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To cite this version:
Andrey Polyakov, Jean-Michel Coron, Lionel Rosier. On Homogeneous Finite-Time Control for Linear Evolution Equation in Hilbert Space. IEEE Transactions on Automatic Control, 2018, 63 (9), pp.3143 - 3150. 10.1109/TAC.2018.2797838 . hal-01695475

HAL Id: hal-01695475
https://inria.hal.science/hal-01695475
Submitted on 29 Jan 2018
On Homogeneous Finite-Time Control for Linear Evolution Equation in Hilbert Space

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Abstract—Based on the notion of generalized homogeneity, a new algorithm of feedback control design is developed for a plant modeled by a linear evolution equation in a Hilbert space with a possibly unbounded operator. The designed control law steers any solution of the closed-loop system to zero in a finite time. Method of homogeneous extension is presented in order to make the developed control design principles to be applicable for evolution systems with non-homogeneous operators. The design scheme is demonstrated for heat equation with the control input distributed on the segment [0,1].

I. INTRODUCTION

A certain type of symmetry of an operator with respect to the so-called dilation group is known as homogeneity [1], [2], [3], [4], [5], [6], [7]. Homogeneity simplifies qualitative analysis of nonlinear dynamical systems. For ordinary differential equations (ODEs) it allows local properties (e.g. local stability) to be extended globally, but asymptotic stability and negative homogeneity degree always imply finite-time stability [8], [9]. In order to ensure finite-time convergence control and/or observer design algorithm can be based on homogeneity [10], [11], [12], [13]. Homogeneous finite-time stabilization of linear plants is studied in [10], [14], [13], [15] but homogeneous finite-time observers are designed in [16], [17], [18]. Homogeneity allows robustness analysis to be done easily for essentially nonlinear systems [19], [20], [21].

Recently [22], a generalized group of dilations was introduced for Banach spaces allowing us to extend all important properties (known before only for ODEs) to homogeneous evolution equations in Banach and Hilbert spaces. The generalized homogeneity can be established for many well-known partial differential equations (PDEs) like heat, wave, Korteweg-de Vries, Saint-Venant, Burgers, Navier-Stocks and fast diffusion equations [22].

The finite-time control of PDEs is still the topic of intensive research [23], [24], [25], [26], [27]. It is related with controllability analysis of evolution systems [28] as well as with sliding mode control method [29], [30].

This technical note deals with homogeneous finite-time control design for plants modeled by linear evolution equations in Hilbert spaces. The design scheme follows the idea of the implicit Lyapunov function method [31], [32], which was recently linked with homogeneity theory [33]. Using the monotone dilations, we introduce the so-called canonical homogeneous norm in a Banach space, which is utilized next for both control design and finite-time stability analysis. The preliminary version of this paper was presented at Conference on Decision and Control [34]. The key differences are as follows: the proof of the main theorem is presented; the results are extended to evolution equations with non-homogeneous operators which admit homogeneous (possibly set-valued) extensions; piecewise linear realization of the homogeneous feedback is studied; the new example of finite-time stabilization of heat system with Dirichlet boundary conditions on the segment [0, 1] is presented.

Notation. Set of reals (integers) is $\mathbb{R}$ ($\mathbb{Z}$ resp.); $\mathbb{R}_+=[0,+\infty)$; $H^n(\Omega,\mathbb{R})=W^{n,2}(\Omega, \mathbb{R})$ is Sobolev space of functions $\Omega \to \mathbb{R}$, $\Omega \subseteq \mathbb{R}$ is an open set; $C_0^\infty(\Omega,\mathbb{R})$ is the set of infinitely smooth functions having compact support in $\Omega$; $H^n_0(\Omega,\mathbb{R})$ is the closure of $C_0^\infty(\Omega,\mathbb{R})$ with respect to the norm in $H^n$; $X$ denotes the closure of the set $X$ of a metric space.

II. PROBLEM STATEMENT AND PRELIMINARIES

Let us consider the following control system

\begin{align}
\dot{u}(t) &= Au(t) + B\xi(u(t)), \quad t > 0 \quad (1) \\
\quad u(0) &= u_0 \in \mathcal{D}(A), \quad (2)
\end{align}

where $A : \mathcal{D}(A) \subset H \to H$ is a (possibly unbounded) closed linear operator with the domain $\mathcal{D}(A)$ dense in $H$, $B : X \to H$ is a linear bounded operator, $X$ is a real Banach space, $H$ is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, $u(t)$ is the system state, $\xi : H \to X$ is a (locally or globally) bounded feedback control.

The control aim is to steer any solution of the closed-loop system to zero in a finite time. The control design procedure is assumed to be based on the generalized homogeneity [22].

A. Homogeneous Dilations in Banach Spaces

Let $\mathcal{L}(B)$ be the space of linear bounded operators $B \to B$, where $B$ is a real Banach space with a norm $\| \cdot \|$. A map $d : \mathbb{R} \to \mathcal{L}(B)$ is called dilation in $B$ if it satisfies

Definition 1 ([22]): A map $d : \mathbb{R} \to \mathcal{L}(B)$ is called dilation in $B$ if it satisfies

- Group property: $d(0) = I \in \mathcal{L}(B)$ and $d(t+s) = d(t)d(s)$ for $t, s \in \mathbb{R}$;
- Strong continuity property: the map $d(\cdot) : \mathbb{R} \to \mathcal{L}(B)$ is continuous for any $u \in B$;
- Limit property: $\lim_{s \to \infty} \|d(s)u\| = 0$ and $\lim_{s \to \infty} \|d(s)u\| = \infty$ uniformly on $u \in S$, where $S$ is the unit sphere in $B$.

