



Hybridizable Discontinuous Galerkin Methods for modelling 3D seismic wave propagation in harmonic domain

M. Bonnasse-Gahot^{1,2}, H. Calandra³, J. Diaz¹ and S. Lanteri²

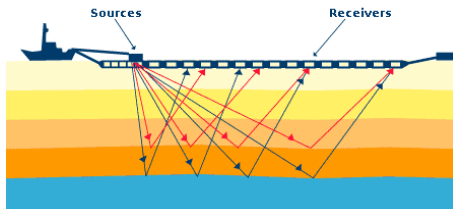
¹ INRIA Bordeaux-Sud-Ouest, team-project Magique 3D

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³ TOTAL Exploration-Production

Motivations

Principles of seismic imaging



Motivations

Examples of seismic imaging campaigns



Motivations

Imaging methods

- ▶ Full Wave Inversion (FWI) : **inversion process** requiring to solve **many forward problems**

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Seismic imaging : time-domain or harmonic-domain ?

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- ▶ **Harmonic-domain** : **imaging condition simple** but **huge computational cost**

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Memory usage



Motivations

Resolution of the forward problem of the inversion process

- ▶ Elastic wave propagation in the frequency domain : **Helmholtz equation**

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Resolution of the forward problem of the inversion process

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First order formulation of Helmholtz wave equations

$$\mathbf{x} = (x, y, z) \in \Omega \subset \mathbb{R}^3,$$

$$\begin{cases} i\omega\rho(\mathbf{x})\mathbf{v}(\mathbf{x}) = \nabla \cdot \underline{\underline{\sigma}}(\mathbf{x}) \\ i\omega\underline{\underline{\sigma}}(\mathbf{x}) = \underline{\underline{C}}(\mathbf{x}) \underline{\underline{\varepsilon}}(\mathbf{v}(\mathbf{x})) + \mathbf{f}_s(\mathbf{x}) \end{cases}$$

- ▶ \mathbf{v} : velocity vector
- ▶ $\underline{\underline{\sigma}}$: stress tensor
- ▶ $\underline{\underline{\varepsilon}}$: strain tensor

Approximation methods

Discontinuous Galerkin Methods

- ✓ unstructured tetrahedral meshes
- ✓ combination between FEM and finite volume method (FVM)
- ✓ *hp*-adaptivity
- ✓ easily parallelizable method

Approximation methods

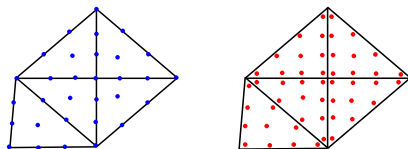
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- ✗ ✗ large number of DOF as compared to classical FEM

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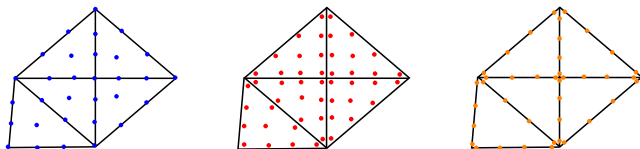
Hybridizable Discontinuous Galerkin Methods

- ✓ same advantages as DG methods : unstructured tetrahedral meshes, *hp*-adaptivity, easily parallelizable method, discontinuous basis functions
- ✓ introduction of a new variable defined only on the interfaces
- ✓ lower number of coupled DOF than classical DG methods

Approximation methods





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Approximation methods

Hybridizable Discontinuous Galerkin Methods

-  B. Cockburn, J. Gopalakrishnan and R. Lazarov. Unified hybridization of discontinuous Galerkin, mixed and continuous Galerkin methods for second order elliptic problems. *SIAM Journal on Numerical Analysis*, Vol. 47 :1319-1365, 2009.
-  S. Lanteri, L. Li and R. Perrussel. Numerical investigation of a high order hybridizable discontinuous Galerkin method for 2d time-harmonic Maxwell's equations. *COMPEL*, 32(3)1112-1138, 2013.
-  N.C. Nguyen, J. Peraire and B. Cockburn. High-order implicit hybridizable discontinuous Galerkin methods for acoustics and elastodynamics. *Journal of Computational Physics*, 230 :7151-7175, 2011
-  N.C. Nguyen and B. Cockburn. Hybridizable discontinuous Galerkin methods for partial differential equations in continuum mechanics. *Journal of Computational Physics* 231 :5955–5988, 2012

