesting to extend the setting of common coupled constraints to account for more complex constraints, while maintaining the analytical tractability of the problem. It would also be interesting to study the tradeoff between the advantages and disadvantages of playing first, since playing first would allow a camp to block certain investments of the competing camp (like in common coupled constraints) but it may also force it to take decision under uncertainty.

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## APPENDIX A PROOF OF PROPOSITION 5

*Proof.* We make the following one-to-one transformation:  $y_i = \mathcal{Y}_i^t$  and  $x_i = \mathcal{X}_i^t$ . So the above optimization problem is equivalent to

$$\begin{aligned} & \max_{\substack{\mathcal{X}_i \geq 0 \\ \sum_i \mathcal{X}_i^t \leq k_g}} \min_{\substack{\mathcal{Y}_i \geq 0 \\ \sum_i \mathcal{Y}_i^t \leq k_b}} \sum_i v_i \\ \text{s.t. } \forall i : & v_i - \sum_{j \neq i} w_{ij} v_j - w_{ii} v_i + w_{ib} \mathcal{Y}_i = w_{ig} \mathcal{X}_i + w_{ii}^0 v_i^0 \end{aligned}$$

Similar to the derivation of Equation (4) in the main text, we get

$$\sum_i v_i = \sum_i r_i (w_{ig} \mathcal{X}_i + w_{ii}^0 v_i^0) - \sum_i r_i w_{ib} \mathcal{Y}_i \quad (22)$$

Owing to the mutual independence between  $\bar{\mathcal{X}}$  and  $\bar{\mathcal{Y}}$  (where  $\bar{\mathcal{X}}$  and  $\bar{\mathcal{Y}}$  are the vectors with components  $\mathcal{X}_i$  and  $\mathcal{Y}_i$  respectively), we can write

$$\begin{aligned} & \max_{\substack{\mathcal{X}_i \geq 0 \\ \sum_i \mathcal{X}_i^t \leq k_g}} \min_{\substack{\mathcal{Y}_i \geq 0 \\ \sum_i \mathcal{Y}_i^t \leq k_b}} \sum_i v_i \\ & = \sum_i r_i w_{ii}^0 v_i^0 + \max_{\substack{\mathcal{X}_i \geq 0 \\ \sum_i \mathcal{X}_i^t \leq k_g}} \sum_i r_i w_{ig} \mathcal{X}_i - \max_{\substack{\mathcal{Y}_i \geq 0 \\ \sum_i \mathcal{Y}_i^t \leq k_b}} \sum_i r_i w_{ib} \mathcal{Y}_i \end{aligned}$$

We now solve for the first optimization term with respect to  $\bar{\mathcal{X}}$ , which can be written as the following convex optimization problem

$$\begin{aligned} & \min - \sum_i r_i w_{ig} \mathcal{X}_i \\ \text{s.t. } & \sum_i \mathcal{X}_i^t - k_g \leq 0 \quad \leftarrow \gamma \\ & \forall i : -\mathcal{X}_i \leq 0 \quad \leftarrow \beta_i \end{aligned}$$



The modified objective function with the Lagrangian multipliers is

$$g = -\sum_i r_i w_{ig} \mathcal{X}_i + \gamma \left( \sum_i \mathcal{X}_i^t - k_g \right) - \sum_i \beta_i \mathcal{X}_i$$

The KKT conditions with the modified objective function  $g$  give the following additional constraints

$$\forall i: \frac{\partial g}{\partial \mathcal{X}_i} = -r_i w_{ig} + t\gamma \mathcal{X}_i^{t-1} - \beta_i = 0 \quad (23)$$

$$\gamma \geq 0 \quad (24)$$

$$\forall i: \beta_i \geq 0 \quad (25)$$

$$\gamma \left( \sum_i \mathcal{X}_i^t - k_g \right) = 0 \quad (26)$$

$$\forall i: \beta_i \mathcal{X}_i = 0 \quad (27)$$

Constraint (23) gives

$$\mathcal{X}_i = \left( \frac{\beta_i + r_i w_{ig}}{t\gamma} \right)^{\frac{1}{t-1}} \quad (28)$$

Note that if  $\gamma = 0$ , Constraint (23) gives  $\beta_i + r_i w_{ig} = 0, \forall i$ . Since  $\beta_i \geq 0$ , this can hold only when  $r_i w_{ig} \leq 0, \forall i$ . For nodes with  $r_i w_{ig} = 0$ , the value of  $r_i w_{ig} \mathcal{X}_i$  stays 0 for any  $\mathcal{X}_i$  and so  $\mathcal{X}_i = 0$  is an optimal solution. For nodes with  $r_i w_{ig} < 0$ , we must have  $\beta_i > 0$  and hence  $\mathcal{X}_i = 0$  (from (27)).