We refer reader to [22], [34] for more details about homogeneous dilations and homogeneous evolution systems.
The dilation $d$ is strongly continuous group of linear bounded invertible operators defined by an infinitesimal generator $G_a : D(G_a) \subset B \rightarrow B$ given by $G_a u = \lim_{s \rightarrow 0} \frac{d(s)u - u}{s}$ with the domain $D(G_a) = \{u \in B : \exists \lim_{s \rightarrow 0} \frac{d(s)u - u}{s}\}$ being the linear subspace dense in $B$. The generator $G_a$ is a closed linear operator satisfying the property [35, Ch. I, Theorem 2.4]:

for any $u \in D(G_a)$ and any $s \in \mathbb{R}$ one holds $d(s)u \in D(G_a)$ and the function $d(s)u : \mathbb{R} \rightarrow D(G_a)$ is continuously differentiable, $\frac{d(s)u}{s} = G_a d(s)u = d(s)G_a u$.

For example, if $B = H^0 \left( \mathbb{R}, \mathbb{R} \right)$ then the dilation can be defined as $\left( d(s)u \right)(x) = e^{su}(e^{-s}x), s \in \mathbb{R}, u \in B, x \in \mathbb{R}$ has the generator $(G_a u)(x) = -xu'(x) + u(x)$ with the domain $D(G_a) = \{u \in H^0 \left( \mathbb{R}, \mathbb{R} \right) : xu' + u \in H^0 \left( \mathbb{R}, \mathbb{R} \right)\}$.

Definition 2 ([22]): A nonempty set $D \subset B$ is said to be d-homogeneous if $D$ is invariant with respect to dilation $d$.

The set $D$ can also be called d-homogeneous cone, since a point $z \in D$ together with the homogeneous curve $\{ d(s)z : s \in \mathbb{R}\}$. The d-homogeneous set $D$ becomes the conventional positive cone in $B$ if $d(s) = e^{s}I \in \mathcal{L}(B)$.

The homogeneous sphere of the radius $r$ is given by

$$S_d(r) = \{u \in B : d(-ln(r))u = S\}, \quad r > 0.$$

One has $S_d(r) = d(ln(r))S_d(1)$. For $g \in \mathcal{L}(B)$ let us denote

$$\|g\|^2_p = \sup_{u \in S_d} \|g(u)\| \quad \text{and} \quad |g|_p = \inf_{u \in S_d} \|g(u)\|.$$

Definition 3 ([34]): A dilation $d$ is (strictly) monotone on $D \subset B$ if $\|d(s)\|^2_p < 1$ (resp. $\exists \beta > 0 : \|d(s)\|^2_p < e^{\beta s}$), $s < 0$.

If the dilation $d$ is monotone on $D \subset B$ then [see [34]]

1. for any $u \in D \backslash \{0\}$ there exists the unique pair $(s_0, u_0) \in \mathbb{R} \times S_d(1)$ such that $u = d(s_0)u_0$;
2. the function $\|d(\cdot)u\| : \mathbb{R} \rightarrow \mathbb{R}^+$ (with $u \in D \backslash \{0\}$) is continuous and strictly increasing on $\mathbb{R}$;
3. the function $\|d(\cdot)\|^2_p : \mathbb{R} \rightarrow \mathbb{R}^+$ (with $\{d(\cdot)\|^2_p : \mathbb{R} \rightarrow \mathbb{R}^+$) is lower (upper) semicontinuous and strictly increasing.

It is worth stressing that monotonicity of dilation depends on the norm $\| \cdot \|$.

For example, the dilation $d(s) = \left( \begin{array}{cc} e^{s} & 0 \\ 0 & e^{-s} \end{array} \right)$ is monotone on $B = \mathbb{R}^2$ equipped with the conventional Euclidean norm, but it is not monotone on $\mathbb{R}^2$ equipped with the weighted norm $\|u\|_p = \sqrt{u^T Pu}$ with $P = \left( \begin{array}{cc} e^{s} & \frac{s}{2} \\ \frac{s}{2} & e^{-s} \end{array} \right)$.

In the latter case, the curve $\{d(s)u : s \in \mathbb{R}\} \subset \mathbb{R}^2$ may cross the sphere $S$ in two different points.

Remark 1 (On dense set): If the dilation $d$ is monotone on a set $D$ dense in $B$ then it is monotone on $B$, since $\overline{D} = B$ and $\|d(s)\|^2_p = \|d(s)\|^2_B$.

Remark 2 (On uniformly continuous dilation): Let the generator $G_a$ be a bounded linear operator. Then the group $d$ is uniformly continuous, $\|d(s)\|^2_p \leq e^{\|G_a\|^p}p(s)$ for $s > 0$ and $\frac{d}{ds}d(s) = G_a d(s) = d(s)G_a$ for any $s \in \mathbb{R}$ (see [35, Ch.1]).

B. Canonical homogeneous norm

Let us introduce the so-called homogeneous norm in $B$.

Definition 4: A continuous functional $p : B \rightarrow \mathbb{R}^+$ is said to be $d$-homogeneous norm in $B$ if $p(u) \rightarrow 0$ as $u \rightarrow 0$ and $p(d(s)u) = e^{sp}(u) > 0$ for $u \in B \backslash \{0\}, s \in \mathbb{R}$.

The functional $p$ may not satisfy triangle inequality $p(u + v) \leq p(u) + p(v)$, so it is not a semi-norm. However, many authors (see e.g. [13], [36], [9]) call functionals satisfying the above definition by "homogeneous norm". We follow this tradition and consider the canonical homogeneous norm

$$\|u\|_a = e^{sp} u, \quad s \in \mathbb{R} : \|d(-s)u\| = 1.$$

Lemma 1: If the dilation $d$ is monotone on $B$ then the functional $\|u\| : B \rightarrow \mathbb{R}$ defined by (4) is single-valued and positive. It can be prolonged to zero by continuity since $\|u\|_a \rightarrow 0$ as $\|u\| \ightarrow 0$. Moreover, if $d$ is strictly monotone on $B$ then $\|u\|_a$ is Lipschitz continuous on $B \backslash \{0\}$.