Principles of the HDG method

1. Introduction of a Lagrange multiplier λ

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- ▶ Weak formulation of waves equations

$$\begin{cases} \int_K i\omega \rho^K \mathbf{v}^K \cdot \mathbf{w} + \int_K \underline{\underline{\sigma}}^K : \nabla \mathbf{w} - \int_{\partial K} \underline{\underline{\sigma}} \cdot \mathbf{n} \cdot \mathbf{w} = 0 \\ \int_K i\omega \underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot (\underline{\underline{C}}^K \underline{\underline{\xi}}) - \int_{\partial K} \mathbf{v} \cdot \underline{\underline{C}}^K \underline{\underline{\xi}} \cdot \mathbf{n} = \int_K f_S^K \cdot \underline{\underline{\xi}} \end{cases}$$

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$\hat{\underline{\underline{\sigma}}}^{\partial K}$ and $\hat{\mathbf{v}}^{\partial K}$ are numerical traces of $\underline{\underline{\sigma}}^K$ and \mathbf{v}^K respectively on ∂K

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$$\begin{array}{l} \text{►} \quad \hat{\mathbf{v}}^{\partial K} = \lambda^F, \quad \forall F \in \mathcal{F}_h, \end{array}$$



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▶

$$\begin{aligned} \hat{\mathbf{v}}^{\partial K} &= \lambda^F, & \forall F \in \mathcal{F}_h, \\ \hat{\underline{\underline{\sigma}}}^{\partial K} \cdot \mathbf{n} &= \underline{\underline{\sigma}}^K \cdot \mathbf{n} - \tau \mathbf{l}(\mathbf{v}^K - \lambda^F), & \text{on } \partial K \end{aligned}$$

where τ is the stabilization parameter ($\tau > 0$)

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- ▶ Local HDG formulation

$$\begin{cases} \int_K i\omega \rho \mathbf{v}^K \cdot \mathbf{w} - \int_K (\nabla \cdot \underline{\underline{\sigma}}^K) \cdot \mathbf{w} + \int_{\partial K} \tau \mathbf{l} (\mathbf{v}^K - \lambda^F) \cdot \mathbf{w} = 0 \\ \int_K i\omega \underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot (\underline{\underline{C}}^K \underline{\underline{\xi}}) - \int_{\partial K} \lambda^F \cdot \underline{\underline{C}}^K \underline{\underline{\xi}} \cdot \mathbf{n} = 0 \end{cases}$$

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- ▶ Local HDG formulation

$$\mathbf{A}^K \mathbf{W}^K + \mathbf{C}^K \lambda = \mathbf{S}^K$$

with $\mathbf{W}^K = (\mathbf{v}^K, \underline{\underline{\sigma}}^K)^T$ and $\lambda = (\lambda^{F_1}, \lambda^{F_2}, \dots, \lambda^{F_{n_f}})^T$, where $n_f = \text{card}(\mathcal{F}_h)$

Principles of the HDG method

1. Introduction of a Lagrange multiplier λ
2. Expressing the initial unknowns \mathbf{w}^K as a function of λ

$$\mathbb{A}^K \mathbf{w}^K + \mathbb{C}^K \lambda = \mathbf{S}^K \longrightarrow \mathbf{w}^K = \left(\mathbb{A}^K\right)^{-1} \mathbf{S}^K - \left(\mathbb{A}^K\right)^{-1} \mathbb{C}^K \lambda$$

