



# Modeling analysis of tidal bore formation in convergent estuaries

Andrea Gilberto Filippini, Luca Arpaia, Philippe Bonneton, Mario Ricchiuto

## ► To cite this version:

Andrea Gilberto Filippini, Luca Arpaia, Philippe Bonneton, Mario Ricchiuto. Modeling analysis of tidal bore formation in convergent estuaries. *European Journal of Mechanics - B/Fluids*, 2018, 10.1016/j.euromechflu.2018.01.001 . hal-01698295

**HAL Id: hal-01698295**

**<https://inria.hal.science/hal-01698295>**

Submitted on 30 Oct 2018

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Modeling analysis of tidal bore formation in convergent estuaries

Andrea Gilberto Filippini (*Corresponding author*)

*Team CARDAMOM, Inria Bordeaux sud-Ouest and IMB, - 200 av. de la vieille tour,  
33405 Talence cedex, France*

Luca Arpaia

*Team CARDAMOM, Inria Bordeaux sud-Ouest and IMB, - 200 av. de la vieille tour,  
33405 Talence cedex, France*

Philippe Bonneton

*CNRS, UMR EPOC, University of Bordeaux, Allée Geoffroy Saint-Hilaire, 33615 Pessac  
cedex, France*

Mario Ricchiuto

*Team CARDAMOM, Inria Bordeaux sud-Ouest and IMB, - 200 av. de la vieille tour,  
33405 Talence cedex, France*

---

## Abstract

Despite the recognized impact of tidal bores on estuarine ecosystems, the large scale mechanism of bore formation in convergent alluvial estuaries is still under investigation. So far, field data exist only for a small number of estuaries, while numerical simulations employ the shallow water equations mainly focusing on the small-scale and local processes. In this work, firstly we apply the fully nonlinear weakly dispersive Serre-Green-Naghdi equations to simulate the tide propagation in a convergent estuary of idealized form, verifying that the local dispersion effects, responsible for the appearance of the secondary waves, do not influence the tidal bore onset, which only results from the large scale processes of amplification/damping and distortion of the incoming wave. In

---

*Email addresses:* `andrea.filippini@inria.fr` (Andrea Gilberto Filippini (*Corresponding author*)), `luca.arpaia@inria.fr` (Luca Arpaia), `p.bonneton@epoc.u-bordeaux1.fr` (Philippe Bonneton), `mario.ricchiuto@inria.fr` (Mario Ricchiuto)

a second part, we numerically investigate (225 runs) the estuarine parameter space in order to identify the physical conditions that lead to tidal bore generation. In this parameter space, we determine a critical curve which divides estuaries according to tidal bore occurrence. As a result of this investigation we have shown that bore formation is controlled by the competition between two physical processes: a) the *knee-shaped* distortion of the tidal wave, with flood dominance and eventually bore inception; b) the dissipation of the tidal wave, which is unfavorable to bore formation. We also provide evidence that amplification due to topographic convergence is not a necessary condition for tidal bore generation and that there exist estuaries which display both wave damping and bore development. Finally, the validity of the results has been also assessed in the presence of freshwater river discharge, showing that for low river discharge, its effect on estuarine dynamics can be neglected.

*Keywords:* Estuarine hydrodynamics; long wave; tidal bore formation; dispersion; undular tidal bore; Serre-Green-Naghdi equations.

---

## 1. Introduction

Tidal wave transformation in convergent alluvial estuaries plays a crucial role in the development of a sustainable management of water resources. For this reason, and to better understand the human impact on the estuarine ecosystem, it has been the subject of intensive scientific research. Parametric studies conducted in [1, 2, 3, 4, 5] have shown that, when neglecting river discharge effects, the estuarine hydrodynamics is controlled by only three dimensionless parameters. These parameters represent a combination of the properties of the tidal forcing at the estuary mouth (wave amplitude and period), the large-scale geometrical characteristics of the channel and the friction coefficient. For particular conditions of the above dimensional variables, of freshwater flow and of river channel bathymetry, the tidal wave may result strongly distorted when the flow turn to rise, and a bore can be observed at the beginning of the flood tide. In the lower part of the estuary, tidal wave propagation can be well described by

15 the so called Saint Venant or Nonlinear Shallow Water (NLSW) equations with  
 16 friction (*cf.* [6]). However the onset of a tidal bore and its evolution upstream  
 17 is controlled by non-hydrostatic dispersive mechanisms (*cf.* [7]). Even if Mad-  
 18 sen *et al.* [8] and Pan and Liu [9] have shown that the non-dispersive NLSW  
 19 equations with shock capturing methods can still be used to simulate breaking  
 20 bore propagation with relatively good results, undular bores requires the use  
 21 of other long wave modeling approaches. If weakly dispersive weakly nonlinear  
 22 Boussinesq-type equations can be adopted to describe the onset of the tidal bore  
 23 (*cf.* [7]), the nonlinear evolution of high-intensity bores requests fully nonlinear  
 24 equations to be employed, as for example the Serre-Green-Naghdi (SGN) ones  
 25 (*cf.* [10, 11, 12]). The first part of this work is thus devoted to reproduce the  
 26 progressive formation of an undular tidal bore inside an idealized estuary using  
 27 the SGN equations.

28 The results obtained in this first part will then justify the use of the NLSW  
 29 model to investigate at large scales how the three dimensionless parameters,  
 30 which control the estuarine long scale dynamics, influence the physical pro-  
 31 cesses of amplification/damping and distortion which may lead to the bore on-  
 32 set. These processes take place at large spatial and temporal scales, where  
 33 the non-hydrostatic effects associated with tidal bores can be neglected. Some  
 34 numerical studies already used the NLSW system to simulate the propagation  
 35 of tidal waves in rivers up to bore formation, but they are limited to a single  
 36 [8, 9] or to a small number [4] of estuaries. A similar limitation concerns also  
 37 field data: since every estuary in nature represents a unique combination of the  
 38 three dimensionless variables, we dispose only of a limited number of points in  
 39 the parameter space coming from *in situ* measurements. Based on the set of  
 40 data available, it is very difficult to understand how such parameters influence  
 41 the bore development process. In this paper, we bypass this major constraint  
 42 through a numerical investigation of the whole space of dimensionless variables.  
 43 This allows us both to characterize the global conditions for tidal bore occur-  
 44 rence and to analyze for each estuarine regime the nonlinear processes associated  
 45 with tidal wave transformation. Our approach is based on a scaling analysis in-

46 introduced in [13], where the reader can also find a collection of estuary data,  
47 which are used here to validate our numerical investigation.

48 The paper is structured as follows: section 2 introduces the idealized ge-  
49 ometry used to accomplish our study and defines the dimensionless parameters  
50 emerging from the scaling analysis of the governing equations; section 3 gives  
51 some details about the solution strategy adopted to solve the SGN system, the  
52 implemented numerical scheme and motivates the choice of the type of bound-  
53 ary conditions used; section 4 introduces a criterion for bore detection based  
54 on experimental observation *in situ*; the numerical simulation of the onset of an  
55 undular tidal bore inside an idealized estuary is presented in section 5; section 6  
56 discusses the numerical investigation on the favorable conditions for tidal bore  
57 occurrence, introducing a criterion for bore detection based on experimental  
58 observation *in situ* and showing the main findings of the study; finally, section  
59 7 treats the effects of water river discharge on tidal bore formation.

## 60 2. Problem setting

61 The study of tidal wave propagation in funnel shaped estuaries is often per-  
62 formed under several simplifying assumptions. The geometry of real alluvial  
63 estuaries in coastal plains can be well-approximated by an exponentially de-  
64 creasing width variation and flat bathymetry [14, 15, 6]. This morphology is  
65 the natural result of a morphodynamic equilibrium of the erosion/sedimentation  
66 process. For this reason, we will perform our study on the simplified case (figure  
67 1) of a tide propagating in an idealized convergent channel of constant depth  
68  $D_0$  and an exponentially decreasing width  $B(x)$ , varying in the longitudinal  
69 direction  $x$  with the law:

$$B(x) = B_0 e^{-x/L_b} \quad (1)$$

70 where  $L_b$  represents the convergence length of the channel. We assume a rect-  
71 angular cross-section suitable, as a first approximation, to describe the behavior  
72 of a real section with the same area.

73

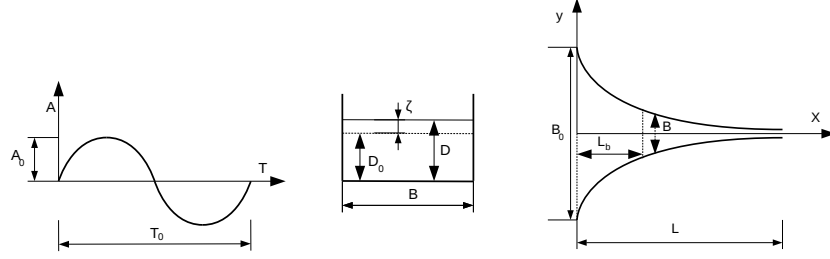


Figure 1: Sketch of the idealized geometry of the channel and basic notations.

It is a common practice in the literature to use quasi-one-dimensional systems of equations to investigate the large scale dynamics of long waves propagating in convergent channels [1, 2, 16, 5, 4]. In the case of open channels with exponentially decreasing cross section areas, the equations assume the following form:

$$\begin{aligned} \frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + D \frac{\partial u}{\partial x} - \frac{uD}{L_b} &= 0 \\ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial \zeta}{\partial x} + C_{f0} \frac{u|u|}{D} &= 0 \end{aligned} \quad (2)$$

where  $D = \zeta + D_0$  denotes the total water depth, with  $\zeta$  the free surface elevation and  $D_0$  the still water depth and  $u$  stands for the cross-sectionally averaged flow velocity. The friction term is modeled by a quadratic law, with  $C_{f0}$  the friction coefficient.

As already remarked by others in the literature [1, 2, 4, 13] three external dimensionless parameters appears to fully control the system (2). Following the scaling of the equations proposed by Bonneton *et al.* [13], these parameters are:

$$\epsilon_0 = \frac{A_0}{D_0}, \quad \delta_0 = \frac{L_{w0}}{L_b}, \quad \phi_0 = \frac{C_{f0}(gD_0)^{1/2}}{\omega_0 D_0}; \quad (3)$$

where  $D_0$ ,  $A_0$  and  $\omega_0$  form a set of reference external parameters, respectively the water depth, the amplitude and the angular frequency of the tidal wave. Here  $\epsilon_0$  represents the standard nonlinearity parameter,  $\delta_0$  is the convergence ratio,  $\phi_0$  is a friction parameter and  $L_{w0} = (gD_0)^{1/2} \omega_0^{-1}$  is the frictionless tidal-

91 wave length scale.

92 Field observations reported in [3] show that in tidal bore estuaries the tidal  
93 wave nonlinearity is mainly characterized by the dimensionless parameter

$$D_i^* = \epsilon_0 \phi_0 / \delta_0 , \quad (4)$$

94 being always one order of magnitude greater than  $\epsilon_0$ .