In [34] such a homogeneous norm was called canonical since it is induced by the canonical norm $\| \cdot \|$ in $B$ and $\|u\|_a = 1 \Leftrightarrow \|u\| = 1$. Obviously, $\|u\|_a = r$ implies $u \in S_d(r), r \geq 0$ and for any $u \in B$ we have

$$\|d(ln(\|u\|))\|^2_B \leq \|u\| \leq \|d(ln(\|u\|))\|^2_B$$

(5)

The canonical homogeneous norm $\| \cdot \|$ is not a norm in a usual sense, but it defines a sort of topology in $B$. Indeed, the homogeneous sphere is given by (3) and the homogeneous ball can be defined as $B_q(r) = \{u \in B : \|u\|_a < r\}$.

Lemma 2: Let $d$ be a dilation in a Hilbert space $H$ and $\exists \beta > 0$ such that $\langle G_d v, v \rangle \geq \beta \|v\|^2$, $v \in D \subset D(G_a)$, where the set $D$ is dense in $H$. Then the dilation $d$ is strictly monotone on $H$, the homogeneous norm $\| \cdot \|$ is Fréchet differentiable on $D \backslash \{0\}$ and the Fréchet derivative of $\| \cdot \|_a$ at $u \in D \backslash \{0\}$ is given by

$$\langle D \|u\|_a \rangle (u) = \frac{\langle d(-ln(\|u\|))u, d(-ln(\|u\|))u \rangle}{\langle G_d(-ln(\|u\|))u, d(-ln(\|u\|))u \rangle} \|u\|.$$  

(6)

Proofs of all claims are given in Appendix.

C. Homogeneous Operator

Operators to be studied in this paper are introduced by

Definition 5 (Homogeneous Operator): An operator $f : D(f) \subset B \rightarrow B$ is said to be d-homogeneous of degree $\nu$ if $D(f)$ is d-homogeneous set and

$$f(d(s)u) = e^{s\nu}d(s)f(u), \quad s \in \mathbb{R}, u \in D(f),$$

(7)

where $d$ is a dilation in $B$ and $\nu \in \mathbb{R}$.

Any linear operator is always d-homogeneous of zero degree if $d(s) = e^{s}I \in \mathcal{L}(B)$. However, in some cases linear operator may also have negative or positive homogeneity degree dependently of the dilation group. For example, the operator $A = \frac{\partial^2}{\partial x^2} : H^2(\mathbb{R}, \mathbb{R}) \subset H^0(\mathbb{R}, \mathbb{R}) \rightarrow H^0(\mathbb{R}, \mathbb{R})$ is d-homogeneous of degree $\nu \in \mathbb{R}$ provided that the dilation $d$ is defined as follows $d(s)(u)(x) = e^{s\nu}u(\nu^{0.5}x)$, $s \in \mathbb{R}$, $u \in H^0(\mathbb{R}, \mathbb{R})$, $x \in \mathbb{R}$ and $\nu < 4\alpha$.

Remark 3 (On set-valued homogeneous operator): Definition 5 is applicable to a set-valued operator $F : D(F) \subset B \rightarrow B$ provided that the identity (7) is understood in the set-theoretic sense (as equality of sets). It remains meaningful even for unbounded sets (values of $F$ since $d(s)$ is a linear bounded invertible operator on $B$. In this paper we introduce the relaxed set-theoretic condition of homogeneity of the operator $F$ given by the inclusion

$$e^{s\nu}d(s)F(u) \subset F(d(s)u), \quad s \in \mathbb{R}, u \in D(F).$$
Evolution equations with d-homogeneous operators were studied in [22] and [34]. Below we basically use the next property: if a uniformly asymptotically stable evolution system is d-homogeneous with negative degree then it is finite-time stable.

D. Homogeneous Extension

Homogeneous extension allows some homogeneity-based methods to be applied for non-homogeneous systems.

**Definition 6:** (Homogeneous extension) A (possibly) set-valued d-homogeneous operator \( F : \mathcal{D}(F) \subset \mathbb{R} \rightarrow \mathbb{R} \) is said to be d-homogeneous extension of an operator \( f : \mathcal{D}(f) \subset \mathbb{R} \rightarrow \mathbb{R} \) if \( f(u) \subset F(u) \) for any \( u \in \mathcal{D}(f) \subset \mathcal{D}(F) \).

Set-valued homogeneous extensions appear, for example, as a result of Filippov regularization procedure applied to discontinuous vector field [12], [14]. It also allows us to simplify robust stability analysis of uncertain non-linear affine system

\[
\text{Robust stability analysis of uncertain non-linear affine system.}
\]

Using the homogeneous extension for Lyapunov analysis of a well-posed non-homogeneous evolution system, below we utilize the homogeneous extension for Lyapunov function. In this paper, we follow the same idea and use the norm \( || \cdot ||_d \) defined by (4) as a Lyapunov function.

**Theorem 1:** If \( A : \mathcal{D}(A) \subset \mathbb{H} \rightarrow \mathbb{H} \) is a generator of a strongly continuous semigroup on \( \mathbb{H} \), then the feedback control \( \xi : \mathbb{H} \rightarrow \mathbb{H} \) is defined by

\[
\xi(u) = \left\{ \begin{array}{ll}
||u||d^{-\mu} Kd_\alpha &(\ln||u||d)u & \text{if } u \neq 0, \\
0 & \text{if } u = 0
\end{array} \right.
\]

is Lipschitz continuous on \( \mathbb{H} \{0\} \). Fréchet differentiable on \( \mathcal{D}(Gd) \setminus \{0\} \) and \( B\xi(D(A)) \subset \mathcal{D}(A) \). If \( \beta > |\| Gd ||_H \) then \( \xi \) is continuous at 0 \( \in \mathbb{H} \). If \( \exists M \geq 1 \) such that \( ||d(s)||_H \leq Me^{(\mu + \gamma)s} \) for \( s > 0 \) then \( ||\xi(\cdot)|| \leq M||K||z \).