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3. Transmission condition : $\int_F \llbracket \underline{\hat{\sigma}}^{\partial K} \cdot \mathbf{n} \rrbracket \cdot \eta = 0$

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4. $\mathbb{M} \lambda = \mathbf{S}$
5. Computation of the solutions of the initial problem \mathbf{w}^K , element by element

Contents

2D Numerical results : performances comparison of the HDG method

- Anisotropic test case

- Marmousi test-case

3D numerical results

Main steps of the algorithms

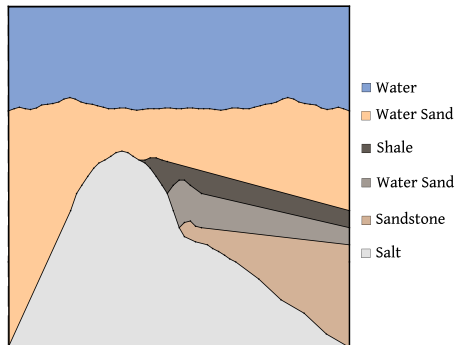
HDG algorithm	IPDG & FE algorithm
<ol style="list-style-type: none">1. Construction of the linear system in λ2. Resolution of the linear system3. Reconstruction of the initial solution \mathbf{W}	<ol style="list-style-type: none">1. Construction of the linear system in \mathbf{u}2. Resolution of the linear system

Main steps of the algorithms

HDG algorithm	IPDG & FE algorithm
1. Construction of the linear system in λ	1. Construction of the linear system in \mathbf{u}
2. Resolution of the linear system	2. Resolution of the linear system
3. Reconstruction of the initial solution \mathbf{W}	

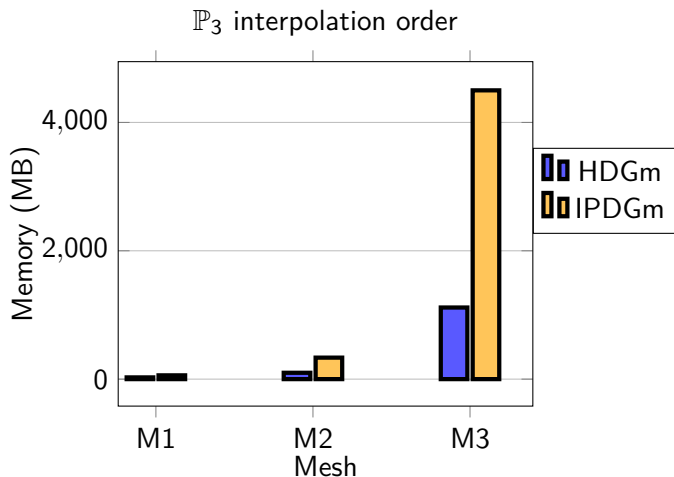
embarrassingly parallel

Anisotropic test case



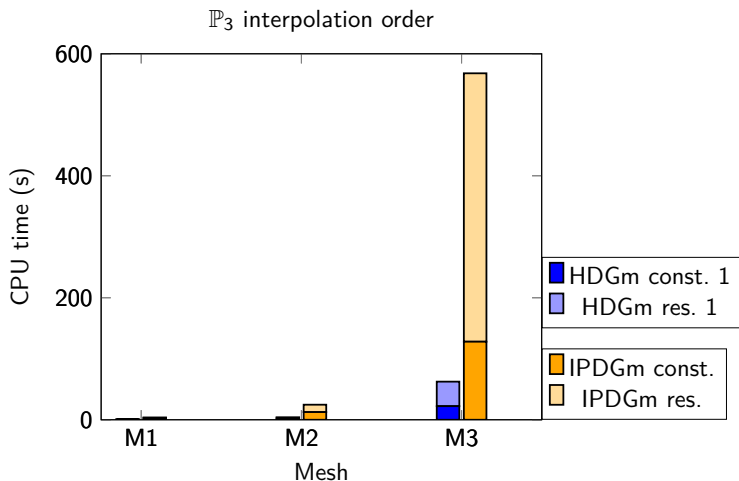
- ▶ Three meshes :
 - ▶ 600 elements
 - ▶ 3000 elements
 - ▶ 28000 elements