95 Both theoretical studies [1] and experimental observations [3] agree that large  
96 values of the dissipation parameter  $D_i^*$  produce great distortion and peaking of  
97 the free surface and velocity profiles of the tidal wave, leading to flood domi-  
98 nance; characteristics which correspond to necessary conditions for tidal bore  
99 formation ( $D_i^* \geq 1.7$  in [3]). However, large values of  $D_i^*$  correspond also to  
100 high energy dissipation, leading to tidal damping; unfavorable to tidal bore for-  
101 mation. For this reason some natural estuaries, despite having high values of  
102  $D_i^*$ , don't display a bore. In order to evaluate the relative importance of friction  
103 in the momentum balance, several definitions of the friction parameter  $\phi_0$  have  
104 been introduced in the literature [1, 2, 17, 4]. Although all of these definitions  
105 are analogous from a physical point of view, only the one introduced by the  
106 present authors in [13] allows to investigate separately the opposite effects of  
107 peaking and dissipation taking place for high values of  $D_i^*$ . In this paper we  
108 describe the details of the numerical investigation behind the scaling proposed  
109 in [13]. We will thus explore numerically the space of the external parameters  
110  $\epsilon_0$ ,  $\delta_0$  and  $\phi_0$ , quantifying which range of values is in favour of tidal bore for-  
111 mation.

112 Considering that, for most alluvial estuaries, the convergence ratio  $\delta_0$  is close to  
113 2 (as showed in figure 2 of [13]), we have chosen to perform our analysis using  
114 a constant value  $\delta_0 = 2$ . Due to such a simplification, the expression of the  
115 dissipation parameter (4) reduces to:  $D_i^* = \alpha \epsilon_0 \phi_0$  (with  $\alpha$  constant). Thus, it  
116 is possible to investigate the separate effects of peaking and dissipation, both  
117 contained in  $D_i^*$ , by numerically exploring only the plane of the dimensionless  
118 parameters  $(\phi_0, \epsilon_0)$ . The effects of nonlinear dispersion and of the discharge are  
119 also investigated, while we leave out for the moment the influence of bathymetric

120 variations.

### 121 **3. Numerical model**

#### 122 *3.1. Governing equations*

123 For nonlinear long waves, a reasonably general description is provided by  
 124 the fully nonlinear weakly dispersive Serre-Green-Naghdi equations. As already  
 125 mentioned, this model allows to simulate the phenomenon in its entirety, includ-  
 126 ing the nonlinear dispersive effects active on the smaller scales. To the authors'  
 127 knowledge, it does not exist so far any mathematical model allowing to account  
 128 for such effects in quasi-1D setting. The closest work is the very recent paper  
 129 of Winckler and Liu [18], who developed a set of weakly nonlinear Boussinesq  
 130 equations. This justifies the use of a full two-dimensional model. In particular,  
 131 the SGN equations can be recast as [19, 20]:

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot (D \mathbf{u}) = 0 , \quad (5a)$$

$$\frac{\partial u}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + g \nabla \zeta + C_{f0} \frac{\mathbf{u} |\mathbf{u}|}{D} = \boldsymbol{\psi} , \quad (5b)$$

$$(I + T) \boldsymbol{\psi} = T \left( g \nabla \zeta - C_{f0} \frac{\mathbf{u} |\mathbf{u}|}{D} \right) - Q(\mathbf{u}) ; \quad (5c)$$

132 where  $\mathbf{u}$  is the depth averaged velocity vector of horizontal component  $u$  and  
 133 transversal one  $v$ ,  $\boldsymbol{\psi}$  characterizes the non-hydrostatic and dispersive effects,  
 134 and the linear operators  $T(\cdot)$  and  $Q(\cdot)$  are the same defined in [20] and their  
 135 expressions are given in appendix A.

136 It is worth noting that, when applying the scaling proposed in [13] and recalled  
 137 in section 2 to system (5) (details are reported in appendix A), an additional  
 138 dimensionless parameter  $\mu^2 = (D_0/L_0)^2$  appears into the equations, multiply-  
 139 ing all the dispersive terms (*cf.* equation (17b)). This parameter is responsible  
 140 for the fact that the non-hydrostatic effects become negligible when the charac-  
 141 teristic scale of the phenomenon is large, transforming the original SGN system  
 142 of equations into the non-conservative form of the Non-Linear Shallow Water  
 143 (NLSW) system. For this reason, dispersion does not influence significantly



the large scales of tidal wave propagation, which can be well described by the NLSW equations. A direct verification of this is given and discussed in section 5.

### 3.2. Numerical strategy

To solve system (5) we adopt the strategy illustrated in [20]. Given an initial solution, the Serre-Green-Naghdi system is solved in two independent steps. First, the elliptic equation (5c) is solved for the non-hydrostatic term  $\psi$ . Then, an hyperbolic step is performed for equations (5a)-(5b), evolving the flow variables in time. This strategy solution has been shown to be very flexible and robust, producing accurate results with different combinations of discretization schemes in space and time. For this study, in the hyperbolic phase a third order MUSCL finite volume scheme has been used, together with a third order SSP Runge-Kutta method for the evolution in time. The elliptic phase is solved, instead, with a classical second order finite element approach. The overall hybrid FV-FE scheme obtained is thus characterized by a dispersion error of the same order, or smaller, than those produced by fourth order finite differencing, providing that at least third order of accuracy is guaranteed for the hyperbolic component. The interested reader can refer to [20] for more details.

### 3.3. Boundary conditions

The choice of the boundary conditions affects the flow field significantly [21]. While exact boundary conditions can be generally imposed in the case of homogeneous hyperbolic problems, this remains an active research topic when source terms are included inside the equations. The task becomes even harder working with dispersive models. Therefore, while the fully reflective wall boundary condition can be obtained by imposing some symmetric conditions stemming from the satisfaction of  $\mathbf{u} \cdot \mathbf{n} = 0$  on the wall line (being  $\mathbf{n}$  the normal direction to the wall), inflow and outflow boundaries require more complex treatments (*cf.* [22, 23, 24, 25]).

173 In this work, the interest in long scale dynamics justifies the use of the NLSW  
 174 invariants also when the SGN equations are solved inside the computational  
 175 domain [26]. The illustrative results contained in section 5 show that the effects  
 176 of this approximation on the large scale is completely negligible on both coarse  
 177 and fine computational meshes.

178

179 In practice, the seaward boundary condition is applied by imposing in the  
 180 incoming NLSW Riemann invariant the free surface given by the sea level tidal  
 181 oscillation:

$$\zeta(t) = A_0 \sin\left(\frac{2\pi t}{T_0}\right) \quad (6)$$

182 We consider here the case of semi-diurnal tides with period  $T_0 = 12.41$  [h].

183 On the landward boundary, the outflow condition is set imposing the informa-  
 184 tion of still water (we recall that the river discharge is neglected at this level)  
 185 coming from far on the right. However, the imposition of the homogeneous  
 186 invariant cannot take into account the rise of the mean (tidally averaged) wa-  
 187 ter level landward caused by the friction (*cf.* [5]), introducing a non-physical  
 188 behavior of the solution in the outlet proximity. For this reason, the outflow  
 189 boundary conditions has been coupled with an extension of the computational  
 190 domain, to reduce the spurious influences on the solution in the region of study.  
 191 In practice it has been enlarged up to  $x = 6L_b$ , to further limit non-physical  
 192 effects due to the boundary condition.

193 In order to guarantee the reliability of the results, a sensitivity study has been  
 194 performed in comparison with the technique of boundary imposition applied by  
 195 [21] and a reference solution computed using a very long computational domain.  
 196 For the interested reader the results of such study are reported in appendix B.

#### 197 4. Computations setting

198 Tide propagation up to an estuary is, in nature, partly limited by an in-  
 199 creasing bottom slope and by discharge. Considering that the effects of both  
 200 the bathymetric variations and the river discharge have been, at first, neglected

201 in our simulations, even the weakest and linear tidal wave would be able to  
 202 propagate landward and, being dissipated by only friction, it would reaches  
 203 non-physical distances with also possible generation of unnatural bores. For  
 204 this reason, and for later use, a characteristic physical length  $L_c$  has been intro-  
 205 duced to limit the region in which the tidal signal is assumed to be physically  
 206 relevant. In particular, examining real estuaries data, we observed that tidal  
 207 bores occur, before reaching a distance of  $3L_b$  from the estuary mouth, see table  
 208 1. Thus, we have chosen to limit the region of interest for our simulations to  
 $L_c = 3L_b$ .

Estuaries	$L_b$ [km]	$x_c$ [km]	$x_c/L_b$
Gironde/Garonne	43	90	2.09
Hooghly	25	60	2.4
Humber	25	75	3.0
Pungue	17	50	2.94
Qiantang	40	90	2.25
Severn	41	55	1.34

Table 1: Ratio between the location of tidal bore inception  $x_c$  and convergence length  $L_b$  for some alluvial estuaries. Data taken from [3].

209  
 210 Moreover, tidal bore inception is a continuous process which takes place as  
 211 a gradual increase of the free surface slope at the beginning of the flood phase.  
 212 The same continuous increase of steepness is observed in our simulations. A  
 213 criterion is thus needed to detect the bore onset, within the domain  $[0, L_c]$   
 214 just defined. During the two measurement campaigns on the Garonne river,  
 215 Bonneton et al.[3] observed that bore passage was associated to an increase in  
 216 the value of the free surface spatial gradient, at the start of the flood, with at  
 217 least one order of magnitude (from  $\mathcal{O}(10^{-4})$  in the smooth case without tidal  
 218 bore). For each numerical simulation we compute, in a post-processing phase,  
 219 the spatial slope of the free surface all along the domain, up to  $L_c$ , retaining for  
 220 each position  $x$  the maximum value in the tidal period. The quantity  $A_{max}$  has

221 been defined as:

$$A_{max} = \max_{x \in L_c} \left[ \max_{t \in T_0} \left( \frac{\partial \zeta(x, t)}{\partial x} \right) \right] \quad (7)$$

222 and the following criterion to determine tidal bore onset is used:  $A_{max} \geq 10^{-3}$ .

## 223 5. Undular tidal bore formation

224 Preliminary simulations concerning the development of an undular tidal bore  
 225 into the idealized channel of figure 1 are reported in this section. We consider a  
 226 strongly nonlinear and weakly dissipative channel of  $\epsilon_0 = 0.7$  and  $\phi_0 = 10$ . To  
 227 visualize the results we consider pointwise plots and distributions of the relevant  
 228 quantities along the channel axis. In developed flow conditions, deviations in  
 229 the transversal direction have been quantified to be less than 1.5%.

230

231 Figure 2 shows the time series over one tidal cycle of the computed free sur-  
 232 face elevation, measured at 11 gauges positions along the channel. It appears  
 233 evident that the results provided by the SGN model do not differ respect to  
 234 those obtained by using the NLSW model. Differences appear, instead, focus-  
 235 ing on the region of the shock formation. The results obtained are then reported  
 236 in figure 3 and show the formation process of a undular tidal bore. Also in this  
 237 case, the two models give identical results in the whole domain except when the  
 238 shock is formed (*cf.* top of figure 3). On the discontinuity, in fact, the SGN  
 239 model provides an undular bore with local formation of a train of secondary  
 240 waves. These waves are characterized by a wavelength of  $50[m]$  and a period of  
 241  $6[s]$  (*cf.* bottom of figure 3). However, it can be remarked that the position of  
 242 the bore inception, the mean bore jump and its celerity are well-described by  
 243 the NLSW model. This justifies its application for investigations on the large  
 244 scale estuary dynamics. Studies concerning the local effects on tidal bores or  
 245 flooding problems cannot, instead, avoid the use of a dispersive model to cor-  
 246 rectly reproduce the peaks of water height due to the secondary waves.

