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► **To cite this version:**

Juliette Chabassier, Marc Duruflé, Victor Péron. Equivalent boundary conditions for acoustic media with exponential densities. Eighth Singular Days , Jun 2016, Nancy, France. hal-01691768

**HAL Id: hal-01691768**

**<https://inria.hal.science/hal-01691768v1>**

Submitted on 24 Jan 2018

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# Equivalent boundary conditions for acoustic media with exponential densities

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Eighth Singular Days, Nancy

June 26-30, 2016

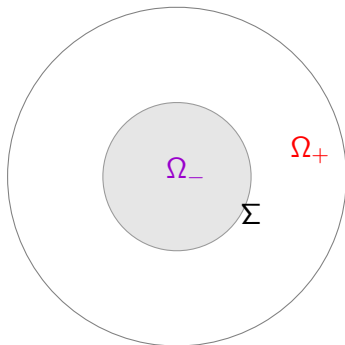


# Introduction

## A 3-D Problem set in heterogeneous acoustic media

We solve the Helmholtz equation (with a Dirichlet B.C.) :

$$-\operatorname{div} \left( \frac{1}{\rho} \nabla u \right) - \frac{\omega^2}{\rho c^2} u = f \quad \text{in } \Omega$$



- $\Omega_- = \{r < r_t\}$  : Density  $\rho^-$ , Velocity  $c^-$
- $\Sigma = \{r = r_t\}$  : Interface
- $\Omega_+ = \Omega \setminus \overline{\Omega_-}$  : Density  $\rho^+$ , Velocity  $c^+$

# Framework

## Data and right-hand side

### Assumption

- (i) The density  $\rho$  and the velocity  $c$  are **smooth** and strictly **positive** functions
- (ii) The density  $\rho^+$  satisfies

$$\rho^+(r) = \gamma \exp(-\alpha(r - r_t))$$

where *the parameter  $\alpha$  is large*

- (iii) The domain  $\Omega_+$  represents a **dissipative medium** :

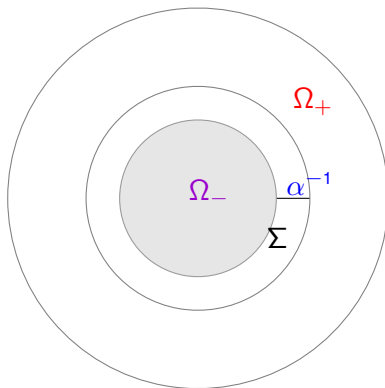
$$\text{Im } \omega^2 \neq 0 \quad \text{in } \Omega_+$$

### Assumption

The right-hand side  $f$  has a support in  $\Omega_-$

## Context

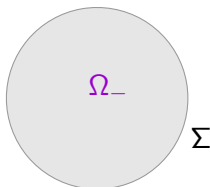
A **boundary layer phenomenon** occurs when  $\alpha$  is large : rapid decay of acoustic fields (pressure and velocity) inside the medium  $\Omega_+$



**Difficulty** : Applying a FEM on a mesh with **thin cells** in the **boundary layer** and much larger **elsewhere**

## Approach : an Asymptotic Method

- 1 Describing the **boundary layer phenomenon** with a multiscale expansion in power series of  $\alpha^{-1}$
- 2 Deriving Equivalent Boundary Conditions on  $\Sigma$  to replace the **boundary layer** inside  $\Omega_+$
- 3 Applying a Finite Element Method to solve the acoustic equation in  $\Omega_-$  with an Equivalent Boundary Condition



## References on **skin effect** in electromagnetism



S. M. RYTOV

Calcul du skin effect par la méthode des perturbations.

*Journal of Physics* (1940).



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Solution procedures for interface problems in [...] electromagnetics.

*CISM Courses and Lectures*. **277** 291–348 (1983).



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A skin effect approximation for eddy current problems.

*Arch. Rational Mech. Anal.* **90**(1) (1985) 87–98.



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Generalized impedance [...] by strongly absorbing obstacles [...].

