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## Trefftz-DG Approximation for the Elasto-Acoustics

Henri Calandra, Elvira Shishenina, Julien Diaz, H el ene Barucq

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# TREFFTZ-DG APPROXIMATION FOR THE ELASTO-ACOUSTICS

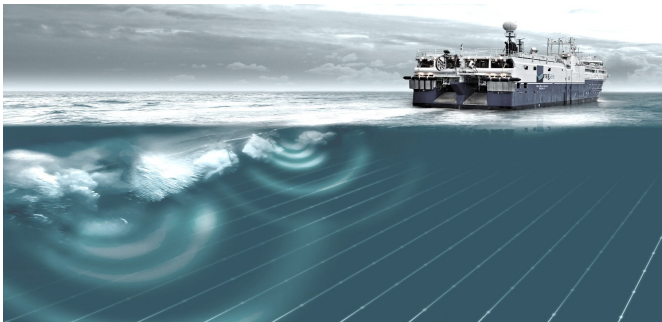
WAVES17 | 13th International Conference on Mathematical and Numerical Aspects of Wave Propagation

# ABSTRACT

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# SEISMIC SURVEY

FIGURE 1: Ocean Bottom Seismic (OBS) data acquisition



# BASIC NUMERICAL METHODS

TABLE 1: Generic properties of the most widely used numerical methods\*

Numerical method	Complex geometries	High-order accuracy and $hp$ -adaptivity	Explicit semi-discrete form	Conservation laws	Elliptic problems
FDM	●	●	●	●	●
FVM					
FEM					
DG-FEM					

\* J.S.Hesthaven T.Warburton. *Nodal DG methods. Algorithms, analysis, and applications*. 2007

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FEM	●	●	●	⊙	●
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FDM	●	●	●	●	●
FVM	●	●	●	●	⊙
FEM	●	●	●	⊙	●
DG-FEM	●	●	●	●	⊙

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# DISCONTINUOUS GALERKIN METHODS

## PROS AND CONS



- Adapted to the complex geometries
- High-order accuracy and hp-adaptivity
- Explicit semi-discrete form
- Conservation laws

# DISCONTINUOUS GALERKIN METHODS

## PROS AND CONS



- Adapted to the complex geometries
- High-order accuracy and hp-adaptivity
- Explicit semi-discrete form
- Conservation laws



- Higher number of degrees of freedom  
comparing to the methods with continuous approximation

# TREFFTZ METHOD

## DEFINITION \*

Given a region of an Euclidean space of some partitions of that region,

a **Trefftz method** is any procedure for solving boundary value problems of PDE or systems of PDE, on such region,

**using solutions of that PDE or its adjoint**, defined in its subregions.

\* I.Herrera. *Trefftz method: a general theory*. 2000

# DISCONTINUOUS GALERKIN METHODS

## TIME-HARMONIC FORMULATIONS

O.Cassenat B.Despres. 1998

T.Huttunen P.Monk J.P.Kaipo. 2002

C.Farhat I.Harari U.Hetmaniuk. 2003

R.Tezaur, C.Farhat. 2006

G.Gabard. 2007

## TIME-DOMAIN FORMULATIONS

H.Egger F.Kretzschmar S.M.Schnepf T.Weiland. 2014

F.Kretzschmar S.M.Schnepf I.Tsukerman T.Weiland. 2014

F.Kretzschmar A.Moiola I.Perugia. 2015

# TREFFTZ METHOD

## EXPECTED ADVANTAGES \*



Better order of convergence

Flexibility in the choice of basis functions

Low dispersion

Adaptivity and local space-time mesh refinement

\* I.Herrera. *Trefftz method: a general theory*. 2000

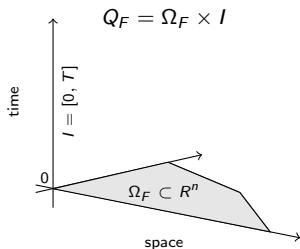
# MATHEMATICAL FORMULATION

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ACOUSTIC SYSTEM

# ACOUSTIC SYSTEM

## PROBLEM EQUATIONS

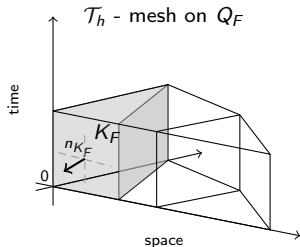


(AS)

$$\left[ \begin{array}{ll} \frac{1}{c_F^2 \rho_F} \frac{\partial p}{\partial t} + \operatorname{div} v_F = f, & \text{in } Q_F \\ \rho_F \frac{\partial v_F}{\partial t} + \nabla p = 0, & \text{in } Q_F \\ v_F(\cdot, 0) = v_{F0}, \quad p(\cdot, 0) = p_0, & \text{on } \Omega_F \\ v_F = g_F^D, & \text{on } \partial\Omega_F \times I \end{array} \right.$$

# ACOUSTIC SYSTEM

## MESHING



$\mathcal{F}_h = \cup_{K_F \in \mathcal{T}_h} \partial K_F$  - mesh skeleton

$K_F \in \mathcal{Q}_F$  ( $c_F, \rho_F \equiv \text{const}$  in  $K_F$ )

$v_F, p \in H^1(K_F)$

$\omega_F, q \in H^1(K_F)$



# ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION IN THE ELEMENT  $K_F$

AS

$(v_F, p)$

# ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION IN THE ELEMENT  $K_F$

$$\begin{array}{|c|} \hline \mathbf{AS} \\ \hline (v_F, p) \\ \hline \end{array} \times \begin{array}{|c|} \hline \text{test} \\ \text{functions} \\ \hline (\omega_F, q) \\ \hline \end{array}$$

# ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION IN THE ELEMENT  $K_F$

$$\underbrace{\begin{array}{|c|} \hline \text{AS} \\ \hline (v_F, p) \\ \hline \end{array}} \times \begin{array}{|c|} \hline \text{test} \\ \text{functions} \\ \hline (\omega_F, q) \\ \hline \end{array} =$$

SPACE-TIME INTEGRATION ON  $K_F$

# ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION IN THE ELEMENT  $K_F$

$$\underbrace{\left[ \begin{array}{c} AS \\ (v_F, p) \end{array} \right] \times \left[ \begin{array}{c} \text{test} \\ \text{functions} \\ (\omega_F, q) \end{array} \right]}_{\text{SPACE-TIME INTEGRATION ON } K_F} = \left[ \begin{array}{c} \text{VOLUME INTEGRATION TERM} \\ AS \times (v_F, p) \\ (\text{test functions}) \end{array} \right]$$

# ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION IN THE ELEMENT  $K_F$

$$\underbrace{\boxed{\begin{array}{c} AS \\ (v_F, p) \end{array}} \times \boxed{\begin{array}{c} \text{test functions} \\ (\omega_F, q) \end{array}}}_{\text{SPACE-TIME INTEGRATION ON } K_F} = \boxed{\begin{array}{c} \text{VOLUME INTEGRATION TERM} \\ AS \times (v_F, p) \\ (\text{test functions}) \end{array}} + \boxed{\begin{array}{c} \text{SURFACE INTEGRATION TERM} \\ (\text{test functions}) \\ (v_F, p) \end{array}}$$

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# ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION IN THE ELEMENT  $K_F$

$$\underbrace{\left[ \begin{array}{c} \text{AS} \\ (v_F, p) \end{array} \right] \times \left[ \begin{array}{c} \text{test functions} \\ (\omega_F, q) \end{array} \right]}_{\text{SPACE-TIME INTEGRATION ON } K_F} = \begin{array}{c} \text{VOLUME INTEGRATION TERM} \\ \left[ \begin{array}{c} \text{test functions in } \mathbf{T} \end{array} \right] \\ + \\ \text{SURFACE INTEGRATION TERM} \\ \left[ \begin{array}{c} (\text{test functions}) \\ (v_F, p) \end{array} \right] \end{array}$$

# ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION IN THE ELEMENT  $K_F$

$$\underbrace{\int_{K_F} AS(v_F, p) \times \int_{K_F} \text{test functions}(\omega_F, q)}_{\text{SPACE-TIME INTEGRATION ON } K_F} = \int_{K_F} \text{VOLUME INTEGRATION TERM} + \int_{K_F} \text{SURFACE INTEGRATION TERM}$$

$\text{VOLUME INTEGRATION TERM} = 0$

$\text{SURFACE INTEGRATION TERM} = (\text{test functions})(v_F, p)$



# ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION IN THE ELEMENT  $K_F$

$$\underbrace{\left[ \begin{array}{c} \text{AS} \\ (v_F, p) \end{array} \right] \times \left[ \begin{array}{c} \text{test} \\ \text{functions} \\ (\omega_F, q) \end{array} \right]}_{\text{SPACE-TIME INTEGRATION ON } K_F} = \text{SURFACE INTEGRATION TERM} \left[ \begin{array}{c} (\text{test functions}) \\ (v_F, p) \end{array} \right]$$