Anisotropic case : Memory consumption



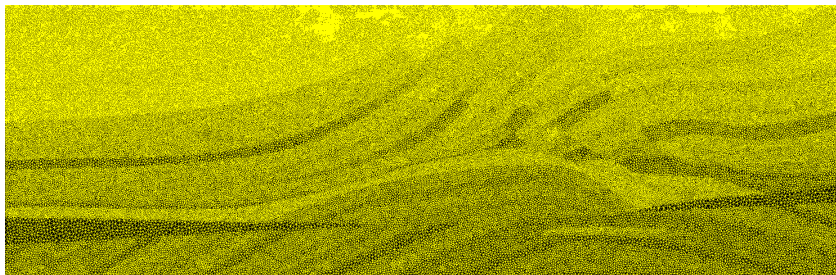
$$\text{IPDG}_{\text{memory}} \simeq 4 \times \text{HDG}_{\text{memory}}$$

Anisotropic case : CPU time (s)



$$\text{IPDG}_{\text{CPUtime}} \simeq 9 \times \text{HDG}_{\text{CPUtime}}$$

Marmousi test-case



Computational domain Ω composed of 235000 triangles

Cluster configuration

- Hardware specification : 16 nodes, 12 cores by nodes
- Characteristics of computing nodes :
 - ▶ 2 Hexa-core Westmere Intel[®] Xeon[®] X5670
 - ▶ Frequency : 2,93 GHz
 - ▶ Cache L3 : 12 Mo
 - ▶ RAM : 96 Go
 - ▶ Infiniband DDR : 20Gb/s
 - ▶ Ethernet : 1Gb/s

Efficiency of the parallelism of the construction of the global matrix

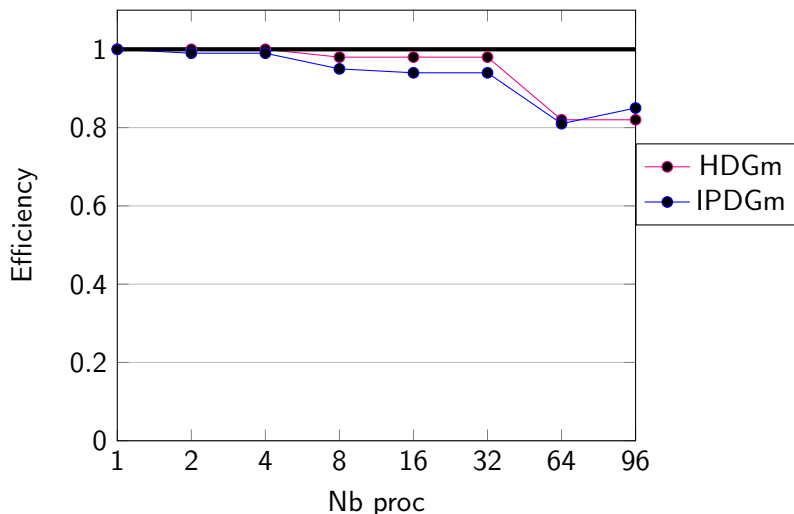
$$\text{Efficiency} = \frac{t_{\text{ref}}}{\text{nb_cores} \times t_{\text{one core}}}$$

where :