*Math. Models Methods Appl. Sci.* **15**(8) (2005) 1273–1300.



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On the influence of the geometry on skin effect in electromagnetism.

*Comput. Methods Appl. Mech. Engrg.* 200 (2011), no. 9-12, 1053–1068.

# Outline

- 1 **Uniform Estimates**
- 2 **Asymptotic Expansion**
- 3 **Equivalent Conditions**
- 4 **Numerical Results**



# Outline

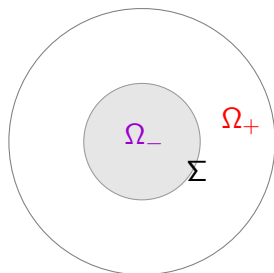
- 1 **Uniform Estimates**
- 2 Asymptotic Expansion
- 3 Equivalent Conditions
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## Problem ( $P_\alpha$ )

Helmholtz equation :

$$-\operatorname{div} \left( \frac{1}{\rho} \nabla u_\alpha \right) - \frac{\omega^2}{\rho c^2} u_\alpha = f \quad \text{in } \Omega$$

with an homogeneous Dirichlet boundary condition on  $\partial\Omega$ .



Here  $\rho^+(r) = \gamma \exp(-\alpha(r - r_t))$  and  $\alpha > 0$ .

## Problem ( $\mathbf{P}_\alpha$ )

**Issue** : *Uniform*  $H^1$  estimates for solutions  $u_\alpha$  of ( $\mathbf{P}_\alpha$ ) w.r.t.  $\alpha > 0$  ?

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**Issue :** *Uniform*  $H^1$  estimates for solutions  $u_\alpha$  of ( $\mathbf{P}_\alpha$ ) w.r.t.  $\alpha > 0$  ?

### Assumption (SA)

*The angular frequency  $\omega$  is not an eigenfrequency of the problem*

$$\begin{cases} \operatorname{div} \left( \frac{1}{\rho} \nabla u^- \right) + \frac{\omega^2}{\rho c^2} u^- = 0 & \text{in } \Omega_- \\ u^- = 0 & \text{on } \Sigma \end{cases}$$

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**Issue :** *Uniform  $H^1$  estimates for solutions  $u_\alpha$  of ( $\mathbf{P}_\alpha$ ) w.r.t.  $\alpha > 0$  ?*

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### Theorem

*Under Assumption (SA), for all  $\alpha > 0$  the problem ( $\mathbf{P}_\alpha$ ) with a right-hand side  $f \in L^2(\Omega)$  has a unique solution  $u_\alpha \in H_0^1(\Omega)$*

$$\|u_\alpha^-\|_{1,\Omega_-} + \left\| \frac{1}{\sqrt{\rho_+}} u_\alpha^+ \right\|_{0,\Omega_+} + \left\| \frac{1}{\sqrt{\rho_+}} \nabla u_\alpha^+ \right\|_{0,\Omega_+} \leq C \|f\|_{0,\Omega}$$

**Application: Convergence of asymptotic expansion for large parameter  $\alpha$**

## Problem ( $P_\delta$ )

Small parameter  $\delta$  :

$$\delta = \frac{1}{\alpha} \longrightarrow 0 \quad \text{when} \quad \alpha \rightarrow \infty$$

## Problem ( $\mathbf{P}_\delta$ )

Small parameter  $\delta$  :

$$\delta = \frac{1}{\alpha} \longrightarrow 0 \quad \text{when} \quad \alpha \rightarrow \infty$$

Problem ( $\mathbf{P}_\delta$ ) writes :

$$-\operatorname{div}\left(\frac{1}{\rho}\nabla u_\delta^-\right) - \frac{\omega^2}{\rho c^2}u_\delta^- = f \quad \text{in } \Omega_-$$

$$-\Delta u_\delta^+ - \frac{1}{\delta}\partial_r u_\delta^+ - \frac{\omega_+^2}{c^2}u_\delta^+ = 0 \quad \text{in } \Omega_+$$

$$u_\delta^+ = u_\delta^- \quad \text{on } \Sigma$$

$$\partial_n u_\delta^+ = \partial_n u_\delta^- \quad \text{on } \Sigma$$

$$u_\delta^+ = 0 \quad \text{on } \partial\Omega$$

# Outline

- 1 Uniform Estimates
- 2 Asymptotic Expansion**
- 3 Equivalent Conditions
- 4 Numerical Results