# ACOUSTIC SYSTEM

## TREFFTZ-DG FORMULATION

$$\begin{array}{c}
 \text{SUM OVER ALL } K_F \\
 \hline
 \boxed{\begin{array}{c} \text{AS} \\ (v_F, p) \end{array}} \times \boxed{\begin{array}{c} \text{test} \\ \text{functions} \\ (\omega_F, q) \end{array}} \\
 \hline
 \text{SPACE-TIME INTEGRATION ON } K_F
 \end{array}
 =
 \begin{array}{c}
 \text{SUM OVER ALL } \partial K_F \\
 \hline
 \text{SURFACE INTEGRATION TERM} \\
 \boxed{\begin{array}{c} (\text{test functions}) \\ (v_F, p) \end{array}}
 \end{array}$$

# ACOUSTIC SYSTEM

## SPACE-TIME DISCRETIZATION

$$\begin{aligned}
 & - \sum_{K_F} \int_{K_F} \left[ p_h \left( \frac{1}{c_F^2 \rho_F} \frac{\partial q}{\partial t} + \operatorname{div} \omega_F \right) + v_{Fh} \cdot \left( \rho_F \frac{\partial \omega_F}{\partial t} + \nabla q \right) \right] dv \\
 & + \sum_{K_F} \int_{\partial K_F} \left[ \left( \frac{1}{c_F^2 \rho_F} \hat{p}_h q + \rho_F v_{Fh} \cdot \omega_F \right) n_{K_F}^t + \left( \hat{p}_h \omega_F + v_{Fh} q \right) \cdot n_{K_F}^x \right] ds = \sum_{K_F} \int_{K_F} f q dv
 \end{aligned}$$

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 & + \sum_{K_F} \int_{\partial K_F} \left[ \left( \frac{1}{c_F^2 \rho_F} \hat{\rho}_h q + \rho_F \mathbf{v}_{Fh} \cdot \omega_F \right) n_{K_F}^t + \left( \hat{\rho}_h \omega_F + \mathbf{v}_{Fh} q \right) \cdot \mathbf{n}_{K_F}^x \right] ds = \sum_{K_F} \int_{K_F} f q dv
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 \end{aligned}$$

## TREFFTZ SPACE

$$\mathbf{T}_F(\mathcal{T}_h) := \left\{ (\omega_F, q) \in H^1(\mathcal{T}_h)^2 : \forall K_F \in \mathcal{T}_h, \rho_F \frac{\partial \omega_F}{\partial t} + \nabla q = \frac{1}{c_F^2 \rho_F} \frac{\partial q}{\partial t} + \operatorname{div} \omega_F = 0 \right\}$$

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 & + \sum_{K_F} \int_{\partial K_F} \left[ \left( \frac{1}{c_F^2 \rho_F} \hat{p}_h q + \rho_F v_{Fh} \omega_F \right) n_{K_F}^t + \left( \hat{p}_h \omega_F + v_{Fh} q \right) n_{K_F}^x \right] ds = \sum_{K_F} \int_{K_F} f q dv
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## SPACE-TIME DISCRETIZATION IN $\mathbf{T}_F$

$$\sum_{K_F} \int_{\partial K_F} \left[ \left( \frac{1}{c_F^2 \rho_F} \hat{p}_h q + \rho_F v_{Fh} \cdot \omega_F \right) n_{K_F}^t + \left( \hat{p}_h \omega_F + v_{Fh} q \right) \cdot n_{K_F}^x \right] ds = \sum_{K_F} \int_{K_F} f q dv$$

# ACOUSTIC SYSTEM

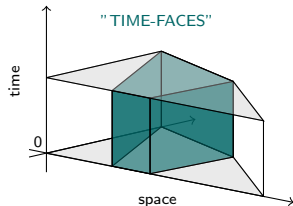
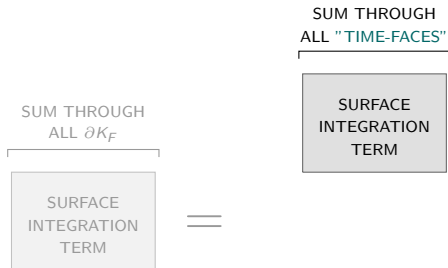
## TREFFTZ-DG FORMULATION

SUM OVER  
ALL  $\partial K_F$

SURFACE  
INTEGRATION  
TERM

# ACOUSTIC SYSTEM

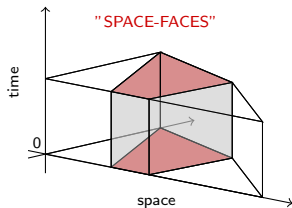
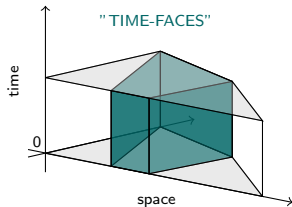
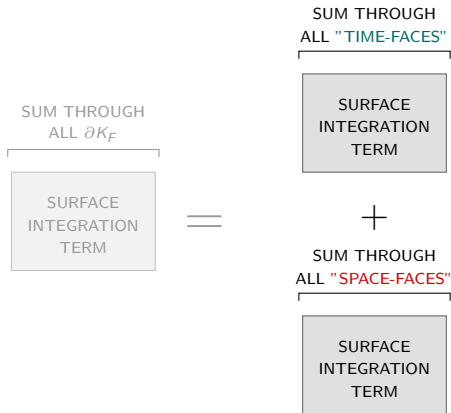
## TREFFTZ-DG FORMULATION





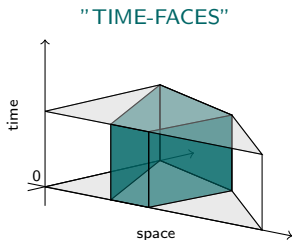
# ACOUSTIC SYSTEM

## TREFFTZ-DG FORMULATION

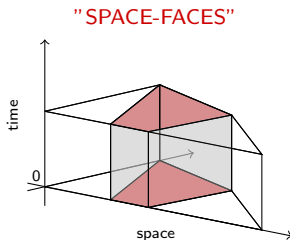


# ACOUSTIC SYSTEM

## MESH NOTATIONS



internal  $\mathcal{F}_h^{I_F}$   
 boundary  $\mathcal{F}_h^{D_F} (\partial\Omega_F \times [0, T])$



internal  $\mathcal{F}_h^{\Omega_F} (t \equiv \text{const.})$   
 initial  $\mathcal{F}_h^{0_F} (\Omega_F \times \{0\})$   
 final  $\mathcal{F}_h^{T_F} (\Omega_F \times \{T\})$

# ACOUSTIC SYSTEM

## TREFFTZ-DG FORMULATION

$$\sum_{\partial K_F} \int_{\partial K_F} \left[ \left( \frac{1}{c_F^2 \rho_F} \hat{p}_h q + \rho_F v_{Fh} \omega_F \right) n_{K_F}^t + \left( \hat{p}_h \omega_F + v_{Fh} q \right) n_{K_F}^x \right] ds = \sum_{K_F} \int_{K_F} f q dv$$

## NUMERICAL FLUXES THROUGH THE ELEMENT FACES

$$\begin{aligned} \mathcal{F}_h^{I_F} : \quad \begin{pmatrix} v_{Fh} \\ \hat{p}_h \end{pmatrix} &:= \begin{pmatrix} \{v_{Fh}\} + \beta [\rho_h]_x \\ \{\rho_h\} + \alpha [v_{Fh}]_x \end{pmatrix} \\ \mathcal{F}_h^{D_F} : \quad \begin{pmatrix} v_{Fh} \\ \hat{p}_h \end{pmatrix} &:= \begin{pmatrix} g \hat{D}_F \\ p_h + \alpha (v_{Fh} \cdot n_{K_F}^x - g_{D_F}) \end{pmatrix} \\ \mathcal{F}_h^{\Omega_F} : \quad \begin{pmatrix} v_{Fh} \\ \hat{p}_h \end{pmatrix} &:= \begin{pmatrix} v_{Fh}^- \\ p_h^- \end{pmatrix} \\ \mathcal{F}_h^{T_F} : \quad \begin{pmatrix} v_{Fh} \\ \hat{p}_h \end{pmatrix} &:= \begin{pmatrix} v_{Fh} \\ p_h \end{pmatrix} \\ \mathcal{F}_h^{0_F} : \quad \begin{pmatrix} v_{Fh} \\ \hat{p}_h \end{pmatrix} &:= \begin{pmatrix} v_{F0} \\ p_0 \end{pmatrix} \end{aligned}$$

# ACOUSTIC SYSTEM

## TREFFTZ-DG FORMULATION

Seek  $(v_{Fh}, p_h) \in \mathbf{V}(\mathcal{T}_h) \subset \mathbf{T}_F(\mathcal{T}_h)$  s.t. for all  $(\omega_F, q) \in \mathbf{V}(\mathcal{T}_h)$  it holds:

$$\mathcal{A}_{TDG_F}((v_{Fh}, p_h); (\omega_F, q)) = \ell_{TDG_F}(\omega_F, q)$$