247 Field observations lead Bonneton *et al.* [3] to consider that the location of the  
 248 tidal bore onset is associated with the appearance of an inflexion point on the

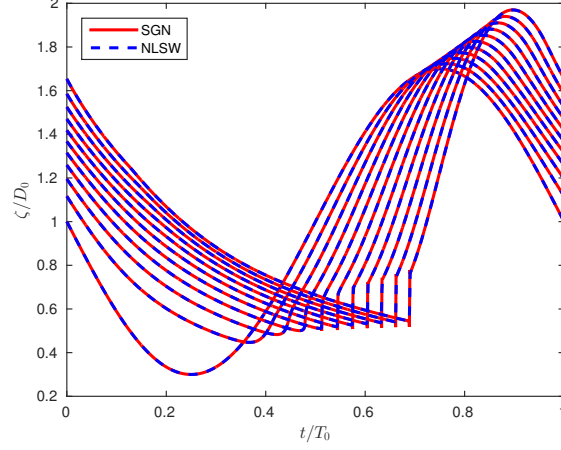


Figure 2: Illustrative result of a tidal propagation into an idealized channel with the shape described in section 2,  $\epsilon_0 = 0.7$ ,  $\phi_0 = 10$ . Computed free surface profile, measured at 11 stations along the channel corresponding to  $x = \alpha L_B$  with  $\alpha = 0 : 1/3 : 3$ . The simulation have been realized using both the NLSW and the SGN models.

249 tidal wave profile. However, these authors stress the difficulty to characterize  
 250 the tidal bore onset on the river, since this would require a high density of water  
 251 elevation measurements over a long distance. By contrast, with our numerical  
 252 simulations we are able to provide a full set of subsequent snapshots (*cf.* figure  
 253 4), describing the progressive transition from a tidal wave regime without bore  
 254 (*cf.* figures 4a and 4b) to a well-developed undular tidal bore regime (*cf.* fig-  
 255 ures 4g and 4h). The gradual evolution of the surface profile towards the bore  
 256 inception passes by the appearance of an inflexion point at the beginning of the  
 257 rising tide.

258 In the middle and upper part of the estuary, the value of  $A_{max}$ , used in our  
 259 tidal bore onset criterion, is registered at the beginning of the rising tide. As  
 260 a consequence, the pictures of figure 4 allow to observe the temporal evolution  
 261 of such quantity as the tide propagates landward. The values of  $\max_{x \in L_c} \left( \frac{\partial \zeta}{\partial x} \right)$  for  
 262 the eight pictures of figure 4 are, thus, showed in table 2 for the two NLSW and  
 263 SGN model. It is worth to note that the dispersive terms becomes relevant after  
 264 the threshold of  $A_{max}$  has been reached, *i.e.* when tidal bore occurrence has

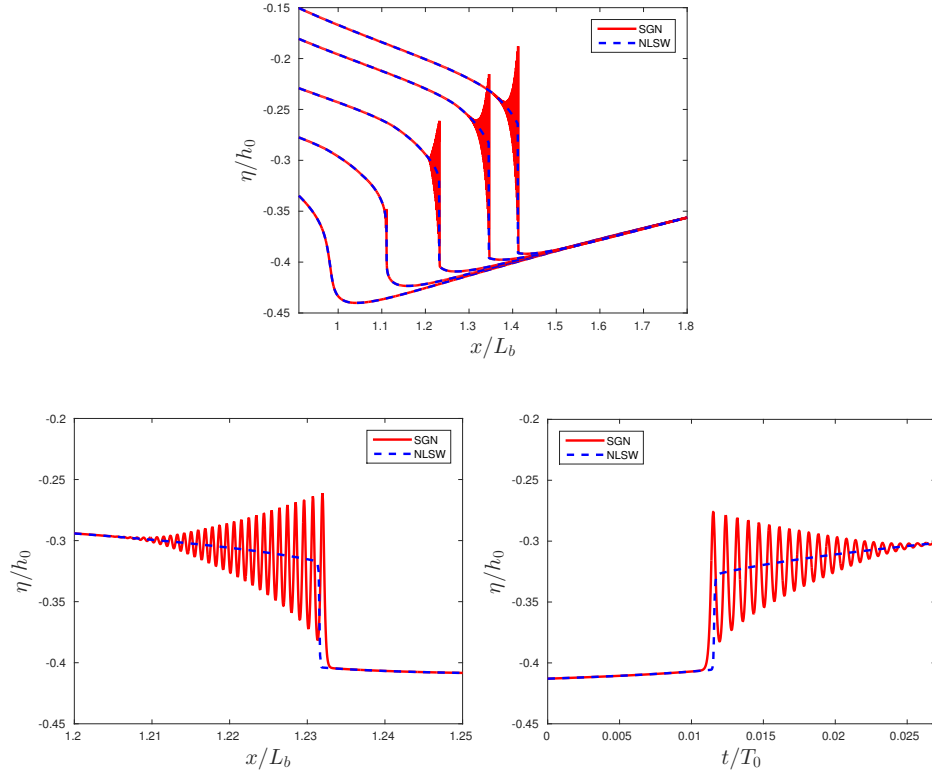


Figure 3: Illustrative result of tidal propagation into an idealized channel with the shape described in section 2,  $\epsilon_0 = 0.7$ ,  $\phi_0 = 10$ . Top: computed free surface profile at different increasing times of the simulation. Bottom-left: free surface signal in space. Bottom-right: free surface signal in time.

265 already been flagged by our criterion. We can, thus, conclude that tidal bore  
 266 appearance is completely determined by the values of the external parameters  
 267  $\epsilon_0$ ,  $\delta_0$  and  $\phi_0$  and is not influenced by the choice of the model used for the  
 268 numerical simulations.

## 269 6. Study of tidal bore occurrence

270 The results of the previous section show that dispersive effects are not neces-  
 271 sary to study the large scales of the propagation and transformation of the tide.  
 272 At these scales the NLSW system is a sufficiently accurate and more efficient

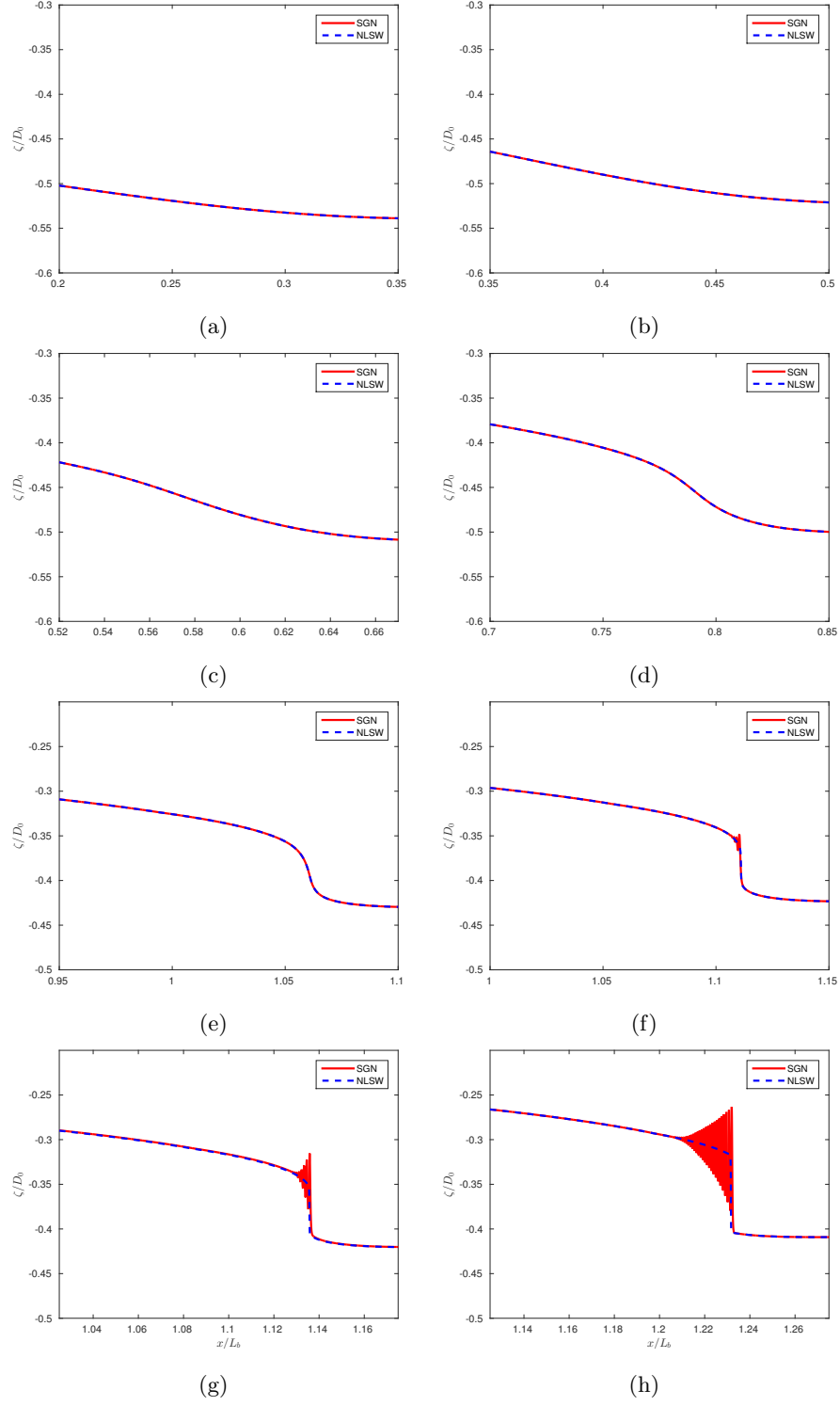


Figure 4: Computed free surface profile at different increasing times of the simulation, showing the undular bore inception from the formation of an inflexion point of the wave profile: from top left to bottom right.

	NLSW	SGN
a)	$7.25 \times 10^{-5}$	$7.25 \times 10^{-5}$
b)	$1.05 \times 10^{-4}$	$1.05 \times 10^{-4}$
c)	$1,74 \times 10^{-4}$	$1,74 \times 10^{-4}$
d)	$4.11 \times 10^{-4}$	$4.11 \times 10^{-4}$
e)	<b><math>1.9 \times 10^{-3}</math></b>	<b><math>1.9 \times 10^{-3}</math></b>
f)	<b><math>2.87 \times 10^{-3}</math></b>	<b><math>1.61 \times 10^{-3}</math></b>
g)	<b><math>5.04 \times 10^{-3}</math></b>	<b><math>3.77 \times 10^{-3}</math></b>
h)	<b><math>9.47 \times 10^{-3}</math></b>	<b><math>7.38 \times 10^{-3}</math></b>

Table 2: Values assumed by the quantity  $\max_{x \in L_c} \left( \frac{\partial \zeta}{\partial x} \right)$  in the several pictures of figure 4 and for the two NLSW and SGN models. Bold numbers represent values which satisfy the tidal bore onset criterion:  $A_{max} \geq 10^{-3}$

model. This is why we will employ it here to perform the numerical investigation of the space  $(\phi_0, \epsilon_0)$  of dimensionless parameters, trying to quantify the favorable conditions for bore inception. However, dispersion simulations will still be used in some cases to provide some quantification of the local order of magnitude of the non hydrostatic terms. This quantitative comparison will allow to comfort our hypothesis and provide further insight into the mechanisms of bore formation.