For any \( u_0 \in \mathcal{D}(A) \setminus \{0\} \) the closed-loop system (1), (2), (12) has a unique classical solution \( u : [0, T(u_0)] \rightarrow \mathbb{H} \) such that \( ||u(t)|| \rightarrow 0 \) as \( t \rightarrow T(u_0) \) and

\[
\frac{d}{dt} ||u(t)||_d \leq -\alpha ||u(t)||_d^{-\mu}, \ \ \ t \in [0, T(u_0)),
\]

and the settling time admits the estimate \( T(u_0) \leq \frac{||u_0||_H^\mu}{\alpha^\mu} \).

The finite-time extinction of classical solutions obviously follows from the inequality (13), which can be derived using conditions of Theorem 1. Indeed, Condition D asks for d-homogeneity of the operator \( f := A + B\xi \) with negative degree \( -\mu < 0 \), so one has \( e^{-\mu^\prime}d(s)f(u) = f(d(s)u) =

1A continuous function \( u : [0, T) \rightarrow \mathbb{H} \) is called classical solution to the initial value problem (1), (2) if it is continuously differentiable on \( (0, T) \), \( u(0) = u_0 \), \( u(t) \in \mathcal{D}(A) \) and \( u(t) = Au(t) + B\xi(u(t)) \) for all \( t \in (0, T) \).
\( (A + BK)du(s)u \) for \( s = -\ln\|u\|_d \) and \( u \in \mathcal{D}(A) \). Hence, Condition C) and the formula (6) yield (13). The detailed proof of Theorem 1 is given in Appendix. If Condition D) does not hold (non-homogeneous case) then the finite-time extinction can be analyzed using extension of A.

**Corollary 1:** Theorem 1 remains true if D) is replaced with D*) there exists a set-valued operator \( F : \mathcal{D}(F) \subset \mathbb{H} \rightarrow H \) such that \( Au + B\xi(t) \in F(u) \) for \( u \in \mathcal{D}(A) \subset \mathcal{D}(F) \), \( F \) is \( d \)-homogeneous of degree \(-\mu \lt 0 < \infty \), and 

\[
\alpha(G_dz, z) + \sup_{y \in F(z)} \langle y, z \rangle \lt 0, \quad z \in \mathcal{D}(F) : \|z\| = 1.
\]

The operator \( F \) is the homogeneous extension of the operator \( A + B\xi \) (see Definition 6). Recall (see Section II-D) that we do not need to study an evolution system \( \tilde{u} \in F(u) \). The only fact \( Au + B\xi(t) \in F(u) \) is utilized for analysis of finite-time extinction of the system (1). Notice that adding a linear feedback \( \xi_{new} = K_0u + \xi, K_0 \in L(H, X) \) we may allow the operator \( A_0 = A + BK_0 \) to become homogeneous with a negative degree (see [33] for more details).

**Corollary 2:** If all conditions of Theorem 1 or Corollary 1 hold then for any \( u_0 \in \mathcal{H} \setminus \{0\} \) the closed-loop system (1), (2), (12) has a unique mild solution \( u : [0, T(u_0)) \rightarrow \mathcal{H} \) given by

\[
u(t) = 0 \quad \text{for} \quad t \geq T(u_0)
\]

for the time instants \( t_j \) and the set \( T(u_0) \), where the settling admits the estimate \( T(u_0) \leq \frac{z_{u_0, T(u_0)}}{\mu} \).

This allows all mild solution of the closed-loop system to be prolonged to zero by continuity \( u(t) = 0 \) for \( t \geq T(u_0) \) since the canonical homogeneous norm \( \|\cdot\|_d \) is the Lyapunov function to the closed-loop system implying uniqueness and finite-stability of the zero solution.

Realization of the control law (12) needs calculation of the homogeneous norm \( \|u\|_d \), which is defined implicitly by the functional equation (4). Sometimes the homogeneous norm can be calculated explicitly (see, [15], [34] and the example below). The control law (12) can also be applied in a sampled way by means of replacing \( \|u(t)\|_d \) in (12) with \( \|u(t_j)\|_d \) for \( t \in [t_j, t_{j+1}) \). The value \( \|u(t_j)\|_d \) can be found by means of an appropriate numerical procedure (see, e.g., [15]). The control (12) becomes linear on each interval of time \( [t_j, t_{j+1}) \).

Let us consider the following switched linear feedback \( \xi_{\sigma(t)}(u(t)) := \frac{-\mu}{\sigma(t)} K_d \xi_{\sigma(t)}(-\ln(r_{\sigma(t)}))u(t) \), with state dependent switching function \( \sigma(t) \) such that \( \sigma(0) \in \mathbb{Z} \),

\[
\sigma(t) = S(\sigma(t^-), u(t^-)), \quad S : \mathbb{Z} \times \mathbb{H} \rightarrow \mathbb{Z},
\]

\[
S(\sigma, u) := \left\{ \begin{array}{ll} i-1 & \text{if } \sigma = i \text{ and } \|u\|_d > r_{i-1}, \\ i & \text{if } \sigma = i \text{ and } r_{i-1} \leq \|u\|_d \leq r_i, \\ i+1 & \text{if } \sigma = i \text{ and } \|u\|_d < r_i, \end{array} \right.
\]

where \( r_i \in \mathbb{R}_+ : r_{i+1} < r_i, \ i \in \mathbb{Z} \) and \( \Omega = \bigcup_{i \in \mathbb{Z}} [r_i, r_{i+1}) \). The switching variable \( \sigma \) has its own evolution equation given by (16) so the pair \((u(t), \sigma(t))\) can be treated as an augmented state of the closed-loop switched system. We refer the reader to [37] for more details about switched systems.