# ACOUSTIC SYSTEM

$L^2(\mathcal{T}_h)$  NORMS IN  $T_F$

$$\begin{aligned}
 |||(\omega_F, q)|||_{TDGF}^2 &:= \frac{1}{2} \left\| \left( \frac{1}{c_F^2 \rho_F} \right)^{1/2} [\mathbf{q}]_t \right\|_{L^2(\mathcal{F}_h^{\Omega F})}^2 + \frac{1}{2} \left\| \rho_F^{1/2} [\omega_F]_t \right\|_{L^2(\mathcal{F}_h^{\Omega F})}^2 + \left\| \alpha^{1/2} [\omega_F]_x \right\|_{L^2(\mathcal{F}_h^{I F})}^2 \\
 &+ \left\| \beta^{1/2} [\mathbf{q}]_x \right\|_{L^2(\mathcal{F}_h^{I F})}^2 + \frac{1}{2} \left\| \left( \frac{1}{c_F^2 \rho_F} \right)^{1/2} q \right\|_{L^2(\mathcal{F}_h^{T F})}^2 + \frac{1}{2} \left\| \rho_F^{1/2} \omega_F \right\|_{L^2(\mathcal{F}_h^{T F})}^2 \\
 &+ \left\| \alpha^{1/2} \omega_F \right\|_{L^2(\mathcal{F}_h^{D F})}^2
 \end{aligned}$$

# ACOUSTIC SYSTEM

$L^2(\mathcal{T}_h)$  NORMS IN  $T_F$

$$\begin{aligned} |||(\omega_F, \mathbf{q})|||_{TDG_F}^2 &:= \frac{1}{2} \left\| \left( \frac{1}{c_F^2 \rho_F} \right)^{1/2} [\mathbf{q}]_t \right\|_{L^2(\mathcal{F}_h^{\Omega_F})}^2 + \frac{1}{2} \left\| \rho_F^{1/2} [\omega_F]_t \right\|_{L^2(\mathcal{F}_h^{\Omega_F})}^2 + \left\| \alpha^{1/2} [\omega_F]_x \right\|_{L^2(\mathcal{F}_h^I F)}^2 \\ &+ \left\| \beta^{1/2} [\mathbf{q}]_x \right\|_{L^2(\mathcal{F}_h^I F)}^2 + \frac{1}{2} \left\| \left( \frac{1}{c_F^2 \rho_F} \right)^{1/2} \mathbf{q} \right\|_{L^2(\mathcal{F}_h^{TF})}^2 + \frac{1}{2} \left\| \rho_F^{1/2} \omega_F \right\|_{L^2(\mathcal{F}_h^{TF})}^2 \\ &+ \left\| \alpha^{1/2} \omega_F \right\|_{L^2(\mathcal{F}_h^{DF})}^2 \end{aligned}$$

$$\begin{aligned} |||(\omega_F, \mathbf{q})|||_{TDG_F^*}^2 &:= |||(\omega_F, \mathbf{q})|||_{TDG_F}^2 \\ &+ \left\| \rho_F^{1/2} \omega_F^- \right\|_{L^2(\mathcal{F}_h^{\Omega_F})}^2 + \left\| \left( \frac{1}{c_F^2 \rho_F} \right)^{1/2} \mathbf{q}^- \right\|_{L^2(\mathcal{F}_h^{\Omega_F})}^2 + \left\| \beta^{-1/2} \{\omega_F\} \right\|_{L^2(\mathcal{F}_h^I F)}^2 \\ &+ \left\| \alpha^{-1/2} \{\mathbf{q}\} \right\|_{L^2(\mathcal{F}_h^I F)}^2 + \left\| \alpha^{-1/2} \omega_F \right\|_{L^2(\mathcal{F}_h^{DF})}^2 \end{aligned}$$

# ACOUSTIC SYSTEM

## COERCIVITY OF TREFFTZ-DG FORMULATION

$$\mathcal{A}_{TDG_F}((\omega_F, \mathbf{q}); (\omega_F, \mathbf{q})) = \|(\omega_F, \mathbf{q})\|_{TDG_F}^2, \quad \forall (\omega_F, \mathbf{q}) \in \mathbf{T}_F(\mathcal{T}_h)$$

# ACOUSTIC SYSTEM

## COERCIVITY OF TREFFTZ-DG FORMULATION

$$\mathcal{A}_{TDG_F}((\omega_F, \mathbf{q}); (\omega_F, \mathbf{q})) = |||(\omega_F, \mathbf{q})|||_{TDG_F}^2, \quad \forall (\omega_F, \mathbf{q}) \in \mathbf{T}_F(\mathcal{T}_h)$$

## CONTINUITY OF TREFFTZ-DG FORMULATION

$$|\mathcal{A}_{TDG_F}((\mathbf{v}_F, \mathbf{p}); (\omega_F, \mathbf{q}))| \leq 2 |||(\mathbf{v}_F, \mathbf{p})|||_{TDG_F^*} |||(\omega_F, \mathbf{q})|||_{TDG_F},$$

$$|\ell_{TDG_F}(\omega_F, \mathbf{q})| \leq \sqrt{2} \left[ \|\rho_F^{1/2} \mathbf{v}_{f0}\|_{L^2(\mathcal{F}_h^{0F})}^2 + \left\| \left( \frac{1}{c_F^2 \rho_F} \right)^{1/2} p_0 \right\|_{L^2(\mathcal{F}_h^{0F})}^2 \right]^{1/2}$$



# ACOUSTIC SYSTEM

## WELL-POSEDNESS

$$\| (v_F - v_{Fh}, p - p_h) \|_{TDG_F} \leq 3 \inf_{(\omega_F, q) \in \mathbf{V}(\mathcal{T}_h)} \| (v_F - \omega_F, p - q) \|_{TDG_F^*}$$

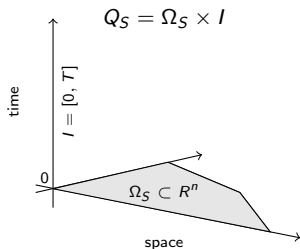
# MATHEMATICAL FORMULATION

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ELASTODYNAMIC SYSTEM

# ELASTODYNAMIC SYSTEM

## PROBLEM EQUATIONS

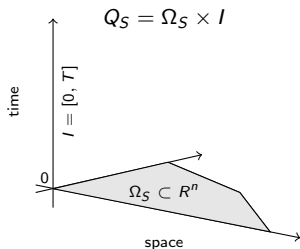


(ES)

$$\left[ \begin{array}{ll} \frac{\partial \underline{\underline{\sigma}}}{\partial t} - \underline{\underline{C}} \underline{\underline{\varepsilon}}(v_S) = 0 & \text{in } Q_S \\ \rho_S \frac{\partial v_S}{\partial t} - \operatorname{div} \underline{\underline{\sigma}} = 0 & \text{in } Q_S \\ v_S(\cdot, 0) = v_{S0}, \underline{\underline{\sigma}}(\cdot, 0) = \underline{\underline{\sigma}}_0 & \text{on } \Omega_S \\ v_S = g_{D_S} & \text{on } \partial \Omega_S \times I \end{array} \right.$$

# ELASTODYNAMIC SYSTEM

## PROBLEM EQUATIONS

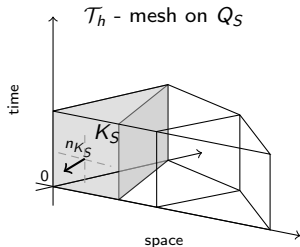


(ES)

$$\left[ \begin{array}{ll} \underline{A} \frac{\partial \underline{\sigma}}{\partial t} - \underline{\underline{\epsilon}}(v_S) = 0 & \text{in } Q_S \\ \rho_S \frac{\partial v_S}{\partial t} - \operatorname{div} \underline{\underline{\sigma}} = 0 & \text{in } Q_S \\ v_S(\cdot, 0) = v_{S0}, \underline{\underline{\sigma}}(\cdot, 0) = \underline{\underline{\sigma}}_0 & \text{on } \Omega_S \\ v_S = g_{D_S} & \text{on } \partial\Omega_S \times I \end{array} \right.$$

# ELASTODYNAMIC SYSTEM

## MESHING



$\mathcal{F}_h = \cup_{K_S \in \mathcal{T}_h} \partial K_S$  - mesh skeleton

$K_S \in Q_S$  ( $A, \rho_S \equiv \text{const}$  in  $K_S$ )

$v_S, \underline{\underline{\sigma}} \in H^1(K_S)$

$\omega_S, \underline{\underline{\xi}} \in H^1(K_S)$

# ELASTODYNAMIC SYSTEM

## TREFFTZ SPACE

$$\mathbf{T}_S(\mathcal{T}_h) := \left\{ (\omega_S, \underline{\underline{\xi}}) \in H^1(\mathcal{T}_h)^2 : \forall K_S \in \mathcal{T}_h, \rho_S \frac{\partial \omega_S}{\partial t} - \operatorname{div} \underline{\underline{\xi}} = \underline{\underline{A}} \frac{\partial \underline{\underline{\xi}}}{\partial t} - \underline{\underline{\xi}}(\omega_S) = 0 \right\}$$

# ELASTODYNAMIC SYSTEM

## TREFFTZ SPACE

$$\mathbf{T}_S(\mathcal{T}_h) := \left\{ (\omega_S, \underline{\underline{\xi}}) \in H^1(\mathcal{T}_h)^2 : \forall K_S \in \mathcal{T}_h, \rho_S \frac{\partial \omega_S}{\partial t} - \operatorname{div} \underline{\underline{\xi}} = \underline{\underline{A}} \frac{\partial \underline{\underline{\xi}}}{\partial t} - \underline{\underline{\varepsilon}}(\omega_S) = 0 \right\}$$

