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Equivalent boundary conditions for acoustic media with exponential densities

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Eighth Singular Days, Nancy

June 26-30, 2016

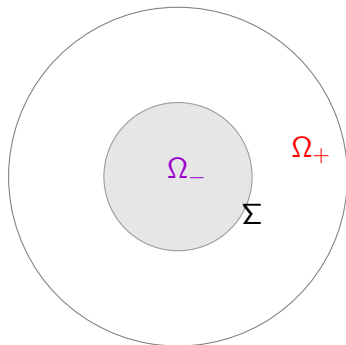


Introduction

A 3-D Problem set in heterogeneous acoustic media

We solve the Helmholtz equation (with a Dirichlet B.C.) :

$$-\operatorname{div} \left(\frac{1}{\rho} \nabla u \right) - \frac{\omega^2}{\rho c^2} u = f \quad \text{in } \Omega$$



- $\Omega_- = \{r < r_t\}$: Density ρ^- , Velocity c^-
- $\Sigma = \{r = r_t\}$: Interface
- $\Omega_+ = \Omega \setminus \overline{\Omega_-}$: Density ρ^+ , Velocity c^+

Framework

Data and right-hand side

Assumption

- (i) The density ρ and the velocity c are **smooth** and strictly **positive** functions
- (ii) The density ρ^+ satisfies

$$\rho^+(r) = \gamma \exp(-\alpha(r - r_t))$$

where *the parameter α is large*

- (iii) The domain Ω_+ represents a **dissipative medium** :

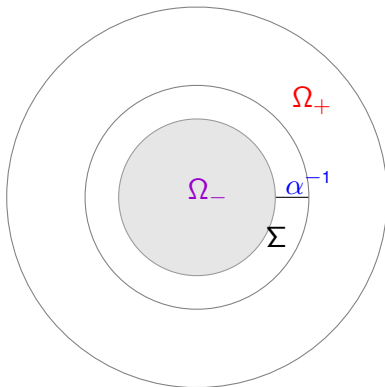
$$\text{Im } \omega^2 \neq 0 \quad \text{in } \Omega_+$$

Assumption

The right-hand side f has a support in Ω_-

Context

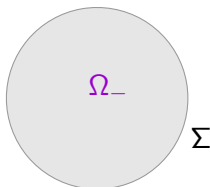
A **boundary layer phenomenon** occurs when α is large : rapid decay of acoustic fields (pressure and velocity) inside the medium Ω_+



Difficulty : Applying a FEM on a mesh with **thin cells** in the **boundary layer** and much larger **elsewhere**

Approach : an Asymptotic Method

- 1 Describing the **boundary layer phenomenon** with a multiscale expansion in power series of α^{-1}
- 2 Deriving Equivalent Boundary Conditions on Σ to replace the **boundary layer** inside Ω_+
- 3 Applying a Finite Element Method to solve the acoustic equation in Ω_- with an Equivalent Boundary Condition



References on **skin effect** in electromagnetism



S. M. RYTOV

Calcul du skin effect par la méthode des perturbations.

Journal of Physics (1940).



E. STEPHAN

Solution procedures for interface problems in [...] electromagnetics.

CISM Courses and Lectures. **277** 291–348 (1983).



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A skin effect approximation for eddy current problems.

Arch. Rational Mech. Anal. **90**(1) (1985) 87–98.



H. HADDAR, P. JOLY, H.-N. NGUYEN

Generalized impedance [...] by strongly absorbing obstacles [...].

Math. Models Methods Appl. Sci. **15**(8) (2005) 1273–1300.



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On the influence of the geometry on skin effect in electromagnetism.

Comput. Methods Appl. Mech. Engrg. 200 (2011), no. 9-12, 1053–1068.

Outline

- 1 **Uniform Estimates**
- 2 **Asymptotic Expansion**
- 3 **Equivalent Conditions**
- 4 **Numerical Results**

Outline

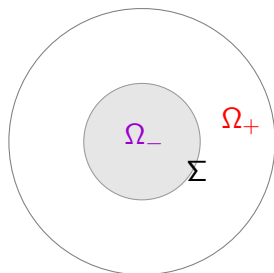
- 1 **Uniform Estimates**
- 2 Asymptotic Expansion
- 3 Equivalent Conditions
- 4 Numerical Results

Problem (P_α)

Helmholtz equation :

$$-\operatorname{div} \left(\frac{1}{\rho} \nabla u_\alpha \right) - \frac{\omega^2}{\rho c^2} u_\alpha = f \quad \text{in } \Omega$$

with an homogeneous Dirichlet boundary condition on $\partial\Omega$.