**Corollary 3 (On switched linear finite-time feedback law):** If all conditions of Theorem 1 hold and the control is given by (16), (17) then \( r_0 = 1, r_{i+1} = e^{-\mu}r_i, i \in \mathbb{Z} \) and \( q_i > 0 \) such that \( \sum_{i=\infty}^\infty q_i = +\infty \) and \( \lim_{i \rightarrow \infty} e^{-\mu q_i} / q_i < 1 \) then

- For \( u_0 \in \mathcal{D}(A) \setminus \{0\} \) the closed-loop system (1), (2), (16), (17) has a unique strong solution and \( \|u(t)\| \rightarrow 0 \) as \( t \rightarrow T(u_0) \), where \( T(u_0) < +\infty \) and \( \|u(t)\| \leq M \|K\|_\mathbb{Z} \) for \( s < 0 \).

Existence, uniqueness and finite-time extinction of mild solutions to the closed-loop system (1), (2), (16), (17) with \( u_0 \in \mathcal{H} \setminus \{0\} \) can be shown repeating the proof of Corollary 2.

Corollary 3 proposes a scheme for homogeneity-based finite-time control design in the form of switched linear feedback. Its applicability to boundary control case looks like a promising research direction, especially, in the context of the papers [26], [38], where a finite-time linear switched boundary control has been designed for heat system using the backstepping approach [39].

**IV: Example: Distributed Finite-Time Control for Heat System.**

Application of Theorem 1 for ODEs can be found in [33] and for PDEs defined on unbounded domains in [34], where, in particular, homogeneous finite-time feedback control is designed for wave equation. To demonstrate a possible way of application of Corollary 1, in this paper, we design a distributed finite-time control for heat system on \([0, 1]\), namely,

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \phi(x) \xi(x, t), \quad u(t, 0) = 0, \quad u(t, 1) = 0,
\]

where \( u \) is the state, \( \xi \) is the distributed control, \( \phi : [0, 1] \rightarrow \mathbb{R}_+ \) is a continuous function such that

\[
\frac{c}{x^2} < \phi(x) \quad \text{for} \quad x \in [0, 1],
\]

for some \( c > 0 \). The function \( \phi \) represents possibly non-uniform feedback gain on \([0, 1]\). The problem of distributed finite-time control design for the heat system with \( \phi(x) = 1 \) has been studied in [30]. Initially, we design a homogeneous finite-time control for a system virtually extended to \( \mathbb{R}_+ \). Next, we reduce the extended closed-loop system to the original one provided that the initial condition has a support only in \((0, 1)\).

**A. Homogeneous finite-time control design on \( \mathbb{R}_+ \).**

Let \( H = H^1(\mathbb{R}_+, \mathbb{R}) \) and the operator \( A : \mathcal{D}(A) \subset \mathcal{H} \rightarrow \mathcal{H} \) be the connection of the Laplace operator with Dirichlet boundary conditions on \([0, 1]\) and the zero operator on \([1, +\infty)\), i.e., \( (Au)(x) = \frac{\partial^2 u}{\partial x^2} \) if \( x \in (0, 1) \) and \( (Au)(x) = 0 \) if \( x \geq 1 \) for \( u \in \mathcal{D}(A) = \left\{ z \in \mathcal{H} : z(0, 1) \in H^1_0(0, 1, \mathbb{R}) \cap H^2((0, 1), \mathbb{R}) \right\} \), where \( z_{a,b} \) denotes the restriction of \( z \) to the interval \((a, b)\). Operator

\[
2 \text{A continuous function } u : [0, T) \rightarrow \mathcal{H} \text{ which is differentiable almost everywhere on } [0, T) \text{ such that } u \in L^1((0, T), \mathcal{H}) \text{ is called strong solution to the initial value problem (1), (2), (16), (17) if } u(t) \in \mathcal{D}(A) \text{ for } t \in (0, T), \text{ and there exists a } (locally) \text{ piecewise constant function } \\
\sigma : [0, T) \rightarrow \mathbb{Z} \text{ such that } \sigma \text{ satisfies (17) on } (0, T) \text{ and } u(t) = Au(t) + B\xi_{\sigma(t)}(u(t)) \text{ almost everywhere on } (0, T).\]
A generates the strongly continuous semigroup \( \{ \Phi(t) \}_{t \geq 0} \):
\[
(\Phi(t)u)(x) = \left\{ \begin{array}{ll}
2 \sum_{n=1}^{\infty} e^{-n^2 \pi^2} \sin(\pi nx) \frac{1}{\sin(\pi ny)} u(y) dy & \text{if } x \in (0,1), \\
0 & \text{if } x \in [1,\infty).
\end{array} \right.
\]

Let us consider the equation (1) with \( A \) given above and \( B : H \to H \) defined as follows:
\( B(x) = \phi(x)z(x) \) if \( x \in (0,1) \) and \( (Bz)(x) = \phi(1)z(x) \) if \( x \geq 1 \) for \( z \in H \). Let us show that conditions A)–C) of Theorem 1 and the condition D*) of Corollary 1 hold for \( K = -\rho I \in L(H) \) with a sufficiently large scalar \( \rho > 0 \).

A) Let the uniformly continuous dilation \( d_{1} \) be defined as \( d_{1}(s)u(x) = e^{s}u(x) \) if \( x \in (0,1) \) and \( (d_{1}(s)u)(x) = u(x) \) if \( x \geq 1 \). Hence, \( 0 \leq \langle G_{d_{1}}u, u \rangle \in H \) and \( \| G_{d_{1}} \|_{L}^{H} = \mu > 0 \).

B) Let the dilation \( d(s) \) in \( H \) be defined as \( d(s)u(x) = e^{s}u(e^{-0.5\mu s}x) \). It has the generator \( G_{d}u = u - 0.5\mu xu' \) with the domain \( D(G_{d}) = \{ u \in H : u - 0.5\mu xu' \in H \} \). Obviously,
\[
\|d(s)u\| = \int_{\mathbb{R}} e^{2s^{2}}(e^{-0.5\mu s}x)^{2} dx = e^{(1+0.5\mu s)}\|u\|
\]
and \( (1+0.5\mu s)|u|^2 \leq \langle G_{d}u, u \rangle \) if \( u \in D(C) \). Hence, we derive
\[
\|u\| = \|d(s)u\|^{2} + \text{formula (12) gives}
\]
\( \mu < 0 \) if \( 0 < \mu < 4/7 \) and globally bounded if \( \mu = 4/7 \).