Henceforth we assume  $\gamma > 0$  and so Equation (28) is valid. This, with (26) and (24), gives

$$\sum_i \mathcal{X}_i^t = k_g$$

From (28), we get

$$\sum_i \left( \frac{\beta_i + r_i w_{ig}}{t\gamma} \right)^{\frac{t}{t-1}} = k_g$$

$$\iff \left( \frac{1}{t\gamma} \right)^{\frac{1}{t-1}} = \frac{k_g^{\frac{1}{t}}}{\left( \sum_i (\beta_i + r_i w_{ig})^{\frac{t}{t-1}} \right)^{\frac{1}{t}}}$$

This with (28) gives

$$\mathcal{X}_i^* = k_g^{\frac{1}{t}} \left( \frac{(\beta_i + r_i w_{ig})^{\frac{1}{t-1}}}{\left( \sum_j (\beta_j + r_j w_{jg})^{\frac{t}{t-1}} \right)^{\frac{1}{t}}} \right) \quad (29)$$

Now, the following cases are possible depending on the sign of  $r_i w_{ig}$ .

Case 1: If  $r_i w_{ig} < 0$ , we must have  $\beta_i > 0$  (from the constraint  $\mathcal{X}_i^* \geq 0$ ). This with (27) gives  $\mathcal{X}_i^* = 0$ .

Case 2: If  $r_i w_{ig} = 0$ , we must have  $\mathcal{X}_i^* = \beta_i = 0$  (from (27) and (28)).

Furthermore in both Case 1 and Case 2, from (28), we have  $\beta_i + r_i w_{ig} = 0$ . So the  $i$ -terms for which  $r_i w_{ig} \leq 0$  vanish from the denominator of (29) and it transforms into

$$\mathcal{X}_i^* = k_g^{\frac{1}{t}} \left( \frac{(\beta_i + r_i w_{ig})^{\frac{1}{t-1}}}{\left( \sum_{j:r_j w_{jg} > 0} (\beta_j + r_j w_{jg})^{\frac{t}{t-1}} \right)^{\frac{1}{t}}} \right), \text{ if } r_i w_{ig} \geq 0$$

$$\text{and } \mathcal{X}_i^* = 0, \text{ if } r_i w_{ig} < 0 \quad (30)$$

Case 3: If  $r_i w_{ig} > 0$ , we would get  $\mathcal{X}_i^* > 0$  (from (25)). This with (27) gives  $\beta_i = 0$ , and hence from (30), we get

$$\mathcal{X}_i^* = k_g^{\frac{1}{t}} \left( \frac{(r_i w_{ig})^{\frac{1}{t-1}}}{\left( \sum_{j:r_j w_{jg} > 0} (r_j w_{jg})^{\frac{t}{t-1}} \right)^{\frac{1}{t}}} \right)$$

The above cases can be concisely written as

$$\mathcal{X}_i^* = k_g^{\frac{1}{t}} \left( \frac{(r_i w_{ig})^{\frac{1}{t-1}}}{\left( \sum_{j:r_j w_{jg} > 0} (r_j w_{jg})^{\frac{t}{t-1}} \right)^{\frac{1}{t}}} \right) \cdot \mathbb{I}_{r_i w_{ig} > 0}$$

$$\iff x_i^* = k_g \left( \frac{(r_i w_{ig})^{\frac{t}{t-1}}}{\sum_{j:r_j w_{jg} > 0} (r_j w_{jg})^{\frac{t}{t-1}}} \right) \cdot \mathbb{I}_{r_i w_{ig} > 0} \quad (31)$$

where  $\mathbb{I}_{r_i w_{ig} > 0} = 1$  if  $r_i w_{ig} > 0$ , and 0 otherwise. Similarly, it can be shown that

$$y_i^* = k_b \left( \frac{(r_i w_{ib})^{\frac{t}{t-1}}}{\sum_{j:r_j w_{jb} > 0} (r_j w_{jb})^{\frac{t}{t-1}}} \right) \cdot \mathbb{I}_{r_i w_{ib} > 0} \quad (32)$$

From the optimization problem, it can be directly seen that, if  $\forall i, r_i w_{ig} \leq 0$ , then  $x_i^* = 0, \forall i$ . Similarly, if  $\forall i, r_i w_{ib} \leq 0$ , then  $y_i^* = 0, \forall i$ .  $\square$

## APPENDIX B

### PROOF OF PROPOSITION 6

*Proof.* This proof goes as an extension to the proof of Proposition 5, with the additional constraint

$$\forall i: \mathcal{X}_i \leq 1 \leftarrow \xi_i \quad (33)$$

Equation (23) changes to

$$\forall i: -r_i w_{ig} + t\gamma \mathcal{X}_i^{t-1} - \beta_i + \xi_i = 0 \quad (34)$$

and Equation (28) changes to

$$\mathcal{X}_i = \left( \frac{\beta_i - \xi_i + r_i w_{ig}}{t\gamma} \right)^{\frac{1}{t-1}} \quad (35)$$

