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Abstract

Wave propagation in heterogeneous media is accurately approximated on coarse meshes through a novel multiscale finite element method. Such a numerical method originates from the primal hybridization of the Helmholtz equation wherein the continuity of the pressure is released on the skeleton of a partition. As a result, the method is driven by the face degree of freedoms defined on faces of the partition, and independent local problems are responsible for the multiscale basis function computation. A two-level version of the method is also proposed in the case the basis functions are not promptly available. Well-posedness and a best approximation result is established for the one- and two-level MHM methods. Also, the MHM method is proved to be super convergent and is shown to recover other numerical methods. We assess theoretical results through a sequence of numerical tests.

1 Introduction

Numerical approximation of waves deserves particular attention by the scientific community. Indeed, though wave propagation problems arise in a wide range of applications, solving realistic 3D problems in the high frequency regime is still either very costly, or impossible.

Discretization methods for wave problems face the so-called "pollution effect": the number of discretization points per wavelength must be increased when the frequency is high. The pollution effect leads to restrictive conditions on the mesh step in the high frequency regime, making the discretization of high frequency problems computationally intensive.

The pollution effect has been vastly studied and is well-understood when the medium of propagation is homogeneous [1, 4, 17, 18, 23]. In order to reduce the pollution effect, the idea to capture *a priori* knowledge of the solution in the basis functions has recently emerged. The key ingredient is to use local solutions to the Helmholtz equation, usually plane waves, as basis functions. Because plane waves are less flexible than polynomials, special techniques are required to construct a conforming discretization space, or to stabilize the method. These techniques include include the Partition of Unity Method (PUm) [29], the Least Squares Method (LSM) [24], the Ultra Weak Variational Formulation (UWVF) [7], the Variational Theory of Complex Rays (VTCR) [28] and the Discontinuous Enrichment Method (DEm) [9].

The above-mentioned methods are established under the assumption that the medium of propagation is homogeneous. Indeed, these methods require the use of local solutions and plane waves can be used when the medium is homogeneous. When the propagation medium is highly heterogeneous, the wave speed might change inside the mesh cells, so that local analytical solutions are no longer available.

Extensions of plane wave methods to heterogeneous media have been investigated. When the propagation medium varies smoothly, so-called generalized plane wave methods can be used [19,30]. Generalized plane waves rely on the assumption that the wavespeed can be approximated by a

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polynomial function. Then, an analytical expression for the solution to the Helmholtz equation with polynomial wavespeed is used to construct the basis functions. On the other hand, when the wavespeed is piecewise constant, plane wave methods can still apply, but it is required that the wavespeed is constant in each cell and special care is taken to account for evanescent waves [20, 21, 31].

Unfortunately, some applications, like seismic wave propagation, feature highly heterogeneous media with jumps in the wavespeed. In this context, it is not clear that generalized plane waves can be used. Besides, restricting the mesh so that the wavespeed is constant in each cell is not always possible. A multiscale strategy based on high order polynomial basis functions has been developed [5, 10, 11]: the Multiscale Medium Approximation Method (MMAm). The MMAm has been applied to geophysical benchmarks and the results are very good if the acoustic medium has a constant density. Unfortunately, the cases of acoustic medium with non-constant density and elastic medium are more tricky to discretize. Though the MMAm outperforms standard finite element discretization in these cases, the results are not fully satisfactory, especially in the case elastic problems in the high frequency regime.

In this paper, we propose a multiscale method for the Helmholtz equation, which is adapted to highly heterogeneous media featuring jumps in the wavespeed. Our multiscale strategy is an adaptation of the Multiscale Hybrid Mixed method (MHMm) originally developed for Darcy flows in highly heterogeneous media [3,15]. Like the DEm [9], the MHMm is based on the primal hybrid formulation of the Helmholtz equation. As a result, we retrieve the lowest-order DEm elements as a particular case of the MHMm when the medium of propagation is homogeneous. When the medium of propagation is heterogeneous, the basis functions are computed as the solutions to a cell-level heterogeneous Helmholtz problem, leading to a fully multiscale strategy.

The objective of this paper is twofold. First, we present the MHMm for the Helmhotz equation and show that it is a promising solution for highly heterogeneous wave problems. Second, in the context of homogeneous media, we give a full frequency-explicit convergence analysis of the MHMm. It turns out that MHMm as a lot in common with the DEm when the medium is homogeneous. To the best of our knowledge, convergence analysis of the DEm is currently limited to the lowest-order elements [2] and the material presented hereafter might be helpful for the analysis of the DEm as well.

1.1 Statement and preliminaries

In this work, we focus on the acoustic Helmholtz equation set in a heterogeneous medium $\Omega \subset \mathbb{R}^2$, characterized by its bulk modulus $\kappa \in L^\infty(\Omega, \mathbb{R})$ and its density $\rho \in L^\infty(\Omega, \mathbb{R})$. Considering an angular frequency $\omega \in \mathbb{R}_+^*$ and a load term $f \in L^2(\Omega)$, the pressure $u \in H^1(\Omega)$ satisfies

$$-\frac{\omega^2}{\kappa} u - \operatorname{div}\left(\frac{1}{\rho}\nabla u\right) = f \text{ in } \Omega.$$

Having in mind seismic wave propagation, we divide the boundary $\partial\Omega$ into two subsets Γ_F and Γ_A. Γ_F represents the Earth surface. We impose a Dirichlet boundary condition $u = 0$ on Γ_F. The Dirichlet boundary condition corresponds to a free surface condition. A transparent boundary condition is prescribed on Γ_A in order to simulate a semi-infinite propagation medium. For the sake of simplicity, we consider a first-order absorbing boundary condition on Γ_A [8]:

$$\frac{1}{\rho}\nabla u \cdot n - \frac{1}{\sqrt{\kappa\rho}} u = 0.$$

It is worth noting that more general transparent conditions, like high order absorbing conditions [13] or perfectly matched layers [6], can be handled by the MHMm with minor modifications.

In the remaining, we focus on the boundary value problem to find $u \in H^1(\Omega)$ solution to

$$\left\{ \begin{array}{rcll} -\dfrac{\omega^2}{\kappa}u - \operatorname{div}\left(\dfrac{1}{\rho}\nabla u\right) & = & f & \text{in } \Omega \\ u & = & 0 & \text{on } \Gamma_F \\ \dfrac{1}{\rho}\nabla u \cdot n - \dfrac{1}{\sqrt{\kappa\rho}}u & = & 0 & \text{on } \Gamma_A. \end{array} \right. \tag{1}$$

Like the DEm [9], the MHMm is based on the primal hybrid formulation of the Helmholtz equation. In this formulation, the pressure is sought in a "broken space" of piecewise continuous fonctions. Thus, we introduce a family of meshes $(\mathcal{T}_H)_{H>0}$ parametrized by the small parameter H. For all $H > 0$, we assume that each cell $K \in \mathcal{T}_H$ is either a triangle or a square such that $\operatorname{diam}(K) \leq H$. We also assume that the meshes are regular in the sense that there exists a constant $\gamma > 0$ such that

$$\frac{\rho(K)}{\operatorname{diam}(K)} \geq \gamma \tag{2}$$

for all $H > 0$ and $K \in \mathcal{T}_h$. Considering $H > 0$, we introduce the space

$$V = \left\{ v \in L^2(\Omega) \quad | \quad v|_K \in H^1(K);\ \forall K \in \mathcal{T}_h \right\}. \tag{3}$$

The continuity of functions in V is not ensured across adjacent cells. In order to weakly enforce the continuity of the solution, we consider a space Λ of Lagrange multipliers. This space is defined by

$$\Lambda = \left\{ \mu \in \prod_{K \in \mathcal{T}_h} H^{-1/2}(\partial K) \quad \middle| \quad \begin{array}{l} \exists \sigma \in H^1(\operatorname{div}, \Omega); \\ \mu|_{\partial K} = \operatorname{div} \sigma \cdot n_K \ \forall K \in \mathcal{T}_h \\ \\ \mu|_{\Gamma_A} = 0 \end{array} \right\}. \tag{4}$$

We introduce the sesquilinear forms $a : V \times V \to \mathbb{C}$ and $b : \Lambda \times V \to \mathbb{C}$ by

$$a(u,v) = \sum_{K \in \mathcal{T}_h} \left\{ -\omega^2 \int_K \frac{1}{\kappa} u\overline{v} - i\omega \int_{\partial K \cap \partial\Omega} \frac{1}{\sqrt{\kappa\rho}} u\overline{v} + \int_K \frac{1}{\rho} \nabla u \cdot \overline{\nabla v} \right\}, \tag{5}$$

for all $u, v \in V$ and

$$b(\mu, v) = \sum_{K \in \mathcal{T}_h} \int_{\partial K} \mu\overline{v}, \tag{6}$$

for $\mu \in \Lambda$ and $v \in V$. The integral on ∂K is understood as the duality product between $H^{-1/2}(\partial K)$ and $H^{1/2}(\partial K)$.

Given $H > 0$, the primal hybrid formulation consists in finding a couple $(u, \lambda) \in V \times \Lambda$ solution to

$$\left\{ \begin{array}{rcll} a(u,v) + b(\lambda, v) & = & (f, v) & \forall v \in V, \\ b(\mu, u) & = & 0 & \forall \mu \in \Lambda. \end{array} \right. \tag{7}$$

The first equation of (7) is obtained by piecewise integration by parts of (1). It also corresponds to an optimality condition where λ plays the role of the Lagrange multiplier. On the other, the second equation of (7) enforces the continuity of u and is equivalent to $u \in H^1(\Omega)$. The couple $(u, \lambda) \in V \times \Lambda$ solves (7), if and only if u is a weak solution to (1) and $\lambda = \nabla u \cdot n_K$ on $\partial K \setminus \Gamma_A$ for all $K \in \mathcal{T}_h$. We refer the reader to [9, 27] for more details about the primal hybrid formulation.

At this point, it is possible to discretize problem (7) directly by selecting finite dimensional subspaces of V and Λ. This approach is taken in the reference work of Raviart and Thomas [27] for the discretization of the Laplace problem. In their work, the authors derive a family of stable pairs of polynomial discretization spaces. The DEm is also directly based on formulation (7). In the standard version of DEm, the Lagrange multiplier is discretized with piecewise polynomial functions and the pressure is discretized using plane waves. Of course, the novelty of the DEm resides in the plane wave basis functions used to discretize V.

2 The MHM method

2.1 The one-level MHM method

The novelty of the MHMm as compared to the DEm, is to substitute the pressure u for the Lagrange multiplier λ. Indeed, rewriting the first equation of (7) as

$$a(u, v) = (f, v) - b(\lambda, v), \quad \forall v \in V,$$

we observe that the pressure can be expressed as $u = T\lambda + \hat{T}f$, where $T : \Lambda \to V$ and $\hat{T} : L^2(\Omega) \to V$ are two linear bounded operators defined by

$$a(T\mu, v) = -b(\mu, v), \quad a(\hat{T}f, v) = (f, v), \tag{8}$$

for all $\mu \in \Lambda$, $f \in L^2(\Omega)$ and $v \in V$.

Of course, the analytical expression of these operators is not available. However, since there is no build-in compatibility condition in V for u, T and $\hat{T}$ can be defined thanks to local boundary value problems in each cell. Indeed, considering a cell $K \in \mathcal{T}_h$ and test functions $v \in H^1(K)$ in (8), we see that

$$\left\{ \begin{array}{rcll} -\dfrac{\omega^2}{\kappa}\left(T\mu|_K\right) - \operatorname{div}\left(\dfrac{1}{\rho}\nabla\left(T\mu|_K\right)\right) & = & 0 & \text{in } K \\ \dfrac{1}{\rho}\nabla\left(T\mu|_K\right)\cdot n & = & -\mu & \text{on } \partial K \setminus \Gamma_A \\ \dfrac{1}{\rho}\nabla\left(T\mu|_K\right)\cdot n - \dfrac{1}{\sqrt{\kappa\rho}}\left(T\mu|_K\right) & = & 0 & \text{on } \partial K \cap \Gamma_A, \end{array} \right. \tag{9}$$

and

$$\left\{ \begin{array}{rcll} -\dfrac{\omega^2}{\kappa}\left(\hat{T}f|_K\right) - \operatorname{div}\left(\dfrac{1}{\rho}\nabla\left(\hat{T}f|_K\right)\right) & = & f & \text{in } K \\ \dfrac{1}{\rho}\nabla\left(\hat{T}f|_K\right)\cdot n & = & 0 & \text{on } \partial K \setminus \Gamma_A \\ \dfrac{1}{\rho}\nabla\left(\hat{T}f|_K\right)\cdot n - \dfrac{1}{\sqrt{\kappa\rho}}\left(\hat{T}f|_K\right) & = & 0 & \text{on } \partial K \cap \Gamma_A. \end{array} \right. \tag{10}$$

Thus, the operators T and $\hat{T}$ are defined locally in each cell K as the solutions to local boundary value problems. The key ingredient of the MHMm is to approximate these operators using a second-level discretization scheme (for instance, Lagrangian finite elements). Since all computations are local, the second-level approximation is naturally parallelized and corresponds to a pre-processing step before solving the global problem.

Assuming that the operators T and $\hat{T}$ are available, we can substitute u for λ by plugging $u = T\lambda + \hat{T}f$ in the second equation of (7). We obtain

$$b(\mu, T\lambda) = -b(\mu, \hat{T}f), \quad \forall \mu \in \Lambda. \tag{11}$$

In contrast to (9) and (10), problem (11) is global. The one-level Multiscale Hybrid Mixed method is obtained by introducing a finite dimensional subspace $\Lambda_H \subset \Lambda$. The Multiscale Hybrid Mixed formulation consist in finding $\lambda_H \in \Lambda_H$ such that

$$b(\mu_H, T\lambda_H) = -b(\mu_H, \hat{T}f), \quad \forall \mu_H \in \Lambda_H. \tag{12}$$

The MHMm is based on a mesh characterized by the step H. One of the aim of the MHMm is to account for highly heterogeneous media. As depicted in [3, 15], one asset the MHMm is to incorporate heterogeneities of characteristic length $\epsilon << H$ on a coarse mesh of characteristic size H. In this situation, it is usually required that to use a mesh of size ϵ for the second-level discretization. However, since the second-level computations are local, using a fine mesh which fit the heterogeneities is affordable.

2.2 The two-level MHM method

In the one-level MHM formulation, we assumed that the operators T and $\hat{T}$ are available and exactly computed. In practice, these operators are only analytically available in some exceptional cases. It is thus of interest to introduce a two-level MHM method in which the operators T and $\hat{T}$ are replaced by discrete counter parts T_h and $\hat{T}_h$ obtained thanks to a second-level method.

In this case, a discretization space $V_h \subset V$ is introduced, and the discrete operators are defined by

$$a(T_h \mu, v_h) = -b(\mu, v_h), \quad a(\hat{T}_h f, v_h) = (f, v_h),$$

for all $\mu \in \Lambda$, $f \in L^2(\Omega)$ and $v_h \in V_h$.

The definition of the operators T_h and $\hat{T}_h$ actually decouple over each cell $K \in \mathcal{T}_{\mathcal{H}}$. As a result, evaluating $T_h \mu$ and $\hat{T}_h f$ for particular arguments $\mu \in \Lambda$ and $f \in L^2(\Omega)$ acutaly amounts to solve a sequence of local cell-wise discrete Galerkin problems.

We would like to point out that the DEm can be seen as a particular case of the MHMm. Indeed, when the propagation medium is homogeneous, the DEm consists in using a set of plane waves in the second-level discretizations space.