C)D*) Consider the homogeneous extension (9), i.e.
\[
F(u) = \bigcup_{s \in \mathbb{R}} \{ A_{s}u + q_{s}(u) \},
\]
where \( A_{s} = e^{s}d_{s}A(\cdot - s) \) and \( q_{s}(u) = e^{s}d_{s}(-s)B(\cdot)(u) \).
It is easy to see that \( (A_{s}u)(x) = u'(x) \) if \( x \in (0, e^{-0.5\mu s}) \) and \( (A_{s}u)(x) = 0 \) if \( x \geq e^{-0.5\mu s} \).

\( q_{s}(u)(x) = -\rho \phi_{1}(x) \rho \phi_{2}(x) \) with homogeneous and switched linear finite-time controls (i.e., feedback law \( B \)). Applying Corollary 3 for \( q_{i} = \ln 2 \) we derive \( \xi(t) := -\lambda(t)u(t) \), where the switched gain \( \lambda(t) = \rho r_{0}^{2}w_{0}^{2} \) with \( r_{0} = 2^{-1} \) and \( w_{0} \) defined by (17). Therefore, for any \( u_{0} \in H_{0} \) (i) \( H_{1} \) and \( H_{2} \) (ii) \( L^{2} \) and \( \|u_{0}\| \) the equation
\[
\|u_{0}\|_{H_{\infty}} \leq \frac{\|u_{0}\|_{L^{\infty}}}{r_{0}^{2}w_{0}^{2}} + \|w_{0}\|_{H_{\infty}} \]

V. Conclusions and Discussions

The universal scheme of finite-time control design for a plant modeled by an evolution equation in a Hilbert space with a linear (possibly unbounded) operator is developed. The design procedure is essentially based on the concept of generalized homogeneity of operators in Banach/Hilbert spaces. The developed framework looks promising for extension of other homogeneity-based results (e.g., robustness (input-to-state stability) analysis [21] or finite/fixed-time observer design [16, 18]) to evaluation equations in Hilbert spaces.

VI. Appendix

A. Proof of Lemma 1

Since the dilation is monotone \( B \) then for any \( u \in B \backslash \{0\} \) there exists a unique pair \( s_{0} \in \mathbb{R} \) and \( u_{0} \in S \) such that \( u = d(s_{0})u_{0} \) (see, Property 1 on the page 2). This means that \( \|d(s_{0})u\|_{H_{\infty}} = 1 \).
Dense in $L^2$-norm \( \|u\| \geq 1 \) one has \( \|\xi(u)\| \leq M \|K\|_{L^2} \) if 
\( \|d(\ln \|u\|)\|_{L^2} \leq M \|u\|_{d}^{\nu + \gamma} \), but for \( \|u\| \leq 1 \) we derive 
\( \|\xi(u)\| \leq \|u\|^{\beta - \mu} \|G_u\|_{L^2} \), since \( \|d_1(s)\|_{L^2} \leq e^{\|G_u\|_{L^2}} \) if \( s \geq 0 \).

II. Existence of Solutions. Let us prove that for any initial condition \( u(0) = u^* \in D(A) \setminus \{0\} \) the system (1), (12) has a classical solution \( u(t) \in D(A) \) defined at least locally.

a) Let function \( \xi : (0, +\infty) \times H \to H \) be defined as 
\( \xi(q, u) = q^{\nu} BK d_1(-\ln(q)) u \).

Since the group \( d_1 \) is uniformly continuous and \( \frac{d}{dt} d_1(s) = G_d d_1(s) \) for all \( s \in \mathbb{R} \) then the function \( \xi \) is continuously Fréchet differentiable on \( (0, +\infty) \times H \).

Let \( T^* \) be an arbitrary positive real number and let \( g : [0, T^*] \to \mathbb{R} \) be an arbitrary continuous function such that 
\( g(t) > 0 \) for all \( t \in [0, T^*] \). To prove that Cauchy problem 
\( u(t) = A(g(t)) u(t) := Au(t) + \xi(g(t), u(t)), \)
\( u(0) = u^* \in D(A) \)
(20)
has a unique classical solution \( u_g(t) \in D(A), t \in [0, T^*] \), we use [35, Chapter 5, Theorem 3.1], which needs to check the following three conditions:

H1) \( \hat{A}(g(t)) \) is a stable family of generators;
H2) \( Y \) is \( \hat{A}(g(t)) \)-admissible, where \( Y = D(A) \) is a Banach space with the norm \( \|u\| = \|u\| + \|Au\| \);
H3) \( \hat{A}(g(t)) \in \mathcal{L}(Y, H) \) for all \( t \in [0, T^*] \) and \( \hat{A}(g(t)) \) is continuous on \( [0, T^*] \) in the norm \( \|\cdot\|_{\mathcal{L}} = \sup_{y \in Y, \|y\| \leq 1} \|y\| \|\cdot\|_{Y} \).