### 6.1. Investigation of the plane $(\phi_0, \epsilon_0)$

We have performed 225 simulations of the idealized case of study defined in section 2, corresponding to an equivalent number of estuaries. We recall that the value of convergence ratio is constant,  $\delta_0 = 2$ , and thus each simulation represents a unique combination of the parameters  $\epsilon_0, \phi_0$ , corresponding to precise conditions of the tidal forcing at the mouth and to specific geometrical and physical properties of the channel. In such a way we have systematically investigated the plane  $(\phi_0, \epsilon_0)$ , applying the criterion described in section 4 in



289 order to detect bore formation. Figure 5(left) shows the contour lines of the  
 290 quantity  $A_{max}$  in the plane of the parameters  $(\phi_0, \epsilon_0)$ , performed by collecting  
 291 and linearly interpolating the simulations' results. The red color denotes the re-  
 292 gion where the bore detection criterion is satisfied, while the blue one represents  
 293 cases characterized by a smooth solution. Figure 5(right) will be discussed later  
 294 on. The results show that there exists a critical curve  $\epsilon_c(\phi_0)$  that can be traced  
 295 in this plane, dividing tidal-bore and no-tidal-bore estuaries. This curve is the  
 white dashed line traced in figure 5(left). Observing the shape of the isolines

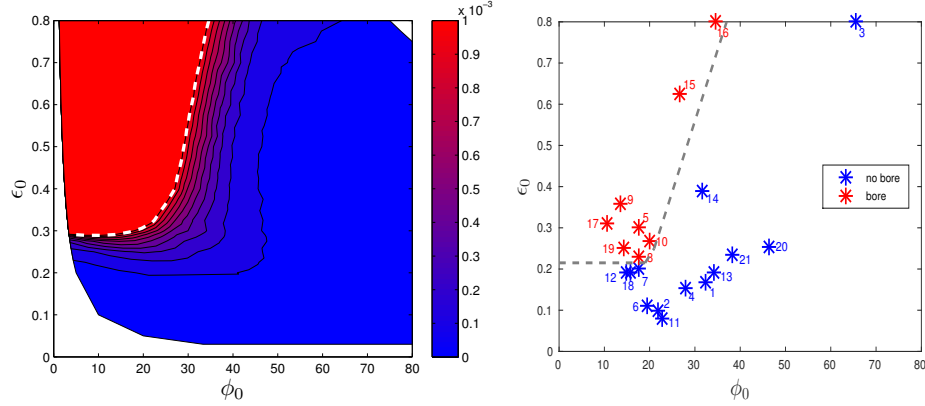


Figure 5: (Left): Isocurves of the quantity  $A_{max}$  in the plane of the parameters  $(\phi_0, \epsilon_0)$ , the white dashed line represents the  $\epsilon_c(\phi_0)$  curve, namely the limit for tidal bore appearance following the criterion  $A_{max} \geq 10^{-3}$ . (Right) : Projection of real alluvial estuaries data on the plane  $(\phi_0, \epsilon_0)$ ; with a grey dashed line (- -) we have represented by hand the transition between the two regimes, inspired by the similar trend in the left picture. The database used to generate the picture can be found in table 2 of [13], for brevity and completeness we list here the name and number of the estuaries included in the picture: 1. Chao Phya; 2. Columbia; 3. Conwy; 4. Corantijn; 5. Daly; 6. Delaware; 7. Elbe; 8. Gironde; 9. Hooghly; 10. Humber; 11. Limpopo; 12. Loire; 13. Mae Klong; 14. Maputo; 15. Ord; 16. Pungue; 17. Qiantang; 18. Scheldt; 19. Severn; 20. Tha Chin; 21. Thames. Sources: [6] for estuaries 1, 4, 11, 13, 14, 18, 20; [1] for 2, 3, 6, 7, 15, 19, 21; [3] for 8, 9, 10, 16, 17; [27] for 5; [28] for 12.

296

297 in figure 5(left), two different behaviors can be distinguished, depending on the  
 298 values assumed by the friction parameter  $\phi_0$ .

299 The first region is characterized by values of the friction parameter in the

range  $1 < \phi_0 \leq \sim 20$ . Here the mechanism of bore inception appears to be fully  
 controlled by the nonlinear parameter  $\epsilon_0$ . Figure 6 shows a comparison between  
 two numerical results computed using two different values of  $\epsilon_0$ , one just above  
 the critical curve and one just below it ( $\epsilon_0 = 0.3$  and  $\epsilon_0 = 0.225$  respectively),  
 at a constant value of  $\phi_0$  equal to 13.33. The comparison is made in terms of  
 the free surface and velocity time series, respectively in figure 6(a) and 6(b),  
 measured at the location  $x = L_c$ . In both figures, the signals associated to the  
 case at higher  $\epsilon_0$  are characterized by a greater distortion at the beginning of  
 the flood phase, leading to bore formation according to our threshold. The two  
 zooms, displayed in figures 6(c) and (d), allow to better appreciate the different  
 time gradients exhibited by the water wave on the time scale of the tidal bore  
 (around 20 minutes). It is also interesting to note that the free surface profile  
 of figure 6(a) does not display a Burger's like shock, but rather a *knee* shape  
 is observed. This is mainly due to the fact that the nonlinear effect of friction  
 prevails on the advective one, remaining the dominant nonlinearity for the major  
 part of the wave transformation. Figure 7 displays the relative importance of  
 the various terms in the momentum equation during a tidal cycle in the previous  
 case of bore development ( $\epsilon_0 = 0.3$  and  $\phi_0 = 13.33$ ). In the figures, the time  
 evolution of these quantities is represented at three equispaced locations along  
 the channel, in particular at  $x = \frac{1}{3}L_c$ ,  $x = \frac{2}{3}L_c$  and  $x = L_c$ . It emerges that  
 the advective term remains negligible in the momentum equation and the bore  
 formation results from a balance between acceleration, friction and hydrostatic  
 terms. Only in the presence of incipient bore, the relative importance of the  
 advective term rises up overcoming the frictional nonlinearity. This behaviour  
 can be also recovered when a dispersive bore development is simulated. For  
 the previous case of undular bore formation of section 5, the time variations  
 of the relative magnitude for the momentum equation's terms are compared  
 in figure 8. For sake of clarity, data are shown for the small time of the bore  
 passage at two space locations: just before (left) and after (center and right)  
 the appearance of the secondary waves. It can be seen that, as in the previous  
 case of figure 7, the advective term becomes the relevant nonlinear term on the

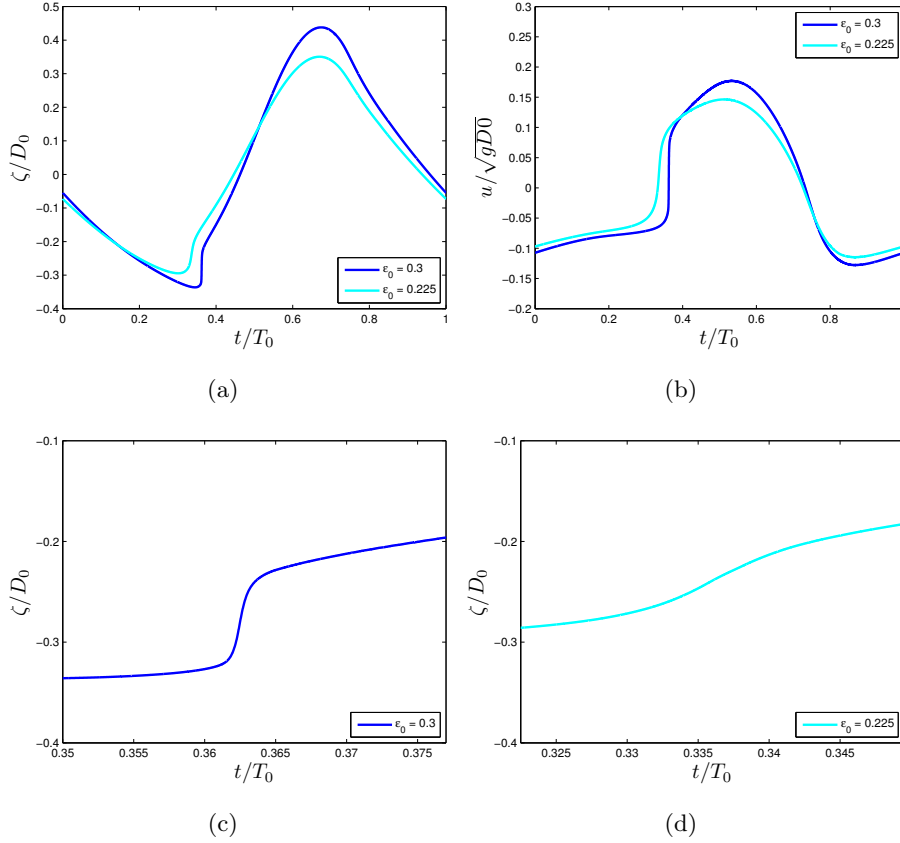


Figure 6: Time variation of the free surface elevation (a) and the velocity (b) signals measured at  $x = L_c$  for a fixed value of  $\phi_0 = 13.33$  and the two different values of  $\epsilon_0 = 0.225$  (—) and  $\epsilon_0 = 0.3$  (—). (c) and (d) are two zooms on the water wave profile at the beginning of the flood phase, on the time scale of the tidal bore (around 20 minutes).

331 bore location. However, in this case, its increase is partially limited by the rise  
 332 of the dispersive term in a general context dominated by the balance between  
 333 the inertial and the hydrostatic contributions.

334

335 For higher values of  $\phi_0$  ( $\phi_0 \geq \sim 20$ ), the isolines of  $A_{max}$  spread out form-  
 336 ing a wider transition region but, more important, display an almost vertical  
 337 slope. This implies an increasing role of the friction parameter in the physical  
 338 mechanism of bore formation. In figure 9, we show the solutions computed for

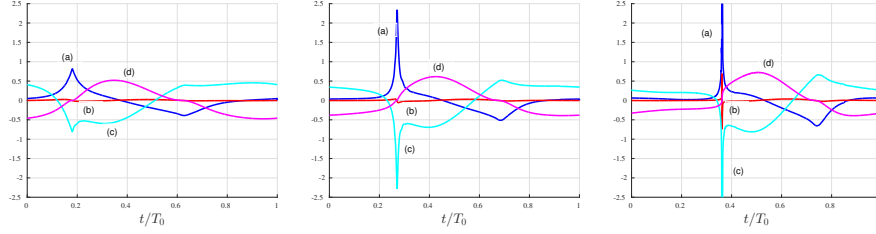


Figure 7: The order of magnitude of the several dimensionless terms in the momentum equation of system (17) without dispersion; three test sections are presented: at  $x = \frac{1}{3} L_c$  (left),  $x = \frac{2}{3} L_c$  (center) and  $x = L_c$  (right): (a)  $\frac{\partial \mathbf{u}}{\partial t}$ , (b)  $\frac{K}{\mathcal{L}} \epsilon_0 (\mathbf{u} \cdot \nabla) \mathbf{u}$ , (c)  $\frac{1}{K\mathcal{L}} \delta_0^2 \nabla \zeta$  and (d)  $K \frac{\epsilon_0 \phi_0}{\delta_0} \frac{\mathbf{u} |\mathbf{u}|}{D}$ . Upper and lower peaks on the figure (right) reach values respectively of 30.65 and  $-28.61$ .