2.3 Practical implementation of the two-level MHM method

We briefly scheme the main steps of the two-level MHM algorithm from a computational point of view

1. We consider a mesh step $H > 0$ and a finite dimensional discretization space $(\mu_k)_{k=1}^n = \Lambda_H \subset \Lambda$ for λ.

2. We compute the images of the basis functions μ_k ($k \in \{1, \ldots, n\}$) by the operator T_h and the image of f by $\hat{T}_h$. This is done by searching functions $\eta_k = T\mu_k \in V_h$ for $i \in \{1, \ldots, n\}$ and $\eta = \hat{T}_h f \in V_h$ solutions to

$$a(\eta_k, v_h) = -b(\mu_k, v_h), \quad a(\eta, v_h) = (f, v_h), \quad \forall v_h \in V_h.$$

   We would like to emphasize that all the computations are local to each cell. This computations correspond to local (cell-wise) Helmholtz problems (9) and (10) and are a pre-processing step before solving the global problem in the next stage.

3. The MHMm approximation λ_H of λ is defined by

$$\lambda_h = \sum_{k=1}^{n} c_k \mu_k,$$

   where $c \in \mathbb{C}^n$ is a vector of unknown coefficients. We obtain c by solving the $n \times n$ linear system stemming from (12):

$$\sum_{k=1}^{n} b(\mu_l, \eta_k) \bar{c_k} = -b(\mu_l, \eta), \quad \forall l \in \{1, \ldots, n\}.$$

   The coefficient of this system simply involve boundary integral of the basis functions, that can be easily computated from their underlying second-level Galerkin representation.

3 Well-posedness of the continuous MHM formulation

In the following, we equip the space V and Λ with the following norms

$$\|v\|_{V,\omega}^2 = \sum_{K \in \mathcal{T}_H} \left\{ \omega^2 |v|_{0,K}^2 + \omega |v|_{0,\partial K \cap \partial \Omega}^2 + |\nabla v|_{0,K}^2 \right\}, \quad \forall v \in V,$$

and

$$\|\mu\|_{\Lambda,\omega} = \sup_{v \in V \setminus \{0\}} \frac{\operatorname{Re} b(\mu, v)}{\|v\|_{V,\omega}}, \quad \forall \mu \in \Lambda.$$

In order to derive our main results concerning the MHM formulation and discretization, we need to assume that the problem we solve is well-posed. Thus, we introduce

Assumption 1. *For all $f \in L^2(\Omega)$, there exists a unique $u \in H^1(\Omega)$ solution to problem* (1). *Furthermore, $u \in H^2(\Omega)$ and we have*

$$\omega|u|_{0,\Omega} + |u|_{1,\Omega} + \omega^{-1}|u|_{2,\Omega} \lesssim |f|_{0,\Omega}. \tag{13}$$

We mentioned that assumption (1) is fullfiled in a range of applications. For instance, if κ and ρ are constant, (13) holds for some configurations of Γ_F and Γ_A as described in [16]. Other sitatutions in which κ and/or ρ vary are analyzed in [5, 10].

3.1 Well-posedness of the local problems

We start by proving that the local problems defining the operators T and $\hat{T}$ are well-posed. This is classically done by showing that the sesquilinear form a is continuous and satisfies an inf-sup condition.

The continuity of a is rather straightforward, and is recorded in Lemma 1.

Lemma 1. *For all $u, v \in V$, it holds that*

$$|a(u, v)| \lesssim \|u\|_{V,\omega}\|v\|_{V,\omega}.$$

Proof. Let $u, v \in V$, we have

$$\begin{aligned}
|a(u,v)| &\lesssim \sum_{K \in \mathcal{T}_H} \left\{ \omega^2 |(\kappa^{-1}u, v)_K| + \omega |\langle (\kappa\rho)^{-1/2} u, v\rangle_{\partial K \cap \Gamma_A}| + |(\rho^{-1}\nabla u, \nabla v)_K| \right\} \\
&\lesssim \sum_{K \in \mathcal{T}_H} \left\{ \omega^2 |u|_{0,K}|v|_{0,K} + \omega |u|_{0,\partial K \cap \partial\Omega}|v|_{0,\partial K \cap \partial\Omega} + |u|_{1,K}|v|_{1,K} \right\} \\
&\lesssim \sum_{K \in \mathcal{T}_H} \|u|_K\|_{V,\omega}\|v|_K\|_{V,\omega} \\
&\lesssim \left(\sum_{K \in \mathcal{T}_H} \|u|_K\|_{V,\omega}^2 \right)^{1/2} \left(\sum_{K \in \mathcal{T}_H} \|v|_K\|_{V,\omega}^2 \right)^{1/2} \\
&\lesssim \|u\|_{V,\omega}\|v\|_{V,\omega}.
\end{aligned}$$

□

The proof of the inf-sup condition is more subtle. We need to separate the case of interior and boundary cells. Two special Poincaré-like inequality are required and presented in Propositions 1 and 2. The proof of the first proposition can be found in [25] and the second is presented in Appendix A.

Proposition 1. *Let K be a triangle or a rectangle with* $\operatorname{diam}(K) \leq H$. *Then we have*

$$|u|_{0,K} \leq \frac{H}{\pi}|u|_{1,K}, \tag{14}$$

for all $u \in H^1(K)$ such that

$$\int_K u = 0.$$

Proposition 2. *Let K be a triangle or a rectangle with* $\operatorname{diam}(K) \leq H$. *Then we have*

$$\omega^2 |u|_{0,K}^2 \leq 2\omega H(\omega |u|_{0,\Gamma}^2) + (2\omega H)^2 |u|_{1,K}^2, \tag{15}$$

for all $u \in H^1(K)$.

The inf-sup condition is constructed cell-wise. There are two different cases two consider, depending on whether or not $\partial K \cap \partial\Omega = \emptyset$. These two cases are considered in Propositions 3 and 4.

Proposition 3. *Assume $K \in \mathcal{T}_H$ is such that $\partial K \cap \partial\Omega = \emptyset$. Then, for all $u \in H^1(K)$, we have*

$$\operatorname{Re} a(u, u^\star) \gtrsim \|u\|_{V,\omega}^2,$$

where

$$u^\star = u - 2\frac{1}{|K|}\int_K u$$

Proof. Consider $u \in H^1(K)$. We write

$$u_0 = \frac{1}{|K|}\int_K u,$$

and $u^\perp = u - u_0$. Then, we have $u = u^\perp + u_0$ and $u^\star = u^\perp - u_0$. It follows that

$$\begin{aligned}
a(u, u^\star) &= a(u^\perp + u_0, u^\perp - u_0) \\
&= a(u^\perp, u^\perp) - a(u_0, u_0) + a(u_0, u^\perp) - a(u^\perp, u_0) \\
&= a(u^\perp, u^\perp) - a(u_0, u_0) - \overline{a(u^\perp, u_0)} - a(u^\perp, u_0) \\
&= a(u^\perp, u^\perp) - a(u_0, u_0) - 2\operatorname{Re} a(u^\perp, u_0). \\
&= -\omega^2 |\kappa^{-1/2} u^\perp|_{0,K}^2 + |\rho^{-1/2}\nabla u^\perp|_{0,K}^2 + \omega^2 |\kappa^{-1/2} u_0|_{0,K}^2 - 2\omega^2 \operatorname{Re}(\kappa^{-1} u^\perp, u_0)
\end{aligned}$$

As a result, we have

$$\operatorname{Re} a(u, u^\star) \gtrsim \left\{ -\omega^2 |u^\perp|_{0,K}^2 + |u^\perp|_{1,K}^2 + \omega^2 |u_0|_{0,K}^2 - 2\omega^2 |u^\perp|_{0,K} |u_0|_{0,K} \right\}.$$

With the algebraic inequality $2|u^\perp|_{0,K}|u_0|_{0,K} \leq \frac{1}{2}|u_0|_{0,K}^2 + 2|u^\perp|_{0,K}^2$, we obtain

$$\operatorname{Re} a(u, u^\star) \gtrsim \left\{ -3\omega^2 |u^\perp|_{0,K}^2 + |u^\perp|_{1,K}^2 + \frac{\omega^2}{2} |u_0|_{0,K}^2 \right\},$$

that we simply rewrite

$$\operatorname{Re} a(u, u^\star) \gtrsim \left\{ -\omega^2 |u^\perp|_{0,K}^2 + |u^\perp|_{1,K}^2 + \omega^2 |u_0|_{0,K}^2 \right\}. \tag{16}$$

To conclude, we need a Poincaré-Wirtinger inequality from [25]. Since K is convex and

$$\int_K u^\perp = 0,$$

we have

$$|u^\perp|_{1,K} \geq \frac{\pi}{\operatorname{diam} K} |u^\perp|_{0,K}.$$

Since $\operatorname{diam} K \leq H$, it follows that

$$|u^\perp|_{1,K}^2 \geq \frac{\pi^2}{H^2} |u^\perp|_{0,K},$$

and therefore

$$\begin{aligned}
-\omega^2|u^\perp|_{0,K}^2 + |u^\perp|_{1,K}^2 &= -\omega^2|u^\perp|_{0,K}^2 + \frac{1}{2}|u^\perp|_{1,K}^2 + \frac{1}{2}|u^\perp|_{1,K}^2 \\
&\geq \left(\frac{\pi^2}{2H^2} - \omega^2\right)|u^\perp|_{0,K}^2 + \frac{1}{2}|u^\perp|_{1,K}^2 \\
&\geq \omega^2\left(\frac{2\pi^2}{2H^2\omega^2} - 1\right)|u^\perp|_{0,K}^2 + \frac{1}{2}|u^\perp|_{1,K}^2.
\end{aligned}$$

Hence, assuming that ωH is small enough, we have

$$-\omega^2|u^\perp|_{0,K}^2 + |u^\perp|_{1,K}^2 \gtrsim \|u^\perp\|_{V,\omega}^2. \tag{17}$$

Furthermore, since $\nabla u_0 = 0$ and $(u_0, u^\perp) = 0$, we have

$$\|u\|_{V,\omega}^2 = \omega^2|u_0|_{0,K}^2 + \|u^\perp\|_{V,\omega}^2,$$

and we conclude since from (16) and (17), we have

$$\operatorname{Re} a(u, u^\star) \gtrsim \left\{\|u^\perp\|_{V,\omega}^2 + \omega^2|u_0|_{0,K}^2\right\} \gtrsim \|u\|_{V,\omega}^2.$$

□

Proposition 4. *Assume $K \in \mathcal{T}_H$ is such that $\partial K \cap \partial\Omega \neq \emptyset$. Then, for all $u \in H^1(K)$, we have*

$$\operatorname{Re} a(u, u^\star) \gtrsim \|u\|_{V,\omega}^2,$$

where $u^\star = (1+i)u$.

Proof. Let $u \in H^1(K)$, and write $\Gamma = \partial K \cap \partial\Omega$. Observe that we have

$$\operatorname{Re} a(u, iu) = \omega|(\kappa\rho)^{-1/4}u|_{0,\Gamma}^2| \gtrsim \omega|u|_{0,\Gamma}^2,$$

and

$$\operatorname{Re} a(u, u) = -\omega^2|\kappa^{-1/2}u|_{0,K}^2 + |\rho^{-1/2}\nabla u|_{0,K}^2 \gtrsim \left\{\omega^2|u|_{0,K}^2 + |u|_{1,K}^2\right\}.$$

As a result, we have

$$\operatorname{Re} a(u, u^\star) \gtrsim \left\{-\omega^2|u|_{0,K}^2 + |u|_{1,K}^2 + \omega|u|_{0,\Gamma}^2\right\}$$

Since Γ contains at least one side of H, we can conclude with the Poincaré inequality (15) of Lemma 2. Indeed, we have

$$\omega^2|u|_{0,K}^2 \leq (2\omega H)^2|u|_{1,K}^2 + (2\omega H)|u|_{0,\Gamma}^2,$$

that can be rewritten as

$$-\omega^2|u|_{0,K}^2 \geq \omega^2|u|_{0,K}^2 - 2(2\omega H)^2|u|_{1,K}^2 - 2(2\omega H)\omega|u|_{0,\Gamma}^2,$$

and it follows that

$$\operatorname{Re} a(u, u^\star) \gtrsim \left\{\omega^2|u|_{0,K}^2 + (1 - 2(2\omega H)^2)|u|_{1,K}^2 + (1 - 2(2\omega H))\omega|u|_{0,\Gamma}^2\right\},$$

and the result follows, assuming that ωH is small enough. □

We are now ready to establish the inf-sup condition.

Lemma 2. *We have*

$$\sup_{v \in V \setminus \{0\}} \frac{\operatorname{Re} a(u, v)}{\|v\|_{V,\omega}} \gtrsim \|u\|_{V,\omega},$$

for all $u \in V$.

Proof. For each cell $K \in \mathcal{T}_{\mathcal{H}}$, let us construct $(u|_K)^\star \in H^1(K)$ like in Proposition 3 and 4. We define $v \in V$ by

$$v|_K = (u|_K)^\star.$$

Then we have

$$\begin{aligned}
\operatorname{Re} a(u,v) &= \sum_{K\in\mathcal{T}_H} \operatorname{Re} a(u|_K, v|_K) \\
&= \sum_{K\in\mathcal{T}_H} \operatorname{Re} a(u|_K, (u|_K)^\star) \\
&\gtrsim \sum_{K\in\mathcal{T}_H} \|u|_K\|^2_{V,\omega} \\
&\gtrsim \|u\|^2_{V,\omega},
\end{aligned}$$

and the proof follows, because clearly, $\|v\|_{V,\omega} \lesssim \|u\|_{V,\omega}$. □

We are now ready to establish the well-posedness of the local problems in Theorem 1.

Theorem 1. *Assume that ωH is small enough. Then, for all $\mu \in \Lambda$ and $f \in L^2(\Omega)$ there exist unique elements $T\mu, \hat{T}f \in V$ such that*

$$a(T\mu, v) = -b(\mu, v), \quad a(\hat{T}f, v) = (f, v), \quad \forall v \in V.$$

Furthermore, we have

$$\|T\mu\|_{V,\omega} \lesssim \|\mu\|_{\Lambda,\omega}, \quad \|\hat{T}f\|_{V,\omega} \lesssim \omega^{-1}|f|_{0,\Omega}, \tag{18}$$

for all $\mu \in \Lambda$ and $f \in L^2(\Omega)$.

Proof. The continutiy of the sesquilinear form a established in Lemma 1 together with the inf-sup condition of Lemma 2 ensure the existence and uniqueness of $T\lambda$ and $\hat{T}f$ for all $\lambda \in \Lambda$ and $f \in L^2(\Omega)$.