## SPACE-TIME DISCRETIZATION IN $T_S$

$$\sum_{K_S} \int_{\partial K_S} \left[ (\underline{\underline{A}} \hat{\underline{\underline{\sigma}}}_h : \underline{\underline{\xi}} + \rho_S v \hat{S}_h \omega_S) n_{K_S}^t - (v \hat{S}_h \underline{\underline{\xi}} + \hat{\underline{\underline{\sigma}}}_h \omega_S) \cdot n_{K_S}^x \right] ds = 0$$

# ELASTODYNAMIC SYSTEM

## TREFFTZ-DG FORMULATION

$$\sum_{K_S} \int_{\partial K_S} \left[ (\underline{\underline{A}} \underline{\underline{\hat{\sigma}}}_h : \underline{\underline{\xi}} + \rho_S v \hat{S}_h \omega_S) n_{K_S}^\dagger - (v \hat{S}_h \underline{\underline{\xi}} + \underline{\underline{\hat{\sigma}}}_h \omega_S) \cdot n_{K_S}^x \right] ds = 0$$

## NUMERICAL FLUXES THROUGH THE ELEMENT FACES

$$\begin{aligned} \mathcal{F}_h^{I_S} : \quad \begin{pmatrix} v \hat{S}_h \\ \underline{\underline{\hat{\sigma}}}_h \end{pmatrix} &:= \begin{pmatrix} \{v_S h\} - \delta [\underline{\underline{\sigma}}_h]_x \\ \{ \underline{\underline{\sigma}}_h \} - \gamma [v_S h]_x \end{pmatrix} \\ \mathcal{F}_h^{D_S} : \quad \begin{pmatrix} v \hat{S}_h \\ \underline{\underline{\hat{\sigma}}}_h \end{pmatrix} &:= \begin{pmatrix} v_S h - \delta (\underline{\underline{\sigma}}_h \cdot n_{K_S}^x - g_{D_S}) \\ g_{D_S} \end{pmatrix} \\ \mathcal{F}_h^{\Omega_S} : \quad \begin{pmatrix} v \hat{S}_h \\ \underline{\underline{\hat{\sigma}}}_h \end{pmatrix} &:= \begin{pmatrix} v_S h^- \\ \underline{\underline{\sigma}}_h^- \end{pmatrix} \\ \mathcal{F}_h^{T_S} : \quad \begin{pmatrix} v \hat{S}_h \\ \underline{\underline{\hat{\sigma}}}_h \end{pmatrix} &:= \begin{pmatrix} v_S h \\ \underline{\underline{\sigma}}_h \end{pmatrix} \\ \mathcal{F}_h^{0_S} : \quad \begin{pmatrix} v \hat{S}_h \\ \underline{\underline{\hat{\sigma}}}_h \end{pmatrix} &:= \begin{pmatrix} v_S 0 \\ \underline{\underline{\sigma}}_0 \end{pmatrix} \end{aligned}$$



# ELASTODYNAMIC SYSTEM

## TREFFTZ-DG FORMULATION

Seek  $(v_{S_h}, \underline{\underline{\sigma}}_h) \in \mathbf{V}(\mathcal{T}_h) \subset \mathbf{T}_S(\mathcal{T}_h)$ , s.t. for all  $K_S \in \mathcal{T}_h$ ,  $(\omega_S, \underline{\underline{\xi}}) \in \mathbf{V}(\mathcal{T}_h)$  it holds:

$$\mathcal{A}_{TDG_S}((v_{S_h}, \underline{\underline{\sigma}}_h); (\omega_S, \underline{\underline{\xi}})) = \ell_{TDG_S}(\omega_S, \underline{\underline{\xi}})$$

# ELASTODYNAMIC SYSTEM

$L^2(\mathcal{T}_h)$  NORMS IN  $T_S$

$$\begin{aligned}
 |||(\omega_S, \underline{\xi})|||_{TDG_S}^2 &:= \frac{1}{2} \|(\underline{A})^{1/2} [\underline{\xi}]_t\|_{L^2(\mathcal{F}_h^{\Omega_S})}^2 + \frac{1}{2} \|\rho_S^{1/2} [\omega_S]_t\|_{L^2(\mathcal{F}_h^{\Omega_S})}^2 + \|\gamma^{1/2} [\omega_S]_x\|_{L^2(\mathcal{F}_h^{I_S})}^2 \\
 &+ \|\delta^{1/2} [\underline{\xi}]_x\|_{L^2(\mathcal{F}_h^{I_S})}^2 + \frac{1}{2} \|(\underline{A})^{1/2} \underline{\xi}\|_{L^2(\mathcal{F}_h^{T_S})}^2 + \frac{1}{2} \|\rho_S^{1/2} \omega_S\|_{L^2(\mathcal{F}_h^{T_S})}^2 \\
 &+ \|\delta^{1/2} \underline{\xi}\|_{L^2(\mathcal{F}_h^{D_S})}^2
 \end{aligned}$$

# ELASTODYNAMIC SYSTEM

$L^2(\mathcal{T}_h)$  NORMS IN  $T_S$

$$\begin{aligned} |||(\omega_S, \underline{\xi})|||_{TDG_S}^2 &:= \frac{1}{2} \|(\underline{A})^{1/2} [\underline{\xi}]_t\|_{L^2(\mathcal{F}_h^{\Omega_S})}^2 + \frac{1}{2} \|\rho_S^{1/2} [\omega_S]_t\|_{L^2(\mathcal{F}_h^{\Omega_S})}^2 + \|\gamma^{1/2} [\omega_S]_x\|_{L^2(\mathcal{F}_h^{I_S})}^2 \\ &+ \|\delta^{1/2} [\underline{\xi}]_x\|_{L^2(\mathcal{F}_h^{I_S})}^2 + \frac{1}{2} \|(\underline{A})^{1/2} \underline{\xi}\|_{L^2(\mathcal{F}_h^{T_S})}^2 + \frac{1}{2} \|\rho_S^{1/2} \omega_S\|_{L^2(\mathcal{F}_h^{T_S})}^2 \\ &+ \|\delta^{1/2} \underline{\xi}\|_{L^2(\mathcal{F}_h^{D_S})}^2 \end{aligned}$$

$$\begin{aligned} |||(\omega_S, \underline{\xi})|||_{TDG_S^*}^2 &:= |||(\omega_S, \underline{\xi})|||_{TDG_S}^2 \\ &+ \|\rho_S^{1/2} \omega_S^-\|_{L^2(\mathcal{F}_h^{\Omega_S})}^2 + \|(\underline{A})^{1/2} \underline{\xi}^-\|_{L^2(\mathcal{F}_h^{\Omega_S})}^2 + \|\delta^{-1/2} \{\omega_S\}\|_{L^2(\mathcal{F}_h^{I_S})}^2 \\ &+ \|\gamma^{-1/2} \{\underline{\xi}\}\|_{L^2(\mathcal{F}_h^{I_S})}^2 + \|\delta^{-1/2} \underline{\xi}\|_{L^2(\mathcal{F}_h^{D_S})}^2. \end{aligned}$$

# ELASTODYNAMIC SYSTEM

## COERCIVITY OF TREFFTZ-DG FORMULATION

$$\mathcal{A}_{TDG_S}((\omega_S, \underline{\underline{\xi}}); (\omega_S, \underline{\underline{\xi}})) = |||(\omega_S, \underline{\underline{\xi}})|||_{TDG_S}^2, \quad \forall (\omega_S, \underline{\underline{\xi}}) \in \mathbf{T}_S(\mathcal{T}_h)$$

# ELASTODYNAMIC SYSTEM

## COERCIVITY OF TREFFTZ-DG FORMULATION

$$\mathcal{A}_{TDG_S}((\omega_S, \underline{\xi}); (\omega_S, \underline{\xi})) = |||(\omega_S, \underline{\xi})|||_{TDG_S}^2, \quad \forall (\omega_S, \underline{\xi}) \in \mathbf{T}_S(\mathcal{T}_h)$$

## CONTINUITY OF TREFFTZ-DG FORMULATION

$$|\mathcal{A}_{TDG_S}((v_S, \underline{\sigma}); (\omega_S, \underline{\xi}))| \leq 2 |||(v_S, \underline{\sigma})|||_{TDG_S^*} |||(\omega_S, \underline{\xi})|||_{TDG_S}$$

$$|\ell_{TDG_S}(\omega_S, \underline{\xi})| \leq \sqrt{2} \left[ \|\rho_S^{1/2} v_{s0}\|_{L^2(\mathcal{F}_h^0 S)}^2 + \|\underline{A}^{1/2} \underline{\sigma}_0\|_{L^2(\mathcal{F}_h^0 S)}^2 \right]^{1/2}$$

# ELASTODYNAMIC SYSTEM

## WELL-POSEDNESS

$$\| (v_S - v_{S_h}, \underline{\underline{\sigma}} - \underline{\underline{\sigma}}_h) \|_{TDG_S} \leq 3 \inf_{(\omega_S, \underline{\underline{\xi}}) \in \mathbf{V}(\mathcal{T}_h)} \| (v_S - \omega_S, \underline{\underline{\sigma}} - \underline{\underline{\xi}}) \|_{TDG_S^*}$$