# Asymptotic Expansion

## Overview

Deriving an Asymptotic Expansion for the solution  $(u_\delta^-, u_\delta^+)$  of  $(\mathbf{P}_\delta)$  when  $\delta \rightarrow 0$  :

$$u_\delta^-(\mathbf{x}) = u_0^-(\mathbf{x}) + \delta u_1^-(\mathbf{x}) + \delta^2 u_2^-(\mathbf{x}) + \delta^3 u_3^-(\mathbf{x}) + \dots \quad \text{in } \Omega_-$$

$$u_\delta^+(\mathbf{x}) = u_0^+(\mathbf{x}; \delta) + \delta u_1^+(\mathbf{x}; \delta) + \delta^2 u_2^+(\mathbf{x}; \delta) + \delta^3 u_3^+(\mathbf{x}; \delta) + \dots \quad \text{in } \Omega_+,$$

$$\text{with } u_j^+(\mathbf{x}; \delta) = \chi(r) \mathfrak{U}_j \left( \theta, \phi, \frac{r - r_t}{\delta} \right)$$

Method based on the Scaling :

$$s = \frac{r - r_t}{\delta}$$

# Equations for the terms $\mathfrak{U}_n$ and $u_n^-$

$$\begin{cases} -\partial_S^2 \mathfrak{U}_n - \partial_S \mathfrak{U}_n &= \sum_{p=1}^n A_p \mathfrak{U}_{n-p} & \text{in } \Sigma \times (0, +\infty), \\ \partial_S \mathfrak{U}_n &= \partial_n u_{n-1}^- & \text{on } \Sigma, \end{cases}$$

and

$$\begin{cases} -\operatorname{div}\left(\frac{1}{\rho} \nabla u_n^-\right) - \frac{\omega^2}{\rho c^2} u_n^- &= f \delta_n^0 & \text{in } \Omega_-, \\ u_n^- &= \mathfrak{U}_n & \text{on } \Sigma. \end{cases}$$

# First terms

1

$$\mathfrak{U}_0 = 0$$

2

$$\begin{cases} -\operatorname{div}\left(\frac{1}{\rho}\nabla u_0^-\right) - \frac{\omega^2}{\rho c^2}u_0^- = f & \text{in } \Omega_- \\ u_0^- = 0 & \text{on } \Sigma \end{cases}$$

3

$$\mathfrak{U}_1(\cdot, s) = -\partial_n u_0^- e^{-s}, \quad s \in (0, \infty)$$

4

$$\begin{cases} -\operatorname{div}\left(\frac{1}{\rho}\nabla u_1^-\right) - \frac{\omega^2}{\rho c^2}u_1^- = 0 & \text{in } \Omega_- \\ u_1^- = -\partial_n u_0^- & \text{on } \Sigma \end{cases}$$

## Higher order terms

- 

$$\mathfrak{L}_2(\cdot, S) = (a_2 + b_2 S) e^{-S}, \quad S \in (0, \infty)$$

Here

$$\begin{cases} a_2 = -\partial_n u_1^- + \frac{2}{r_t} \partial_n u_0^-, \\ b_2 = \frac{2}{r_t} \partial_n u_0^-. \end{cases}$$

- 

$$\begin{cases} -\operatorname{div}\left(\frac{1}{\rho} \nabla u_2^-\right) - \frac{\omega^2}{\rho c^2} u_2^- = 0 & \text{in } \Omega_-, \\ u_2^- = -\partial_n u_1^- + \frac{2}{r_t} \partial_n u_0^- & \text{on } \Sigma. \end{cases}$$