Let us show (18). Consider $\lambda \in \Lambda$. Then for $v \in V \setminus \{0\}$, we have

$$\frac{\operatorname{Re} a_{w,h}(T\lambda, v)}{\|v\|_{V,\omega}} = -\frac{\operatorname{Re} b(\lambda, v)}{\|v\|_{V,\omega}} \leq \|\lambda\|_{\Lambda,\omega},$$

and the first estimate of (18) follows from Lemma 2. Now, if $f \in L^2(\Omega)$, we have

$$\operatorname{Re} a(\hat{T}f, v) = \operatorname{Re}(f, v) \leq |f|_{0,\Omega}|v|_{0,\Omega} \leq \omega^{-1}|f|_{0,\Omega}\|v\|_{V,\omega}, \quad \forall v \in V$$

so that

$$\|\hat{T}f\|_{V,\omega} \lesssim \sup_{v\in V\setminus\{0\}} \frac{\operatorname{Re} a(\hat{T}f, v)}{\|v\|_{V,\omega}} \leq \omega^{-1}|f|_{0,\Omega},$$

and the second estimate of (18) follows. □

Corollary 1. *The applications $\|.\|_{\Lambda,\omega}$ and $\|T.\|_{V,\omega}$ define equivalent norms on Λ:*

$$\|\mu\|_{\Lambda,\omega} \lesssim \|T\mu\|_{V,\omega} \lesssim \|\mu\|_{\Lambda,\omega} \quad \forall \mu \in \Lambda. \tag{19}$$

Proof. From Theorem 1, the upper bound of (19) is already established. To prove the lower bound, we only need to observe that

$$\|\mu\|_{\Lambda,\omega} = \sup_{v\in V\setminus\{0\}} \frac{\operatorname{Re} b(\mu, v)}{\|v\|_{V,\omega}} = \sup_{v\in V\setminus\{0\}} \frac{\operatorname{Re} a(T\mu, v)}{\|v\|_{V,\omega}} \lesssim \|T\mu\|_{V,\omega},$$

by Lemma 1 □

3.2 Well-posedness of the global MHM problem

In this subsection, we focus on proving that the MHM global formulation is well-posed. We start by showing existence, uniqueness and stability of the solution in Theorem 2.

Theorem 2. *Assume that ωH is small enough. Then, for all $f \in L^2(\Omega)$, there exists a unique solution $\lambda \in \Lambda$ such that*

$$b(\mu, T\lambda) = -b(\mu, \hat{T}f), \quad \forall \mu \in \Lambda.$$

Furthermore, it holds that

$$\|\lambda\|_{\Lambda,\omega} \lesssim |f|_{0,\Omega}. \tag{20}$$

Proof. From Assumption 1, we know that there exists a couple $(u, \lambda) \in V \times \Lambda$ such that

$$\left\{ \begin{array}{rcll} a(u,v) + b(\lambda, v) & = & (f,v) & \forall v \in V \\ b(\mu, u) & = & 0 & \forall \mu \in \Lambda, \end{array} \right. \tag{21}$$

where $u \in H^1(\Omega)$ is the usual solution to the Helmholtz equation and λ is defined as the normal derivative of u on the boundary of each $K \in \mathcal{T}_H$.

Now, by Theorem 1, the operators T and $\hat{T}$ are well-defined and invertible. As a result, the first equation of (21) shows that

$$u = T\lambda + \hat{T}f. \tag{22}$$

Injecting (22) into the second equation of (21) shows that λ is the solution to the continuous MHM formulation, and existence follows. Also, we have uniqueness, as the couple (u, λ) is unique.

Finally, we have

$$\|T\lambda\|_{V,\omega} \leq \|u\|_{V,\omega} + \|\hat{T}f\|_{V,\omega},$$

and (20) follows. We conclude with norm equivalence (19). □

The next results are devoted to the analysis of the MHM sesquilinear form $\Lambda \ni \mu, \lambda \to b(\mu, T\lambda)$. In Proposition 5, we give a symmetry result.

Proposition 5. *For all $\mu, \lambda \in \Lambda$, we have*

$$b(\mu, T\lambda) = b(\overline{\lambda}, T\overline{\mu}). \tag{23}$$

Proof. Let $\mu, \lambda \in \Lambda$. We have

$$\overline{b(\mu, T\lambda)} = b(\overline{\mu}, \overline{T}\lambda) = -a(T\overline{\mu}, \overline{T}\lambda) = -a(T\lambda, \overline{T}\overline{\mu}) = b(\lambda, \overline{T}\overline{\mu}),$$

and (23) follows by taking the complex conjugate. □

In Lemma 3, we associate to each $\lambda \in \Lambda$ and element $\eta \in \Lambda$ that plays a crucial role in inf-sup estimates.

Lemma 3. *For $\lambda \in \Lambda$, define $\eta \in \Lambda$ as the unique solution to*

$$b(\mu, T\overline{\eta}) = -b\left(\mu, \hat{T}\left(\overline{T}\lambda\right)\right) \tag{24}$$

Then, we have

$$b(\eta, T\lambda) = |T\lambda|^2_{0,\Omega}. \tag{25}$$

Proof. Consider $\lambda \in \Lambda$. The existence and uniqueness result of Theorem 2 ensures that definition (24) makes sense for η.

Then, because of Proposition 5, it holds that

$$b(\eta, T\lambda) = b\left(\overline{\lambda}, T\overline{\eta}\right) = -b\left(\overline{\lambda}, \hat{T}\left(\overline{T}\lambda\right)\right). \tag{26}$$

Using the definitions of T and $\hat{T}$, we derive

$$\begin{aligned} \overline{-b(\overline{\lambda}, \hat{T}(\overline{T}\lambda))} &= -b(\lambda, \overline{\hat{T}}(\overline{T}\lambda)) \\ &= a(T\lambda, \overline{\hat{T}}(\overline{T}\lambda)) \\ &= a(\hat{T}(\overline{T}\lambda), \overline{T}\lambda) \\ &= (\overline{T}\lambda, \overline{T}\lambda) \\ &= |\overline{T}\lambda|^2_{0,\Omega}. \end{aligned} \tag{27}$$

Taking the complex conjugate of (27), we conclude the demonstration. Indeed, since the right-hand-side of (27) is real, we have

$$-b(\overline{\lambda}, \hat{T}(\overline{T}\lambda)) = \overline{|\overline{T}\lambda|^2_{0,\Omega}} = |\overline{T}\lambda|^2_{0,\Omega} = |T\lambda|^2_{0,\Omega},$$

which conclude the proof using (26). □

We close this section with an inf-sup condition for the MHM sesquilinear form, which is established in Theorem 3.

Theorem 3. *Assume that ωH is small enough. Then, the following inf-sup condition holds:*

$$\sup_{\mu \in \Lambda \setminus \{0\}} \frac{\operatorname{Re} b(\mu, T\lambda)}{\|T\mu\|_{V,\omega}} \gtrsim \frac{1}{\omega} \|T\lambda\|_{V,\omega}, \tag{28}$$

for all $\lambda \in \Lambda$.

Proof. Fix $\lambda \in V$. We have

$$\operatorname{Re} b(\lambda, T\lambda) = \operatorname{Re} a(T\lambda, T\lambda) = \|T\lambda\|^2_{\Lambda,\omega} - 2\omega^2 |T\lambda|^2_{0,\Omega}.$$

Recalling Lemma 3, if we define $\eta \in \Lambda$ as the unique solution to

$$b(\mu, T\overline{\eta}) = -2\omega^2 b\left(\mu, \hat{T}\left(\overline{T}\lambda\right)\right), \quad \forall \mu \in \Lambda,$$

and we have

$$\operatorname{Re} b(\eta, T\lambda) = 2\omega^2 |T\lambda|^2_{0,\Omega}.$$

We define $\mu = \lambda + \eta \in \Lambda$ and we have

$$\operatorname{Re} b(\mu, T\lambda) = \|T\lambda\|^2_{V,\omega}.$$

We apply stability estimate (20) from Theorem 2. It follows that

$$\|T\mu\|_{V,\omega} \lesssim \omega^2 |T\lambda|_{0,\Omega} \lesssim \omega \|T\lambda\|_{V,\omega}.$$

Thus we have

$$\operatorname{Re} b(\mu, T\lambda) \gtrsim \frac{1}{\omega} \|T\lambda\|_{V,\omega} \|T\mu\|_{V,\omega},$$

and the result follows. □

4 Well-posedness and convergence of the one-level MHMm

In this section, we analyse the one-level MHM method, where the space Λ is replaced by an internal approximation subspace $\Lambda_H \subset \Lambda$. We show that if the element of Λ_H can accuratly reproduce the continuous solution to (12), then the discrete problem is well-posed, and the MHMm solution is quasi-optimal.

Here, we derive a general theory under the mere assumption that Λ_H is a finite-dimensional subspace of Λ. Applications to particular discretization spaces Λ_H are discussed later in the article.

We introduce the real number $\alpha_{\omega,H}$ in order to characterize the approximation properties of Λ_H. It is defined by

$$\alpha_{\omega,H} = \sup_{f \in L^2(\Omega)\setminus\{0\}} \inf_{\mu_h \in \Lambda_H} \frac{\|\lambda_f - \mu_H\|_{\Lambda,\omega}}{|f|_{0,\Omega}}, \tag{29}$$

where $\lambda_f \in \Lambda$ is the unique solution to

$$b(\mu, T\lambda_f) = -b(\mu, \hat{T}f), \quad \forall \mu \in V.$$

Since Λ_H is finite-dimensional, as a direct consequence of (29), for all $f \in L^2(\Omega)$, there exists an element $\mu_H \in \Lambda_H$ such that

$$\|\lambda_f - \mu_H\|_{\Lambda,\omega} \leq \alpha_{\omega,H}|f|_{0,\Omega}. \tag{30}$$

Theorem 4. *Assume that ωH and $\omega\alpha_{\omega,H}$ are small enough. Then it holds that*

$$\sup_{\mu_H \in \Lambda\setminus\{0\}} \frac{\operatorname{Re} b(\mu_H, T\lambda_H)}{\|T\mu_H\|_{V,\omega}} \gtrsim \frac{1}{\omega}\|T\lambda_H\|_{V,\omega}, \tag{31}$$

for all $\lambda_H \in \Lambda_H$.

Proof. Pick an arbitrary $\lambda_H \in \Lambda_H$. Recalling Lemma 3, we define $\eta \in \Lambda$ as the unique solution to

$$b(\mu, T\overline{\eta}) = -2\omega^2 b\left(\mu, \hat{T}\left(\overline{T}\lambda\right)\right) \quad \forall \mu_H \in \Lambda_H,$$

and we have

$$b(\eta, T\lambda_H) = 2\omega^2|T\lambda_H|_0^2.$$

Unfortunately, as $\eta \notin \Lambda_H$, we can not set $\mu_H = \lambda_H + \eta$ to obtain the inf-sup condition as in Theorem 3. Instead, we introduce $\eta_H \in \Lambda_H$ as the best approximation of η and set $\mu_H = \lambda_H + \eta_H \in \Lambda_H$. Recalling the definition of $\alpha_{\omega,H}$ and (30), we have

$$\|T(\eta - \eta_H)\|_{V,\omega} \leq C\omega^2\alpha_{\omega,H}|T\lambda_H|_{0,\Omega} \leq C\omega\alpha_{\omega,H}\|T\lambda_H\|_{V,\omega}$$

As a result, letting $\mu = \lambda_H + \eta \in \Lambda$ it follows that

$$\begin{aligned}
\operatorname{Re} b(\mu_H, T\lambda_H) &= \operatorname{Re} b(\mu, T\lambda_H) - \operatorname{Re} b(\mu - \mu_H, T\lambda_H) \\
&= \operatorname{Re} b(\mu, T\lambda_H) - \operatorname{Re} b(\eta - \eta_H, T\lambda_H) \\
&= \|T\lambda_H\|_{V,\omega}^2 - \operatorname{Re} a(T(\eta - \eta_H), T\lambda_H) \\
&\gtrsim \|T\lambda_H\|_{V,\omega}^2 - \|T(\eta - \eta_H)\|_{V,\omega}\|T\lambda_H\|_{V,\omega} \\
&\gtrsim (1 - C\omega\alpha_{\omega,H})\,\|T\lambda\|_{V,\omega}^2,
\end{aligned}$$

for some constant C that is independent of ω and H. Then, assuming that $\omega\alpha_{\omega,H}$ is small enough, we obtain that

$$\operatorname{Re} b(\mu_H, T\lambda_H) \gtrsim \|T\lambda_H\|_{V,\omega}^2,$$

and it remains to show that

$$\|T\mu_H\|_{V,\omega} \lesssim \omega\|T\lambda_H\|_{V,\omega}.$$

But we have

$$\begin{aligned}
\|T\mu_H\|_{V,\omega} &\leq \|T\mu\|_{V,\omega} + \|T(\mu - \mu_H)\|_{V,\omega} \\
&\leq \|T\lambda_H\|_{V,\omega} + \|T\eta\|_{V,\omega} + \|T(\eta - \eta_H)\|_{V,\omega} \\
&\lesssim \|T\lambda_H\|_{V,\omega} + \omega\|T\lambda_H\|_{V,\omega} + \omega\alpha_{\omega,H}\|T\lambda_H\|_{V,\omega},
\end{aligned}$$

and the proof follows, using again the assumption that $\omega\alpha_{\omega,H}$ is small enough. □

Theorem 4 shows that the one-level MHM method is well-posed as soon as the discretization subspace Λ_H reproduces solution to (12) accurately. This corresponds to the condition that $\omega\alpha_{\omega,H}$ must be sufficiently small.

In the remaining of this section, we derive an error-estimate for the discrete solution of the one-level problem. Though the error-estimate can be directly obtained from the inf-sup condition, we perform an additional analysis to obtain a sharper bound. We start by showing an Aubin-Nitsche type inequality in Lemma 4.

Lemma 4. *Consider $f \in L^2(\Omega)$. Let $\lambda \in \Lambda$ and $\lambda_H \in \Lambda_H$ solve the continuous and one-level MHM formulations, that is:*

$$b(\mu, T\lambda) = -b(\mu, \hat{T}f), \quad \forall \mu \in \Lambda,$$

and

$$b(\mu_H, T\lambda_H) = -b(\mu_H, \hat{T}f) \quad \forall \mu_H \in \Lambda_H.$$

Then it holds that

$$|T(\lambda - \lambda_H)|_{0,\Omega} \lesssim \alpha_{\omega,H} \|(T\lambda - \lambda_H)\|_{V,\omega}. \tag{32}$$

Proof. We use again Lemma 3, and define $\eta \in \Lambda$ as

$$b(\mu, T\overline{\eta}) = -b\left(\mu, \hat{T}\left(\overline{T}(\lambda - \lambda_H)\right)\right)$$

so that we have

$$b(\eta, T(\lambda - \lambda_H)) = |T\lambda - T\lambda_H|^2_{0,\Omega}.$$

Then, by Galerkin orthoganlity and by the definition of the norm $\|.\|_{\Lambda,\omega}$, it is clear that

$$\begin{aligned} |T\lambda - T\lambda_H|^2_{0,\Omega} &= b(\eta, T(\lambda - \lambda_H)) \\ &= b(\eta - \overline{\eta_H}, T(\lambda - \lambda_H)) \\ &= b(\overline{\eta} - \eta_H, \overline{T}(\lambda - \lambda_H)) \\ &= a(T(\overline{\eta} - \eta_H), \overline{T}(\lambda - \lambda_H)) \\ &\lesssim \|T(\overline{\eta} - \eta_H)\|_{V,\omega} \|T(\lambda - \lambda_H)\|_{V,\omega}, \end{aligned} \tag{33}$$

for all $\eta_H \in \Lambda_H$.

By definition of $\alpha_{\omega,H}$, recalling (30), there exists a $\eta_H \in \Lambda_H$ such that

$$\|T(\overline{\eta} - \eta_H)\|_{V,\omega} \lesssim \alpha_{\omega,H} |\overline{T}(\lambda - \lambda_H)|_{0,\Omega} = \alpha_{\omega,H} |T\lambda - T\lambda_H|_{0,\Omega},$$

and it follows that

$$|T(\lambda - \lambda_H)|^2_{0,\Omega} \lesssim \alpha_{\omega,H} |T(\lambda - \lambda_H)|_{0,\Omega} \|T(\lambda - \lambda_H)\|_{V,\omega},$$

which conclude the proof. $\square$

We are now ready to establish our main convergence result for the one-level MHMm in Theorem **??**.