Since the function \( g \) is continuous and positive on \( [0, T^*] \) then \( \xi(g(t), \cdot) \in \mathcal{L}(H, H) \), the conditions H1) and H2) hold and \( \hat{A}(g(t)) \in \mathcal{L}(Y, H) \) for all \( t \in [0, T^*] \). Finally, the inequalities 
\( \|A(g(t) + \delta) - \|A(g(t))\| \leq \|A(g(t))\| \)
\( \|\xi(g(t), \cdot)\|_{\mathcal{L}} = \sup_{y \in Y, \|y\| \leq 1} \|y\| \|\xi(g(t), \cdot)\|_{\mathcal{L}} \)
\( g^\nu(t) \) is a Banach semigroup generated by \( A \) and uniform continuity of the group \( d_1 \) yield that condition H3) also holds. Therefore, the considered Cauchy problem (20) has the unique classical solution \( u_g(t) \) on \( [0, T^*] \).

b) Let the numbers \( q_{\min}, q_{\max}, r_{\min}, r_{\max} \) be defined as follows:
\( q_{\min} = \inf \{ \|u\|_d : \|u\|_d = \|u^*\| \}, \sup \{ \|u\|_d : \|u\|_d = \|u^*\| \} = \|u^*\| < q_{\max} < +\infty, r_{\min} = \sup \{ \|u\|/\|u^*\| : \|u\|_d = q_{\min} \}, r_{\max} = \inf \{ \|u\|/\|u^*\| : \|u\|_d = q_{\max} \}. \) From definition of homogeneous norm we have \( r_{\min} \to 0 \) as \( q_{\min} \to 0 \) and \( r_{\max} \to +\infty \) as \( q_{\max} \to +\infty \). Let us select \( q_{\min} \) and \( q_{\max} \) such that \( r_{\min} < 1 < r_{\max} \). Let \( \{T(t)\}_{t>0} \) be strongly continuous semigroup generated by \( A \) then there exist \( \kappa > 0 \) and \( R > 0 \) such that 
\( \|\Phi(t)u\| \leq Re^{\kappa t} \|u\| \) for any \( u \in H \).

Let us consider the family of Cauchy problems 
\( \hat{u}_n(t) = Au(t) + \xi(q_n(t), u(t)), \quad t \in (0, T_n), \quad u_n(0) = u^* \in D(A) \setminus \{0\}, \quad (21) \)
where \( n \geq 1, q_n \) and \( T_n \) are defined recursively as follows 
\( T_0 > 0, 2^{\|\Phi(t_0)u^*\|_d} \in [r_{\min}, r_{\max}], \forall t \in [0, T_0], \)
\( T_n = T_{n-1} - 1 \) if \( \|u_n(t)\|_d \in [r_{\min}, r_{\max}], \forall t \in [0, T_{n-1}], \)
\( T_n = \inf \{ t \in (0, T_{n-1}) : d_n(t) \in [r_{\min}, r_{\max}] \} \) otherwise, 
\( g_0(t) = \|u^*_d\| \) and \( g_n(t) = \|u_n(t)\|_d \).

Since \( q_n \) is a continuous positive function then the Cauchy problem (21) has unique classical solution \( u_n(t) \in D(A) \) on \([0, T_n]\). So the sequences \( \{q_n\} \) and \( \{u_n\} \) are well defined and 
\( q_{\min} \leq q_n(t) \leq q_{\max}, r_{\min} \|u^*\| \leq \|u_n(t)\| \leq r_{\max} \|u^*\| \).

Fig. 1. Evolution of \( L^2 \)-norm of the heat equation with linear, switched linear and homogeneous control laws plotted in the logarithmic scale.
c) Let us show that $T^* := \inf\{n > 0 : T_n > 0\}$. Initially, note that 
$$
\left\| \xi'(q_{n-1}(s), u_n(s)) \right\| = \left\| u_{n-1}(s) \right\| B \|K\| \left( -\ln \|u_{n-1}(s)\| \right) \leq \|B K\| \|\xi_{q_{n-1}}\| B \|q_{n-1}(s)\| \left\| u_{n-1}(s) \right\| \leq c \|u\|, 
$$
where $c = \|B K\| L$. Hence, we derive that $\max \{\xi_{q_{n-1}} : 0 < n < m\}, \min \{\xi_{q_{n-1}} : 0 < n < m\}$ are bounded, and for sufficiently small $t$ divergent $\frac{c}{m}$. Let $u(t) \in D(\xi) \cap D(\xi_{q_{n-1}})$, then for sufficiently small $t$ divergent $\frac{c}{m}$.

D. Proof of Corollary 1

The existence and uniqueness of solutions is proven in Theorem 1 (using Conditions A)-C only. Since due to homogeneity $d(-s(t))u(t) \in D(-s(t))A \frac{d}{u(t)} + B K d(-s(t))u(t)$ and $\|d(-s(t))u(t)\| = 1$ then $D^*$ implies $\frac{d}{u(t)} \|u(t)\| \leq \|u(t)\| = 1$ (see Proof of Theorem 1).

E. Proof of Corollary 2

Since the control $\xi$ is Lipschitz continuous on $H_1(\xi)$ then using [35, Theorems 1.2 and 1.4, Chapter 6] we derive that for any $u_0 \in H_1(\xi)$ the closed-loop system has a unique mild solution $u$ given by (14) and defined at least locally on $[0, T(u_0))$ such that $u(t) \in H_1(\xi)$ if $t \in [0, T(u_0))$ and $\|u(t)\| \rightarrow 0$ as $t \rightarrow T(u_0)$, where $T(u_0)$ is a finite instant of time or $T(u_0) = +\infty$. Since $D(\xi)$ is dense in $H_1(\xi)$ then for any $u_0 \in H_1(\xi)$ and any $\varepsilon > 0$ there exists $T(\varepsilon) > 0$ such that $u(t) \in H_1(\xi)$ if $t \in [0, T(\varepsilon))$ and $\|u(t)\| \rightarrow 0$ as $t \rightarrow T(\varepsilon)$, where $T(\varepsilon)$ is continuous and bounded at $0$. We complete the proof taking into account that all classical solutions satisfy Theorem 1 (or Corollary 1) but the inequality (13) is equivalent to (15) if $u$ is a classical solutions.