Here $\rho^+(r) = \gamma \exp(-\alpha(r - r_t))$ and $\alpha > 0$.

Problem (\mathbf{P}_α)

Issue : *Uniform H^1 estimates* for solutions u_α of (\mathbf{P}_α) w.r.t. $\alpha > 0$?

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Issue : *Uniform* H^1 estimates for solutions u_α of (\mathbf{P}_α) w.r.t. $\alpha > 0$?

Assumption (SA)

The angular frequency ω is not an eigenfrequency of the problem

$$\begin{cases} \operatorname{div} \left(\frac{1}{\rho} \nabla u^- \right) + \frac{\omega^2}{\rho c^2} u^- = 0 & \text{in } \Omega_- \\ u^- = 0 & \text{on } \Sigma \end{cases}$$

Problem (\mathbf{P}_α)

Issue : *Uniform H^1 estimates for solutions u_α of (\mathbf{P}_α) w.r.t. $\alpha > 0$?*

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Theorem

Under Assumption (SA), for all $\alpha > 0$ the problem (\mathbf{P}_α) with a right-hand side $f \in L^2(\Omega)$ has a unique solution $u_\alpha \in H_0^1(\Omega)$

$$\|u_\alpha^-\|_{1,\Omega_-} + \left\| \frac{1}{\sqrt{\rho_+}} u_\alpha^+ \right\|_{0,\Omega_+} + \left\| \frac{1}{\sqrt{\rho_+}} \nabla u_\alpha^+ \right\|_{0,\Omega_+} \leq C \|f\|_{0,\Omega}$$

Application: Convergence of asymptotic expansion for large parameter α

Problem (P_δ)

Small parameter δ :

$$\delta = \frac{1}{\alpha} \longrightarrow 0 \quad \text{when} \quad \alpha \rightarrow \infty$$

Problem (\mathbf{P}_δ)

Small parameter δ :

$$\delta = \frac{1}{\alpha} \longrightarrow 0 \quad \text{when} \quad \alpha \rightarrow \infty$$

Problem (\mathbf{P}_δ) writes :

$$\begin{aligned} -\operatorname{div}\left(\frac{1}{\rho}\nabla u_\delta^-\right) - \frac{\omega^2}{\rho c^2}u_\delta^- &= f && \text{in } \Omega_- \\ -\Delta u_\delta^+ - \frac{1}{\delta}\partial_r u_\delta^+ - \frac{\omega_+^2}{c^2}u_\delta^+ &= 0 && \text{in } \Omega_+ \\ u_\delta^+ &= u_\delta^- && \text{on } \Sigma \\ \partial_n u_\delta^+ &= \partial_n u_\delta^- && \text{on } \Sigma \\ u_\delta^+ &= 0 && \text{on } \partial\Omega \end{aligned}$$

Outline

- 1 Uniform Estimates
- 2 Asymptotic Expansion**
- 3 Equivalent Conditions
- 4 Numerical Results

Asymptotic Expansion

Overview

Deriving an Asymptotic Expansion for the solution (u_δ^-, u_δ^+) of (\mathbf{P}_δ) when $\delta \rightarrow 0$:

$$u_\delta^-(\mathbf{x}) = u_0^-(\mathbf{x}) + \delta u_1^-(\mathbf{x}) + \delta^2 u_2^-(\mathbf{x}) + \delta^3 u_3^-(\mathbf{x}) + \dots \quad \text{in } \Omega_-$$

$$u_\delta^+(\mathbf{x}) = u_0^+(\mathbf{x}; \delta) + \delta u_1^+(\mathbf{x}; \delta) + \delta^2 u_2^+(\mathbf{x}; \delta) + \delta^3 u_3^+(\mathbf{x}; \delta) + \dots \quad \text{in } \Omega_+,$$

$$\text{with } u_j^+(\mathbf{x}; \delta) = \chi(r) \mathfrak{U}_j \left(\theta, \phi, \frac{r - r_t}{\delta} \right)$$