Theorem 5. *Assume that ωH and $\omega\alpha_{\omega,H}$ are small enough, and consider $f \in L^2(\Omega)$. Let $\lambda \in \Lambda$ and $\lambda_H \in \Lambda_H$ solve the continuous and one-level MHM formulations. The following error-estimates hold:*

$$\|T(\lambda - \lambda_H)\|_{V,\omega} \lesssim \alpha_{\omega,H} |f|_{0,\Omega} \tag{34}$$

and

$$\|u - u_H\|_{V,\omega} \lesssim \alpha_{\omega,H} |f|_{0,\Omega}, \tag{35}$$

where $u = T\lambda + \hat{T}f$ and $u_H = T\lambda_H + \hat{T}f$.

Proof. By definition of a and b, we have that

$$\begin{aligned}\operatorname{Re} b(\lambda - \lambda_H, T(\lambda - \lambda_H)) &= \operatorname{Re} a(T(\lambda - \lambda_H), T(\lambda - \lambda_H)) \\ &= \|T\lambda - T\lambda_H\|_{V,\omega}^2 - 2\omega^2 |T\lambda - T\lambda_H|_{0,\Omega}^2.\end{aligned}$$

Recalling Lemma 4, we have

$$2\omega^2 |T\lambda - T\lambda_H|_{0,\Omega}^2 \leq 2\omega^2 \alpha_{\omega,H}^2 \|T\lambda - T\lambda_H\|_{V,\omega}^2,$$

and it follows that

$$\operatorname{Re} b(\lambda - \lambda_H, T(\lambda - \lambda_H)) \geq \left(1 - 2\omega^2 \alpha_{\omega,H}^2\right) \|T\lambda - T\lambda_H\|_{V,\omega}^2.$$

Assuming that $\omega\alpha_{\omega,H} \leq 1/2$, we obtain that

$$\operatorname{Re} b(\lambda - \lambda_H, T(\lambda - \lambda_H)) \gtrsim \|T\lambda - T\lambda_H\|_{V,\omega}^2. \tag{36}$$

We can now end the proof of error-estimate (34) using Galerkin orthogonality and the definition of $\alpha_{\omega,H}$. Indeed, using Galerkin orthogonality in (36), it holds that

$$\begin{aligned}\|T\lambda - T\lambda_H\|_{V,\omega}^2 &\lesssim \operatorname{Re} b(\lambda - \lambda_H, T(\lambda - \lambda_H)) \\ &\lesssim \operatorname{Re} b(\lambda - \mu_H, T(\lambda - \lambda_H)) \\ &\lesssim \operatorname{Re} a(T(\lambda - \mu_H), T(\lambda - \lambda_H)) \\ &\lesssim \|T\lambda - T\mu_H\|_{V,\omega} \|T\lambda - T\lambda_H\|_{V,\omega},\end{aligned} \tag{37}$$

for all $\mu_H \in \Lambda_H$.

It follows from (30) and (37) that

$$\|T\lambda - T\lambda_H\|_{V,\omega} \lesssim \inf_{\mu_H \in \Lambda_H} \|T(\lambda - \mu_H)\|_{V,\omega} \lesssim \alpha_{\omega,H} |f|_{0,\Omega},$$

and error-estimate (34) follows. Then, error-estimate (35) is a direct consequence of (34). □

Remark 1. *If we use directly the inf-sup condition to derive error-estimate* (34)*, we obtain*

$$\|T(\lambda - \lambda_H)\|_{V,\omega} \lesssim \omega\alpha_{\omega,H} \|f\|_{0,\Omega},$$

which is less sharp than (34)*, because of the factor* ω *in the right-hand-side.*

5 Well-posedness and convergence of the two-level MHM method

In the one-level MHM method, we assume that the local operators T and $\hat{T}$ are known analytically. However, apart in some particular cases, they are defined as solutions to local boundary value problems for which the solution is not analytically available. Then, these operators need to replaced by some approximations T_h and $\hat{T}_h$, that can be computed.

Here, we consider the case where these approximations are Galerkin approximations. It means that the space V is replaced by a discrete space V_h in the definition of the discrete operators. Hence, the discrete operators $T_h : \Lambda \to V_h$ and $\hat{T}_h : L^2(\Omega) \to V_h$ are formally defined by

$$a(T_h\lambda, v_h) = -b(\lambda, v_h), \quad a(\hat{T}_h f, v_h) = -(f, v_h),$$

for all $\lambda \in \Lambda$, $f \in L^2(\Omega)$ and $v_h \in V_h$.

For each $\mu_H \in \Lambda_H$ and $f \in L^2(\Omega)$, evaluating $T_h\mu_H$ and $\hat{T}_h f$ amounts to solve a set of local discrete boundary value problems. Then, the global two-level MHM problem reads: find $\lambda_H \in \Lambda_H$ such that

$$b(\mu_H, T_h\lambda_H) = -b(\mu_H, \hat{T}_h f), \quad \forall \mu_H \in \Lambda_H.$$

Because, by definition, the space V is defined independently in each cell $K \in \mathcal{T}_H$, we will naturally assume that the same holds for V_h, that is

$$V_h = \bigoplus_{K \in \mathcal{T}_H} V_{h,K},$$

where, each $v_h \in V_{h,K}$ satisfies $\operatorname{supp} v_h \subset \overline{K}$. Note that in particular, it implies that $a(v_+, v_-) = 0$ as soon as $v_\pm \in V_{h,K_\pm}$, with $K_+ \neq K_-$.

5.1 Well-posedness of the (discrete) local problems

We start by showing the well-posedness of the discrete local problems, which implies that the definition of the local operators T_h and $\hat{T}_h$ makes sense. This is equivalent to show and inf-sup condition for the sesquilinear form a on the discrete space V_h.

Theorem 6. *Assume that*

$$\mathbf{1}_K \in V_{h,K},$$

for all $K \in \mathcal{T}_H$. Then, the inf-sup condition

$$\sup_{v_h \in V_h \setminus \{0\}} \frac{\operatorname{Re} a(u_h, v_h)}{\|v_h\|_{V,\omega}} \geq C \|u_h\|_{V,\omega}, \quad \forall u_h \in V_h \tag{38}$$

holds.

Proof. Let $u_h \in V_h$. For each $K \in \mathcal{T}_H$, let write $u_K = u_h|_K \in V_h$.

Consider $K \in \mathcal{T}_H$. If $\partial K \cap \partial\Omega \neq \emptyset$, let us put $v_K = 2(1+i)u_K$. Then, since V_h is a vector space, it is clear that $v_K \in V_h$. Furthermore, from Proposition 4, we have

$$\operatorname{Re} a(u_K, v_K) \gtrsim \|u_K\|^2_{V,\omega},$$

and

$$\|v_K\|^2_{V,\omega} \lesssim C\|u_K\|^2_{V,\omega}.$$

Suppose now that $\partial K \cap \partial\Omega = \emptyset$. Recalling 3, we define

$$v_K = u_K - 2\int_K u_K = u_K + \alpha \mathbf{1}_K,$$

with $\alpha = -2\int_K u_K$. By assumption, we have $\mathbf{1}_K \in V_h$, so that we clearly have $v_K \in V_h$. Furthermore, we have

$$\operatorname{Re} a(u_K, v_K) \gtrsim \|u_K\|^2_{V,\omega},$$

and

$$\|v_K\|^2_{V,\omega} \lesssim \|u_K\|^2_{V,\omega}.$$

Then, we set

$$v_h = \sum_{K \in \mathcal{T}_H} v_K,$$

and we see that

$$\operatorname{Re} a(u_h, v_h) = \sum_{K \in \mathcal{T}_H} \|u_K\|^2_{V,\omega} = \|u_h\|^2_{V,\omega}.$$

On the other hand, we have

$$\|v_h\|^2_{V,\omega} = \sum_{K \in \mathcal{T}_H} \|v_K\|^2_{V,\omega} \lesssim \sum_{K \in \mathcal{T}_H} \|u_K\|^2_{V,\omega} \lesssim \|u_h\|^2_{V,\omega},$$

so that $\|u_h\|_{V,\omega} \gtrsim \|v_h\|_{V,\omega}$. Thus, it holds that

$$\operatorname{Re} a(u_h, v_h) \gtrsim \|u_h\|_{V,\omega} \|v_h\|_{V,\omega},$$

and the result follows. ∎

5.2 Well-posedness of the two-level global problem

As in Section 4, we consider fairly general discretization subspaces V_h. Hence, the main results will be stated under the assumption that V_h can represent accurately the solution of the local problems. We thus introduce the quantities

$$\beta_{\omega,H,h} = \sup_{\mu_H \in \Lambda_H \setminus \{0\}} \frac{\|T\mu_H - T_h\mu_H\|_{V,\omega}}{\|\mu_H\|_{\Lambda,\omega}},$$

and

$$\gamma_{\omega,H,h} = \sup_{f \in L^2(\Omega) \setminus \{0\}} \frac{\|\hat{T}f - \hat{T}_h f\|_{V,\omega}}{\|f\|_{0,\Omega}}.$$

For a fixed H, we know that $T\mu_H, \hat{T}f \in H^1(\mathcal{T}_H)$. As a result, if a finite element type method is used at the second level, we know that $\beta_{\omega,H,h}, \gamma_{\omega,H,h} \to 0$ as $h \to 0$. For particular method, the precise relation between H, h and ω required to ensure stability can then be recovered by a careful analysis of $\beta_{\omega,H,h}$ and $\gamma_{\omega,H,h}$.

Remark that for any one-level discretization scheme Λ_H, we know that the constants $\beta_{\omega,H,h}$ and $\gamma_{\omega,H,h}$ can be made arbitrarily small, if h is small enough. As a result, if the one-level MHMm is well-posed, the main results of this section show that the two-level method is also well-posed, if the second-level discretization scheme is sufficiently refined. As only few assumptions are required on Λ_H and V_h in this section, we can not be specific on how h must be small. However, as shown later on, we can be more specific for particular choices of Λ_H and V_h.

Theorem 7. *Assume that ωH, $\omega\alpha_{\omega,H}$ and $\omega\beta_{\omega,H,h}$ are small enough. Then we have*

$$\sup_{\mu_H \in \Lambda_H \setminus \{0\}} \frac{\operatorname{Re} b(\mu_H, T_h\lambda_H)}{\|\mu_H\|_{\Lambda,\omega}} \gtrsim \frac{1}{\omega}\|\lambda_H\|_{\Lambda,\omega}, \quad \forall \lambda_H \in \Lambda_H. \tag{39}$$

Proof. Let $\lambda_H \in \Lambda_H$. For all $\mu_H \in \Lambda_H$, we have

$$b(\mu_H, T_h\lambda_H) = b(\mu_H, T\lambda_H) - b(\mu_H, T\lambda_H - T_h\lambda_H).$$

Since the one-level MHMm is well-posed under the condition that ωH and $\omega\alpha_{\omega,H}$ are small enough, there exists a particular $\mu_H^\star \in \Lambda_H$ such that

$$\operatorname{Re} b(\mu_H^\star, T\lambda_H) \gtrsim \frac{1}{\omega}\|\mu_H^\star\|_{\Lambda,\omega}\|\lambda_H\|_{\Lambda,\omega}.$$

Then, since

$$\operatorname{Re} b(\mu_H^\star, T\lambda_H - T_h\lambda_H) \lesssim \|\mu_H^\star\|_{\Lambda,\omega}\|T\lambda_H - T_h\lambda_H\|_{V,\omega},$$

we see that

$$\operatorname{Re} b(\mu_H^\star, T_h\lambda_H) \gtrsim \frac{1}{\omega}\|\mu_H^\star\|_{\Lambda,\omega}\left(\|\lambda_H\|_{\Lambda,\omega} - \omega\|T\lambda_H - T_h\lambda_H\|_{V,\omega}\right).$$

Recalling the definition of $\beta_{\omega,H,h}$, we have

$$\operatorname{Re} b(\mu_H^\star, T_h\lambda_H) \gtrsim \frac{1}{\omega}\left(1 - \omega\beta_{\omega,H,h}\right)\|\mu_H^\star\|_{\Lambda,\omega}\|\lambda_H\|_{\Lambda,\omega},$$

and the result follows assuming that $\omega\beta_{\omega,H,h} \leq 1/2$. □

Like the analysis of the one-level MHMm in Section 4, we could directly obtain error-estimates from the inf-sup condition. However, we perform a refined analysis to obtain sharper error-estimates. The first step is an Aubin-Nitsche type error-estiamte.

Lemma 5. *Consider $f \in L^2(\Omega)$ and assume that $\alpha_{\omega,H}$ is small enough. Let $\lambda \in \Lambda$ and $\lambda_{H,h} \in \Lambda_H$ solve the continuous and two-level MHM formulations, that is:*

$$b(\mu, T\lambda) = -b(\mu, \hat{T}f), \quad \forall \mu \in \Lambda$$

and

$$b(\mu_H, T_h\lambda_{H,h}) = -b(\mu_H, \hat{T}_h f), \quad \forall \mu_H \in \Lambda_H.$$

Then it holds that

$$\|T\lambda - T\lambda_{H,h}\|_{0,\Omega} \lesssim \alpha_{\omega,H}\|T\lambda - T\lambda_{H,h}\|_{V,\omega} + (\beta_{\omega,H,h} + \gamma_{\omega,H,h})\|f\|_{0,\Omega},$$

Proof. Recalling Lemma 3, we define $\eta \in \Lambda$ as

$$b(\mu, T\overline{\eta}) = -b\left(\mu, \hat{T}\left(\overline{T}(\lambda - \lambda_{H,h})\right),\right)$$

so that we have

$$b(\eta, T\lambda - T\lambda_H) = \|T\lambda - T\lambda_{H,h}\|^2_{0,\Omega}.$$

It follows that

$$\begin{aligned} \|T\lambda - T\lambda_{H,h}\|^2_{0,\Omega} &= b(\eta, T\lambda - T\lambda_{H,h}) \\ &= b(\eta - \eta_H, T\lambda - T\lambda_{H,h}) + b(\eta_H, T\lambda - T\lambda_{H,h}) \\ &= b(\eta - \eta_H, T\lambda - T\lambda_H) + b(\eta_H, T\lambda) - b(\eta_H, T_h\lambda_{H,h}) \\ &- b(\eta_H, T\lambda_{H,h} - T_h\lambda_{H,h}), \end{aligned} \tag{40}$$

for all $\eta_H \in \Lambda_H$.