# MATHEMATICAL FORMULATION

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ELASTO-ACOUSTIC SYSTEM

# ELASTO-ACOUSTIC SYSTEM

## NUMERICAL COUPLING ADVANTAGES



Computing 1 unknown scalar pressure instead of 6 components of stress in fluid



# ELASTO-ACOUSTIC SYSTEM

## NUMERICAL COUPLING ADVANTAGES

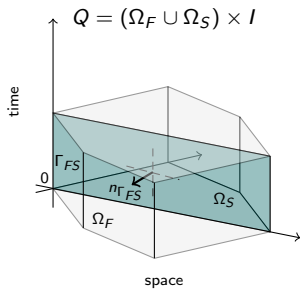


Computing 1 unknown scalar pressure instead of 6 components of stress in fluid

Avoiding the numerical artefacts caused by the slow  $S$  - waves

# ELASTO-ACOUSTIC SYSTEM

## TRANSMISSION CONDITIONS



(TC) 
$$\begin{cases} v_F n_{\Gamma_{FS}} = v_S n_{\Gamma_{FS}} & \text{on } \Gamma_{FS} \\ \underline{\underline{\sigma}} \cdot n_{\Gamma_{FS}} = -p n_{\Gamma_{FS}} & \text{on } \Gamma_{FS} \end{cases}$$

# ELASTO-ACOUSTIC SYSTEM

## TREFFTZ SPACE

$$\mathbf{T}(\mathcal{T}_h) := \left\{ (\omega_F, \mathbf{q}, \omega_S, \underline{\underline{\xi}}) \in H^1(\mathcal{T}_h)^4 : \rho_F \frac{\partial \omega_F}{\partial t} + \nabla \mathbf{q} = \frac{1}{c_F^2 \rho_F} \frac{\partial \mathbf{q}}{\partial t} + \operatorname{div} \omega_F = 0, \right. \\ \left. \rho_S \frac{\partial \omega_S}{\partial t} - \operatorname{div} \underline{\underline{\xi}} = \frac{\partial \underline{\underline{A}} \underline{\underline{\xi}}}{\partial t} - \underline{\underline{\varepsilon}}(\omega_S) = 0, \forall K_F, K_S \in \mathcal{T}_h \right\}$$

# ELASTO-ACOUSTIC SYSTEM

## TREFFTZ SPACE

$$\mathbf{T}(\mathcal{T}_h) := \left\{ (\omega_F, \mathbf{q}, \omega_S, \underline{\underline{\xi}}) \in H^1(\mathcal{T}_h)^4 : \rho_F \frac{\partial \omega_F}{\partial t} + \nabla \mathbf{q} = \frac{1}{c_F^2 \rho_F} \frac{\partial \mathbf{q}}{\partial t} + \operatorname{div} \omega_F = 0, \right. \\ \left. \rho_S \frac{\partial \omega_S}{\partial t} - \operatorname{div} \underline{\underline{\xi}} = \frac{\partial \underline{\underline{A}} \underline{\underline{\xi}}}{\partial t} - \underline{\underline{\xi}}(\omega_S) = 0, \forall K_F, K_S \in \mathcal{T}_h \right\}$$

## SPACE-TIME DISCRETIZATION IN T

$$\sum_{K_F} \int_{\partial K_F} \left[ \left( \frac{1}{c_F^2 \rho_F} \hat{\rho}_h \mathbf{q} + \rho_F \mathbf{v}_{\hat{F}h} \omega_F \right) n_{K_F}^{\dagger} + \left( \hat{\rho}_h \omega_F + \mathbf{v}_{\hat{F}h} \mathbf{q} \right) n_{K_F}^{\times} \right] ds \\ + \sum_{K_S} \int_{\partial K_S} \left[ \left( \underline{\underline{A}} \hat{\sigma}_h : \underline{\underline{\xi}} + \rho_S \mathbf{v}_{\hat{S}h} \omega_S \right) n_{K_S}^{\dagger} - \left( \mathbf{v}_{\hat{S}h} \underline{\underline{\xi}} + \underline{\underline{\sigma}}_h \omega_S \right) \cdot n_{K_S}^{\times} \right] ds = \sum_{K_F} \int_{K_F} f \mathbf{q} dv$$

# ELASTO-ACOUSTIC SYSTEM

## TREFFTZ-DG FORMULATION

$$\sum_{K_F} \int_{\partial K_F} \left[ \left( \frac{1}{c_F^2 \rho_F} \hat{p}_h \mathbf{q} + \rho_F v \hat{F}_h \cdot \omega_F \right) n_{K_F}^t + (\hat{p}_h \omega_F + v \hat{F}_h \mathbf{q}) \cdot n_{K_F}^x \right] ds,$$

$$+ \sum_{K_S} \int_{\partial K_S} \left[ (\underline{\underline{A}} \underline{\underline{\hat{\sigma}}}_h : \underline{\underline{\xi}} + \rho_S v \hat{S}_h \cdot \omega_S) n_{K_S}^t - (\underline{\underline{\xi}} v \hat{S}_h + \underline{\underline{\hat{\sigma}}}_h \omega_S) \cdot n_{K_S}^x \right] ds = \sum_{K_F} \int_{K_F} f q dv$$

## NUMERICAL FLUXES THROUGH THE FLUID-SOLID INTERFACE

$$\mathcal{F}_h^{FS} : \begin{pmatrix} v \hat{F}_h n_{K_F}^x \\ \hat{p}_h \\ v \hat{S}_h \\ \underline{\underline{\hat{\sigma}}}_h \cdot n_{K_S}^x \end{pmatrix} := \begin{pmatrix} v_{S_h} n_{K_F}^x \\ p_h + \alpha (v_{F_h} n_{K_F}^x - v_{S_h} n_{K_F}^x) \\ v_{S_h} - \delta (\underline{\underline{\sigma}}_h \cdot n_{K_S}^x - p_h n_{K_S}^x) \\ -p_h n_{K_S}^x \end{pmatrix}$$

# ELASTO-ACOUSTIC SYSTEM

## TREFFTZ-DG FORMULATION

Seek  $(v_{Fh}, p_h, v_{Sh}, \underline{\underline{\sigma}}_h) \in \mathbf{V}(\mathcal{T}_h)$ :  $\forall K_F, K_S \in \mathcal{T}_h, (\omega_F, \mathbf{q}, \omega_S, \underline{\underline{\xi}}) \in \mathbf{T}(\mathcal{T}_h)$  it holds:

$$\mathcal{A}_{TDG}((v_{Fh}, p_h, v_{Sh}, \underline{\underline{\sigma}}_h); (\omega_F, \mathbf{q}, \omega_S, \underline{\underline{\xi}})) = \ell_{TDG}(\omega_F, \mathbf{q}, \omega_S, \underline{\underline{\xi}})$$

# ELASTO-ACOUSTIC SYSTEM

$L^2(\mathcal{T}_h)$  NORMS IN T

$$\|(\omega_F, \mathbf{q}, \omega_S, \underline{\underline{\xi}})\|_{TDG}^2 := \|(\omega_F, \mathbf{q})\|_{TDG_F}^2 + \|(\omega_S, \underline{\underline{\xi}})\|_{TDG_S}^2 + 2\|\delta^{1/2}\underline{\underline{\xi}}\|_{L^2(\mathcal{F}_h^{FS})}^2$$

# ELASTO-ACOUSTIC SYSTEM

$L^2(\mathcal{T}_h)$  NORMS IN T

$$\|(\omega_F, \mathbf{q}, \omega_S, \underline{\underline{\xi}})\|_{TDG}^2 := \|(\omega_F, \mathbf{q})\|_{TDG_F}^2 + \|(\omega_S, \underline{\underline{\xi}})\|_{TDG_S}^2 + 2\|\delta^{1/2}\underline{\underline{\xi}}\|_{L^2(\mathcal{F}_h^{FS})}^2$$

$$\|(\omega_F, \mathbf{q}, \omega_S, \underline{\underline{\xi}})\|_{TDG^*}^2 := \|(\omega_F, \mathbf{q})\|_{TDG_F^*}^2 + \|(\omega_S, \underline{\underline{\xi}})\|_{TDG_S^*}^2 + \frac{1}{2}\|\delta^{-1/2}\underline{\underline{\xi}}\|_{L^2(\mathcal{F}_h^{FS})}^2$$



# ELASTO-ACOUSTIC SYSTEM

## COERCIVITY OF TREFFTZ-DG FORMULATION

$$\mathcal{A}_{TDG}((\omega_F, \mathbf{q}, \omega_S, \underline{\underline{\xi}}); (\omega_F, \mathbf{q}, \omega_S, \underline{\underline{\xi}})) = |||(\omega_F, \mathbf{q}, \omega_S, \underline{\underline{\xi}})|||_{TDG}^2$$

# ELASTO-ACOUSTIC SYSTEM

## COERCIVITY OF TREFFTZ-DG FORMULATION

$$\mathcal{A}_{TDG}((\omega_F, \mathbf{q}, \omega_S, \underline{\underline{\xi}}); (\omega_F, \mathbf{q}, \omega_S, \underline{\underline{\xi}})) = |||(\omega_F, \mathbf{q}, \omega_S, \underline{\underline{\xi}})|||_{TDG}^2$$