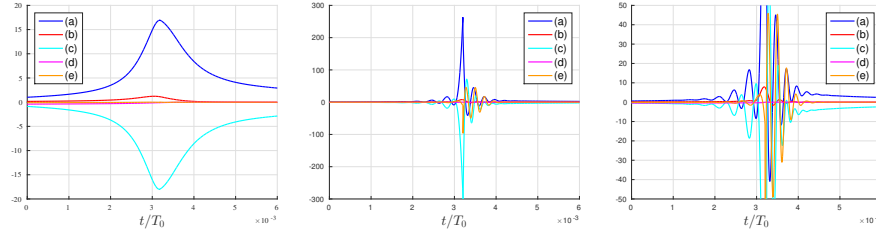


Figure 8: The order of magnitude of the several dimensionless terms in the momentum equation of system (17), including the dispersion; two test sections are presented: at  $x = \frac{1.02}{3} L_c$  (left) and  $x = \frac{1.12}{3} L_c$  (center), focusing on the time scale of the bore passage, while the third picture (right) represents, for clearness, a zoom of the central one: (a)  $\frac{\partial \mathbf{u}}{\partial t}$ , (b)  $\frac{K}{\mathcal{L}} \epsilon_0 (\mathbf{u} \cdot \nabla) \mathbf{u}$ , (c)  $\frac{1}{K\mathcal{L}} \delta_0^2 \nabla \zeta$  and (d)  $K \frac{\epsilon_0 \phi_0}{\delta_0} \frac{\mathbf{u} |\mathbf{u}|}{D}$  and (e)  $\mu^2 \Psi$ .

339 two cases across the transition zone ( $\phi_0 = 20$  and  $\phi_0 = 40$  respectively), keep-  
 340 ing constant  $\epsilon_0 = 0.4$ . Once again, we look to the free surface and velocity  
 341 time signals to compare the two results, figures 9(a) and 9(b) respectively. It  
 342 clearly appears that an increase of the value of the friction parameter  $\phi_0$  is  
 343 directly associated to both potential and kinetic energy dissipation, leading to  
 344 more damped profiles. This process decreases the local nonlinearity of the wave  
 345 which, in turn, is smeared out, as one can see from the free surface zooms on the  
 346 time scale of the bore (figures 9(c) and 9(d)). Moreover, figures 9(b) displays  
 347 the particular tendency of the tidal current to become constant during the ebb  
 348 tide for large values of the friction parameter (strongly dissipative estuaries), in

349 agreement with previous studies (e.g. [1] and [3]).

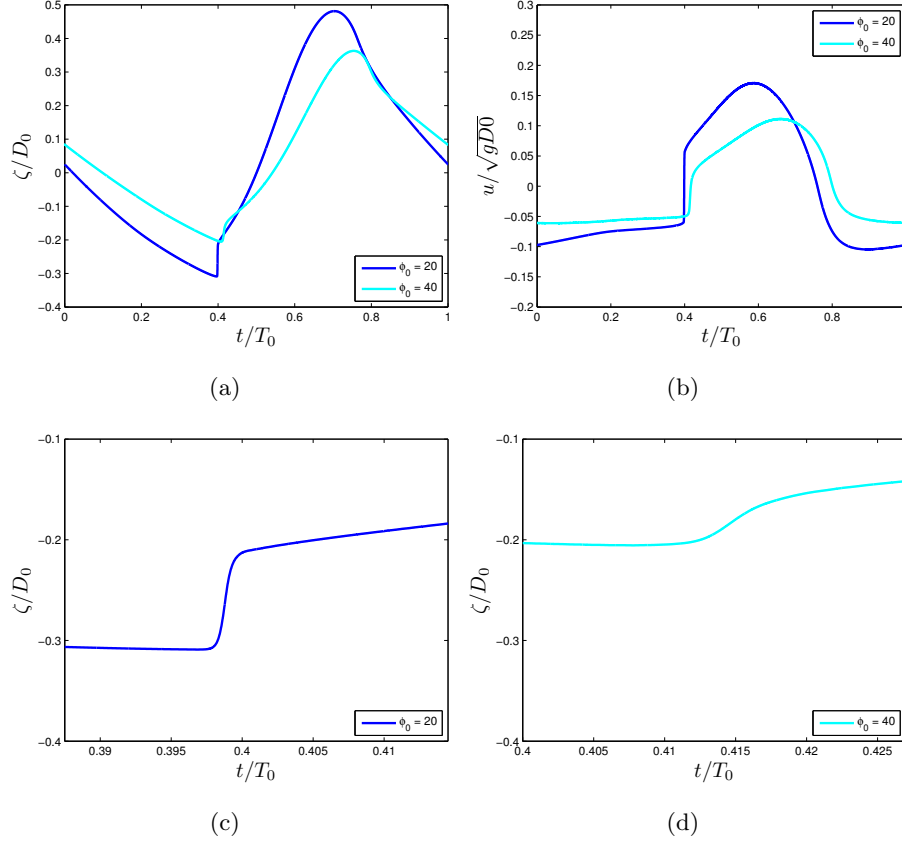


Figure 9: Time variation of the free surface elevation (a) and the velocity (b) signals measured at  $x = L_c$  for a fixed value of  $\epsilon_0 = 0.4$  and the two different values of  $\phi_0 = 20$  (—) and  $\phi_0 = 40$  (---). (c) and (d) are two zooms on the water wave profile at the beginning of the flood phase, on the time scale of the tidal bore (around 20 minutes).

350 We can schematically summarize that the nonlinear parameter  $\epsilon_0$  mainly  
 351 relates to the distortion mechanism which leads to bore formation, while the  
 352 friction parameter  $\phi_0$  mainly relates the dissipation of the tidal wave, unfavor-  
 353 able to bore generation.

354 The separation between estuaries displaying or not a tidal bore, which emerges  
 355 from our numerical results, is in qualitative agreement with real-estuary obser-  
 356 vations analyzed in [13]. A list of tidal and geometrical properties for 21 conver-

gent alluvial estuaries (8 of them displaying a bore and 13 not) has been used in [13] to produce figure 5(right), reported here for sake of clarity. In the picture, a grey dashed line dividing tidal-bore and no-tidal-bore estuaries has been traced by hand, inspired by the trend of the critical curve emerged in figure 5(left). We can notice that the numerical critical curve  $\epsilon_c(\phi_0)$  and the data-based one only slightly differ considering the several simplifying modeling assumptions. In particular, in real alluvial estuaries the general decrease of depth landward is favorable to bore inception and this can cause the critical data-based curve (figure 2b of [13]) to be located slightly below with respect to the modeling one on figure 5(left).

From the observations made in the previous paragraph, a close relation between bore formation and tidal damping emerges. A standard parameter used in the literature [29, 30] to measure the amplification/damping of the tidal wave during its propagation along the estuary is the rate of change of the tidal range  $\text{Tr}$ , defined in accordance to reference [30] as:

$$\delta_{\text{Tr}} = \frac{1}{\text{Tr}} \frac{d\text{Tr}}{dx} . \quad (8)$$

In the present work we integrate equation (8), from the estuary mouth to  $x = L_c$  (end of our region of study), and we compute, for each simulation performed, the quantity:

$$\Delta_{\text{Tr}} = \frac{\text{Tr}(L_c) - \text{Tr}(0)}{\text{Tr}(0)} \quad (9)$$

using the tidal range at the estuary mouth  $\text{Tr}(0)$  as a scaling factor.

Figure 10(left) shows, on the same plane  $(\phi_0, \epsilon_0)$  of figure 5, the contour lines of the computed quantity  $\Delta_{\text{Tr}}$ , obtained by linearly interpolating the values of each simulation. The black dashed line is the contour line for  $\Delta_{\text{Tr}} = 0$ , namely the marginal curve for tidal range amplification, where all the estuaries with unamplified and undamped wave lie. It represents an ideal situation for which the amplification effect associated to funneling is exactly balanced by friction. The marginal curve divides the plane into two regions; estuaries located below are characterized by a tidal range amplification while estuaries located above are affected by damping. A simple analytical model, derived by Savenije et al.

[29] in the linearized case, allows to explicitly compute the damping factor of an estuary as a function of the three external independent parameters of the estuarine dynamics. According to this model, Toffolon et al. [2] found that synchronous estuaries ( $\delta_{Tr} = 0$ ) lie on a curve, which can be recast in terms of our parameters as:

$$\epsilon_0 = \frac{\delta_0(\delta_0^2 + 1)}{\phi_0} . \quad (10)$$

Using the value of  $\delta_0 = 2$  in (10), we get the red curve plotted in figure 10(left). A good agreement is observed between the marginal curve obtained through numerical simulations (black dashed line) and the one of the analytical model of Savenije (red line), especially for low values of  $\epsilon_0$ , namely the linear regime.

From figure 10(left) it is interesting to note that  $\epsilon_0$  plays a fundamental role also in the damping/amplification process. If we fix the physical and geometrical properties  $L_b$ ,  $C_{f0}$ ,  $D_0$ , this corresponds to a specific estuary configuration with  $\phi_0$  constant in addition to  $\delta_0 = 2$ . In this context, the variations of  $\epsilon_0$  can be considered as associated with neap-spring tide cycles. We can, thus, conclude that strong tides lead to weaker tidal wave amplification values and, for particularly strong tides, the wave most likely will be damped. This result is in qualitative agreement with the physical observation made in [3] for the Garonne river. Their measurements showed that, for such estuary always characterized by tidal wave amplification, minor amplifications were related to higher values of  $\epsilon_0$ , observed during spring tides.

The superimposition of the computed marginal curve on the contour lines of the quantity  $A_{max}$ , in figure 10(right), shows that a large part of the red region in the figure, lies in the part of the plane  $(\phi_0, \epsilon_0)$  characterized by damping of the tidal range. This means that, despite a reduction of the local nonlinearity of the wave, this remains high enough to develop distorted profiles and to produce bores. We can conclude that, contrary to what is generally accepted, tidal range amplification along the estuary is not a necessary condition for tidal bore formation.

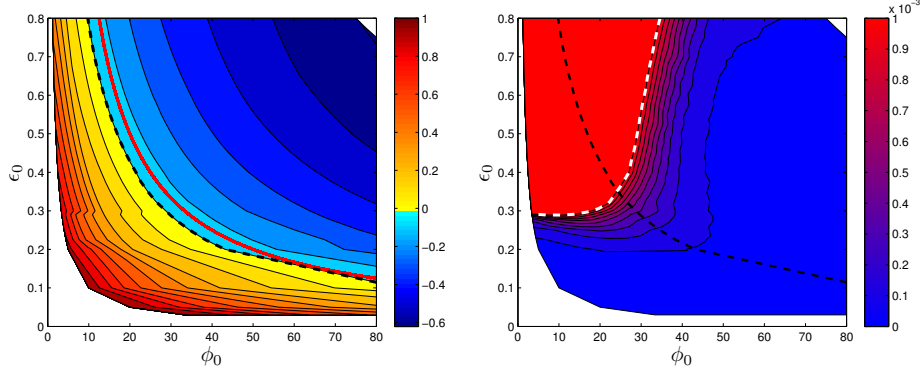


Figure 10: (Left): contour lines of the quantity  $\Delta_{Tr}$  on the plane  $(\phi_0, \epsilon_0)$ ; the black dashed line (---) represents the marginal curve resulting from the computations, while the continuous red one (—) is the analytical marginal curve of the Savenije model [30]. Hot colors cover the region of amplification of the tidal wave during propagation; cold colors represent damping. (Right): the computed marginal curve (---) is superimposed on the  $A_{max}$  contour lines.