F. Proof of Corollary 3

Let us denote $\mathcal{K}(\alpha, \beta) = \{u \in D(\alpha) : \alpha < \|u\| \leq \beta\}$. Existence and Uniqueness of Solution. Since $B K_1 \in L(H)$ for any $i \in \mathbb{Z}$, where $K_1 = r_1^{-\mu} K d(\ln r_1)$, then for any initial condition $u(0) = u^* \in D(\alpha)$ the equation evolution

$$
u(t) = (A + B K_1)u(t)$$

has a unique classical solution $u(t, u^*) \in D(\alpha)$ defined for all $t \geq 0$. We have $r_1 + r_2 > R$, $i \in \mathbb{Z}$ by construction.

Since $\|u(t, u^*)\| : \mathbb{R} \rightarrow \mathbb{R}^+$ is continuous (see, property 4 on page 2) then for $u(t, u^*) = 0 \in K(r_{i+1}^+) \cap \sigma(t) = i$ the closed-loop system (1), (2), (16), (17) has classical solution $u$ at least on $[t_i, t_{i+1})$ and $u(t) = u(t - t_i, u_0)$ for $t \in [t_i, t_{i+1})$, where $t_i = +\infty$ and/or $\lim\limits_{t \rightarrow t_{i+1}} \|u(t)\| = r_i + 1$. Let us show the only latter case is possible for some $t_{i+1} < +\infty$. Since $\frac{d}{u(t)} \|d(\ln r_i)u(t)\|^2 = 2(\|d(-\ln r_i)u(t)\|) = 2e^{-\|\ln r_i\|} \|A + B K \| d(\ln r_i)u(t)\| = 2e^{-\|\ln r_i\|} \|A + B K \| d(\ln r_i)u(t)\| 

then the continuous function $\|d(-\ln r_i)u(t)\|^2$ decays exponentially. Taking into account $\|d(-\ln r_i)u(t)\|^2 = 1$
if \( \|u\|_{\mathcal{A}} = r_i \) and \( \|d(\ln(r_i))\|_{\mathcal{A}} < 1 \) if \( \|u\|_{\mathcal{A}} < r_i \), we derive \( \|u(t)\|_{\mathcal{A}} < r_i \) for \( t > t_i \) and \( \lim_{t \to t_i} \|u(t)\|_{\mathcal{A}} = r_i \). Moreover, \( t_i + 1 \leq t_i + r_i^\beta - \alpha_i \ln \left( \frac{\|d(\ln(r_i))\|_{\mathcal{A}} \|u(t_i)\|_{\mathcal{A}}}{\alpha_i} \right) \), where \( \alpha_i > 0 \) is such that \( \|d(\ln(r_i))\|_{\mathcal{A}} \leq \alpha_i \Rightarrow \|u\|_{\mathcal{A}} \leq r_i \).

Therefore, \( u(t_{i+1}) = u_i(t_{i+1} - t_i, u_0) \in \mathcal{D}(A) \). Repeating the same considerations for \( t_{i+1}, t_{i+2} \), etc, and taking into account uniqueness and continuity of solutions of the equation (\*) we derive that for any initial condition \( u(0) \in \mathcal{D}(A) \), \( \sigma(0) = i_0 \) such that \( r_i < 1 < \|u(0)\|_{\mathcal{A}} < r_i - 1 \) the closed-loop system (1), (2), (16), (17) has a unique strong solution defined on \( [0, T(u_0)] \), which may not have time derivative only at isolated instances of time \( t_i : \|u(t_i)\|_{\mathcal{A}} = r_i \), where

\[
T(u_0) = \sum_{j=i_0}^{+\infty} t_{i+1} - t_i
\]

is the unique accumulation point of the set \( \{t_i\} \). In addition, \( \sigma(t) = const \) for \( t \in (t_i, t_{i+1}) \).

Since \( \sum_{j=i_0}^{+\infty} q_j = +\infty \) then \( r_i = \frac{1}{e^{\sum_{j=i_0}^{+\infty} q_j}} \to 0 \) as \( i \to +\infty \) and \( \|u(t)\|_{\mathcal{A}} \to 0 \) as \( t \to T(u_0) \).

II. Setting Time Estimate. Taking into account \( \beta_i \|d(s)\|_{\mathcal{A}} \leq \|Gd(s)\|_{\mathcal{A}} \|s\|_{\mathcal{A}} \) and \( \|d(\ln(r_i))\|_{\mathcal{A}} = \|Gd(\ln(r_i))\|_{\mathcal{A}} = \|Gd(\ln(r_i))\|_{\mathcal{A}} \) we derive \( \|u(t)\|_{\mathcal{A}} \leq e^{\beta_i t} \|u(0)\|_{\mathcal{A}} \) for \( t \geq 0 \). Selecting \( \epsilon_i = e^{\beta_i t_i} \) we obtain \( t_{i+1} \leq t_i + \epsilon_i^{-1} r_i \) and the settling time \( T(u_0) \) is finite since

\[
limit_{t \to +\infty} r_i e^{\beta_i t} q_i^{-1} = \lim_{t \to +\infty} q_i^{-1} e^{-\beta_i t} i \leq 1.
\]

III. Boundedness of Control. Since for \( u(t) \in \mathcal{C}(r_i, r_i) \) and \( \sigma(t) = i \) the control law can be rewritten as follows \( \xi_i u(t_i) = r_i^{-1} Kd_i(\ln(r_i))u(t_i) \). Taking into account (5),

\[
\|d(s)\|_{\mathcal{H}} \leq e^{\beta_i t} \|d(s)\|_{\mathcal{H}} \leq e^{\beta s} \leq s, \quad \|d(s)\|_{\mathcal{H}} \leq M e^{(\mu+\gamma)^{K_i}},
\]

with \( M \geq 1 \) and \( \|A(s)\|_{\mathcal{H}} \leq e^{\|Gd(s)\|_{\mathcal{H}}} \) if \( s \geq 0 \) we derive \( \|\xi_i(t)\|_{\mathcal{H}} \leq M e^{\|K_i\|_{\mathcal{H}}} \) since \( \beta - \mu - \|Gd\|_{\mathcal{H}} \geq 0 \) (see Theorem 1).