The complementary slackness conditions for the additional constraints are

$$\forall i: \xi_i (\mathcal{X}_i - 1) = 0 \quad (36)$$

Further note that the constraints  $\mathcal{X}_i \geq 0$  and  $\mathcal{X}_i \leq 1$  cannot both be tight, and so at least one of  $\beta_i$  or  $\xi_i$  must be 0. That is,

$$\forall i: \beta_i \xi_i = 0 \quad (37)$$

When  $\gamma = 0$ , Equation (34) transforms into  $\beta_i - \xi_i + r_i w_{ig} = 0, \forall i$ . Now if  $r_i w_{ig} = 0$ , we must have  $\beta_i - \xi_i = 0$  and hence  $\beta_i = \xi_i = 0$  (from Equation (37)). It is further clear from the objective function that if  $r_i w_{ig} = 0$ , it is optimal to have  $\mathcal{X}_i = 0$  and hence  $x_i^* = 0$ . If  $r_i w_{ig} < 0$ , we must have  $\beta_i > 0$  (since  $\beta_i - \xi_i + r_i w_{ig} = 0$ ) and so  $\mathcal{X}_i = x_i^* = 0$  (from Equation (27)). If  $r_i w_{ig} > 0$ , we must have  $\xi_i > 0$  and so  $\mathcal{X}_i = x_i^* = 1$  (from Equation (36)). That is, when  $\gamma = 0$ , we

invest an amount of 1 on all nodes  $i$  with positive values of  $r_i w_{ig}$  and 0 on all other nodes.

$$\gamma = 0 \implies x_i^* = 1 \cdot \mathbb{I}_{r_i w_{ig} > 0} \quad (38)$$

For  $\gamma > 0$ , we have

$$\begin{aligned} \sum_i \mathcal{X}_i^t &= k_g \\ \iff \sum_i \left( \frac{\beta_i - \xi_i + r_i w_{ig}}{t\gamma} \right)^{\frac{t}{t-1}} &= k_g \end{aligned}$$

When  $\beta_i > 0$ , we have  $\xi_i = 0$  (from (37)) and  $\mathcal{X}_i = 0$  (from (27)). From the above equation, this corresponds to  $r_i w_{ig} < 0$ . Further note that for  $r_i w_{ig} = 0$ ,  $\mathcal{X}_i = 0$  is an optimal solution. So we have

$$\sum_{i:r_i w_{ig} > 0} \left( \frac{-\xi_i + r_i w_{ig}}{t\gamma} \right)^{\frac{t}{t-1}} = k_g$$

We can have  $\mathcal{X}_i = 1$  or if not, we should have  $\xi_i = 0$ . So for any  $i$  such that  $r_i w_{ig} > 0$ , we have  $\mathcal{X}_i = \min \left\{ \left( \frac{r_i w_{ig}}{t\gamma} \right)^{\frac{t}{t-1}}, 1 \right\}$ . Let  $J_\gamma = \{i : 0 < r_i w_{ig} \leq t\gamma\}$ . So the above equation results in

$$\begin{aligned} \sum_{i \in J_\gamma} \left( \frac{r_i w_{ig}}{t\gamma} \right)^{\frac{t}{t-1}} + \sum_{\substack{i \notin J_\gamma \\ i:r_i w_{ig} > 0}} 1 &= k_g \quad (39) \\ \iff \left( \frac{1}{t\gamma} \right)^{\frac{t}{t-1}} = \frac{k_g - \sum_{i \notin J_\gamma, i:r_i w_{ig} > 0} 1}{\sum_{i \in J_\gamma} (r_i w_{ig})^{\frac{t}{t-1}}} \end{aligned}$$

Note that if  $i \in J_\gamma$ , then any  $j$  for which  $r_j w_{jg} < r_i w_{ig}$  belongs to  $J_\gamma$ . So  $J_\gamma$  and hence  $\gamma$  can be determined by adding nodes to  $J_\gamma$ , one at a time in increasing order of  $r_i w_{ig}$ , subject to  $r_i w_{ig} > 0$ . Let  $\hat{\gamma}$  be the value of  $\gamma$  so obtained. It can be seen that as  $\gamma$  decreases, the left hand side of (39) increases. Since the right hand side is a constant, we would obtain a unique  $\hat{\gamma}$  satisfying the equality. Furthermore, for  $\gamma > 0$ , this budget constraint is tight and so we are ensured the existence of  $\hat{\gamma}$ . Once  $\hat{\gamma}$  and hence  $J_{\hat{\gamma}}$  are obtained, an optimal solution for the good camp can be expressed as follows (recall that  $x_i = \mathcal{X}_i^t$ ):