## CONTINUITY OF TREFFTZ-DG FORMULATION

$$|\mathcal{A}_{TDG}((v_F, \mathbf{p}, v_S, \underline{\underline{\sigma}}); (\omega_F, \mathbf{q}, \omega_S, \underline{\underline{\xi}}))| \leq 2 |||(v_F, \mathbf{p}, v_S, \underline{\underline{\sigma}})|||_{TDG} * |||(\omega_F, \mathbf{q}, \omega_S, \underline{\underline{\xi}})|||_{TDG}$$

$$|\ell_{TDG}(\omega_F, \mathbf{q}, \omega_S, \underline{\underline{\xi}})| \leq \sqrt{2} \left[ \|\rho_F^{1/2} v_{F0}\|_{L^2(\mathcal{F}_h^{0F})}^2 + \left\| \left( \frac{1}{c_F^2 \rho_F} \right)^{1/2} \mathbf{p}_0 \right\|_{L^2(\mathcal{F}_h^{0F})}^2 \right. \\ \left. + \|\rho_S^{1/2} v_{S0}\|_{L^2(\mathcal{F}_h^{0S})}^2 + \|\underline{\underline{A}}^{1/2} \underline{\underline{\sigma}}_0\|_{L^2(\mathcal{F}_h^{0S})}^2 \right]^{1/2}$$

# ELASTO-ACOUSTIC SYSTEM

## WELL-POSEDNESS

$$\| (v_F - v_{Fh}, p - p_h, v_S - v_{Sh}, \underline{\underline{\sigma}} - \underline{\underline{\sigma}}_h) \|_{TDG}$$

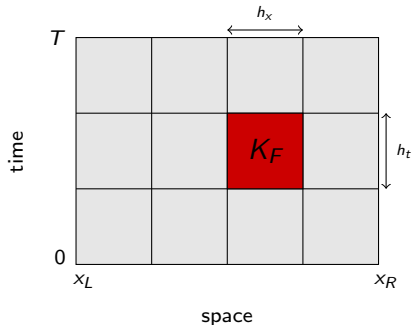
$$\leq 3 \inf_{(\omega_F, q, \omega_S, \underline{\underline{\xi}}) \in \mathbf{V}(\mathcal{T}_h)} \| (v_F - \omega_F, p - q, v_S - \omega_S, \underline{\underline{\sigma}} - \underline{\underline{\xi}}) \|_{TDG^*}.$$

# IMPLEMENTATION OF THE ALGORITHM

1D ACOUSTIC SYSTEM EXAMPLE

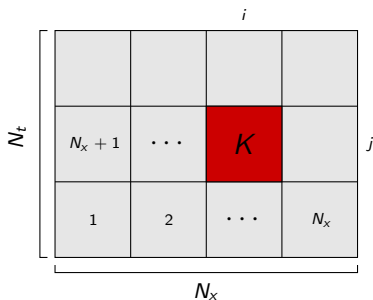
# IMPLEMENTATION OF THE ALGORITHM

## MESH AND ELEMENT NUMBERING



# IMPLEMENTATION OF THE ALGORITHM

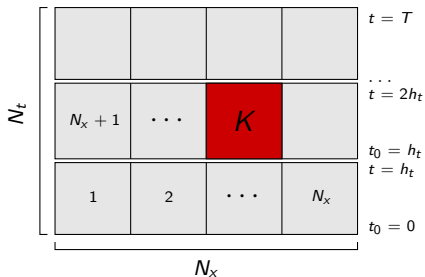
## MESH AND ELEMENT NUMBERING



$$K = (j - 1) \times N_x + i$$

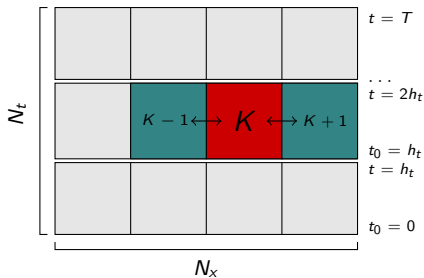
# IMPLEMENTATION OF THE ALGORITHM

## MESH AND ELEMENT NUMBERING



# IMPLEMENTATION OF THE ALGORITHM

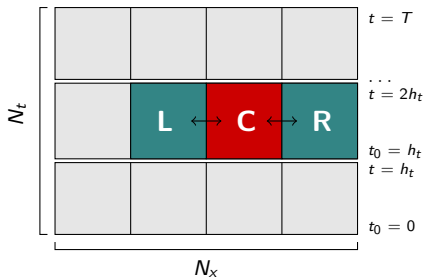
## MESH AND ELEMENT NUMBERING





# IMPLEMENTATION OF THE ALGORITHM

## MESH AND ELEMENT NUMBERING



## IMPLEMENTATION OF THE ALGORITHM

GLOBAL MATRIX ( $1^{ST}$  LAYER)

$$\begin{bmatrix}
 \boxed{C} & \boxed{R} & & \dots & & & \boxed{L} \\
 \boxed{L} & \boxed{C} & \boxed{R} & & & & \\
 & \boxed{L} & \boxed{C} & \boxed{R} & & & \\
 \vdots & & & \ddots & & & \vdots \\
 & & & & \boxed{L} & \boxed{C} & \boxed{R} \\
 \boxed{R} & & \dots & & \boxed{L} & \boxed{C} & 
 \end{bmatrix}
 \times
 \begin{bmatrix}
 \boxed{u_1} \\
 \boxed{u_2} \\
 \boxed{u_3} \\
 \vdots \\
 \boxed{u_{N_x-1}} \\
 \boxed{u_{N_x}}
 \end{bmatrix}
 =
 \begin{bmatrix}
 \boxed{u_1^0} \\
 \boxed{u_2^0} \\
 \boxed{u_3^0} \\
 \vdots \\
 \boxed{u_{N_x-1}^0} \\
 \boxed{u_{N_x}^0}
 \end{bmatrix}$$

# IMPLEMENTATION OF THE ALGORITHM

GLOBAL MATRIX ( $1^{ST}$  LAYER)

$$\mathbf{M} \times \mathbf{U} = \mathbf{U}^0$$

# IMPLEMENTATION OF THE ALGORITHM

## 1. INITIATION

COMPUTE

$$\left[ \begin{array}{c} \mathbf{M} \end{array} \right] ; \left[ \begin{array}{c} \mathbf{M}^{-1} \end{array} \right] ; \left[ \begin{array}{c} \mathbf{U}^0 \end{array} \right]$$

## 2. PROPAGATION

FOR  $t = 1 : N_t$

$$\left[ \begin{array}{c} \mathbf{U} \end{array} \right] = \left[ \begin{array}{c} \mathbf{M}^{-1} \end{array} \right] \times \left[ \begin{array}{c} \mathbf{U}^0 \end{array} \right]$$

$$\left[ \begin{array}{c} \mathbf{U}^0 \end{array} \right] = \left[ \begin{array}{c} \text{projection} \\ \text{operator} \end{array} \right] \times \left[ \begin{array}{c} \mathbf{U} \end{array} \right]$$

ENDFOR

# IMPLEMENTATION OF THE ALGORITHM

## POLYNOMIAL BASIS (ORDER 3)

$$\phi_1^v = 0$$

$$\phi_2^v = 1$$

$$\phi_3^v = x$$

$$\phi_4^v = c_F t$$

$$\phi_5^v = -\frac{x^2}{2} - \frac{c_F^2 t^2}{2}$$

$$\phi_6^v = -c_F x t$$

$$\phi_7^v = -\frac{x^3}{6} - \frac{x c_F^2 t^2}{2}$$

$$\phi_8^v = -\frac{c_F^3 t^3}{6} - \frac{x^2 c_F t}{2}$$

$$\phi_1^p = -c_F$$

$$\phi_2^p = 0$$

$$\phi_3^p = -c_F^2 t$$

$$\phi_4^p = -c_F x$$

$$\phi_5^p = c_F^2 x t$$

$$\phi_6^p = c_F \left( \frac{x^2}{2} + \frac{c_F^2 t^2}{2} \right)$$

$$\phi_7^p = c_F \left( \frac{c_F^3 t^3}{6} + \frac{x^2 c_F t}{2} \right)$$

$$\phi_8^p = c_F \left( \frac{x^3}{6} + \frac{x c_F^2 t^2}{2} \right)$$

# IMPLEMENTATION OF THE ALGORITHM

## NUMERICAL TESTS

FIGURE 1: 1D Acoustic system.  
Exact and numerical velocities  $v_F(x, t = 1)$ .  
Periodical boundaries.  $\alpha = \beta = 0$

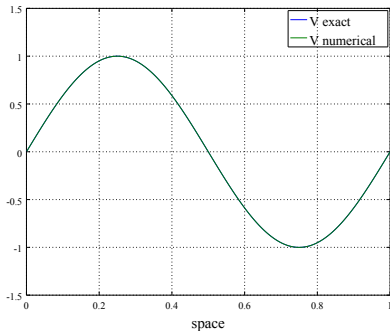
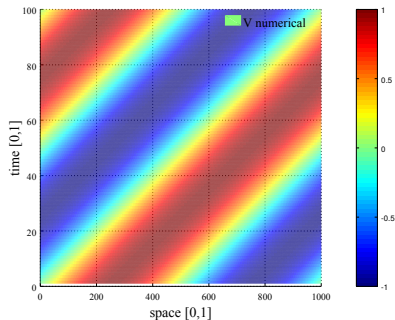


FIGURE 2: 1D Acoustic system.  
Numerical velocity  $v_F(x, t)$ .  
Periodical boundaries.  $\alpha = \beta = 0$