Let us pick $\eta_H \in \Lambda_H$ such that

$$\|\eta - \eta_H\|_{\Lambda,\omega} \lesssim \alpha_{\omega,H}|T\lambda - T\lambda_{H,h}|_{0,\Omega}$$

and treat in three steps the right-hand-side of (40). First, we have

$$\begin{aligned} |b(\eta - \eta_H, T\lambda - T\lambda_{H,h})| &\lesssim \|\eta - \eta_H\|_{\Lambda,\omega}\|T\lambda - T\lambda_{H,h}\|_{V,\omega} \\ &\lesssim \alpha_{\omega,H}|T\lambda - T\lambda_{H,h}|_{0,\Omega}\|T\lambda - T\lambda_{H,h}\|_{V,\omega}. \end{aligned}$$

Then, we derive

$$\begin{aligned} b(\eta_H, T\lambda) - b(\eta_H, T_h\lambda_{H,h}) &= b(\eta_H, \hat{T}f) - b(\eta_H, \hat{T}_h f) \\ &= b(\eta_H, \hat{T}f - \hat{T}_h f) \\ &= b(\eta, \hat{T}f - \hat{T}_h f) - b(\eta - \eta_H, \hat{T}f - \hat{T}_h f), \end{aligned}$$

so that

$$|b(\eta_H, T\lambda) - b(\eta_H, T_h\lambda_{H,h})| \lesssim (\|\eta\|_{\Lambda,\omega} + \|\eta - \eta_H\|_{\Lambda,\omega})\,\|\hat{T}f - \hat{T}_h f\|_{V,\omega}.$$

We have

$$\|\eta\|_{\Lambda,\omega} \lesssim |T\lambda - T\lambda_{H,h}|_{0,\Omega}$$

and

$$\|\hat{T}f - \hat{T}hf\|_{V,\omega} \lesssim \gamma_{\omega,H,h}|f|_{0,\Omega},$$

so that

$$|b(\eta_H, T\lambda) - b(\eta_H, T_h\lambda_{H,h})| \lesssim \gamma_{\omega,H,h}|f|_{0,\Omega}|T\lambda - T\lambda_{H,h}|_{0,\Omega}.$$

Finally, we see that

$$\begin{aligned} |b(\eta_H, T\lambda_{H,h} - T_h\lambda_{H,h})| &\lesssim \|\eta_H\|_{\Lambda,\omega}\|T\lambda_{H,h} - T_h\lambda_{H,h}\|_{V,\omega} \\ &\lesssim \beta_{\omega,H,h}\|\eta_H\|_{\Lambda,\omega}\|\lambda_{H,h}\|_{\Lambda,\omega} \\ &\lesssim \beta_{\omega,H,h}(\|\eta\|_{\Lambda,\omega} + \|\eta - \eta_H\|_{\Lambda,\omega})(\|\lambda\|_{\Lambda,\omega} + \|\lambda - \lambda_{H,h}\|_{\Lambda,\omega}) \\ &\lesssim \beta_{\omega,H,h}(1 + \alpha_{\omega,H})^2\|T\lambda - T\lambda_{H,h}\|_{0,\Omega}\|f\|_{0,\Omega} \\ &\lesssim \beta_{\omega,H,h}\|T\lambda - T\lambda_{H,h}\|_{0,\Omega}\|f\|_{0,\Omega}. \end{aligned}$$

Then, plugging everything into (40) and dividing by $\|T\lambda - T\lambda_{H,h}\|_{0,\Omega}$, we obtain

$$\|T\lambda - T\lambda_{H,h}\|_{0,\Omega} \lesssim \alpha_{\omega,H}\|T\lambda - T\lambda_{H,h}\|_{V,\omega} + (\beta_{\omega,H,h} + \gamma_{\omega,H,h})\|f\|_{0,\Omega},$$

and the conclusion follows. □

With Lemma 5, we can establish the main error-estimate of this section.

Theorem 8. *Assume that ωH, $\omega\alpha_{\omega,H}$ and $\omega\beta_{\omega,H,h}$ are small enough. Then, there exists a unique $\lambda_{H,h} \in \Lambda_H$ such that*

$$b(\mu_H, T_h\lambda_{H,h}) = -b(\mu_H, \hat{T}_h f), \quad \forall \mu_H \in \Lambda_H.$$

In addition, the following error estimate holds

$$\|\lambda - \lambda_{H,h}\|_{\Lambda,\omega} \lesssim (\alpha_{\omega,H} + \omega\beta_{\omega,H,h} + \omega\gamma_{\omega,H,h})\,\|f\|_{0,\Omega}. \tag{41}$$

Proof. We have

$$\begin{aligned}\|T\lambda - T\lambda_{H,h}\|^2_{V,\omega} &\lesssim \omega^2|T\lambda - T\lambda_{H,h}|^2_{0,\Omega} + \operatorname{Re} b(\lambda - \lambda_{H,h}, T\lambda - T\lambda_{H,h}) \\ &\lesssim \omega^2\alpha^2_{\omega,H}\|T\lambda - T\lambda_{H,h}\|_{V,\omega} + \omega^2(\beta_{\omega,H,h} + \gamma_{\omega,H,h})^2|f|^2_{0,\Omega} + \operatorname{Re} b(\lambda - \lambda_{H,h}, T\lambda - T\lambda_{H,h}),\end{aligned}$$

so that

$$\left(1 - C\omega^2\alpha^2_{\omega,H}\right)\|T\lambda - T\lambda_{H,h}\|^2_{V,\omega} \lesssim \omega^2(\beta_{\omega,H,h} + \gamma_{\omega,H,h})^2|f|^2_{0,\Omega} + \operatorname{Re} b(\lambda - \lambda_{H,h}, T\lambda - T\lambda_{H,h}),$$

for an appropriate constant C, and we have

$$\|T\lambda - T\lambda_{H,h}\|^2_{V,\omega} \lesssim (\omega\beta_{\omega,H,h} + \omega\gamma_{\omega,H,h})^2|f|^2_{0,\Omega} + \operatorname{Re} b(\lambda - \lambda_{H,h}, T\lambda - T\lambda_{H,h}),$$

assuming that $\omega\alpha_{\omega,H}$ is small enough.

Now we also have

$$\begin{aligned}\operatorname{Re} b(\lambda - \lambda_{H,h}, T\lambda - T\lambda_{H,h}) &= \operatorname{Re} b(\lambda - \lambda_{H,h}, T\lambda - T_h\lambda_{H,h}) - \operatorname{Re} b(\lambda - \lambda_{H,h}, T\lambda_{H,h} - T_h\lambda_{H,h}) \\ &= \operatorname{Re} b(\lambda - \mu_H, T\lambda - T_h\lambda_{H,h}) - \operatorname{Re} b(\lambda - \lambda_{H,h}, T\lambda_{H,h} - T_h\lambda_{H,h})\end{aligned}$$

for all $\mu_H \in \Lambda_H$. Hence, we have

$$\begin{aligned}|\operatorname{Re} b(\lambda - \lambda_{H,h}, T\lambda - T\lambda_{H,h})| &\lesssim \|\lambda - \mu_H\|_{\Lambda,\omega}\|T\lambda - T_h\lambda_{H,h}\|_{V,\omega} + \|\lambda - \lambda_{H,h}\|_{\Lambda,\omega}\|T\lambda_{H,h} - T_h\lambda_{H,h}\|_{V,\omega} \\ &\lesssim \alpha_{\omega,H}\|T\lambda - T_h\lambda_{H,h}\|_{V,\omega}|f|_{0,\Omega} + \omega\beta_{\omega,H,h}\|T\lambda - T\lambda_{H,h}\|_{V,\omega}\|\lambda_{H,h}\|_{\Lambda,\omega}.\end{aligned}$$

Then, we have

$$\|\lambda_{H,h}\|_{\Lambda,\omega} \lesssim \|\lambda - \lambda_{H,h}\|_{\Lambda,\omega} + \|\lambda\|_{\Lambda,\omega} \lesssim \|T\lambda - T\lambda_{H,h}\|_{V,\omega} + |f|_{0,\Omega},$$

so that

$$\begin{aligned}&|\operatorname{Re} b(\lambda - \lambda_{H,h}, T\lambda - T\lambda_{H,h})| \lesssim \\ &\quad \alpha_{\omega,H}\|T\lambda - T_h\lambda_{H,h}\|_{V,\omega}|f|_{0,\Omega} + \omega\beta_{\omega,H,h}\|T\lambda - T\lambda_{H,h}\|^2_{V,\omega} + \omega\beta_{\omega,H,h}\|T\lambda - T\lambda_{H,h}\|_{V,\omega}|f|_{0,\Omega},\end{aligned}$$

and

$$\begin{aligned}&(1 - \omega\beta_{\omega,H,h})\|T\lambda - T\lambda_{H,h}\|^2_{V,\omega} \lesssim \\ &\quad (\omega\beta_{\omega,H,h} + \omega\gamma_{\omega,H,h})^2|f|^2_{0,\Omega} + \alpha_{\omega,H}\|T\lambda - T_h\lambda_{H,h}\|_{V,\omega}|f|_{0,\Omega} + \omega\beta_{\omega,H,h}\|T_\lambda - T\lambda_{H,h}\|_{V,\omega}|f|_{0,\Omega},\end{aligned}$$

We obtain the desired result by using the Young inequality. □

6 Analysis of an example: polynomial discretizations

In Sections 4 and 5 we derive general inf-sup conditions and error-estimates for the one-level and two-level MHMm. Few assumptions are made on the discretization subspace Λ_H and V_h, so that the main results involve abstract quantities $\alpha_{\omega,H}$, $\beta_{\omega,H,h}$ and $\gamma_{\omega,H,h}$. These quantities depend on the approximation properties of the discretization subspace and thus, implicitly depends on the mesh sizes H and h.

In this section, we consider a particular case of discretization subspace Λ_H and V_h. Specifically, we consider the case where Λ_H consists of polynomial functions on each edge of $\mathcal{T}_H$. These functions are continuous along each edge, but are allowed to jump across vertices.

For the two-level MHMm, it turns out that obtaining fully explicit expressions for $\beta_{\omega,H,h}$ and $\gamma_{\omega,H,h}$ is rather tricky. For this reason, we focus on the simplest case in which Λ_H is made of edgewise constant functions. In addition, V_h is though to be a linear Lagrange finite-element space.

Also, to simplify the analysis, we assume that the medium of propagation is homogeneous. Without loss of generality, we set $\kappa = \rho = 1$ uniformly in Ω. In addition, we consider the case where the boundary of $\Gamma_D = \emptyset$, and $\Gamma_A = \partial\Omega$ is a convex polygon. Under these assumptions, we can state Theorem 4.10 from [23]:

Proposition 6. *Let $f \in L^2(\Omega)$. For all $\omega \geq 1$, there exists a unique $u \in H^1(\Omega)$ such that*

$$\begin{cases} -\omega^2 u - \Delta u &= f \quad \text{in } \Omega \\ \nabla u \cdot n - i\omega u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

and we have

$$\omega \|u\|_{0,\Omega} + |u|_{1,\Omega} \lesssim \|f\|_{0,\Omega}.$$

In addition, for any integer $p \geq 2$ there exists two functions $u_2 \in H^2(\Omega)$ and $u_p \in H^p(\Omega)$ such that $u = u_2 + u_p$ and

$$\|u_2\|_{2,\Omega} \lesssim \|f\|_{0,\Omega}, \quad |\phi \nabla^p u_p|_{p,\Omega} \lesssim \omega^p \|f\|_{0,\Omega}, \tag{42}$$

where $\phi \in L^\infty(\Omega)$ is weighting function that "compensate" the singularities of u_p close to the corners of Ω and $\nabla^p u_p$ stands for all p-th weak derivatives of u_p.

As depicted in Proposition 6, the solution might exhibit singularities close to the corners of Ω. Following [23], we employ meshes that are geometrically refined in these locations. Such geometric refinements permits to compensate the singular behaviour of the solution, and preserve optimal convergence rates.

Remark 2. *The case where Ω is smoother can be handled without locally refined meshes. However, we need to employ curved finite element in this case. Such curved element are complex to analyze in our current settings. Specifically, the authors are not aware of a rigorous derivation of an interpolation operator, together with optimal interpolation error-estimates in this case. Nevertheless, the authors believe the main results will apply in the case of a smooth domain Ω.*

6.1 One-level MHM method

We analyze the case of the one-level MHMm where Λ_H is made of edgewise polynomial functions. This family of spaces is extensively described in [27]. In the following, l denotes a fixed natural integer, and Λ_H is defined as

$$\Lambda_H = \{\mu_H \in \Lambda \mid \mu_H|_e \in \mathcal{P}_l \ \forall e \in \mathcal{E}_H\}, \tag{43}$$

where $\mathcal{P}_l$ denotes the space of polynomial of degree at most l depending on one variable.

The main task this section is to establish an upper bound of $\alpha_{\omega,H}$ for this particular choice of Λ_H. To do so, we take advantage of interpolation properties of Λ_H. In Proposition 7, we derive an interpolation error estimate based on [27].

Proposition 7. *Assume $\lambda \in \Lambda$ is such that there exists a function $u \in H^{l+1}(\Omega)$ satisfying*

$$\nabla u \cdot n|_{\partial K \setminus \partial \Omega} = \lambda_{\partial K \setminus \partial \Omega} \quad \forall K \in \mathcal{T}_H.$$

Then there exists an element $\Pi_H \lambda \in \Lambda_H$ such that

$$\|\lambda - \Pi_H \lambda\|_{\Lambda,\omega} \lesssim h^l |u|_{l+1,\Omega}.$$

Proof. Under the assumptions of the Proposition, we know from [27] that there exists a $\Pi_H \lambda \in \Lambda_H$ such that

$$\|\lambda - \Pi_H \lambda\|_{\Lambda} \lesssim h^l |u|_{l+1,\Omega},$$

where

$$\|\lambda - \Pi_H \lambda\|_{\Lambda} = \sup_{v \in V \setminus \{0\}} \frac{\operatorname{Re} b(\lambda, v)}{\|v\|_V},$$

and

$$\|v\|_V^2 = \sum_{K \in \mathcal{T}_H} \|v\|_{1,K}^2.$$

Hence it remains to show that $\|.\|_{\Lambda,\omega} \lesssim \|.\|_{\Lambda}$, but this is obvious from the definition of the $\|.\|_{V,\omega}$ norm, as $\omega \geq 1$. □

Theorem 9. *We have*

$$\alpha_{\omega,H} \lesssim H + \omega^{l+1} H^{l+1}. \tag{44}$$

We conclude this section with an error-estimate for the one-level MHMm that is explicitly stated in terms of ω and H. As the proof is direct consequence of Theorems 5 and 9, it is omitted.

Corollary 2. *Assume that ωH and $\omega^{l+2} H^{l+1}$ are sufficiently small. Then, there exists a unique $\lambda_H \in \Lambda_H$ solution to the one-level MHM problem. Furthermore, it holds that*

$$\|\lambda - \lambda_H\|_{\Lambda,\omega} \leq C \left(H + \omega^{l+1} H^{l+1} \right) \|f\|_{0,\Omega}, \tag{45}$$

and

$$\|u - u_H\|_{V,\omega} \leq C \left(H + \omega^{l+1} H^{l+1} \right) \|f\|_{0,\Omega},$$

where $u = T\lambda + \hat{T} f$ and $u_H = T\lambda_H + \hat{T} f$.

6.2 Two-level MHM method

Here, we focus on the case where $\mathcal{T}_H$ is made of squares. In addition, we assume that Λ_H consists of edgewise constant Lagrange multipliers. This corresponds to the particular case where $l = 0$. Then, if $\mu_H \in \Lambda_H^0$, we can write

$$\mu_H = \sum_{e \in \mathcal{E}_H^{int}} \alpha_e \mathbf{1}_e,$$

for some $\alpha_e \in \mathbb{C}$. We define a norm on Λ_H^0 by

$$\|\mu_H\|_{\mathcal{E}_H}^2 = \sum_{e \in \mathcal{E}_H^{int}} |\alpha_e|^2.$$

Equivalence properties of the $\|.\|_{\mathcal{E}_H}$ norm are derived in Lemma 6.

Lemma 6. *For all $\mu_H \in \Lambda_H^0$, we have*

$$\|\mu_H\|_{\mathcal{E}_H} \lesssim H^{-1} \|\mu_H\|_{\Lambda,\omega} \tag{46}$$

and

$$\left(\sum_{K \in \mathcal{T}_H} \|\mu_H\|_{0,\partial K}^2 \right)^{1/2} \lesssim H^{-1/2} \|\mu_H\|_{\Lambda,\omega}. \tag{47}$$

Proof. Let us first consider an edge $e \in \mathcal{E}_H^{int}$. For the sake of simplicity, we assume that e is an internale edge shared by two elements $K_\pm \in \mathcal{T}_H$. In addition, we consider the case where e is the right edge of K_- and the left edge of K_+. We do not detail the other cases here (external edge, or horizontal edge), but the proof are very similar.