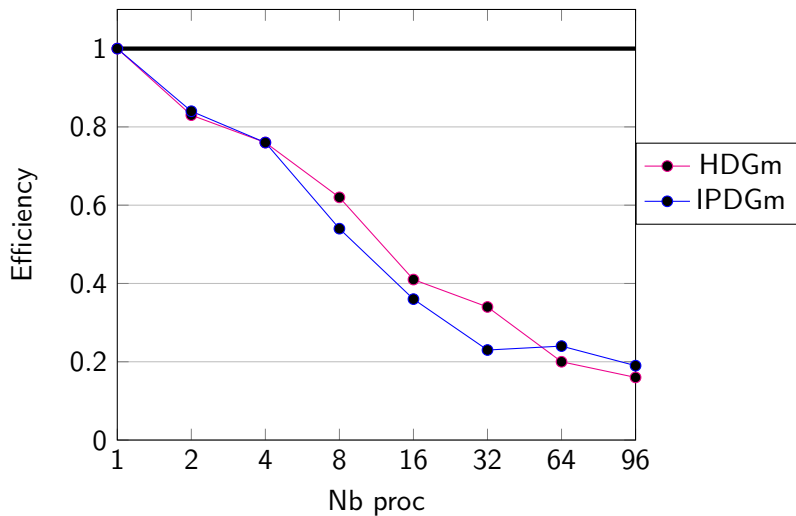
$t_{\text{ref}} = t_{\text{seq}}$ = sequential computational time

$t_{\text{one core}}$ = computational time with one core

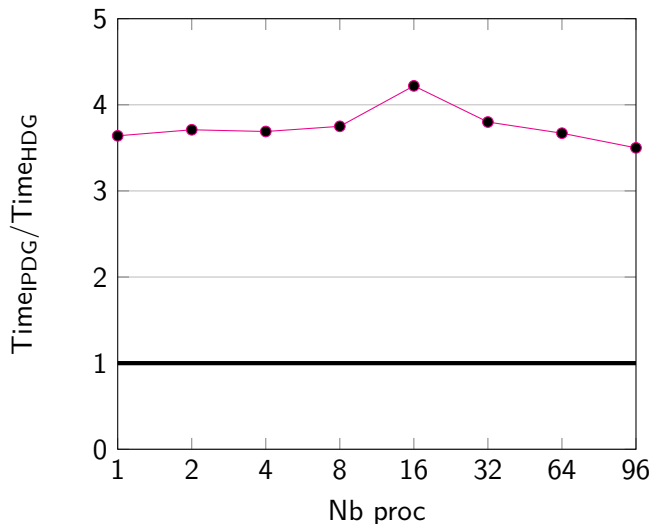
Efficiency of the parallelism of the construction of the global matrix



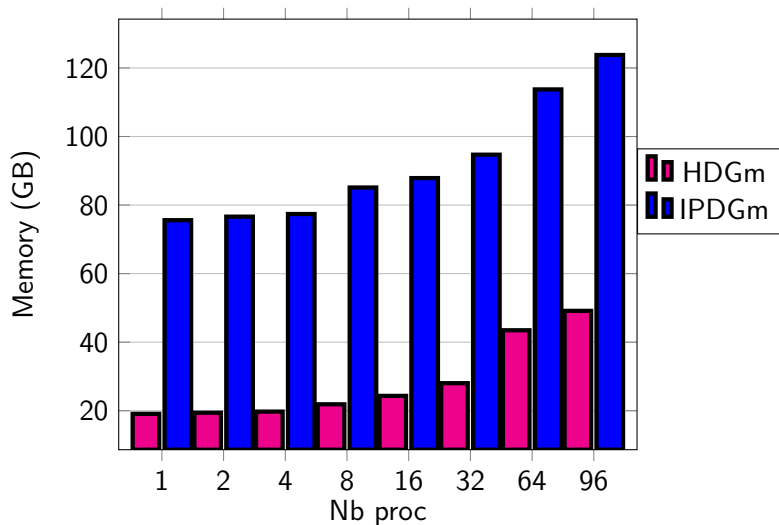
Efficiency of the parallelism of the whole algorithm