# IMPLEMENTATION OF THE ALGORITHM

## NUMERICAL TESTS

FIGURE 3: 1D Acoustic system.  
Exact and numerical velocities  $v_F(x, t = 1)$ .  
Periodical boundaries.  $\alpha = \beta = 0$

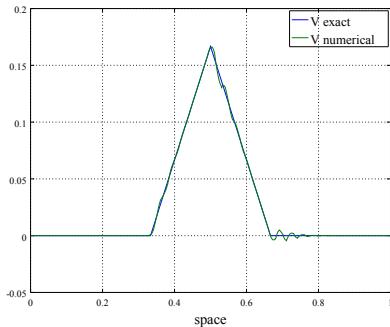
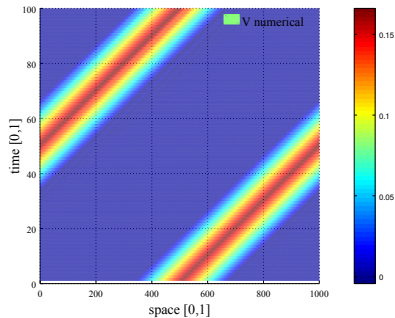


FIGURE 4: 1D Acoustic system.  
Numerical velocity  $v_F(x, t)$ .  
Periodical boundaries.  $\alpha = \beta = 0$



# IMPLEMENTATION OF THE ALGORITHM

## NUMERICAL TESTS

FIGURE 5: 1D Acoustic system.  
Exact and numerical velocities  $v_F(x, t = 1)$ .  
Periodical boundaries.  $\alpha = \beta = 0.5$

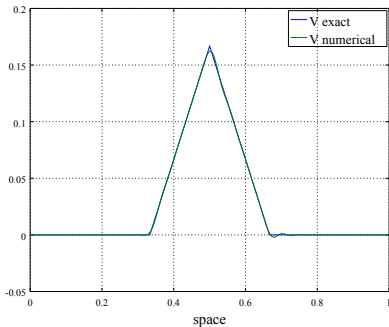
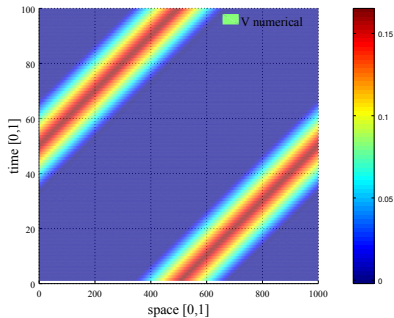


FIGURE 6: 1D Acoustic system.  
Numerical velocity  $v_F(x, t)$ .  
Periodical boundaries.  $\alpha = \beta = 0.5$





## IMPLEMENTATION OF THE ALGORITHM

## NUMERICAL TESTS

FIGURE 7: 1D Acoustic system.  
Convergence of velocity  $v_F$  in function of cell size  $h$ .  
Periodical boundaries.  $\alpha = \beta = 0.0$

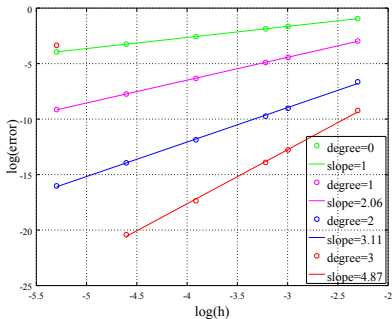
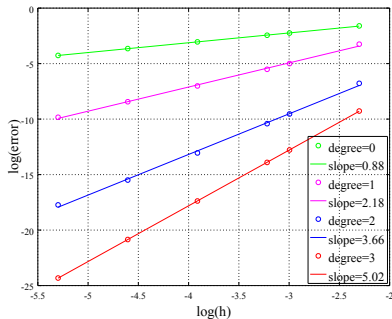


FIGURE 8: 1D Acoustic system.  
Convergence of velocity  $v_F$  in function of cell size  $h$ .  
Periodical boundaries.  $\alpha = \beta = 0.5$



## IMPLEMENTATION OF THE ALGORITHM

## NUMERICAL TESTS

FIGURE 9: 1D Acoustic system.  
Numerical error in function of ratio  $h_t/h_x$ .  
Periodical boundaries.  $\alpha = \beta = 0.5$

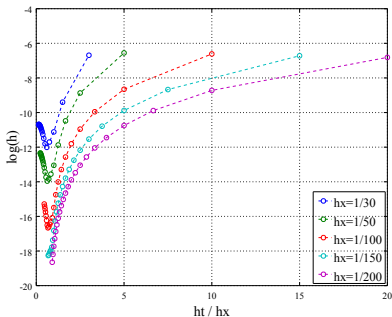
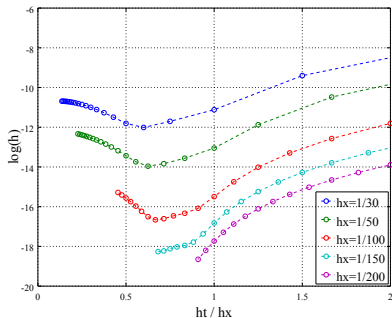


FIGURE 10: 1D Acoustic system.  
Numerical error in function of ratio  $h_t/h_x$ .  
Error minimum corresponds to  $h_t/h_x = 1/(1 + \alpha)$



# NUMERICAL RESULTS

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## 2D SIMULATION

# NUMERICAL RESULTS

## 2D ACOUSTIC AND ELASTODYNAMIC SYSTEMS

FIGURE 11: 2D Acoustic system.

Convergence of velocity  $v_F$  in function of cell size  $h$

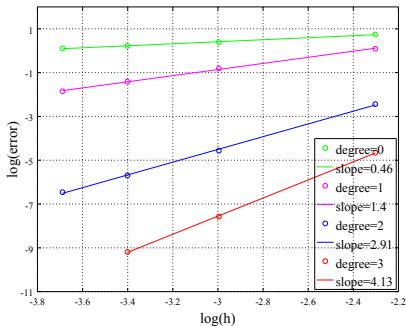
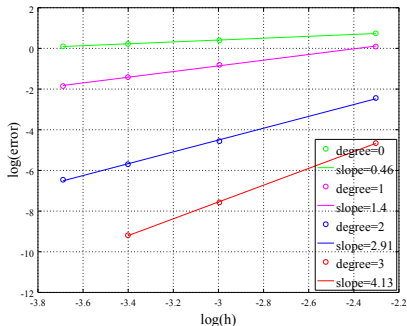


FIGURE 12: 2D Elastodynamic system.

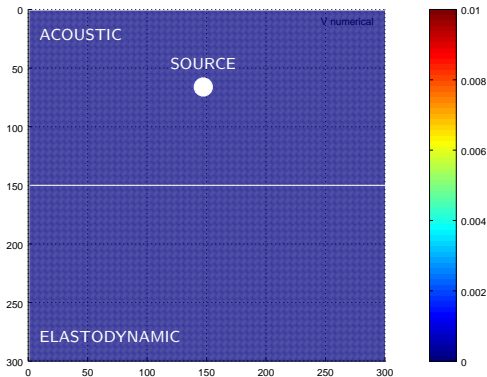
Convergence of velocity  $v_S$  in function of cell size  $h$



# NUMERICAL RESULTS

## 2D ELASTO-ACOUSTIC SYSTEM

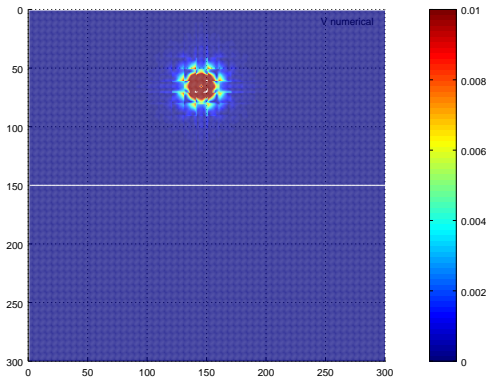
FIGURE 13: 2D Elasto-acoustic system.  
Numerical velocity  $v(x, 0.0)$ . Dirichlet boundaries



# NUMERICAL RESULTS

## 2D ELASTO-ACOUSTIC SYSTEM

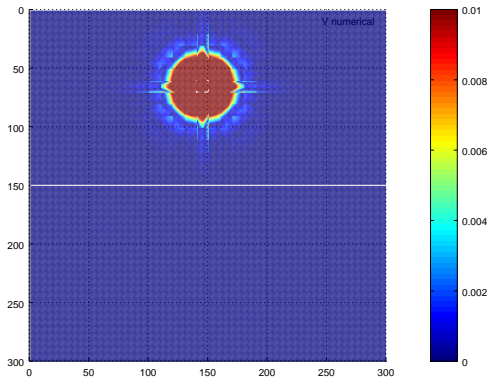
FIGURE 13: 2D Elasto-acoustic system.  
Numerical velocity  $v(x, 0.233)$ . Dirichlet boundaries