Up to translation, we can assume that $K_- = (-H, 0) \times (0, H)$ and $K - + = (0, H) \times (0, H)$. Then, we define $q_e \in H^1(\mathcal{T}_H)$ by

$$q_e|_{K_-}(\mathbf{x}_1, \mathbf{x}_2) = \frac{\mathbf{x}_1 + H}{H} \frac{\mathbf{x}_2(H - \mathbf{x}_2)}{H^2}, \quad q_e|_{K_+}(\mathbf{x}_1, \mathbf{x}_2) = \frac{H - \mathbf{x}_1}{H} \frac{\mathbf{x}_2(H - \mathbf{x}_2)}{H^2},$$

and $q_e|_K = 0$ for the remainings elements $K \in \mathcal{T}_H$. We see that $q_e|_e(\mathbf{x}_2) = H^{-2}\mathbf{x}_2(H - \mathbf{x}_2)$, and $q_e|_{e'} = 0$ for every other edge $e' \in \mathcal{E}_H$. In addition, we have

$$\int_e q_e = H^{-2} \int_0^H \mathbf{x}_2(H - \mathbf{x}_2) d\mathbf{x}_2 = \frac{H}{6}.$$

Then, we have

$$b(\mu_H, q_e) = \sum_{K \in \mathcal{T}_H} \int_{\partial K} \mu_H q_e = 2\mu_e \int_e q_e = \frac{H}{3} \mu_e.$$

Then, direct computations show that

$$\|q_e\|_{0,\Omega} \simeq H, \quad |q_e|_{1,\mathcal{T}_H} \simeq 1.$$

Finally, since $q_e = 0$ on Γ_A

$$\|q_e\|_{V,\omega}^2 = \omega^2 \|q_e\|_{0,\Omega}^2 + |q_e|_{1,\mathcal{T}_H}^2 \lesssim 1 + \omega^2 H^2 \lesssim 1. \tag{48}$$

For an arbitrary $\mu_H \in \Lambda_H$, we set

$$q = \sum_{e \in \mathcal{E}_H^{int}} \mu_e q_e,$$

where q_e is defined for each edge as above. Using (48), we see that

$$\|q\|_{V,\omega} \lesssim \|\mu\|_{\mathcal{E}_H}. \tag{49}$$

We have

$$b(\mu_H, q_e) = \sum_{e \in \mathcal{E}_H^{int}} \overline{\mu_e} b(\mu_H, q_e) = \frac{H}{3} \sum_{e \in \mathcal{E}_H^{int}} |\mu_e|^2 \simeq H \|\mu_H\|_{\mathcal{E}_H}^2.$$

By definition of the $\|.\|_{\Lambda,\omega}$ norm, we obtain

$$H \|\mu_H\|_{\mathcal{E}_H}^2 \simeq b(\mu_H, q) \lesssim \|\mu_H\|_{\Lambda,\omega} \|q\|_{V,\omega},$$

and (46) follows from (49).

To proove (47), we first consider an arbitrary element $K \in \mathcal{T}_H$. We have

$$\begin{aligned} \|\mu_H\|_{0,\partial K}^2 &= \sum_{e \subset \partial K \setminus \partial \Omega} \|\mu_H\|_{0,e}^2 \\ &= \sum_{e \subset \partial K \setminus \partial \Omega} |\mu_e|^2 |e| \\ &= H \sum_{e \subset \partial K \setminus \partial \Omega} |\mu_e|^2. \end{aligned}$$

Then, by summation over $K \in \mathcal{T}_H$, we see that each edge $e \in \mathcal{E}_H^{int}$ is counted twice, so that

$$\|\mu_H\|_{0,\partial K}^2 = 2H \|\mu_H\|_{\mathcal{E}_H}^2,$$

and (47) follows. □

In Lemma 8 and 9, we analyze the scaling behaviour of lifiting and trace inequality over $\mathcal{T}_H$. We first state Lemma 7, which is obtained by elementary scaling arguments.

Lemma 7. *Let $u \in H^1(K)$ and $\phi \in \tilde{H}^{1/2}(\partial K)$ for some $K \in \mathcal{T}_H$. We define the functions $\hat{u}(\hat{\mathbf{x}}) = u(\mathbf{x})$ and $\hat{\phi}(\hat{\mathbf{x}}) = \phi(\mathbf{x})$. Then, $\hat{u} \in H^1(\hat{K})$, $\hat{\phi} \in \tilde{H}^{1/2}(\hat{K})$, and we have*

$$|u|_{0,K} = H|\hat{u}|_{0,\hat{K}}, \quad |u|_{1,K} = |\hat{u}|_{1,\hat{K}},$$

as well as

$$|\phi|_{0,K} = H^{1/2}|\hat{\phi}|_{0,\hat{K}}, \quad |\phi|_{1/2,\partial K} = |\hat{\phi}|_{1/2,\partial\hat{K}}.$$

Lemma 8. *Let $K \in \mathcal{T}_H$ and $\phi \in L^2(\partial K)$ such that*

$$\phi|_e \in H^{1/2}(e),$$

for edge $e \subset \partial K$. Then, there exists a $w \in H^2(K)$ such that

$$\nabla w \cdot n = \phi$$

on ∂K, and

$$|w|_{2,K} \lesssim H^{1/2}\|\phi\|_{0,\mathcal{E}} + \sum_{e\subset\partial K} |\phi|_{1/2,e}. \tag{50}$$

Proof. To start with, we introduce $\hat{\phi}(\hat{\mathbf{x}}) = \phi(\mathbf{x})$ so that $\hat{\phi}|_{\hat{e}} \in H^{1/2}(\hat{e})$ for all $\hat{e} \subset \partial\hat{K}$. In addition, recalling Lemma 7, we have

$$\|\hat{\phi}\|_{0,\partial\hat{K}} = H^{-1/2}\|\phi\|_{0,\partial K}, \quad |\hat{\phi}|_{1/2,\hat{e}} = |\phi|_{1/2,\hat{e}},$$

for all $\hat{e} \subset \partial\hat{K}$.

Theorem 1.6 of [12] ensures the existence of a function $\hat{w} \in H^2(K)$ such that $\nabla\hat{w} \cdot n = \hat{\phi}$ on $\partial\hat{K}$, and

$$\|\hat{w}\|_{2,K} \lesssim \sum_{\hat{e}\subset\partial\hat{K}} \|\hat{\phi}\|_{1/2,\hat{e}} \lesssim H^{-1/2}\|\phi\|_{0,\partial K} + \sum_{e\subset\partial K} |\phi|_{1/2,e}.$$

Then, if we define $w(\mathbf{x}) = H\hat{w}(\hat{\mathbf{x}})$, we see that $w \in H^2(K)$ and $\nabla w \cdot n = \phi$ on ∂K. By simple computations, we show that

$$|w|_{2,K} \lesssim |\hat{w}|_{2,\hat{K}},$$

and (50) follows. □

Lemma 9. *Let $u \in H^1(K)$ for some $K \in \mathcal{T}_H$, then we have*

$$|u|_{1/2,\partial K} \lesssim H^{-1}\|u\|_{0,K} + |u|_{1,K}. \tag{51}$$

Proof. Let us introduce $\hat{u}(\hat{\mathbf{x}}) = u(\mathbf{x})$, so that $u \in H^1(\hat{K})$. Since the reference element $\hat{K}$ does not depend on H, we have

$$|\hat{u}|_{1/2,\partial\hat{K}} \lesssim |u|_{0,K} + |u|_{1,K},$$

and the result follows from the scaling results of Lemma 7. □

Theorem 10. *We have*

$$|T\mu_H|_{2,\mathcal{T}_H} \lesssim H^{-1}\|\mu_H\|_{\Lambda,\omega},$$

for all $\mu_H \in \Lambda_H^0$. Also, it holds that

$$|\hat{T}f|_{2,\mathcal{T}_H} \lesssim (\omega H)^{-1}\|f\|_{0,\Omega},$$

for all $f \in L^2(\Omega)$.

Proof. We recall that for each element $K \in \mathcal{T}_H$, $u = T\mu_H|_K$ is solution to

$$\left\{\begin{array}{rcll} -\dfrac{\omega^2}{\kappa}u - \dfrac{1}{\rho}\Delta u & = & 0 & \text{in } K, \\ \dfrac{1}{\rho}\nabla u \cdot n & = & -\mu_H & \text{on } \partial K \setminus \partial\Omega, \\ \dfrac{1}{\rho}\nabla u \cdot n - \dfrac{i\omega}{\sqrt{\kappa\rho}}u & = & 0 & \text{on } \partial K \cap \Omega, \end{array}\right.$$

so that

$$\left\{\begin{array}{rcll} -\Delta u & = & \dfrac{\omega^2\rho}{\kappa}u & \text{in } K, \\ \nabla u \cdot n & = & -\rho\mu_H & \text{on } \partial K \setminus \partial\Omega, \\ \nabla u \cdot n & = & i\omega\sqrt{\dfrac{\rho}{\kappa}}u & \text{on } \partial K \cap \Omega. \end{array}\right.$$

Let $\phi \in L^2(\partial K)$ defined by

$$\phi|_{\partial K\setminus\partial\Omega} = -\rho\mu_H, \quad \phi|_{\partial K\cap\partial\Omega} = i\omega\sqrt{\frac{\kappa}{\rho}}u,$$

and we have

$$\|\phi\|_{0,\partial K} \lesssim \|\mu_H\|_{0,\partial K} + \omega\|u\|_{0,\partial K\cap\partial\Omega},$$

Since $u \in H^{1/2}(\partial K)$ and $\mu_H|_e \in H^{1/2}(e)$ for all edges $e \subset \partial K$, $\phi|_e \in H^{1/2}(e)$ for all $e \subset \partial K$. Recalling that $|\mu_H|_e|_{1/2,e} = 0$, and using (51), we see that

$$|\phi|_{1/2,e} \lesssim \omega|u|_{1/2,e} \lesssim \omega\left(H^{-1}|u|_{0,K} + |u|_{1,K}\right) \lesssim H^{-1}\left(\omega|u|_{0,K} + |u|_{1,K}\right).$$

Then, there exists an element $w \in H^2(K)$ such that $\nabla w \cdot n = \phi$ on ∂K and

$$\begin{array}{rcl} |w|_{2,K} & \lesssim & H^{1/2}\|\phi\|_{0,\partial K} + \displaystyle\sum_{e\subset\partial K}|\phi|_{1/2,e} \\ & \lesssim & H^{1/2}\|\mu_H\|_{0,\partial K} + \omega H^{1/2}\|u\|_{0,\partial K\cap\partial\Omega} + H^{-1}\left(\omega\|u\|_{0,K} + |u|_{1,K}\right) \\ & \lesssim & H^{1/2}\|\mu_H\|_{0,\partial K} + H^{-1}(1+(\omega H)^{1/2})\left(\omega\|u\|_{0,K} + |u|_{1,K} + \omega^{1/2}\|u\|_{0,\partial K\cap\partial\Omega}\right). \\ & \lesssim & H^{1/2}\|\mu_H\|_{0,\partial K} + H^{-1}\left(\omega\|u\|_{0,K} + |u|_{1,K} + \omega^{1/2}\|u\|_{0,\partial K\cap\partial\Omega}\right). \end{array}$$

By summation, we see that

$$|w|_{2,K} \lesssim H^{1/2}\left(\sum_{K\in\mathcal{T}_H}\|\mu_H\|^2_{0,\partial K}\right)^{1/2} + H^{-1}\|T\mu_H\|_{V,\omega},$$

and we conclude that

$$|w|_{2,K} \lesssim H^{-1}\|\mu_H\|_{\Lambda,\omega},$$

from (47) and Theorem (1).

Then, we let $v = T\mu_H - w$, so that $v|_K$ satifies

$$\left\{\begin{array}{rcll} -\Delta v & = & \dfrac{\omega^2\rho}{\kappa}u + \Delta w & \text{in } K \\ \nabla v \cdot n & = & 0 & \text{on } \partial K, \end{array}\right.$$

in each element $K \in \mathcal{T}_H$

Then, by Theorem 2.2.1 and Lemma 2.2.2 from [14], we have $v|_K \in H^2(K)$ with

$$|v|_{2,K} \lesssim \omega^2\|u\|_{0,K} + |w|_{2,K}.$$

Hence, we obtain that $v \in H^2(\mathcal{T}_H)$ with

$$\begin{aligned} |v|_{2,\mathcal{T}_H} &\lesssim \omega \|T\mu_H\|_{V,\omega} + |w|_{2,\mathcal{T}_H}. \\ &\lesssim \left(\omega + H^{-1}\right) \|\mu_H\|_{\Lambda,\omega} \\ &\lesssim H^{-1}\left(1+\omega H\right) \|\mu_H\|_{\Lambda,\omega} \\ &\lesssim H^{-1}\|\mu_H\|_{\Lambda,\omega}. \end{aligned}$$

We thus conclude that $T\mu_H \in H^2(\mathcal{T}_H)$ with

$$|T\mu_H|_{2,\mathcal{T}_H} \lesssim H^{-1}\|\mu_H\|_{\Lambda,\omega}.$$

Now, let $f \in L^2(\Omega)$. In each element $K \in \mathcal{T}_H$ $u = \hat{T}f|_K$ satisfies

$$\begin{cases} -\Delta u &= f + \dfrac{\omega^2\rho}{\kappa} u \quad \text{in } K, \\ \nabla u \cdot n &= 0 \quad \text{on } \partial K \setminus \partial\Omega, \\ \nabla u \cdot n &= i\omega\sqrt{\dfrac{\rho}{\kappa}} u \quad \text{on } \partial K \cap \Omega. \end{cases}$$

We first construct a lifting w such that $\nabla w \cdot n = 0$ on $\partial K \setminus \Omega$ and $\nabla w \cdot n = i\omega\sqrt{\kappa/\rho}u$ on $\partial K \cap \partial\Omega$. Arguing as in the case of $T\mu_H$, we obtain that $w \in H^2(\mathcal{T}_H)$ and

$$|w|_{2,\mathcal{T}_H} \lesssim H^{-1}\|\hat{T}f\|_{V,\omega}.$$

From Theorem 1, we obtain that

$$|w|_{2,\mathcal{T}_H} \lesssim (\omega H)^{-1}\|f\|_{0,\Omega}.$$

Then, we let $v = \hat{T}f - w$, so that in each element $K \in \mathcal{T}_H$, we have

$$\begin{cases} -\Delta v &= f + \dfrac{\omega^2\rho}{\kappa}\hat{T}f + \Delta w \quad \text{in } K, \\ \nabla u \cdot n &= 0 \quad \text{on } \partial K, \end{cases}$$

and arguing as before, we get that

$$|v|_{2,\mathcal{T}_H} \lesssim \|f\|_{0,\Omega} + \omega\|\hat{T}f\|_{V,\omega} + |w|_{2,\mathcal{T}_H} \lesssim (\omega H)^{-1}\|f\|_{0,\Omega} \lesssim H^{-1}\|f\|_{0,\Omega}.$$

□

As a direct consequence of Theorem 10, we have

Corollary 3. *We have*

$$\beta_{\omega,H,h} \lesssim \frac{h}{H}, \quad \gamma_{\omega,H,h} \lesssim \frac{h}{H}.$$

We conclude with an error-estimate, which is a direct consequence of Theorem 8:

Corollary 4. *Assume that ωH, $\omega^2 H$ and $\omega h/H$ are small enough, then, there exists a unique $\lambda_{H,h} \in \Lambda_H^0$ such that*

$$b(\mu_H, T_h\lambda_{H,h}) = -b(\mu_H, \hat{T}_h f), \quad \forall \mu_H \in \Lambda_H^0.$$

In addition, we have

$$\|\lambda - \lambda_{H,h}\|_{\Lambda,\omega} \lesssim \left(\omega H + \frac{\omega h}{H}\right)\|f\|_{0,\Omega}.$$

7 Numerical examples

We consider 4 test-cases to validate the MHMm approach numerically. We start with 3 experiments based on the academic problem to approximate a plane wave solution. The results presented in this section are based on square elements.