Method based on the Scaling :

$$s = \frac{r - r_t}{\delta}$$

Equations for the terms \mathfrak{U}_n and u_n^-

$$\begin{cases} -\partial_S^2 \mathfrak{U}_n - \partial_S \mathfrak{U}_n &= \sum_{p=1}^n A_p \mathfrak{U}_{n-p} & \text{in } \Sigma \times (0, +\infty), \\ \partial_S \mathfrak{U}_n &= \partial_n u_{n-1}^- & \text{on } \Sigma, \end{cases}$$

and

$$\begin{cases} -\operatorname{div}\left(\frac{1}{\rho} \nabla u_n^-\right) - \frac{\omega^2}{\rho c^2} u_n^- &= f \delta_n^0 & \text{in } \Omega_-, \\ u_n^- &= \mathfrak{U}_n & \text{on } \Sigma. \end{cases}$$

First terms

1

$$\mathfrak{U}_0 = 0$$

2

$$\begin{cases} -\operatorname{div}\left(\frac{1}{\rho}\nabla u_0^-\right) - \frac{\omega^2}{\rho c^2}u_0^- = f & \text{in } \Omega_- \\ u_0^- = 0 & \text{on } \Sigma \end{cases}$$

3

$$\mathfrak{U}_1(\cdot, s) = -\partial_n u_0^- e^{-s}, \quad s \in (0, \infty)$$

4

$$\begin{cases} -\operatorname{div}\left(\frac{1}{\rho}\nabla u_1^-\right) - \frac{\omega^2}{\rho c^2}u_1^- = 0 & \text{in } \Omega_- \\ u_1^- = -\partial_n u_0^- & \text{on } \Sigma \end{cases}$$

Higher order terms



$$\mathfrak{L}_2(\cdot, S) = (a_2 + b_2 S) e^{-S}, \quad S \in (0, \infty)$$

Here

$$\begin{cases} a_2 = -\partial_n u_1^- + \frac{2}{r_t} \partial_n u_0^-, \\ b_2 = \frac{2}{r_t} \partial_n u_0^-. \end{cases}$$



$$\begin{cases} -\operatorname{div}\left(\frac{1}{\rho} \nabla u_2^-\right) - \frac{\omega^2}{\rho c^2} u_2^- = 0 & \text{in } \Omega_-, \\ u_2^- = -\partial_n u_1^- + \frac{2}{r_t} \partial_n u_0^- & \text{on } \Sigma. \end{cases}$$

Higher order terms



$$\mathfrak{L}_3(\theta, \phi, \mathcal{S}) = (a_3(\theta, \phi) + \mathcal{S}b_3(\theta, \phi) + \mathcal{S}^2c_3(\theta, \phi))e^{-\mathcal{S}}, \quad \mathcal{S} \in (0, \infty)$$

Here

$$\begin{cases} a_3 = -\partial_{\mathbf{n}}u_2^- + \frac{2}{r_t}\partial_{\mathbf{n}}u_1^- - \left\{ \frac{6}{r_t^2} + \frac{\omega_+^2}{c^2} + \Delta_{\Sigma} \right\} \partial_{\mathbf{n}}u_0^-, \\ b_3 = \frac{2}{r_t}\partial_{\mathbf{n}}u_1^- - \left\{ \frac{6}{r_t^2} + \frac{\omega_+^2}{c^2} + \Delta_{\Sigma} \right\} \partial_{\mathbf{n}}u_0^-, \\ c_3 = -\frac{3}{r_t^2}\partial_{\mathbf{n}}u_0^-. \end{cases}$$



$$\begin{cases} -\operatorname{div}\left(\frac{1}{\rho}\nabla u_3^-\right) - \frac{\omega^2}{\rho c^2}u_3^- = 0 & \text{in } \Omega_-, \\ u_3^- = -\partial_{\mathbf{n}}u_2^- + \frac{2}{r_t}\partial_{\mathbf{n}}u_1^- - \left\{ \frac{6}{r_t^2} + \frac{\omega_+^2}{c^2} + \Delta_{\Sigma} \right\} \partial_{\mathbf{n}}u_0^- & \text{on } \Sigma. \end{cases}$$