# NUMERICAL RESULTS

## 2D ELASTO-ACOUSTIC SYSTEM

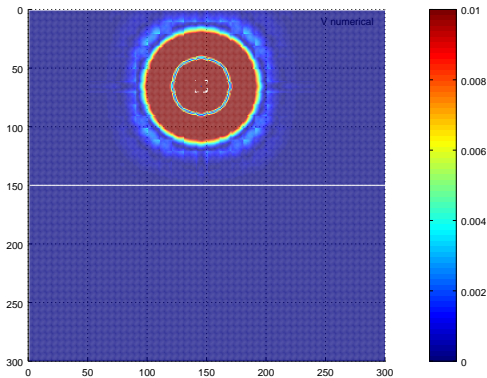
FIGURE 13: 2D Elasto-acoustic system.  
Numerical velocity  $v(x, 0.300)$ . Dirichlet boundaries



# NUMERICAL RESULTS

## 2D ELASTO-ACOUSTIC SYSTEM

FIGURE 13: 2D Elasto-acoustic system.  
Numerical velocity  $v(x, 0.367)$ . Dirichlet boundaries

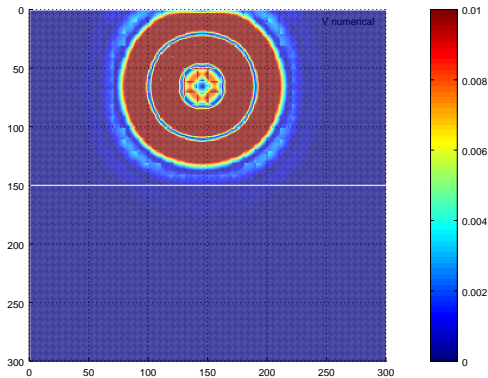




# NUMERICAL RESULTS

## 2D ELASTO-ACOUSTIC SYSTEM

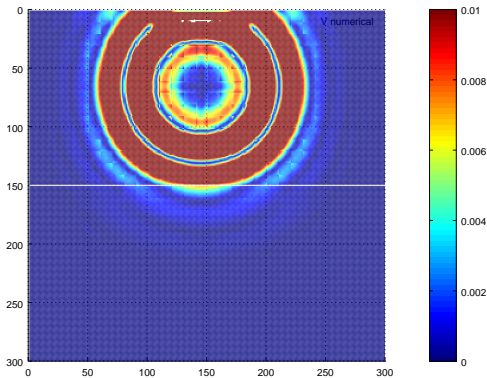
FIGURE 13: 2D Elasto-acoustic system.  
Numerical velocity  $v(x, 0.433)$ . Dirichlet boundaries



# NUMERICAL RESULTS

## 2D ELASTO-ACOUSTIC SYSTEM

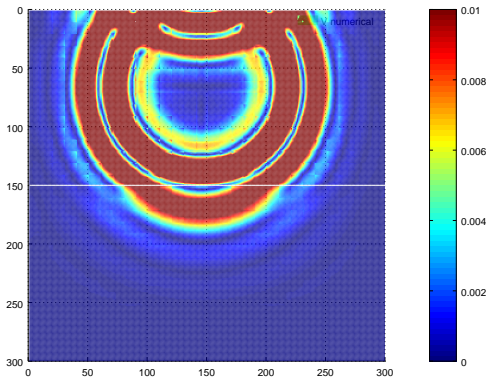
FIGURE 13: 2D Elasto-acoustic system.  
Numerical velocity  $v(x, 0.500)$ . Dirichlet boundaries



# NUMERICAL RESULTS

## 2D ELASTO-ACOUSTIC SYSTEM

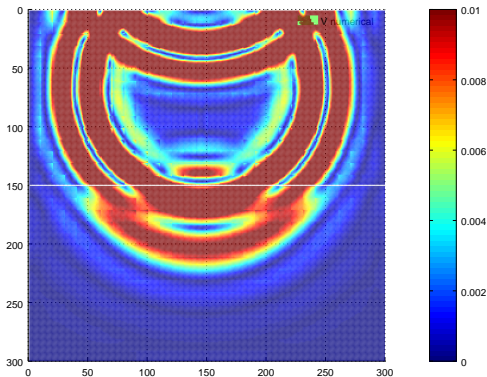
FIGURE 13: 2D Elasto-acoustic system.  
Numerical velocity  $v(x, 0.567)$ . Dirichlet boundaries



# NUMERICAL RESULTS

## 2D ELASTO-ACOUSTIC SYSTEM

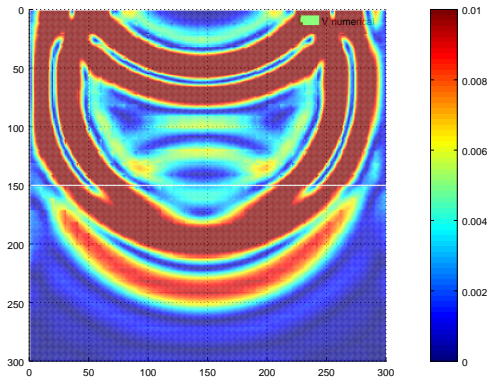
FIGURE 13: 2D Elasto-acoustic system.  
Numerical velocity  $v(x, 0.633)$ . Dirichlet boundaries



# NUMERICAL RESULTS

## 2D ELASTO-ACOUSTIC SYSTEM

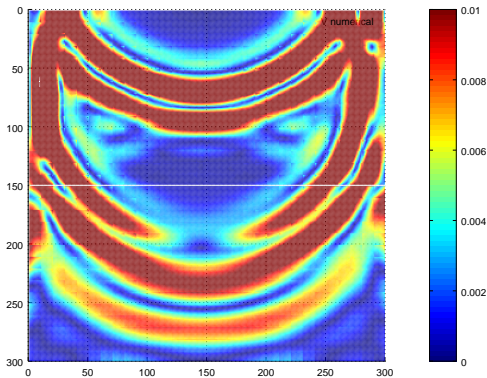
FIGURE 13: 2D Elasto-acoustic system.  
Numerical velocity  $v(x, 0.700)$ . Dirichlet boundaries



# NUMERICAL RESULTS

## 2D ELASTO-ACOUSTIC SYSTEM

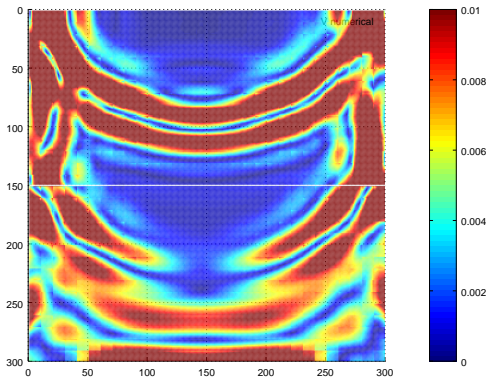
FIGURE 13: 2D Elasto-acoustic system.  
Numerical velocity  $v(x, 0.767)$ . Dirichlet boundaries



# NUMERICAL RESULTS

## 2D ELASTO-ACOUSTIC SYSTEM

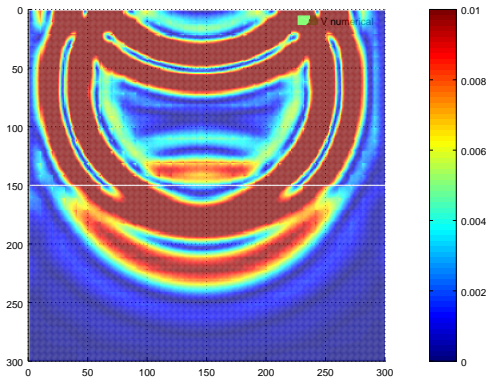
FIGURE 13: 2D Elasto-acoustic system.  
Numerical velocity  $v(x, 0.833)$ . Dirichlet boundaries



# NUMERICAL RESULTS

## 2D ELASTO-ACOUSTIC SYSTEM

FIGURE 13: 2D Elasto-acoustic system.  
Numerical velocity  $v(x, 0.667)$ . Dirichlet boundaries





# NUMERICAL RESULTS

## 2D ELASTO-ACOUSTIC SYSTEM

FIGURE 14: 2D Elasto-acoustic system.  
Numerical and analytical\* seismograms for  $v_x$ .  
Source (150,75), receiver (195, 210)

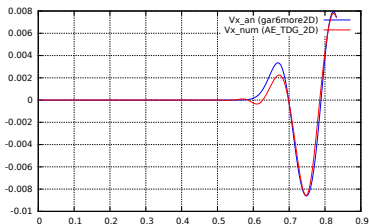
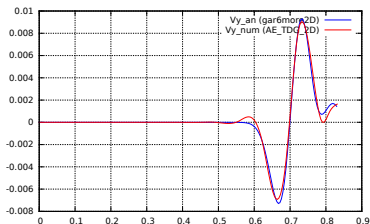


FIGURE 15: 2D Elasto-acoustic system.  
Numerical and analytical\* seismograms for  $v_y$ .  
Source (150,75), receiver (195, 210)



\* J.Diaz. *Garbmore2D*.

# CONCLUSION

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# CONCLUSION



Implementation of the method for 2D acoustic, elastic, elasto-acoustic systems

Numerical tests, validation of the results by analytical solutions

Analytical and numerical study of efficiency of the method

Change-over between time-layers study (initial conditions)

# CONCLUSION

## ON-GOING WORK AND PERSPECTIVES



Change-over between time-layers study (source)

Alternative for the global matrix inversion

Boundary conditions

Hybrid Trefftz-DG + FVM coupling for the elasto-acoustics

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THANK YOU FOR ATTENTION!

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