- If  $r_i w_{ig} \leq 0$ ,  $x_i^* = 0$ .
- If  $r_i w_{ig} > t\hat{\gamma}$ ,  $x_i^* = 1$ .
- If  $0 < r_i w_{ig} \leq t\hat{\gamma}$ ,

$$x_i^* = \left( k_g - \sum_{\substack{i \notin J_{\hat{\gamma}} \\ i:r_i w_{ig} > 0}} 1 \right) \left( \frac{(r_i w_{ig})^{\frac{t}{t-1}}}{\sum_{i \in J_{\hat{\gamma}}} (r_i w_{ig})^{\frac{t}{t-1}}} \right)$$

However, if Equation (39) is not satisfied for any  $\gamma$ , that is, when

$$\sum_{i \in J_\gamma} \left( \frac{r_i w_{ig}}{t\gamma} \right)^{\frac{t}{t-1}} + \sum_{\substack{i \notin J_\gamma \\ i:r_i w_{ig} > 0}} 1 < k_g$$

even for the lowest value of  $\gamma > 0$ , we have  $J_\gamma = \{\}$  and hence  $\sum_{i:r_i w_{ig} > 0} 1 < k_g$ . Here, the number of nodes with  $r_i w_{ig} > 0$  is less than  $k_g$ , meaning that the budget constraint is not tight and so  $\gamma = 0$  (from Equation (26)). The investment is thus as per Equation (38).  $\square$

## APPENDIX C PROOF OF PROPOSITION 7

*Proof.*

$$\begin{aligned} \max_{\substack{\sum_i x_i \leq k_g \\ x_i \geq 0}} \min_{y_i \geq 0} \sum_i y_i \\ \text{s.t. } \sum_i v_i \leq 0 \end{aligned}$$

$$\sum_i v_i = \sum_i r_i (w_{ig} x_i + w_{ii}^0 v_i^0) - \sum_i r_i w_{ib} y_i$$

The inner term of this optimization problem is

$$\begin{aligned} \min \sum_i y_i \\ \text{s.t. } \sum_i r_i w_{ib} y_i \geq \sum_i r_i (w_{ig} x_i + w_{ii}^0 v_i^0) \quad \leftarrow \pi \\ \forall i : y_i \geq 0 \end{aligned}$$

It is clear that if  $\sum_i r_i (w_{ig} x_i + w_{ii}^0 v_i^0) \leq 0$ , it is optimal for the bad camp to have  $y_i = 0, \forall i$ , that is,  $\sum_i y_i = 0$ . So we need to only consider the case where  $\sum_i r_i (w_{ig} x_i + w_{ii}^0 v_i^0) > 0$ . In this case, for the constraint  $\sum_i v_i \leq 0$  to be satisfied, it is necessary that there exists a node  $j$  with positive value of  $r_j w_{jb}$ .

The dual problem of the above optimization problem can be written as

$$\begin{aligned} \max \pi \sum_i r_i (w_{ig} x_i + w_{ii}^0 v_i^0) \\ \text{s.t. } \forall i : \pi r_i w_{ib} \leq 1 \quad \leftarrow y_i \\ \pi \geq 0 \end{aligned}$$

From the dual constraints, we have  $\pi \leq \frac{1}{\max_j r_j w_{jb}}$  (recall that  $\max_j r_j w_{jb} > 0$  in this considered case). Since we aim to maximize the dual objective function, we have

$$\pi = \frac{1}{\max_j r_j w_{jb}}$$

The optimization problem now becomes

$$\begin{aligned} \max_{\substack{\sum_i x_i \leq k_g \\ x_i \geq 0}} \pi \sum_i r_i (w_{ig} x_i + w_{ii}^0 v_i^0) \\ = \pi \left( \max_{\substack{\sum_i x_i \leq k_g \\ x_i \geq 0}} \sum_i r_i w_{ig} x_i + \sum_i r_i w_{ii}^0 v_i^0 \right) \\ = \frac{1}{\max_j r_j w_{jb}} \left( k_g \max\{\max_i r_i w_{ig}, 0\} + \sum_i r_i w_{ii}^0 v_i^0 \right) \end{aligned}$$

The above expression thus gives the total investment made by the bad camp in the case where  $(k_g \max\{\max_i r_i w_{ig}, 0\} + \sum_i r_i w_{ii}^0 v_i^0) > 0$ .

Accounting for the case that it optimal for the bad camp to have  $y_i = 0, \forall i$  when  $(k_g \max\{\max_i r_i w_{ig}, 0\} + \sum_i r_i w_{ii}^0 v_i^0) \leq 0$ , the total investment made by the bad camp is

$$\max \left\{ \frac{1}{\max_j r_j w_{jb}} \left( k_g \max\{\max_i r_i w_{ig}, 0\} + \sum_i r_i w_{ii}^0 v_i^0 \right), 0 \right\}$$

$\square$