## Higher order terms



$$\mathfrak{L}_3(\theta, \phi, \mathcal{S}) = (a_3(\theta, \phi) + \mathcal{S}b_3(\theta, \phi) + \mathcal{S}^2c_3(\theta, \phi))e^{-\mathcal{S}}, \quad \mathcal{S} \in (0, \infty)$$

Here

$$\begin{cases} a_3 = -\partial_{\mathbf{n}}u_2^- + \frac{2}{r_t}\partial_{\mathbf{n}}u_1^- - \left\{ \frac{6}{r_t^2} + \frac{\omega_+^2}{c^2} + \Delta_{\Sigma} \right\} \partial_{\mathbf{n}}u_0^-, \\ b_3 = \frac{2}{r_t}\partial_{\mathbf{n}}u_1^- - \left\{ \frac{6}{r_t^2} + \frac{\omega_+^2}{c^2} + \Delta_{\Sigma} \right\} \partial_{\mathbf{n}}u_0^-, \\ c_3 = -\frac{3}{r_t^2}\partial_{\mathbf{n}}u_0^-. \end{cases}$$



$$\begin{cases} -\operatorname{div}\left(\frac{1}{\rho}\nabla u_3^-\right) - \frac{\omega^2}{\rho c^2}u_3^- = 0 & \text{in } \Omega_-, \\ u_3^- = -\partial_{\mathbf{n}}u_2^- + \frac{2}{r_t}\partial_{\mathbf{n}}u_1^- - \left\{ \frac{6}{r_t^2} + \frac{\omega_+^2}{c^2} + \Delta_{\Sigma} \right\} \partial_{\mathbf{n}}u_0^- & \text{on } \Sigma. \end{cases}$$

## Existence and regularity of the asymptotics

### Proposition

Let  $k \in \mathbb{N}$ . Assume that  $f \in H^k(\Omega_-)$  and  $\rho \in C^\infty(\overline{\Omega_-})$ . Then it is possible to derive the first  $(k + 2)$  terms  $(u_j^-, \mathfrak{U}_j)$ , and

$$u_0^- \in H^{k+2}(\Omega_-), u_1^- \in H^{k+1}(\Omega_-), \dots, u_{k+1}^- \in H^1(\Omega_-)$$

and

$$\mathfrak{U}_1 \in H^{k+\frac{1}{2}}(\Sigma \times \mathbb{R}^+), \mathfrak{U}_2 \in H^{k-\frac{1}{2}}(\Sigma \times \mathbb{R}^+), \dots, \mathfrak{U}_{k+1} \in H^{\frac{1}{2}}(\Sigma \times \mathbb{R}^+)$$

## Validation of the Asymptotic Expansion

Aim : proving Estimates for Remainders

$$r_\delta^N := u_\delta - \sum_{n=0}^N \delta^n u_n \quad \text{in } \Omega$$

Evaluation of the right hand sides in

$$\left\{ \begin{array}{ll} -\operatorname{div} \left( \frac{1}{\rho} \nabla r_\delta^{N,-} \right) - \frac{\omega^2}{\rho c^2} r_\delta^{N,-} = 0 & \text{in } \Omega_- \\ -\operatorname{div} \left( \frac{1}{\rho} \nabla r_\delta^{N,+} \right) - \frac{\omega_\pm^2}{\rho c^2} r_\delta^{N,+} = f_{N,\delta} & \text{in } \Omega_+ \\ r_\delta^{N,+} = r_\delta^{N,-} & \text{on } \Sigma \\ \partial_{\mathbf{n}} r_\delta^{N,+} = \partial_{\mathbf{n}} r_\delta^{N,-} + g_{N,\delta} & \text{on } \Sigma \\ r_\delta^N = 0 & \text{on } \partial\Omega \end{array} \right.$$