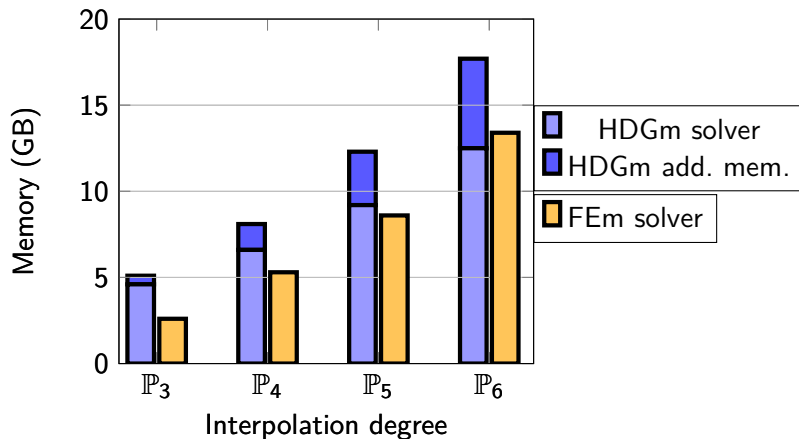
Speed up for the global matrix construction



Memory required (GB) for the simulation

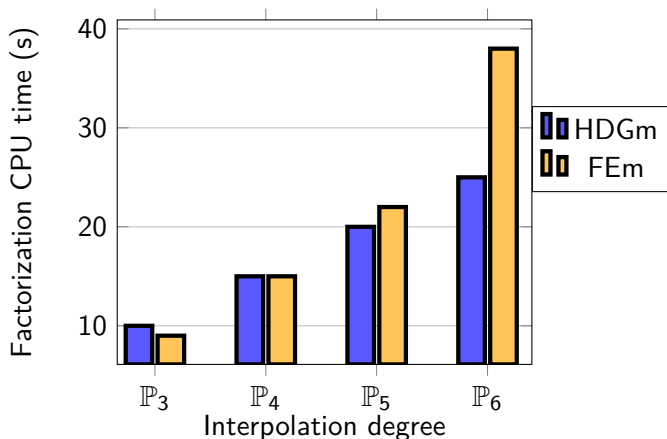


Comparison with Finite Elements method



$$\text{FE}_{\text{memory}} \simeq 0.7 \times \text{HDG}_{\text{memory}}$$

Comparison with Finite Elements method



Contents

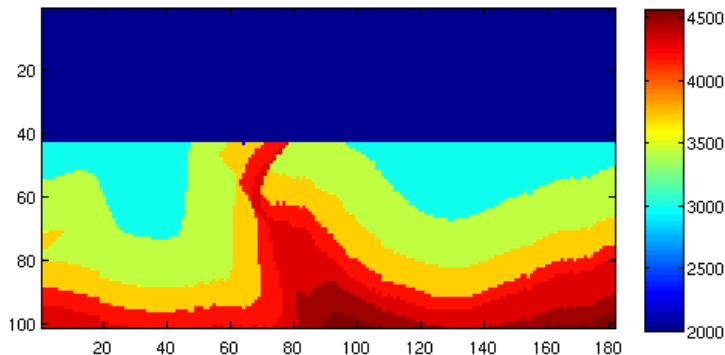
2D Numerical results : performances comparison of the HDG method