## 7. Influence of river discharge

In the previous sections, we have analyzed tidal wave transformation and tidal bore occurrence in a simplified context in which freshwater river discharge was neglected. However, it is well known that tide in estuaries may be significantly affected by the rate of discharge [16, 5]. The effects of river discharge become much more important moving landward from the mouth of the estuary and can influence for bore formation. The present section will provide a qualitative estimation on the effects of discharge, leaving a full quantitative study for future works.

In the experimental campaigns on the Garonne river, Bonneton et al. [3] observed that small river discharges  $Q$  were favorable to tidal range amplification and bore occurrence, while significant freshwater discharges offsets the amplification mechanism related to estuary convergence. Horrevoets et al. [16] described, with an analytical model, that the influence of river discharge on tidal damping takes place mainly through friction. Generally speaking, the tidally averaged free surface elevation along the estuary does not coincide with the mean sea level, due to the nonlinear frictional effect on the averaged water level



430  $\bar{D}(x)$ . In practice there is a monotone increase of  $\bar{D}(x)$  landward, proportional  
 431 to the river discharge  $Q$ , see [31, 5]. Moreover, a damping of the local tidal  
 432 range  $Tr(x)$  has been pointed out by Bonneton et al. [3], hence the local non-  
 433 linear parameter  $\epsilon(x) = Tr(x)/(2\bar{D}(x))$  is a decreasing function of  $Q$ . Due to  
 434 this damping effect, tidal bores are rarely observed for strong freshwater river  
 435 discharges.

436 For a fixed estuary (fixed  $\delta_0$  and  $\phi_0$ ), the dimensionless parameters governing  
 437 the flow dynamics are the amplitude of tidal forcing  $\epsilon_0$  and the dimensionless  
 438 intensity of river discharge  $Q_0$ . The goal of the present section is to explore the  
 439 space of parameters  $(\epsilon_0, Q_0)$  in order to find, for an estuary characterized by  
 440  $\delta_0 = 2$  and  $\phi_0 = 18$  (values closed to the ones of the Garonne river), a critical  
 441 curve  $\epsilon_c(Q_0)$  for tidal bore development, following the criterion  $A_{max} \geq 10^{-3}$ .  
 442 In order to perform our investigation, we have to express  $Q_0$  as a function of the  
 443 external variables of the problem. The scaling analysis proposed in [13] leads  
 444 to the definition of a parameter  $K = \frac{U_0 D_0}{A_0 \omega_0 L_b}$  (see also appendix A for details),  
 445 that has been measured to be  $K \sim 1$  in convergent alluvial estuaries [3]. A  
 446 characteristic velocity scale  $U_0 = \epsilon_0 \omega_0 L_b$  can, thus, be derived. Being  $B_0$  the  
 447 width at the estuary mouth (figure 1), we finally define:

$$Q_0 = \frac{Q}{A_0 B_0 L_b \omega_0} . \quad (11)$$

448 The river discharge is introduced by the boundary condition already described in  
 449 section 3, through the incoming Riemann invariant from far on the right. Figures  
 450 11 shows, for the particular estuary considered, the effect of an increasing river  
 451 flow in terms of normalized free surface elevation and velocity signals at the  
 452 position  $x = L_c$  along the channel. The simulations were performed using  
 453  $\epsilon_0 = 0.32$  and a range of values  $Q_0 \in [0, 4.16 \times 10^{-3}]$ , obtained by scaling  
 454 the typical values of the Garonne river through relation (11). In particular,  
 455 the values  $Q_0 = 4.16 \times 10^{-4}$  and  $Q_0 = 4.16 \times 10^{-3}$  correspond to the low and  
 456 high characteristic fresh water discharges measured in the Garonne (respectively  
 457  $Q = 150 [m^3/s]$  and  $Q = 1500 [m^3/s]$ ). In figure 11 (a), we observe that the  
 458 dimensionless mean water depth  $\bar{D}(L_c)/D_0$  increases with  $Q_0$ , from 1.033 with

459  $Q_0 = 0$  to 1.219 with  $Q_0 = 4.16 \times 10^{-3}$ . We can also measure the damping  
 460 effect of freshwater river discharge on the tidal range; the dimensionless value  
 461  $\Delta_{Tr}(L_c)$  (as defined in (8)) goes from 0.156 with  $Q_0 = 0$  (amplified case) to  
 462 -0.041 with  $Q_0 = 4.16 \times 10^{-3}$  (damped case). An important vertical shift of  
 463 the velocity curve, in agreement with field observations [16], can be observed in  
 464 figure 11 (b), moving towards the condition of unidirectional flow. Moreover,  
 465 it can be noticed that this result confirms the theoretical predictions set by  
 466 Horrevoets et al. [16] concerning the evolution of the phase lag between high  
 467 water and high water slack (and at the same time between low water and low  
 468 water slack). Note that all the effects described are small below the value of  
 469  $Q_0 = 4.16 \times 10^{-4}$  and the discharge does not affect the topology of the tidal  
 470 wave for the river flow values typically observed in the Garonne river at the end  
 471 of the summer season. Thus, the parametric analysis performed in section 2, by  
 472 neglecting  $Q_0$ , can be considered valid in this range of small  $Q_0$ .

473 In figure 11 (c), the zoom on the time scale of the bore displays the ten-  
 474 dency of the free surface profile to become much smoother as the value of river  
 475 discharge increases. In order to explore better this point, we have performed 47  
 476 simulations for different combinations of tidal amplitude and river discharge ( $\epsilon_0$ ,  
 477  $Q_0$ ). The values of  $A_{max}$ , obtained for all the simulations, have been plotted in  
 478 figure 12. Note that we chose to represent in the  $y$ -axis of the figure the product  
 479  $Q_0\epsilon_0$ , rather than simply  $Q_0$ , in order to remove the dependence of  $Q_0$  from  $A_0$   
 480 (11). The figure shows that, in the presence of weak river discharges, estuarine  
 481 dynamics is not influenced by  $Q_0$  and, consequently, the effects of discharge can  
 482 be considered negligible in the bore formation process. A qualitative critical  
 483 curve  $\epsilon_c(Q_0)$  has been traced by hand (grey dashed line in figure 12) according  
 484 to the few computations performed and following the criterion  $A_{max} \geq 10^{-3}$ .  
 485 This trend is in qualitative agreement with experimental data for the Garonne  
 486 river, presented in [13].

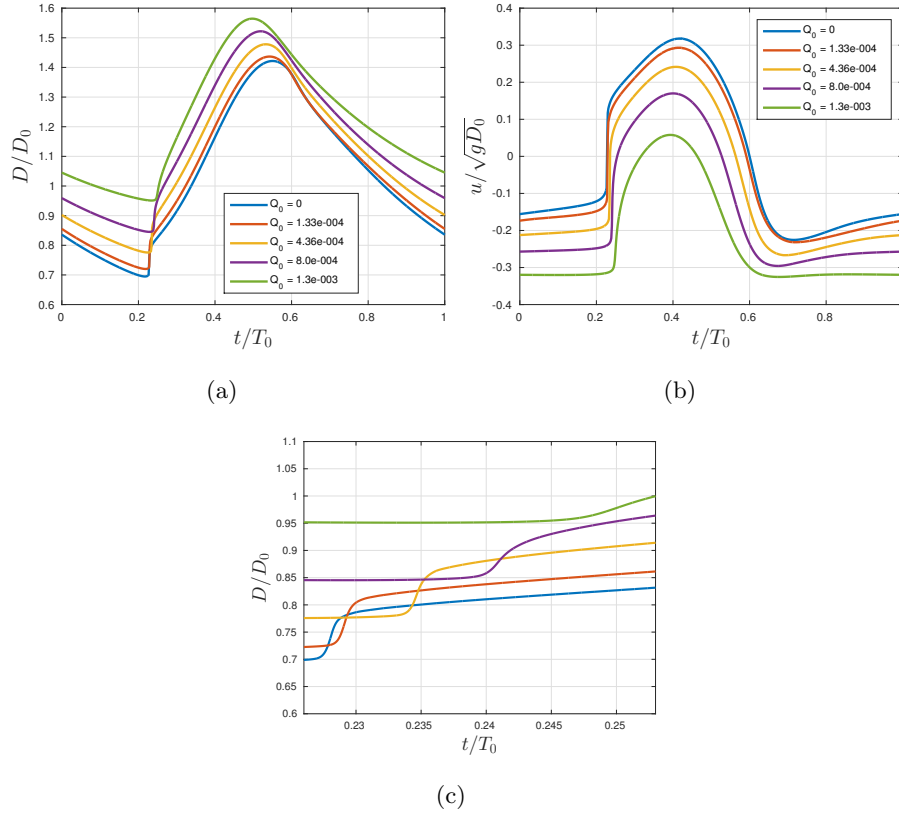


Figure 11: Time variation of the water depth (a) and velocity (b) signals measured at  $x = 3L_b$  for an ideal estuary characterized by  $\epsilon_0 = 0.32$ ,  $\phi_0 = 18$  and increasing values of freshwater discharge from  $Q_0 = 0$  to  $Q_0 = 4.16 \times 10^{-3}$  (in particular the values have been chosen considering the typical range of values displayed by the Garonne river and measured by Bonneton et al. [3]). Figure (c) represents a zoom on (a) in the time scale of the bore (around 20 min).

## 8. Conclusions

The two-dimensional Serre-Green-Naghdi system of equations has been used in order to simulate the inception of tidal bores in convergent alluvial estuaries of idealized geometry. Exploiting the dispersive properties of the model, we were, thus, the first to reproduce the formation of an undular tidal bore inside a channel, to the authors' knowledge. The results obtained have illustrated that dispersion does not impact the large scale of propagation of the tide, as it comes

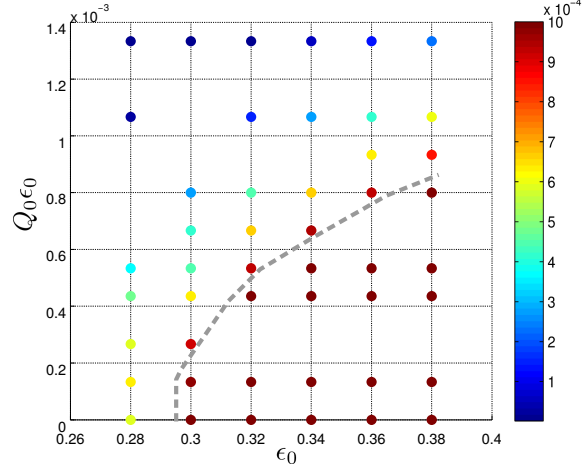


Figure 12: Circles represent computations performed for a fixed estuary ( $\delta_0 = 2$  and  $\phi_0 = 18$ ) varying the values of the tidal forcing amplitude  $A_0$  and river discharge  $Q_0$ ; colors represent the intensity of  $A_{max}$  for each computation; with a grey dashed line we have represented by hand the  $\epsilon_c(Q_0)$  curve, namely the limit for tidal bore appearance following the criterion  $A_{max} \geq 10^{-3}$

also from the scaling analysis of the equations proposed in appendix A, and affects the solution only in the proximity of bore formation. For these reasons, an accurate description of the free surface profile at large scale can be obtained via the simpler NLSW system. Under the hypotheses of constant bathymetry and negligible river discharge, we have thus employed the NLSW equations to numerically investigate the bore occurrence in convergent alluvial estuaries of idealized geometry.