We first analyse the behaviour of the discrete solution with respect to angle of propagation. We are able to show that MHMm is able to exactly reproduce the plane wave in some cases. Then, we estimate numerically the convergence rates of the method. We end the analytical test-cases with a study of the impact of the pollution effect.

We conclude this section with an applied problem. We simulate the propagation of wave from a seismic source in a synthetic geophyscial medium: the Marmousi II model [22]. We give a comparison with a Lagragian polynomial approach: the MMAm [5, 10, 11].

7.1 Anisotropy study and comparison with DEm

We present an anisotropy study of the MHMm on cartesian grids. To this end, we solve the problem

$$\left\{\begin{array}{rcll} -\omega^2 u - \Delta u &=& 0 & \text{in } \Omega, \\ \nabla u \cdot n - i\omega u &=& \nabla e_\theta \cdot n - i\omega e_\theta & \text{on } \partial\Omega, \end{array}\right. \tag{52}$$

where $\Omega = (0,1)^2$, $\theta \in [0, 2\pi]$ and

$$e_\theta(x) = \exp\big(i\omega(\cos(\theta)x_1 + \sin(\theta)x_2)\big), \quad \forall x \in \mathbb{R}^2.$$

The solution to problem (52) is the plane wave $u = e_\theta$. In order to present anisotropy curves, we solve problem (52) for 720 values of θ ranging from 0 to 2π. We select the angular frequency $\omega = 10\pi$. For $l = 0$, we use a 64×64 grid mesh and the results are presented on Figure 1. On Figure 2, we use a 8×8 grid with $l = 2$. The results for $l = 4$ are presented on Figure 3, a 4×4 grid is used.

We first observe that every scheme gives the exact solution when $\theta = k\pi/2$, $k \in \mathbb{Z}$. As we already mentioned, this is because the normal derivatives of a plane wave e_θ with $\theta = k\pi/2$ are constant on the edges of the mesh and can be exactly represented by the space $\Lambda^P_{h,0}$.

Besides, it is clear that the spaces Λ^S_H are able to exactly reproduce plane waves e_θ for angles $\theta = k\pi/4$ for $l = 2$ and $\theta = k\pi/6$ when $l = 4$. This is in agreement with the concept used to construct these spaces.

We also observe that the spaces Λ^S_H based on oscillating functions are more accurate that the polynomial spaces Λ^P_H to approximate plane waves.

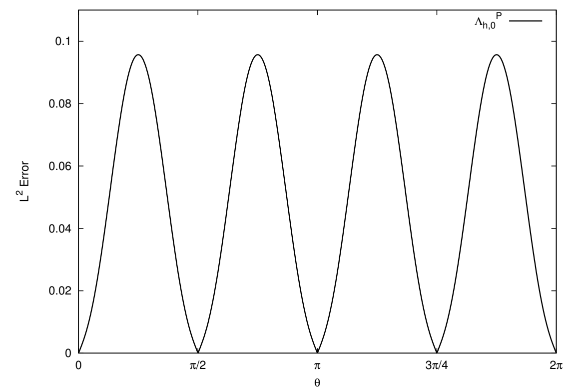


Figure 1: Anisotropy study for $l = 0$

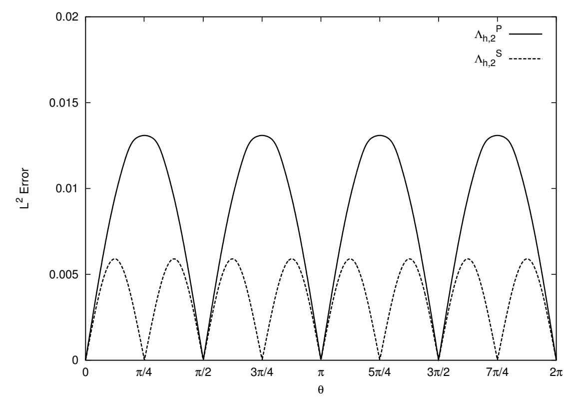


Figure 2: Anisotropy study for $l = 2$

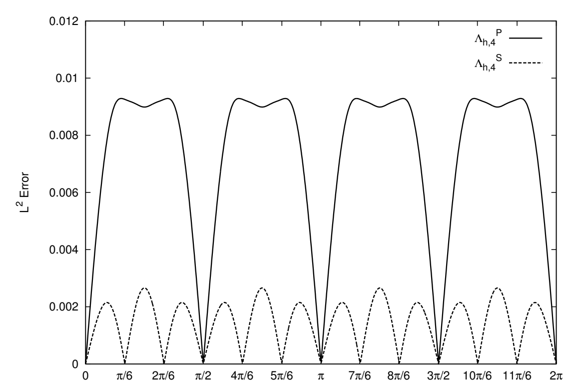


Figure 3: Anisotropy study for $l = 4$

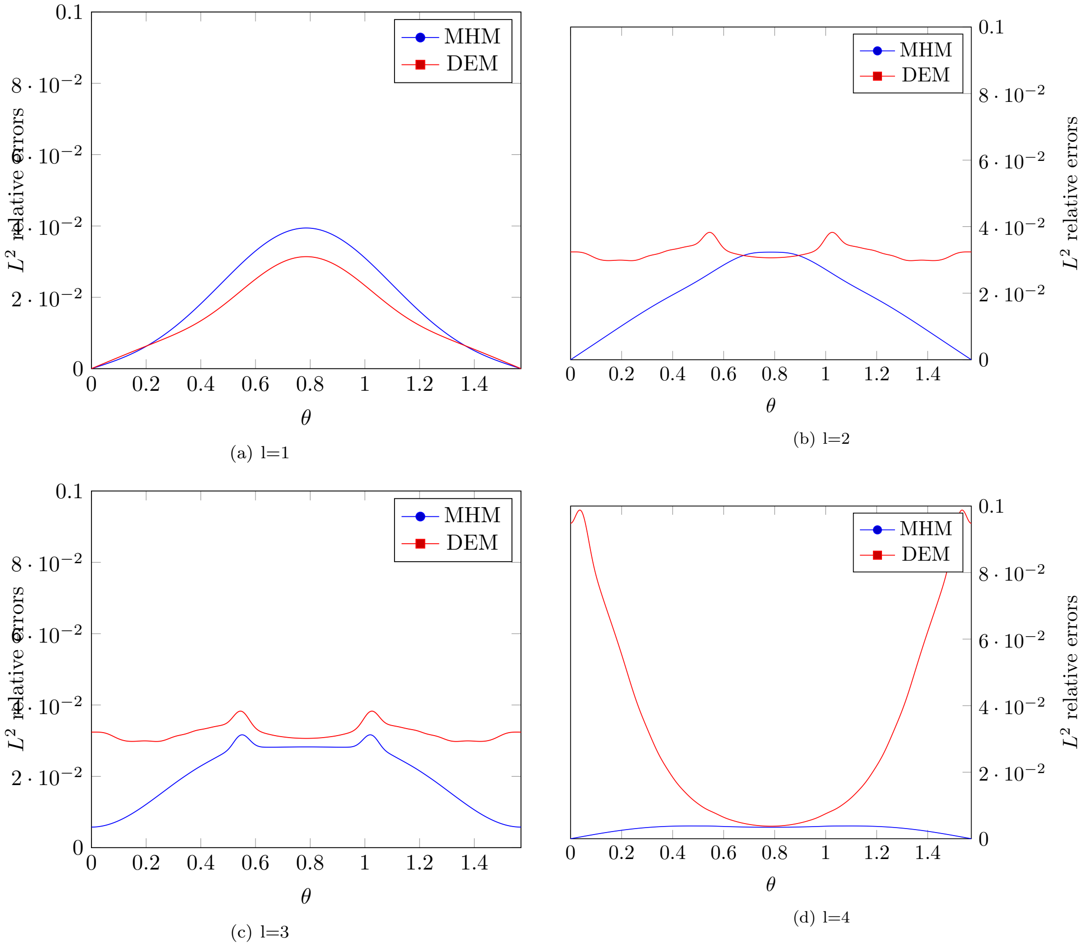
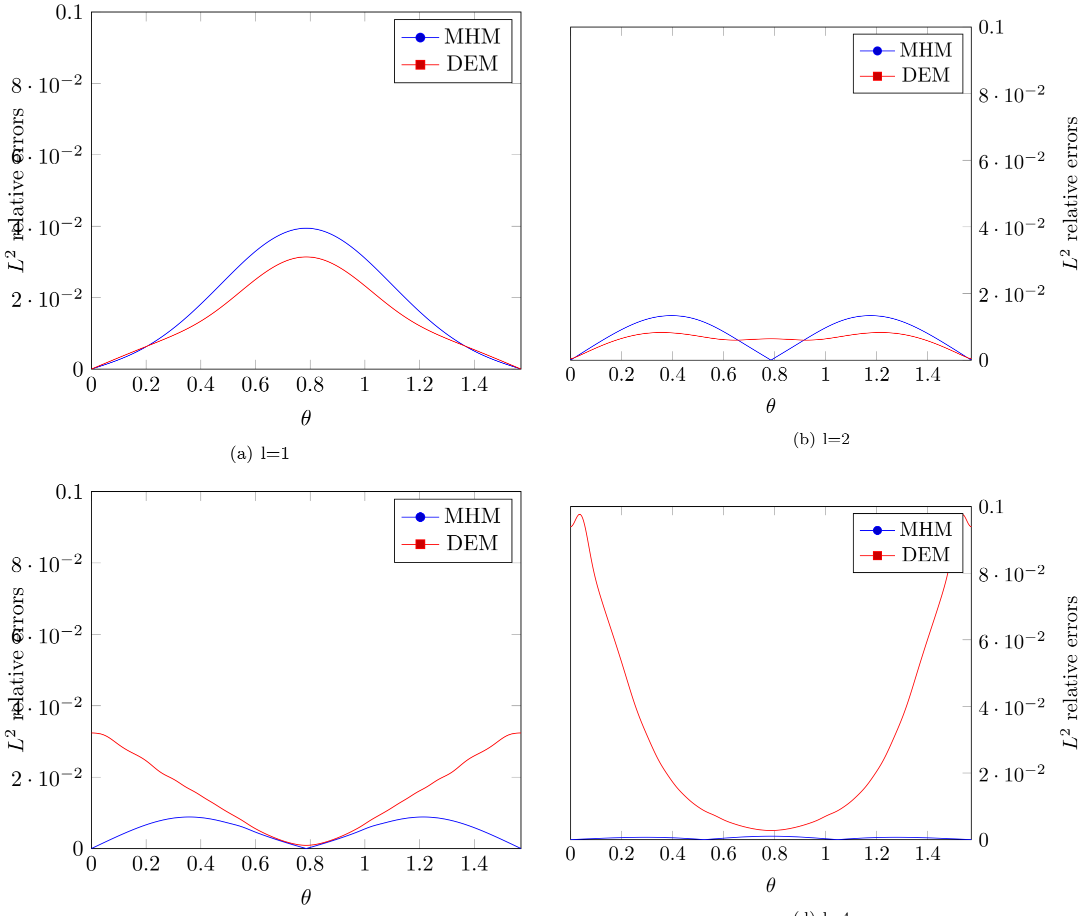


Figure 4: DEM vs MHM: polynomial basis functions



(a) l=1

(b) l=2

(c) l=3

(d) l=4

Figure 5: DEM vs MHM: oscillating basis functions

7.2 Convergence study

We now analyse numerically the convergence rates of the method. We use test-case (52) with $\omega = 6\pi$. Reminding the anisotropic analysis, we select the propagation angle $\theta = \pi/8$ to make sure that the solution is not exactly approximated. In particular, this choice seems to be the "worst case senario" for the space $\Lambda^S_{h,2}$.

We compute the errors in the L^2 norm and in the H^1 semi-norm. The results are presented on Figures 6 to 8. The experimental convergence rates are conform to the expectation: the convergence is in $\mathcal{O}(h^{l+2})$ in the L^2 norm and in $\mathcal{O}(h^{l+1})$ in the H^1 semi-norm. Like in the anisotropy study, the solutions based on oscillating spaces Λ^S_H are more accurate that the solutions based on polynomial spaces Λ^P_H.

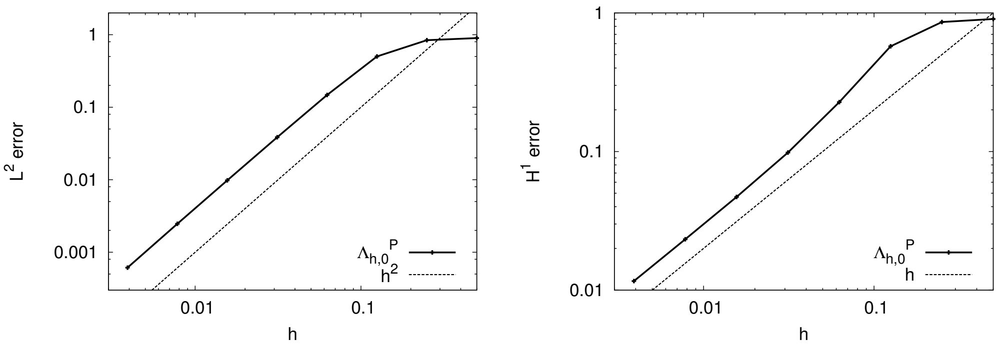


Figure 6: Convergence curves for $l = 0$

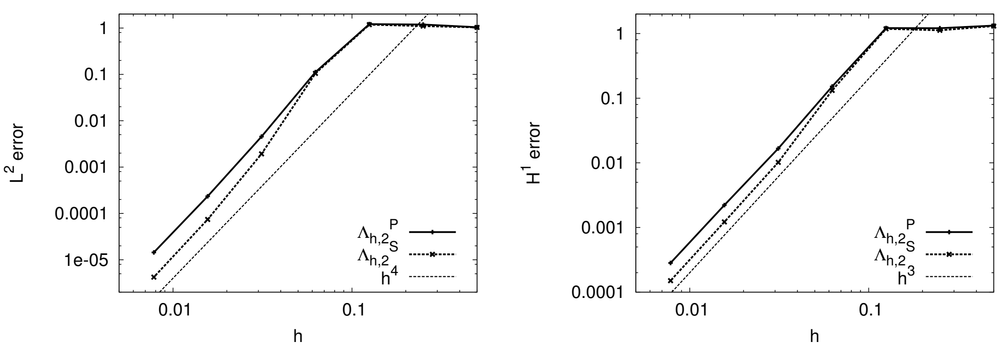


Figure 7: Convergence curves for $l = 2$

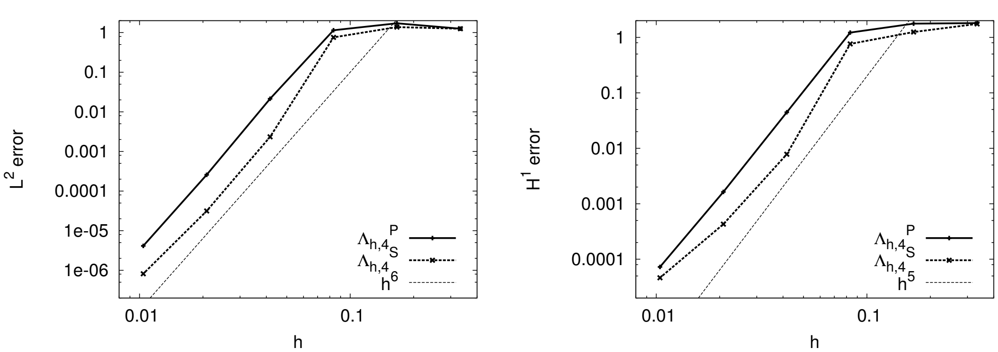


Figure 8: Convergence curves for $l = 4$

7.3 Estimation of the pollution effect

The main problem one encounters when solving for wave problems is the so-called "pollution effect": it is required to increase the number of points per wavelength when the frequency increases to maintain the accuracy of the numerical solution. Thus, if the number of points per wavelength is the same for all frequencies, that is $h \simeq \omega^{-1}$, the numerical solution looses its precision for sufficiently high frequencies.