The RHS are explicit :  $f_{N,\delta} = \mathcal{O}(\delta^{N-1})$  and  $g_{N,\delta} = \delta^N \partial_{\mathbf{n}} u_N^-$

$$\|r_\delta^{N,-}\|_{1,\Omega_-} + \delta^{-1/2} \left\| \frac{1}{\sqrt{\rho_+}} r_\delta^{N,+} \right\|_{0,\Omega_+} + \delta^{1/2} \left\| \frac{1}{\sqrt{\rho_+}} \nabla r_\delta^{N,+} \right\|_{0,\Omega_+} \leq C_N \delta^{N+1}$$

# Outline

- 1 Uniform Estimates
- 2 Asymptotic Expansion
- 3 Equivalent Conditions**
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# Equivalent Conditions on $\Sigma$

## Overview

We identify a simpler problem satisfied by

$$u_{k,\delta}^- := u_0^- + \delta u_1^- + \delta^2 u_2^- + \cdots + \delta^k u_k^- \quad \text{up to } \mathcal{O}(\delta^{k+1})$$

The simpler problem *of order  $k + 1$*  writes

$$(\mathbf{P}_\delta^k) \quad \begin{cases} -\operatorname{div}\left(\frac{1}{\rho}\nabla u_k^\delta\right) - \frac{\omega^2}{\rho c^2} u_k^\delta & = f \quad \text{in } \Omega_-, \\ u_k^\delta + D_{k,\delta}(\partial_{\mathbf{n}} u_k^\delta) & = 0 \quad \text{on } \Sigma, \end{cases}$$

with  $D_{k,\delta}$  a surface differential operator

# Equivalent Conditions

- 1 Order 1 :

$$u_0 = 0 \quad \text{on } \Sigma$$

- 2 Order 2 :

$$u_1^\delta + \delta \partial_{\mathbf{n}} u_1^\delta = 0 \quad \text{on } \Sigma$$

- 3 Order 3 :

$$u_2^\delta + \delta \left( 1 - \frac{2\delta}{r_t} \right) \partial_{\mathbf{n}} u_2^\delta = 0 \quad \text{on } \Sigma$$

- 4 Order 4 :

$$u_3^\delta + \delta \left( 1 - \frac{2\delta}{r_t} + \delta^2 \left\{ \frac{6}{r_t^2} - \frac{\omega_+^2}{c^2} - \Delta_\Sigma \right\} \right) \partial_{\mathbf{n}} u_3^\delta = 0 \quad \text{on } \Sigma$$

## Dirichlet-to-Neumann Conditions

- Order 4 :

$$\delta^{-1} \left( \left( 1 + \frac{2\delta}{r_t} \right) \mathbb{I} - \delta^2 \left\{ \frac{2}{r_t^2} + \frac{\omega_+^2}{c^2} + \Delta_\Sigma \right\} \right) v_3^\delta + \partial_{\mathbf{n}} v_3^\delta = 0 \quad \text{on } \Sigma$$

## Dirichlet-to-Neumann Conditions

- Order 4 :

$$\delta^{-1} \left( \left(1 + \frac{2\delta}{r_t}\right) \mathbb{I} - \delta^2 \left\{ \frac{2}{r_t^2} + \frac{\omega_+^2}{c^2} + \Delta_\Sigma \right\} \right) \mathbf{v}_3^\delta + \partial_{\mathbf{n}} \mathbf{v}_3^\delta = 0 \quad \text{on } \Sigma$$

For all  $k \in \{1, 2, 3\}$ , the simpler problem writes

$$(\mathbf{q}_\delta^k) \quad \begin{cases} -\operatorname{div}\left(\frac{1}{\rho} \nabla \mathbf{v}_k^\delta\right) - \frac{\omega^2}{\rho c^2} \mathbf{v}_k^\delta & = f \quad \text{in } \Omega_-, \\ \mathbf{N}_{k,\delta}(\mathbf{v}_k^\delta) + \partial_{\mathbf{n}} \mathbf{v}_k^\delta & = 0 \quad \text{on } \Sigma, \end{cases}$$

with  $\mathbf{N}_{k,\delta}$  a surface differential operator.