3D numerical results

Performances analysis of the HDG method

Impact of the linear solver

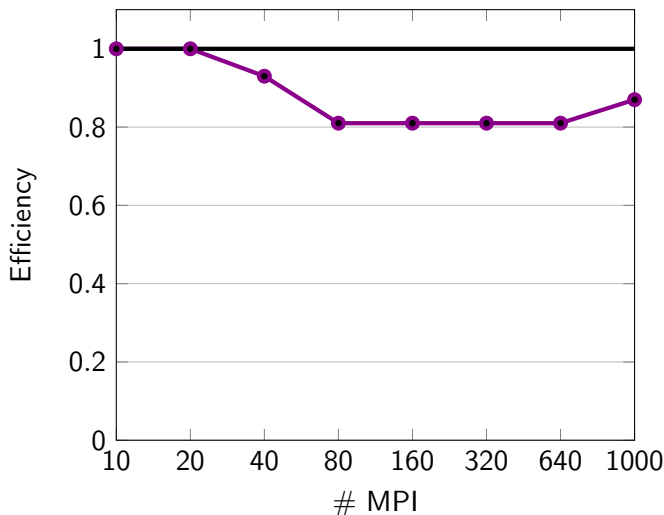
Epati test-case



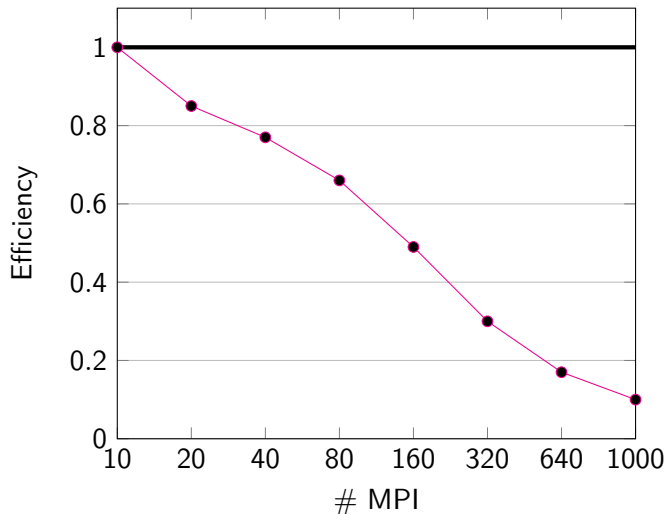
V_p -velocity model (m.s^{-1}), vertical section at $y = 700$ m

Mesh composed of 25 000 tetrahedrons

Epati test-case : Scalability, efficiency of the HDG global system construction



Efficiency of the parallelism of the whole algorithm (construction and resolution)



Resolution of the linear system $\mathbb{M}\lambda = \mathbf{S}$

- ▶ Direct solvers
- ▶ Iterative solvers
- ▶ Hybrid solvers , combination between direct and iterative solvers
- ▶ Direct solvers combined with H -matrices
- ▶ ...

Resolution of the linear system $\mathbb{M}\lambda = \mathbf{S}$

- ▶ **Direct solvers**
- ▶ Iterative solvers
- ▶ **Hybrid solvers**, combination between direct and iterative solvers
- ▶ Direct solvers combined with H -matrices
- ▶ ...

Resolution of the linear system $\mathbb{M}\lambda = \mathbf{S}$

- ▶ **direct solver : MUMPS** (Multifrontal Massively Parallel sparse direct Solver) :

- ▶ Direct factorization $A = LU$ or $A = LDL^T$
- ▶ Multiple RHS



P.R. Amestoy, I.S. Duff and J.-Y. L'Excellent. Multifrontal parallel distributed symmetric and unsymmetric solvers. *Computational Methods in Applied Mechanics and Engineering*, Vol. 184 :501-520, 2000.

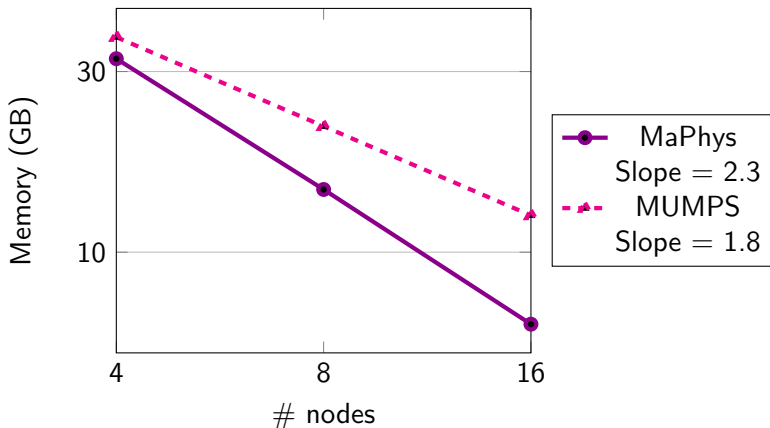
- ▶ **hybrid solver : MaPhys** (Massively Parallel Hybrid Solver) :
 - ▶ combination of direct and iterative methods
 - ▶ non-overlapping algebraic domain decomposition method (Schur complement method)
 - ▶ resolution of each local problem thanks to direct solver such as MUMPS