Existence and regularity of the asymptotics

Proposition

Let $k \in \mathbb{N}$. Assume that $f \in H^k(\Omega_-)$ and $\rho \in C^\infty(\overline{\Omega_-})$. Then it is possible to derive the first $(k + 2)$ terms (u_j^-, \mathfrak{U}_j) , and

$$u_0^- \in H^{k+2}(\Omega_-), u_1^- \in H^{k+1}(\Omega_-), \dots, u_{k+1}^- \in H^1(\Omega_-)$$

and

$$\mathfrak{U}_1 \in H^{k+\frac{1}{2}}(\Sigma \times \mathbb{R}^+), \mathfrak{U}_2 \in H^{k-\frac{1}{2}}(\Sigma \times \mathbb{R}^+), \dots, \mathfrak{U}_{k+1} \in H^{\frac{1}{2}}(\Sigma \times \mathbb{R}^+)$$

Validation of the Asymptotic Expansion

Aim : proving Estimates for Remainders

$$r_\delta^N := u_\delta - \sum_{n=0}^N \delta^n u_n \quad \text{in } \Omega$$

Evaluation of the right hand sides in

$$\left\{ \begin{array}{ll} -\operatorname{div} \left(\frac{1}{\rho} \nabla r_\delta^{N,-} \right) - \frac{\omega^2}{\rho c^2} r_\delta^{N,-} = 0 & \text{in } \Omega_- \\ -\operatorname{div} \left(\frac{1}{\rho} \nabla r_\delta^{N,+} \right) - \frac{\omega_\pm^2}{\rho c^2} r_\delta^{N,+} = f_{N,\delta} & \text{in } \Omega_+ \\ r_\delta^{N,+} = r_\delta^{N,-} & \text{on } \Sigma \\ \partial_{\mathbf{n}} r_\delta^{N,+} = \partial_{\mathbf{n}} r_\delta^{N,-} + g_{N,\delta} & \text{on } \Sigma \\ r_\delta^N = 0 & \text{on } \partial\Omega \end{array} \right.$$

The RHS are explicit : $f_{N,\delta} = \mathcal{O}(\delta^{N-1})$ and $g_{N,\delta} = \delta^N \partial_{\mathbf{n}} u_N^-$

$$\|r_\delta^{N,-}\|_{1,\Omega_-} + \delta^{-1/2} \left\| \frac{1}{\sqrt{\rho_+}} r_\delta^{N,+} \right\|_{0,\Omega_+} + \delta^{1/2} \left\| \frac{1}{\sqrt{\rho_+}} \nabla r_\delta^{N,+} \right\|_{0,\Omega_+} \leq C_N \delta^{N+1}$$

Outline

- 1 Uniform Estimates
- 2 Asymptotic Expansion
- 3 Equivalent Conditions**
- 4 Numerical Results

Equivalent Conditions on Σ

Overview

We identify a simpler problem satisfied by

$$\mathbf{u}_{k,\delta}^- := \mathbf{u}_0^- + \delta \mathbf{u}_1^- + \delta^2 \mathbf{u}_2^- + \cdots + \delta^k \mathbf{u}_k^- \quad \text{up to } \mathcal{O}(\delta^{k+1})$$

The simpler problem *of order $k + 1$* writes

$$(\mathbf{P}_\delta^k) \quad \begin{cases} -\operatorname{div}\left(\frac{1}{\rho}\nabla\mathbf{u}_k^\delta\right) - \frac{\omega^2}{\rho c^2}\mathbf{u}_k^\delta & = f \quad \text{in } \Omega_-, \\ \mathbf{u}_k^\delta + D_{k,\delta}(\partial_{\mathbf{n}}\mathbf{u}_k^\delta) & = 0 \quad \text{on } \Sigma, \end{cases}$$

with $D_{k,\delta}$ a surface differential operator

Equivalent Conditions

- 1 Order 1 :

$$u_0 = 0 \quad \text{on } \Sigma$$

- 2 Order 2 :

$$u_1^\delta + \delta \partial_{\mathbf{n}} u_1^\delta = 0 \quad \text{on } \Sigma$$

- 3 Order 3 :

$$u_2^\delta + \delta \left(1 - \frac{2\delta}{r_t} \right) \partial_{\mathbf{n}} u_2^\delta = 0 \quad \text{on } \Sigma$$

- 4 Order 4 :

$$u_3^\delta + \delta \left(1 - \frac{2\delta}{r_t} + \delta^2 \left\{ \frac{6}{r_t^2} - \frac{\omega_+^2}{c^2} - \Delta_\Sigma \right\} \right) \partial_{\mathbf{n}} u_3^\delta = 0 \quad \text{on } \Sigma$$