The scaling of the equations shows that estuarine dynamics is fully controlled by three dimensionless parameters entirely dependent on the estuary geometrical properties and tidal forcing: the nonlinearity  $\epsilon_0$ , the convergence parameter  $\delta_0$  and the friction parameter  $\phi_0$ , defined by (3). Taking a constant value of  $\delta_0 = 2$ , we have numerically explored the space  $(\phi_0, \epsilon_0)$ . By means of a bore detection criterion, we have traced the critical line  $\epsilon_c(\phi_0)$  dividing estuaries into displaying or not a tidal bore. This curve is in good agreement with real estuaries data, despite the several assumptions made, and provides the necessary

509 conditions for tidal bore formation. These conditions are a result of a complex  
 510 equilibrium between nonlinear distortion and tidal range damping/amplification  
 511 processes both driven by the dissipation parameter  $D_i^* \propto \epsilon_0 \phi_0$  multiplying the  
 512 friction, the dominant nonlinear term for this class of estuaries. The particular  
 513 shape of  $\epsilon_c(\phi_0)$  shows that, for low values of  $\phi_0$  (indicatively  $1 < \phi_0 \leq \sim 20$ ),  
 514 bore formation depends almost exclusively on the nonlinear parameter  $\epsilon_0$ , while  
 515 being, instead, mainly disciplined by the dissipation, related to the value of  $\phi_0$ ,  
 516 for  $\phi_0 \geq \sim 20$ .

The critical curve intersects the computed marginal curve of amplification, di-

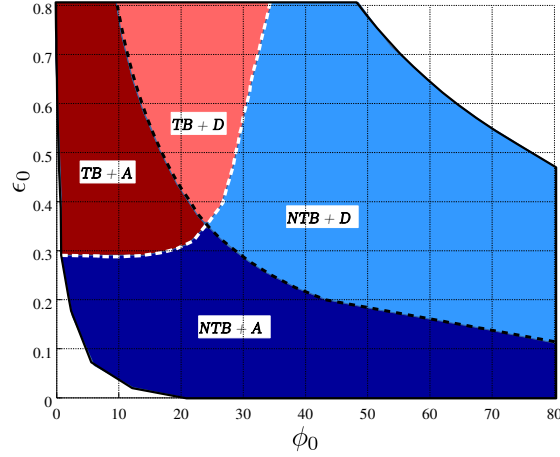


Figure 13: The computed critical line for bore formation (white dashed line) and the computed  
 marginal curve of amplification (black dashed line) divide the  $(\phi_0, \epsilon_0)$  plane into four main  
 areas. In the picture, TB stands for tidal-bore estuaries, while NTB stands for no tidal-bore  
 estuaries; A indicates amplification of the tidal wave along the estuary, D damping.

517  
 518 viding the plane into four main areas. Estuaries will thus experience tidal range  
 519 amplification or damping, tidal bore formation or not, depending on which re-  
 520 gion in the plane they belong to (see figure 13). The existence of a sector  
 521 characterized by tidal bore generation and tidal range damping shows that tidal  
 522 range amplification along the estuary is not a necessary condition for tidal bore  
 523 occurrence, as it is instead commonly assumed in the literature [32]. This result  
 524 is also in agreement with field observations analyzed in [13].

Finally, we have studied the effect of river discharge for estuaries characterized by  $\delta_0 = 2$  and  $\phi_0 = 18$  (which are close to the values displayed by the Gironde/Garonne estuary). We have shown that for low  $Q_0$  (i.e.  $Q_0 < 4.16 \times 10^{-3}$ ), corresponding to the dry season, the effect of river discharge on tidal wave dynamics and bore formation can be neglected.

The above findings are based on several simplifying assumptions, that have allowed a clear understanding of the bore inception mechanism. Other effects influencing the spatial location of bore development have been so far ignored and may arise due to the variable bathymetry, river banks and meanders. The consideration of such effects will require significant attention, but will provide a more thorough comprehension when approaching the analysis of real natural estuaries.

## Appendix A: Scaling of the SGN equations

In this section we report some details concerning the application of the scaling proposed in [13] to the SGN system of equations (5). The following scaling of the physical variables is thus applied:

$$\begin{aligned} x &= L_0 x' ; & y &= L_0 y' ; & t &= \frac{t'}{\omega_0} ; & D &= D_0 D' ; \\ \zeta &= A_0 \zeta' ; & \mathbf{u} &= U_0 \mathbf{u}' . \end{aligned} \quad (12)$$

The governing equations (5a) and (5b), written in dimensionless form, will thus read (dropping the primes for sake of clarity):

$$\frac{\partial \zeta}{\partial t} + \frac{K}{\mathcal{L}} (\epsilon_0 \mathbf{u} \cdot \nabla \zeta + D \nabla \cdot \mathbf{u}) = 0 , \quad (13a)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{K}{\mathcal{L}} \epsilon_0 (\mathbf{u} \cdot \nabla) \mathbf{u} + \frac{1}{K \mathcal{L}} \delta_0^2 \nabla \zeta + K \frac{\epsilon_0 \phi_0}{\delta_0} \frac{\mathbf{u} |\mathbf{u}|}{D} = \psi . \quad (13b)$$

where  $\mathcal{L} = \frac{L_0}{L_b}$  and  $K = \frac{U_0 D_0}{A_0 \omega_0 L_b}$  are functions of the length and velocity scales of the phenomenon [13] and the scale factor  $\frac{2\pi U_0}{T_0}$  has been used for the non-hydrostatic term  $\psi$ .

547 Considering a flat bathymetry, the linear operators  $T(\mathbf{w})$  and  $Q(\mathbf{w})$ , applied to  
 548 a generic vector  $\mathbf{w}$ , are:

$$T(\mathbf{w}) = -\frac{1}{3D} \nabla (D^3 \nabla \cdot \mathbf{w}) , \quad (14a)$$

$$Q(\mathbf{w}) = \frac{2}{3D} \nabla \left[ D^3 (\nabla \mathbf{w}_1 \cdot \nabla^\perp \mathbf{w}_2 + (\nabla \cdot \mathbf{w})^2) \right] ; \quad (14b)$$

549 in which  $\mathbf{w}_1$  and  $\mathbf{w}_2$  indicates respectively the first and second component of the  
 550 vector  $\mathbf{w}$  and  $\nabla^\perp$  stands for the normal gradient operator. Applying the scaling  
 551 (12) to the two expressions above and introducing the dimensionless dispersion  
 552 parameter  $\mu = \frac{D_0}{L_0}$ , we can state that  $T(\mathbf{w}) = \mu^2 T'(\mathbf{w})$  and  $Q(\mathbf{w}) = \frac{\mu^2}{L_0} Q'(\mathbf{w})$ .  
 553 Equation (5c), written in terms of dimensionless variables, then becomes (still  
 554 primes are dropped for clarity):

$$(\mathbf{I} + \mu^2 T) \boldsymbol{\psi} = \mu^2 T \left( \frac{1}{K\mathcal{L}} \delta_0^2 \nabla \zeta - K \frac{\epsilon_0 \phi_0}{\delta_0} \frac{\mathbf{u}|\mathbf{u}|}{D} \right) - \mu^2 Q \left( \frac{K}{\mathcal{L}} \epsilon_0 \mathbf{u} \right) . \quad (15)$$

555 Equation (15) can be rearranged as follows:

$$\boldsymbol{\psi} = \mu^2 (\mathbf{I} + \mu^2 T)^{-1} \left[ T \left( \frac{1}{K\mathcal{L}} \delta_0^2 \nabla \zeta - K \frac{\epsilon_0 \phi_0}{\delta_0} \frac{\mathbf{u}|\mathbf{u}|}{D} \right) - Q \left( \frac{K}{\mathcal{L}} \epsilon_0 \mathbf{u} \right) \right] , \quad (16)$$

556 showing that  $\boldsymbol{\psi} = \mu^2 \boldsymbol{\Psi}$ .

557 The final form of scaled SGN system will thus read:

$$\frac{\partial \zeta}{\partial t} + \frac{K}{\mathcal{L}} (\epsilon_0 \mathbf{u} \nabla \zeta + D \nabla \cdot \mathbf{u}) = 0 , \quad (17a)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{K}{\mathcal{L}} \epsilon_0 (\mathbf{u} \cdot \nabla) \mathbf{u} + \frac{1}{K\mathcal{L}} \delta_0^2 \nabla \zeta + K \frac{\epsilon_0 \phi_0}{\delta_0} \frac{\mathbf{u}|\mathbf{u}|}{D} = \mu^2 \boldsymbol{\Psi} , \quad (17b)$$

$$\boldsymbol{\Psi} = (\mathbf{I} + \mu^2 T)^{-1} \left[ T \left( \frac{1}{K\mathcal{L}} \delta_0^2 \nabla \zeta - K \frac{\epsilon_0 \phi_0}{\delta_0} \frac{\mathbf{u}|\mathbf{u}|}{D} \right) - Q \left( \frac{K}{\mathcal{L}} \epsilon_0 \mathbf{u} \right) \right] . \quad (17c)$$

## 558 **Appendix B: Landward boundary conditions**

559 Imposing correctly the landward boundary condition (b.c.) is not a trivial  
 560 operation. Up to the authors knowledge, there is not an efficient and systematic  
 561 method to impose an outflow conditions in the presence of friction and conver-  
 562 gence. In this appendix a sensitivity analysis has been performed in order to  
 563 quantify the influence of the inexact b.c. that has been implemented for this

564 study (*cf.* section 3.3). Two tests have been conducted: one for a low and  
 565 the other for a high value of the nonlinear parameter  $\epsilon_0$ , respectively  $\epsilon_0 = 0.1$   
 566 and  $\epsilon_0 = 0.7$ . We compare two solutions: one computed by setting the outflow  
 567 b.c. at the location  $x = 5L_b$ , the other, considered as a reference, derived by  
 568 imposing a wall b.c. at the further distance  $x = 8L_b$ , where the tidal wave is as-  
 569 sumed to be completely dissipated. Figure 14 shows the time evolution profiles  
 570 of the non-dimensional free surface and velocity at different positions along the  
 571 channel. In order to measure the deviation from the reference solution we use  
 572 the  $L_2$ -norm  $\frac{\|\zeta - \zeta_{ref}\|_{L_2}}{\|\zeta_{ref}\|_{L_2}}$ . At the station  $x = 3L_b$ , the percentage values of the  
 573 deviation are 3.3% for  $\zeta$  and 3.98% for  $u$  in the case with  $\epsilon_0 = 0.1$ , while being  
 574 respectively 5.27% and 7.67% in the case with  $\epsilon_0 = 0.7$ .

575 Toffolon [21] revealed the difficulties of imposing such conditions. He con-  
 576 sidered two limit cases: the reflecting barrier and the transparent condition.  
 577 On the former, a wall boundary condition was imposed at the end of the chan-  
 578 nel, which determined a complete reflection of the wave. The latter condition,  
 579 instead, referred to a situation where the tidal wave exited from the compu-  
 580 tational domain without being deformed or reflected (transparent b.c.). This  
 581 condition was implemented by replacing, on the last cells of the domain, mass  
 582 and momentum conservation laws with a scalar advection equation for each con-  
 583 servative variable, characterized by an advection speed equal to the outgoing  
 584 eigenvalue. Even though, the need of considering a longer estuary in order to  
 585 vanish the influence of the boundary on the solution, remained.