## Stability and convergence results

### Theorem

Under Assumption (SA), there exists  $\delta_0 > 0$  s.t. for all  $\delta \in (0, \delta_0)$ , the problem  $(\mathbf{Q}_\delta^k)$  with data  $f \in L^2(\Omega_-)$  has a unique solution  $v_k^\delta \in \mathbf{V}_k$ , and

$$\|v_k^\delta\|_{1, \Omega_-} \leq C \|f\|_{0, \Omega_-}$$

$$\|u_\delta - v_k^\delta\|_{1, \Omega_-} \leq C_k \delta^{k+1}$$

**Functional setting :**  $v_k^\delta \in \mathbf{V}_k$  where

$$\mathbf{V}_k = \mathbf{H}^1(\Omega_-) \quad \text{when } k = 1, 2$$

and

$$\mathbf{V}_3 = \{u \in \mathbf{H}^1(\Omega_-) \mid u|_\Sigma \in \mathbf{H}^1(\Sigma)\}$$

## Proof of the stability result

### Lemma

Under Assumption (SA), there exists  $\delta_0 > 0$  s.t. for all  $\delta \in (0, \delta_0)$ , any solution  $v_k^\delta \in V_k$  of problem  $(\mathbf{Q}_\delta^k)$  with a data  $f \in L^2(\Omega_-)$  satisfies

$$\|v_k^\delta\|_{0, \Omega_-} \leq C \|f\|_{0, \Omega_-} .$$

## Sketch of the proof of the Lemma

Assume : there exists  $(u_m) \in (V_k)^{\mathbb{N}}$  satisfying  $(Q_{\delta_m}^k)$  with  $\delta_m \rightarrow 0$  and  $f_m \in L^2(\Omega_-)$  such that

$$\|u_m\|_{0,\Omega_-} = 1 \quad \text{and} \quad \|f_m\|_{0,\Omega_-} \rightarrow 0$$

We prove successively that :

①

$$\|u_m\|_{1,\Omega_-} \leq C$$

② there exists a subsequence  $(u_m)$  s.t.

$$u_m \rightarrow u \quad \text{in} \quad L^2(\Omega_-) \quad \text{and} \quad \nabla u_m \rightharpoonup \nabla u \quad \text{in} \quad L^2(\Omega_-)$$

$$\text{and } u_m \rightarrow 0 \quad \text{in} \quad L^2(\Sigma)$$

③

$$\|u\|_{0,\Omega_-} = 1$$

④ Using Assumption (SA), we prove that  $u = 0$ . Contradiction.

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- 1 Uniform Estimates
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# Numerical method

## Helmholtz equation with radial data

Using a decomposition in spherical harmonics we solve a sequence of 1-D problems with a Finite Element Method.

- 1-D Computational domain :

Approximate models :  $\Omega_- = (0; 1)$

Reference solutions :  $\Omega = (0; R_\Omega)$  such that  $\rho^+(R_\Omega) = 10^{-15}$

- $\mathbb{P}_{10}$ -finite elements (Lagrange) available in the Library Montjoie
- Data :
  - 1 Artificial data
  - 2 Realistic data

# Artificial data

## Framework

- The density is given as

$$\rho(r) = \begin{cases} D_1 & \text{if } r < 1 \\ D_1 e^{-(r-1)/\delta} & \text{otherwise} \end{cases}$$

$$D_1 = 2 \times 10^{-4} \text{ kg/m}^3,$$

- The velocity is constant

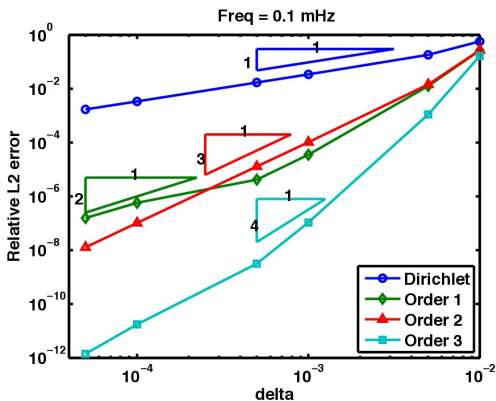
$$c = 8 \times 10^3 \text{ m/s}$$

- **Radial gaussian source** located at the radius  $r = 0.8$
- $\omega = 2\pi f_0$  where  $f_0 \in \{7 \times 10^4 \text{ Hz}, 7 \times 10^5 \text{ Hz}\}$