Dirichlet-to-Neumann Conditions

- Order 4 :

$$\delta^{-1} \left(\left(1 + \frac{2\delta}{r_t} \right) \mathbb{I} - \delta^2 \left\{ \frac{2}{r_t^2} + \frac{\omega_+^2}{c^2} + \Delta_\Sigma \right\} \right) v_3^\delta + \partial_{\mathbf{n}} v_3^\delta = 0 \quad \text{on } \Sigma$$

Dirichlet-to-Neumann Conditions

- Order 4 :

$$\delta^{-1} \left(\left(1 + \frac{2\delta}{r_t}\right) \mathbb{I} - \delta^2 \left\{ \frac{2}{r_t^2} + \frac{\omega_+^2}{c^2} + \Delta_\Sigma \right\} \right) \mathbf{v}_3^\delta + \partial_{\mathbf{n}} \mathbf{v}_3^\delta = 0 \quad \text{on } \Sigma$$

For all $k \in \{1, 2, 3\}$, the simpler problem writes

$$(\mathbf{q}_\delta^k) \quad \begin{cases} -\operatorname{div}\left(\frac{1}{\rho} \nabla \mathbf{v}_k^\delta\right) - \frac{\omega^2}{\rho c^2} \mathbf{v}_k^\delta & = f \quad \text{in } \Omega_-, \\ \mathbf{N}_{k,\delta}(\mathbf{v}_k^\delta) + \partial_{\mathbf{n}} \mathbf{v}_k^\delta & = 0 \quad \text{on } \Sigma, \end{cases}$$

with $\mathbf{N}_{k,\delta}$ a surface differential operator.

Stability and convergence results

Theorem

Under Assumption (SA), there exists $\delta_0 > 0$ s.t. for all $\delta \in (0, \delta_0)$, the problem (\mathbf{Q}_δ^k) with data $f \in L^2(\Omega_-)$ has a unique solution $v_k^\delta \in V_k$, and

$$\|v_k^\delta\|_{1, \Omega_-} \leq C \|f\|_{0, \Omega_-}$$

$$\|u_\delta - v_k^\delta\|_{1, \Omega_-} \leq C_k \delta^{k+1}$$

Functional setting : $v_k^\delta \in V_k$ where

$$V_k = H^1(\Omega_-) \quad \text{when } k = 1, 2$$

and

$$V_3 = \{u \in H^1(\Omega_-) \mid u|_\Sigma \in H^1(\Sigma)\}$$

Proof of the stability result

Lemma

Under Assumption (SA), there exists $\delta_0 > 0$ s.t. for all $\delta \in (0, \delta_0)$, any solution $v_k^\delta \in V_k$ of problem (\mathbf{Q}_δ^k) with a data $f \in L^2(\Omega_-)$ satisfies

$$\|v_k^\delta\|_{0, \Omega_-} \leq C \|f\|_{0, \Omega_-} .$$

Sketch of the proof of the Lemma

Assume : there exists $(u_m) \in (V_k)^{\mathbb{N}}$ satisfying $(\mathbf{Q}_{\delta_m}^k)$ with $\delta_m \rightarrow 0$ and $f_m \in L^2(\Omega_-)$ such that

$$\|u_m\|_{0,\Omega_-} = 1 \quad \text{and} \quad \|f_m\|_{0,\Omega_-} \rightarrow 0$$

We prove successively that :

①

$$\|u_m\|_{1,\Omega_-} \leq C$$

② there exists a subsequence (u_m) s.t.

$$u_m \rightarrow u \quad \text{in} \quad L^2(\Omega_-) \quad \text{and} \quad \nabla u_m \rightharpoonup \nabla u \quad \text{in} \quad L^2(\Omega_-)$$

$$\text{and } u_m \rightarrow 0 \quad \text{in} \quad L^2(\Sigma)$$

③

$$\|u\|_{0,\Omega_-} = 1$$

④ Using Assumption (SA), we prove that $u = 0$. Contradiction.

Outline

- 1 Uniform Estimates
- 2 Asymptotic Expansion
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Numerical method

Helmholtz equation with radial data

Using a decomposition in spherical harmonics we solve a sequence of 1-D problems with a Finite Element Method.