<https://gitlab.inria.fr/solverstack/maphys>

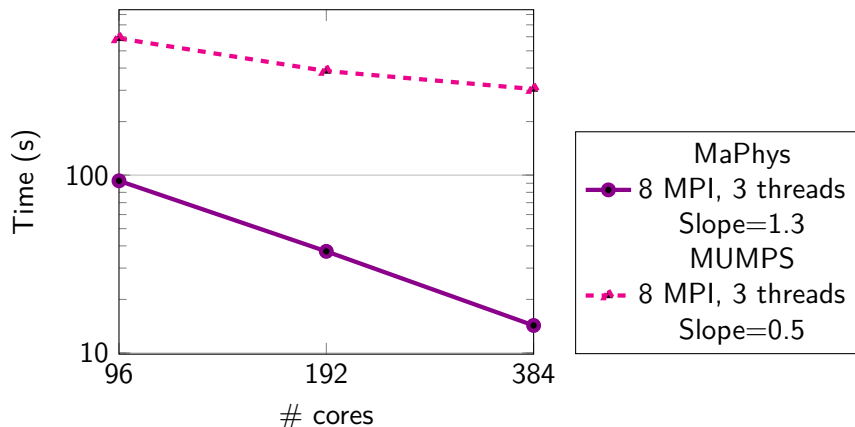
Epati test-case : Memory consumption

Average memory for one node (8 MPI by node and 3 threads by MPI)

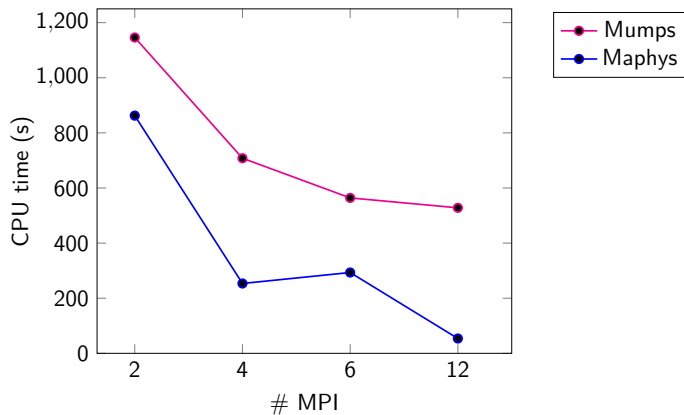


Epati test-case : Execution time

Execution time for the resolution of the HDG- \mathbb{P}_3 system



Epati test-case : Cumulative time of the two solvers using 96 cores



Conclusion-Perspectives

Conclusions

- ▶ HDG method more efficient than classical DG methods for a same accuracy (memory and CPU time)
- ▶ In 2D, HDG method has equivalent computational performances than FE methods
- ▶ Maphys good alternative to MUMPS for the resolution of the HDG system with one RHS
- ✗ Multiple RHS not yet implemented in Maphys

Conclusion-Perspectives

Perspectives

- ▶ more detailed analysis of the comparison between solvers
 - ▶ larger meshes
 - ▶ more powerful clusters
- ▶ extension to H -matrices
- ▶ 3D comparison between FE and HDG methods
- ▶ extension to elasto-acoustic case
- ▶ application to inverse problems

Thank you !