# Convergence results

Freq.  $f_0 = 7 \times 10^4$  Hz

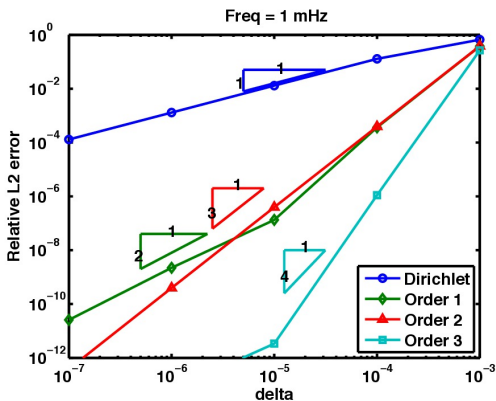
Relative  $L^2$ -error between exact solution and approximate solutions versus  $\delta$



# Convergence results

Freq.  $f_0 = 7 \times 10^5$  Hz

Relative  $L^2$ -error between exact solution and approximate solution versus  $\delta$

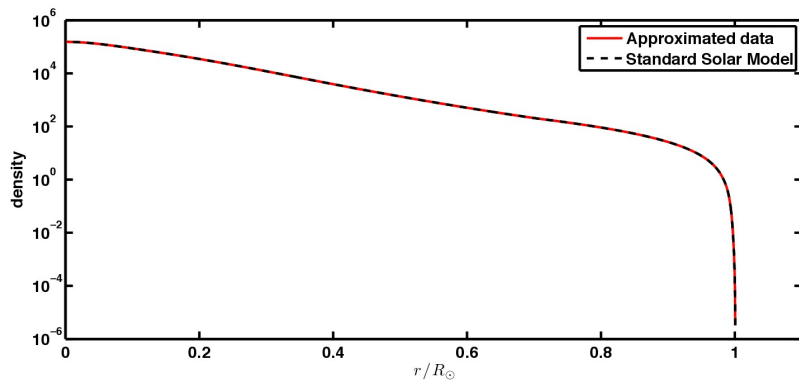


## Realistic data

Using the data given by the Standard Solar Model



CHRISTENSEN-DALSGAARD ET AL, THE CURRENT STATE OF SOLAR MODELING. SCIENCE, 1996

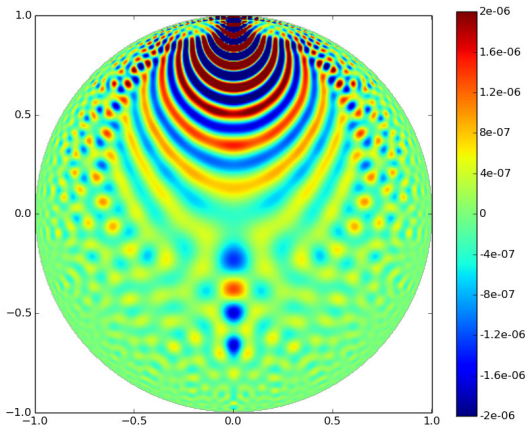


**Figure :** Exact value and approximated value of  $\rho^{\text{sun}}$  versus the relative radius  $r/R_{\odot}$ .

# Realistic data

## Reference solution in the plane $Oxz$

Real part of  $u/\sqrt{\rho}$  when  $\delta = 1/7000$

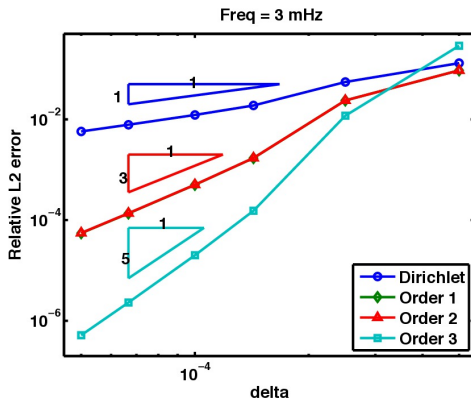


The source is a 3-D gaussian and we use 100 spherical harmonics

# Realistic setting

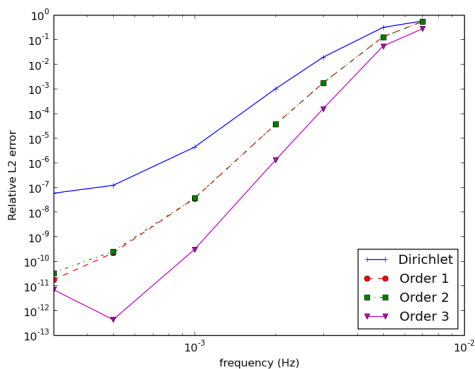
## Convergence results

Relative  $L^2$ -error between reference solution and approximate solution versus  $\delta$



# Realistic setting

Comparison of equivalent boundary conditions for different frequencies from 0.3 mHz until 7 mHz.