586 Figure 15 shows the comparison with the solution performed by implementing  
 587 the transparent boundary condition used in [21]. The percentage deviations  
 588 from the reference are of the same order of magnitude as those recovered us-  
 589 ing our approach and, most importantly, the two approaches provide identical  
 590 topologies for both the water depth and the velocity. The results of our study  
 591 are thus independent from the choice of the method used.



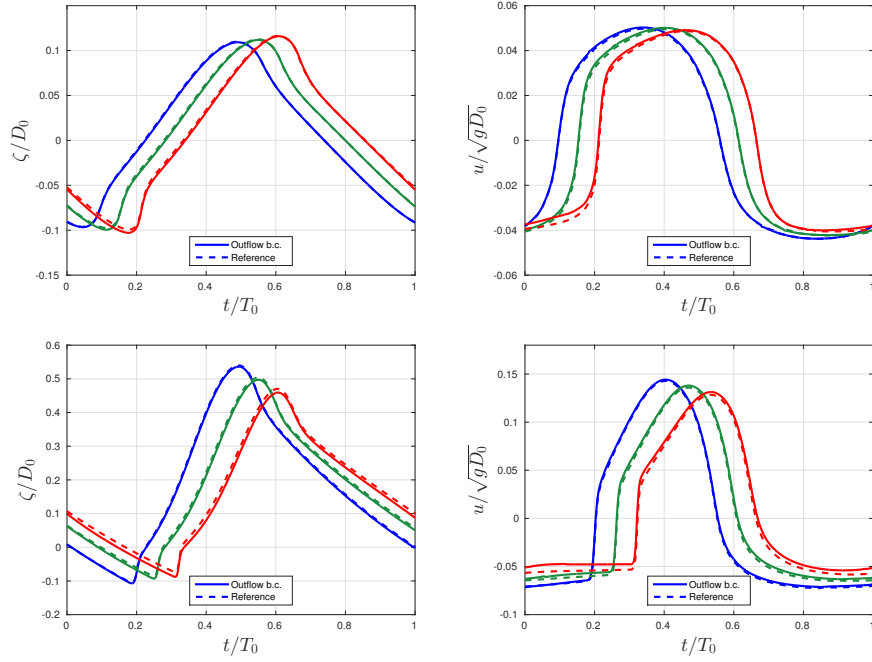


Figure 14: Time evolution profiles of the non-dimensional free surface (left) and velocity (right) measured at  $x = 2L_b$  (—),  $x = 2.5L_b$  (—) and  $x = 3L_b$  (—) for the two test cases performed with  $\epsilon_0 = 0.1$  (top) and  $\epsilon_0 = 0.7$  (bottom). In the two computations  $\phi_0$  has been set constant and equals to  $\phi_0 = 35$ . Continuous lines represent the results obtained by using absorbing landward b.c. by means of the homogeneous NLSW invariants at  $x = 5L_b$ ; while dashed lines were obtained by imposing wall b.c. at  $x = 8L_b$ .

## 592 Acknowledgments

593 Experiments presented in this paper were carried out using the PlaFRIM  
 594 experimental platform, developed under the INRIA PlaFRIM development ac-  
 595 tion with support from LABRI and IMB and other entities: Conseil Régional  
 596 d'Aquitaine, FeDER, Université de Bordeaux and CNRS. This work was par-  
 597 tially funded by the TANDEM contract, reference ANR-11-RSNR-0023-01 of  
 598 the French *Programme Investissements d'Avenir*.

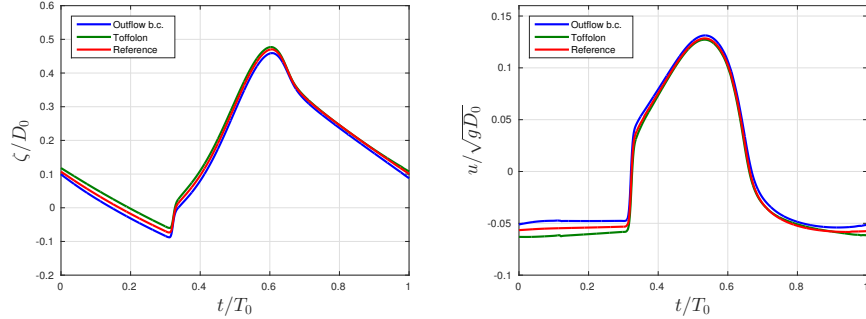


Figure 15: Time evolution profiles of the non-dimensional free surface (left) and velocity (right) measured at  $x = 3L_b$ , obtained by setting the absorbing homogeneous b.c. (—) and the transparent b.c. of [21] (---). The red curve (—) represents the reference solution computed by imposing wall b.c. on a longer domain of  $L = 8L_b$  ( $\epsilon_0 = 0.7$  and  $\phi_0 = 35$ ).

## References

- [1] S. Lanzoni, G. Seminara, On tide propagation in convergent estuaries, J. Geophys. Res. 103 (1998) 30793–30812.
- [2] M. Toffolon, G. Vignoli, M. Tubino, Relevant parameters and finite amplitude effects in estuarine hydrodynamics, J. Geophys. Res. 111 (2006) 1–17.
- [3] P. Bonneton, N. Bonneton, J. P. Parisot, B. Castelle, Tidal bore dynamics in funnel-shaped estuaries, J. Geophys. Res. 120 (2015) DOI: 10.1002/2014JC010267.
- [4] A. Munchow, R. W. Garvin, Nonlinear barotropic tides and bores in estuaries, Tellus 43 (1991) 246–256.
- [5] H. Cai, H. H. G. Savenije, M. Toffolon, Linking the river to the estuary: influence of river discharge on tidal damping, Hydrol. Earth Syst. Sci. 18 (2014) 287–304.
- [6] H. H. G. Savenije, Salinity and tides in alluvial estuaries, 2nd Edition, 2012.

- 614 [7] D. H. Peregrine, Calculations of the development of an undular bore,  
615 J.Fluid Mech. 25 (1966) 321–330.
- 616 [8] P. A. Madsen, H. J. Simonsen, C. H. Pan, Numerical simulation of tidal  
617 bores and hydraulic jumps, Coast. Eng. 52 (2005) 409–433.
- 618 [9] C. H. Pan, H. Lu, 2d numerical simulation of tidal bore on Qiantang river  
619 using KFVS scheme, Vol. 32, 2010.
- 620 [10] A. E. Green, P. M. Naghdi, A derivation of equations for wave propagation  
621 in water of variable depth, J.Fluid Mech. 78 (1976) 237–246.
- 622 [11] M. Tissier, P. Bonneton, F. Marche, F. Chazel, D. Lannes, Nearshore dy-  
623 namics of tsunami-like undular bores using a fully nonlinear Boussinesq  
624 model, J. Coast. Res. 64 (2011) 603–607.
- 625 [12] G. A. El, R. H. Grimshaw, N. F. Smyth, Unsteady undular bores in fully  
626 nonlinear shallow water theory, Phys. Fluids 18(2).
- 627 [13] P. Bonneton, A. G. Filippini, L. Arpaia, N. Bonneton, M. Ricchiuto, Con-  
628 ditions for tidal bore formation in convergent alluvial estuaries, Estuarine,  
629 Coastal and Shelf Science, doi:10.1016/j.ecss.2016.01.019.
- 630 [14] G. Davies, C. D. Woodroffe, Tidal estuary width convergence: Theory  
631 and form in North Australian estuaries, Earth Surf. Process. Landforms 35  
632 (2010) 737749.doi: 10.1002/esp.1864.
- 633 [15] C. T. Friedrichs, D. G. Aubrey, Tidal propagation in strongly  
634 convergent channels, J. Geophys. Res. 99(C2) (1994) 33213336.doi:  
635 10.1029/93JC03219.
- 636 [16] A. C. Horrevoets, H. H. G. Savenije, J. N. Schuurman, S. Graas, The  
637 influence of river discharge on tidal damping in alluvial estuaries, J. Hydrol.  
638 294 (2004) 213–228.
- 639 [17] H. H. G. Savenije, M. Toffolon, J. Haas, M. Veling, Analytical description  
640 of tidal dynamics in convergent estuaries, J. Geophys. Res. 113 (2008) 1–18.

- 641 [18] P. Winckler, P. L. Liu, Long waves in a straight channel with non-uniform  
642 cross-section, *J.Fluid Mech.* 770 (2015) 156–188.
- 643 [19] P. Bonneton, F. Chazel, D. Lannes, F. Marche, M. Tissier, A splitting  
644 approach for the fully nonlinear and weakly dispersive Green-Naghdi model,  
645 *J.Comp.Phys.* 230(4) (2011) 1479–1498.
- 646 [20] A. G. Filippini, M. Kazolea, M. Ricchiuto, A flexible genuinely non-  
647 linear approach for nonlinear wave propagation, breaking and run-up,  
648 *J.Comp.Phys.* 310 (2016) 381417.
- 649 [21] M. Toffolon, Hydrodynamics and morphodynamics of tidal channels, Ph.D.  
650 thesis, Università degli studi di Trento (2002).
- 651 [22] J. T. Kirby, G. Wei, Q. Chen, FUNWAVE 1.0, Fully Nonlinear Boussinesq  
652 Wave Model Documentation and User’s Manual, Tech. Rep. CACR-98-06,  
653 University of Delaware (September 1998).
- 654 [23] G. Wei, J. T. Kirby, A. Sinha, Generation of waves in Boussinesq models  
655 using a source function method, *Coast.Eng.* 36 (1999) 271–299.
- 656 [24] M. A. Walkley, A numerical method for extended Boussinesq shallow-water  
657 wave equations, Ph.D. thesis, University of Leeds (1999).
- 658 [25] Y. Zhang, A. B. Kennedy, N. Panda, C. Dawson, J. J. Westerink,  
659 Generating-absorbing sponge layers for phase-resolving wave models,  
660 *Coast.Eng.* 84 (2014) 1–9.
- 661 [26] R. Cienfuegos, E. Barthelémy, P. Bonneton, A fourth-order com-  
662 pact finite volume scheme for fully nonlinear and weakly dispersive  
663 Boussinesq-type equations. part II: Boundary conditions and validation,  
664 *Int.J.Numer.Meth.Fluids* 53 (2007) 1423–1455.
- 665 [27] E. Wolanski, D. Williams, E. Hanert, The sediment trapping efficiency of  
666 the macro-tidal Daly Estuary, tropical Australia, *Estuarine, Coastal and*  
667 *Shelf Science* 69(1) (2006) 291–298.

- 668 [28] J. C. Winterwerp, Z. B. Wang, A. van Braeckel, G. van Holland, F. Kusters,  
669 Man-induced regime shifts in small estuaries II: a comparison of rivers,  
670 Ocean Dynamics 63(11-12) (2013) 1293–1306.
- 671 [29] H. H. G. Savenije, E. J. M. Veling, Relation between tidal damp-  
672 ing and wave celerity in estuaries, J. Geophys. Res. 101 (2005)  
673 doi:10.1029/2004JC002278.
- 674 [30] H. H. G. Savenije, A simple analytical expression to describe tidal damping  
675 or amplification, J. Hydrol. 243 (2001) 205–215.
- 676 [31] G. Vignoli, M. Toffolon, M. Tubino, Non-linear frictional residual effects  
677 on tide propagation, Proceedings of XXX IAHR Congress (2003) 291–298.
- 678 [32] H. Chanson, Tidal Bores, Aegir, Aegir, Eagre, Mascaret, Pororoca: Theory  
679 and Observations, World Scientific, Singapore, 2012.