There exists an optimal exponant $\alpha > 1$, such that, if $h \simeq \omega^{-\alpha}$, the accuracy of the solution remains bounded independently of the frequency. For the case of Lagrangian polynomial discretizations of degree p, it is well-known that the optimal exponant is $\alpha = 1 + 1/(2p)$. This result has been proven by Babuška and Ihlenburg for 1D problems [18].

We see in particular that the higher p is, the closer α is to one. Thus, the pollution effect is reduced for large p. This is the reason why high order methods perform very well for wave problems.

In the following, we solve model problem (52) for increasing frequencies with the angle of propagation $\theta = \pi/8$. We select $h \simeq \omega^{-\alpha}$, where $\alpha = 1 + 1/(2(l+1))$. This choice is the natural adaptation of the standard α exponant to the MHMm.

The results are presented on Figures 9 to 11 and confirm our expectations: if $h \simeq \omega^{-1-1/(2(l+1))}$, we observe a constant accuracy for all considered frequencies.

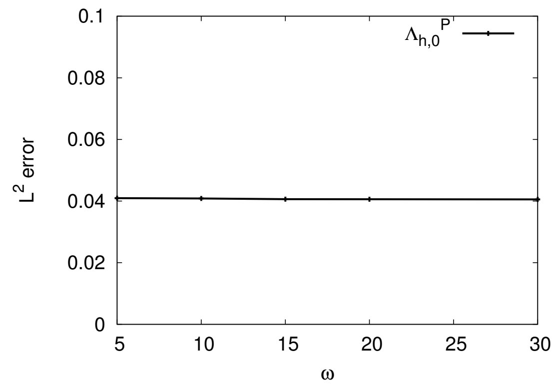


Figure 9: Pollution analysis for $l = 0$ ($h \simeq \omega^{-3/2}$)

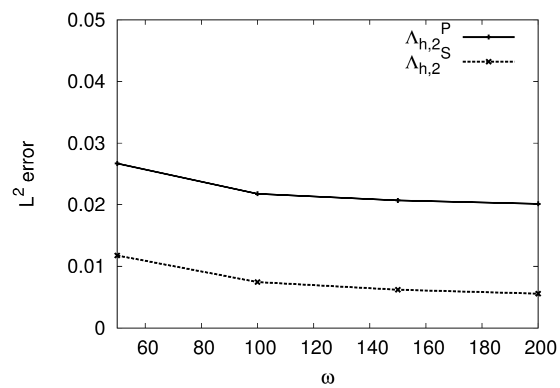


Figure 10: Pollution analysis for $l = 2$ ($h \simeq \omega^{-7/5}$)

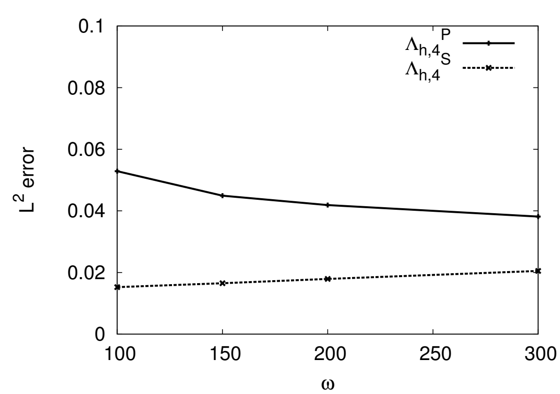


Figure 11: Pollution analysis for $l = 4$ ($h \simeq \omega^{-11/10}$)

7.4 A geophysical model: the Marousi II test-case

We consider the Marmousi II synthetical model [22]. The domain of propagation is 10240 m large and 2560 m deep. The P-wave velocity c_p and the density ρ are given as a 2048×512 grid (see Figure 12 and 13). The bulk modulus is defined from the P-wave velocity and the density with $\kappa = \rho c_p^2$.

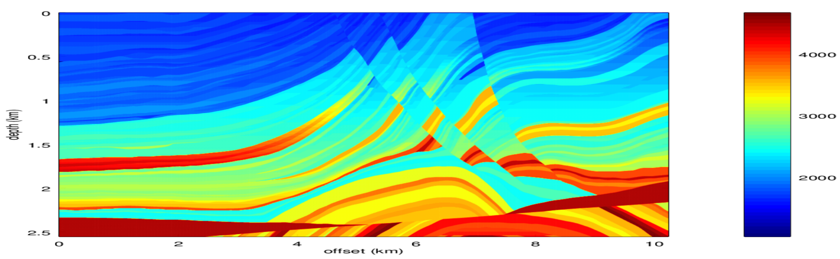


Figure 12: Marmousi II: P-wave velocity model

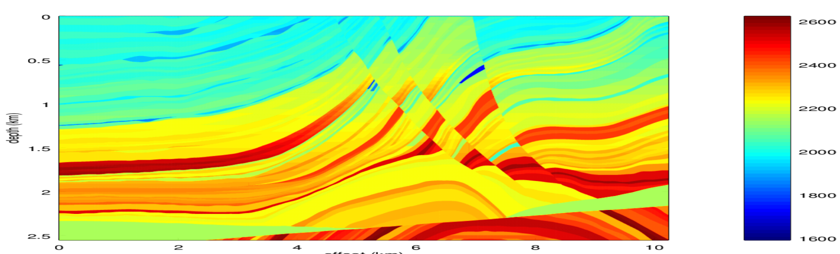


Figure 13: Marmousi II: density model

A Dirichlet boundary condition is imposed on the top of the domain and a first-order absorbing boundary condition is applied on the rest of the boundary to simulate a semi-infinite propagation medium.

The seismic source is represented by a Dirac right-hand-side $\phi \in \mathcal{D}'(\Omega)$ located at $x = 5000$ m and $z = 50$ m:

$$\langle \phi, v \rangle = \overline{v(5000, 50)}, \quad \forall v \in \mathcal{D}(\Omega).$$

There is no analytical solution for this benchmark. Hence, we use a finite element solution u_{ref} computed on a very fine mesh as a reference. More precisely, this solution is computed with triangular Lagrangian elements of degree 4. The mesh is based on a 2048×512 cartesian grid, each square of the grid being subdivided into two triangles. It is worth noting that the mesh is based on the same cartesian grid as the medium parameters. Hence, these parameters are constant on each cell and we can use a standard finite element method to approximate the solution.

The solutions are evaluted on a 513×129 cartesian grid which is used to compute relative L^2 errors. The relative error of a solution u_{mhm} computed with the MHMm is given by

$$E(u_{mhm}) = \left(\frac{\sum_{i=1}^{513} \sum_{j=1}^{129} |u_{ref}(x_i, z_j) - u_{mhm}(x_i, z_j)|^2}{\sum_{i=1}^{513} \sum_{j=1}^{129} |u_{ref}(x_i, z_j)|^2} \right)^{1/2},$$

where x_i and z_j correspond to the offset and depth of the evaluation grid lines.

We tabulate the error of the MHMm solutions for different choices of h and l. We use 3 different values of h: $20, 40$ and 80 m, which corresponds to cartesian grids of size 512×128, 256×64 and 128×32 respectively. Thus, it is clear that the original medium parameters grid is coarsen by a factor 4, 8 or 16.

For the second level methodology, we use square Lagrangian finite-elements of degree 3 based on a 8×8, 16×16 or 32×32 caresian grid for $h = 20, 40$ or 80 m respectively. This choice ensures that the second-level computational scale is twice smaller than the medium parameters grid. In particular, it is clear that the small-scale matches the scale of the heterogeneities.

In order to give a comparison with more standard finite-element methods, we also compute the solutions with the MMAm [5, 10, 11]. The method is based on triangular Lagrangian elements so that we subdivide each squares of the MHMm grids into two triangles to apply the MMAm and we use enough subcells to take into account the medium parameters exactly.

We solve the problem for the frequency $f = 20$ Hz, the angular frequency begin defined as $\omega = 2\pi f$. The results are presented in Table 1.

MHMm						MMAm					
h	$l = 0$	$l = 1$	$l = 2$	$l = 3$	$l = 4$	h	$p = 1$	$p = 2$	$p = 3$	$p = 4$	$p = 5$
20	138	2.90	0.13	0.11	0.03	20	127	90.2	2.41	0.85	x
40	225	45.1	1.47	0.37	0.20	40	121	134	80.3	5.23	1.34
80	161	154	44.8	4.12	0.33	80	101	124	132	124	57.4

Table 1: Relative error (%) in the Marmousi II model

From Table 1, it is clear that the MHMm is able to produce accurate solutions on coarse meshes. Furthermore, when considering the same mesh and the same order of discretization, the MHMm clearly outperforms the MMAm in terms of accuracy. This is very interesting because the global linear system has approximately the same size and filling in both case. The drawback for the MHMm, however, is that it is required to solve local problems to construct the global linear system.

Conclusion

We present a new approach to solve the heterogeneous Helmholtz equation: the Multiscale Hybrid Mixed method (MHMm). The method is originally developed for elliptic problems and is known to be robust in highly heterogeneous media [3, 15].

When the MHMm is applied to the Helmholtz equation, it has a lot of similarities with the DEm of Farhat et al. [9]. Indeed, both methods are based on the primal hybrid formulation of the Helmholtz equation. The difference lies in the fact that the basis functions of the MHMm are taken as local solutions to the Helmholtz equation which need to be computed with a second-level strategy, while the DEm uses plane waves.

In homogeneous propagation media, the MHMm and DEm are closely related. In particular, the lowest-order MHMm and DEm elements are identical. However, the MHMm behaves differently when the propagation medium is heterogeneous. In particular, one advtange of the MHMm is to naturally handle subcell variations of the medium parameters, where the parameters need to be constant in each mesh cell for the DEm. The robustness of the MHMm with respect to small-scale heterogeneities is illustrated by our numerical experiments.

We present a convergence analysis of the MHMm for homogeneous propagation media, for elements of arbitrary orders. The convergence rates are optimal and correspond to the standard finite-element method. Our results also generalize the convergence result for the lowest-order DEm elements presented in [2].

Numerical experiments illustrate the accuracy of the MHMm on analytical solutions and geophysical benchmarks. The analytical experiments validate our convergence analysis. On the other, our experiments on the Marmousi II geophysical benchmark [22] show that the method is very efficient for geophysical applications. In particular, the superiority of the MHMm over polynomial Lagrangian elements is highlighted.

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A Proof of Proposition 2

Lemma 10. *Let $\sigma \in C^\infty(K)^2$ and let $u \in H^1(\Omega)$. Then*

$$\int_K \operatorname{div}(\sigma)|u|^2 \leq \int_{\partial K} |u|^2 \sigma \cdot n_K + 2|\sigma|_{0,\infty,K}|u|_{0,K}|u|_{1,K}. \tag{53}$$

Proof. First, using Stokes formula, we have

$$\int_{\partial K} |u|^2 \sigma \cdot n_K = \int_K \operatorname{div}(\sigma |u|^2).$$

Futhermore, we have

$$\operatorname{div}(\sigma|u|^2) = \operatorname{div}(\sigma)|u|^2 + \sigma \cdot \nabla|u|^2 = \operatorname{div}(\sigma|u|^2) = \operatorname{div}(\sigma)|u|^2 + 2\operatorname{Re} u\sigma \cdot \overline{\nabla u}. \tag{54}$$

It follows that

$$\begin{aligned} \int_K \operatorname{div}(\sigma)|u|^2 &= \int_{\partial K} |u|^2\sigma \cdot n_K - 2\operatorname{Re}\int_K u\sigma \cdot \overline{\nabla u} \\ &\leq \int_{\partial K} |u|^2\sigma \cdot n_K + 2|\sigma|_{0,\infty,K}|u|_{0,K}|u|_{1,K} \end{aligned}$$

□

Lemma 11. *Let K be a rectangle with edge of size h and let Γ be one edge of K. Then, for all $u \in H^1(K)$, there holds*

$$|u|_{0,K}^2 \leq 2h|u|_{0,\Gamma}^2 + 4h^2|u|_{1,K}^2 \tag{55}$$

Proof. It is clear that for any square K of size h, there exist an isometry which project K onto $K_0 = (0,h)^2$ and Γ on $(0,h)\times\{0\}$. We can therefore focus on proving (55) for the case $K = (0,h)^2$ and $\Gamma = (0,h)\times\{0\}$.

We define $\sigma \in C^\infty(K)^2$ by $\sigma(x,y) = (0, y-h)^t$ for all $x, y \in \overline{K}$. It is clear that $\operatorname{div}\sigma = 1$ and $|\sigma|_\infty = h$, and that $\sigma \cdot n_K = h$ on $(0,h)\times\{0\}$ and 0 on the rest of ∂K. Applying trace inequality A.3 from [26], we see that

$$\int_K |u|^2 \leq h\int_\Gamma |u|^2 + 2h|u|_{0,K}|u|_{1,K}. \tag{56}$$

We obtain (55) applying

$$2h|u|_{0,K}|u|_{1,K} \leq \frac{1}{2}|u|_{0,K}^2 + 2h^2|u|_{1,K}^2.$$

in (56). □

Lemma 12. *Let K be a triangle and Γ be one side of K. Then, for all $u \in H^1(K)$, there holds*

$$|u|_{0,K}^2 \leq h|u|_{0,\Gamma}^2 + h^2|u|_{1,K}^2. \tag{57}$$

Proof. Let $a \in \mathbb{R}^2$ be the vertex of K opposite to Γ and define $\sigma(x) = a - x$ for $x \in \mathbb{R}^2$. Then it is clear that $\sigma \in C^\infty(K)^2$, $\operatorname{div}\sigma = 2$, $|\sigma|_{0,\infty,K} \leq h$. It is also clear that $\sigma \cdot n_K = 0$ on $\partial K \setminus \Gamma$ and that $|\sigma \cdot n_K| \leq h$ on Γ. Hence, we have

$$\int_{\partial K} |u|^2\sigma \cdot n_K \leq h\int_\Gamma |u|^2.$$

It follows that

$$2\int_K |u|^2 \leq h|u|_{0,\Gamma}^2 + 2h|u|_{0,K}|u|_{1,K},$$

and we conclude with

$$2h|u|_{0,K}|u|_{1,K} \leq |u|_{0,K}^2 + h^2|u|_{1,K}^2. \tag{58}$$

□