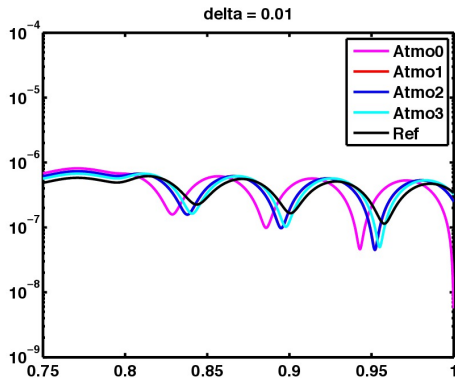




Thank you for your attention

# Numerical tests with artificial data

## Qualitative Comparisons

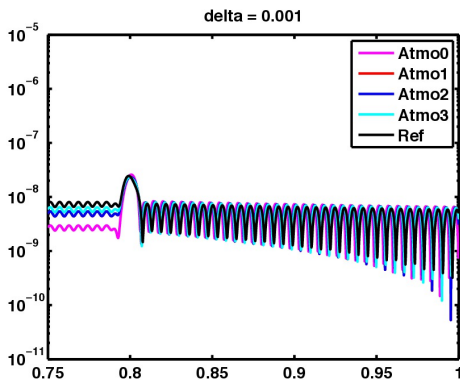


Reference solution and solutions using equivalent boundary conditions.

$$f_0 = 7 \times 10^4 \text{ Hz}$$

# Numerical tests with artificial data

## Qualitative Comparisons



Reference solution and solutions using atmosphere boundary conditions.

$$f_0 = 7 \times 10^5 \text{ Hz}$$

## Realistic setting

- The density is given as

$$\rho_0(r) = \left(1 + \frac{2i\gamma}{\omega}\right) \begin{cases} \rho^{\text{sun}} & \text{if } r/R_{\odot} \leq 1.0007126 \\ D_1 e^{-(r/R_{\odot}-1)/\delta} & \text{otherwise} \end{cases}$$

$$D_1 \approx 3.292 \times 10^{-6} \text{ kg/m}^3, \gamma = \frac{\omega}{100}$$

- The velocity is given as

$$c_0(r) = \begin{cases} c^{\text{sun}} & \text{if } r/R_{\odot} \leq 1.0007126 \\ C_1 & \text{otherwise} \end{cases}$$

$$C_1 \approx 6.865 \times 10^3 \text{ m/s}$$

- The source is a 3-D gaussian:

$$f(x) = \exp(-\beta \|x - x_0\|^2)$$

Here  $x_0 \approx (0, 0, 0.981)$  and  $\beta$  is chosen such that  $\exp(-0.015 \beta) = 10^{-6}$ .