- 1-D Computational domain :

Approximate models : $\Omega_- = (0; 1)$

Reference solutions : $\Omega = (0; R_\Omega)$ such that $\rho^+(R_\Omega) = 10^{-15}$

- \mathbb{P}_{10} -finite elements (Lagrange) available in the Library Montjoie
- Data :
 - 1 Artificial data
 - 2 Realistic data

Artificial data

Framework

- The density is given as

$$\rho(r) = \begin{cases} D_1 & \text{if } r < 1 \\ D_1 e^{-(r-1)/\delta} & \text{otherwise} \end{cases}$$

$$D_1 = 2 \times 10^{-4} \text{ kg/m}^3,$$

- The velocity is constant

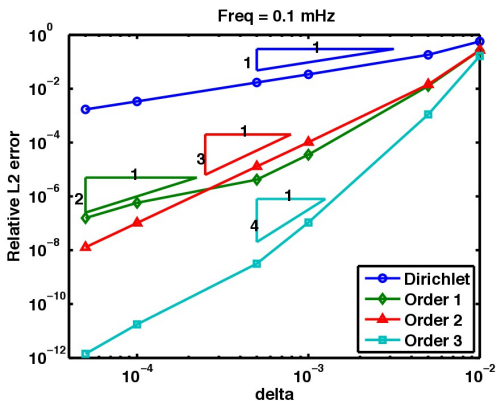
$$c = 8 \times 10^3 \text{ m/s}$$

- **Radial gaussian source** located at the radius $r = 0.8$
- $\omega = 2\pi f_0$ where $f_0 \in \{7 \times 10^4 \text{ Hz}, 7 \times 10^5 \text{ Hz}\}$

Convergence results

Freq. $f_0 = 7 \times 10^4$ Hz

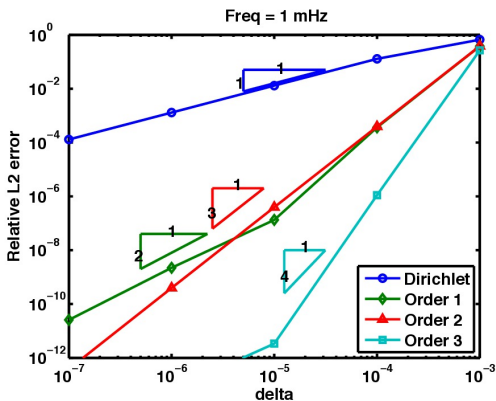
Relative L^2 -error between exact solution and approximate solutions versus δ



Convergence results

Freq. $f_0 = 7 \times 10^5$ Hz

Relative L^2 -error between exact solution and approximate solution versus δ



Realistic data

Using the data given by the Standard Solar Model



CHRISTENSEN-DALSGAARD ET AL, THE CURRENT STATE OF SOLAR MODELING. SCIENCE, 1996

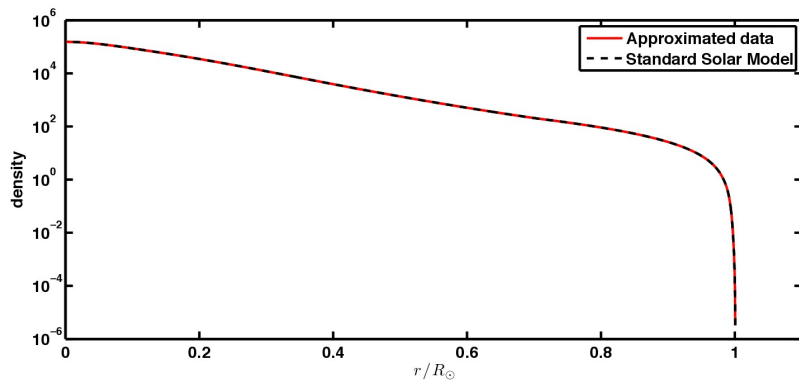
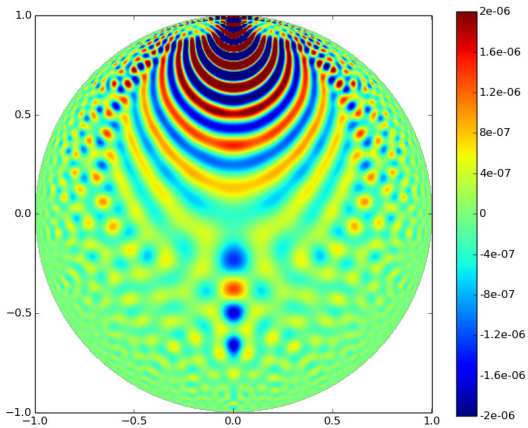


Figure : Exact value and approximated value of ρ^{sun} versus the relative radius r/R_{\odot} .

Realistic data

Reference solution in the plane Oxz

Real part of $u/\sqrt{\rho}$ when $\delta = 1/7000$

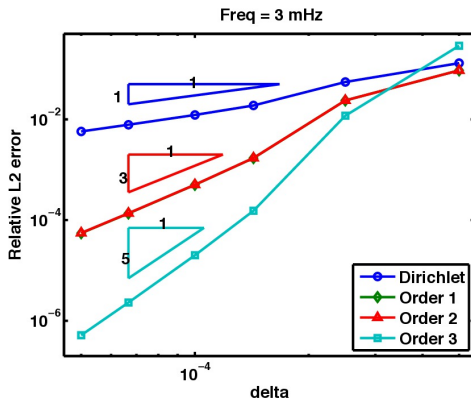


The source is a 3-D gaussian and we use 100 spherical harmonics

Realistic setting

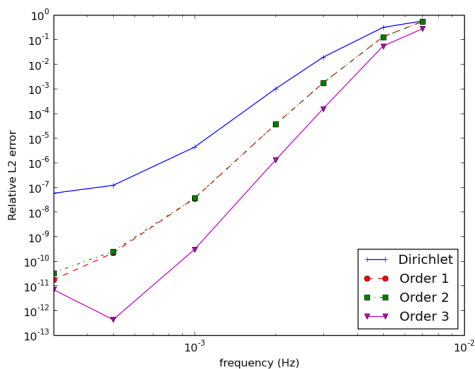
Convergence results

Relative L^2 -error between reference solution and approximate solution versus δ



Realistic setting

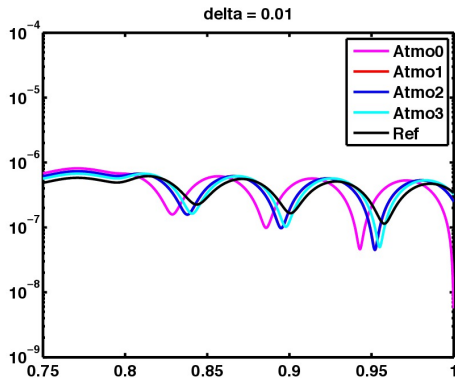
Comparison of equivalent boundary conditions for different frequencies from 0.3 mHz until 7 mHz.



Thank you for your attention

Numerical tests with artificial data

Qualitative Comparisons

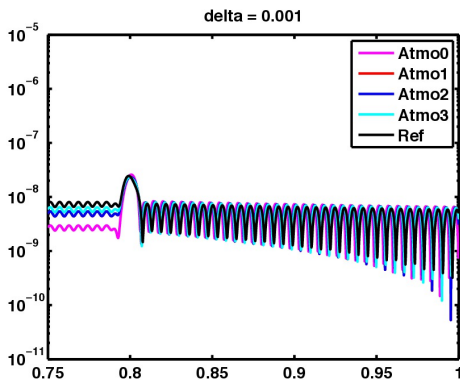


Reference solution and solutions using equivalent boundary conditions.

$$f_0 = 7 \times 10^4 \text{ Hz}$$

Numerical tests with artificial data

Qualitative Comparisons



Reference solution and solutions using atmosphere boundary conditions.

$$f_0 = 7 \times 10^5 \text{ Hz}$$

Realistic setting

- The density is given as

$$\rho_0(r) = \left(1 + \frac{2i\gamma}{\omega}\right) \begin{cases} \rho^{\text{sun}} & \text{if } r/R_{\odot} \leq 1.0007126 \\ D_1 e^{-(r/R_{\odot}-1)/\delta} & \text{otherwise} \end{cases}$$

$$D_1 \approx 3.292 \times 10^{-6} \text{ kg/m}^3, \gamma = \frac{\omega}{100}$$

- The velocity is given as

$$c_0(r) = \begin{cases} c^{\text{sun}} & \text{if } r/R_{\odot} \leq 1.0007126 \\ C_1 & \text{otherwise} \end{cases}$$

$$C_1 \approx 6.865 \times 10^3 \text{ m/s}$$

- The source is a 3-D gaussian:

$$f(x) = \exp(-\beta \|x - x_0\|^2)$$

Here $x_0 \approx (0, 0, 0.981)$ and β is chosen such that $\exp(-0.015 \beta) = 10^{-6}$.