Web supplementary material for the article: Malleable task-graph scheduling with a practical speed-up model

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We provide here the proof of Lemma 2, which is used in the proof of Theorem 1. We first recall its statement.

**Lemma 2.** In any valid schedule \( S \) for \( J \),

i) each pair of chains \( L_{x_1}, L_{x_2} \) is completely processed during time interval \([t_i, M - t_i]\),

ii) one of them is started at time \( t_i \) and the other one at time \( t_i + 1 \),

iii) all tasks of both chains are allocated their threshold, iv) there is no idle time between any two consecutive tasks of each chain.

**Proof.** The proof is done by induction on \( i \), by carefully checking when the first and last tasks of chains \( L_{x_1}, L_{x_2} \) may be scheduled, given the resources which are not used by the previous chains and by \( L_{pro} \).

**Base case:** Consider \( i = 1 \). The critical path of chains \( L_{x_1} \) and \( L_{x_2} \) is \( 4n + 2m - 4i + 3 = 4n + 2m - 1 \). With \( M = 4n + 2m \), both have to start in the interval \([0, 1]\).

The following discussion of the base case is written in general terms (that is for any \( i \)) to reuse it in the inductive step, but applies here for \( i = 1 \), with \( t_1 = 0 \).

We consider the first task of chain \( L_{x_1} \) and the first task of chain \( L_{x_2} \). Both tasks have weight 1. Let \( A \) denote the first of these two tasks to complete (at a time \( t_A \)) and let \( B \) be the other one (which completes at time \( t_B \)). Given the \( 2(i - 1) \) chains already scheduled (none for \( i = 1 \)), the number of processors available during interval \([t_i, t_i + 1]\) is 1 and during interval \([t_i + 1, t_i + 2]\) is \( 1 + \varepsilon \). \( A \) and \( B \) both complete at or after time \( t_i + 1 \). We note \( t_A = t_i + 1 + \Delta_1 \) and \( t_B = t_i + 1 + \Delta_1 + \Delta_2 \). \( \Delta_1 \geq 0 \) and \( \Delta_2 \geq 0 \). Note that because of the critical path length of the remaining tasks of both chains and the limited time span, \( \Delta_1 \leq 1 \) and \( \Delta_1 + \Delta_2 \leq 1 \). The following figure illustrates the previous notations and the amount of processors available for tasks \( A \) and \( B \) (note that after time \( t_A \), \( B \) may use only \( \delta_B = 1 \) processor).

Since \( w_A + w_B = 2 \) work units have to be performed before time \( t_B \), we have

\[ 1 + \Delta_1 (1 + \varepsilon) + \Delta_2 \geq 2 \]

and thus \( \Delta_2 \geq 1 - \Delta_1 (1 + \varepsilon) \) and \( t_B \geq t_i + 2 - \Delta_1 \varepsilon \).

We symmetrically apply the same reasoning to the last tasks \( C \) and \( D \) of these two chains, and their starting times \( t_C \) and \( t_D \), assuming that \( C \) is started before \( D \). By setting \( t_D = M - t_i - 1 - \Delta_1 \varepsilon \), we get \( t_C \leq M - t_i - 2 - \Delta_1 \varepsilon \). We distinguish between two cases, depending on the chains to which \( A, B, C, \) and \( D \) belong to:

i) In the first case, we assume that \( A \) and \( D \) belong to the same chain. We consider the other chain, containing \( B \) and \( C \). Because exactly \( 4(n - i) + 2m + 1 \) tasks need to be processed between these two tasks, we have

\[ t_C \geq t_B + 4(n - i) + 2m + 1 \]

which gives

\[ \Delta_1 \varepsilon \geq 1 - \Delta_1 \varepsilon \]

We have \( \Delta_1 \leq 1 \) and similarly, \( \Delta_1' \leq 1 \). Together with the previous inequality, this gives \( \varepsilon \geq 1/4m \). Hence \( B \) and \( C \) cannot belong to the same chain.

ii) In the second case, we consider that \( A \) and \( C \) belong to the same chain. Because exactly \( 4(n - i) + 2m + 1 \) tasks need to be processed between \( A \) and \( C \) (and between \( B \) and \( D \)), we have

\[ t_C \geq t_A + 4(n - i) + 2m + 1 \quad \text{and} \quad t_D \geq t_B + 4(n - i) + 2m + 1 \]

which gives

\[ 2m + 4n - t_i - 2 + \Delta_1' \varepsilon \geq t_i + 1 + \Delta_1 + 4(n - i) + 2m + 1 \]
and

\[2m + 4n - t_i - 1 - \Delta'_1 \geq t_i + 2 - \Delta_1 \varepsilon + 4(n - i) + 2m + 1,\]

which are simplified (using \( t_i = 2(i - 1) \)) into

\[\Delta'_1 \varepsilon \geq \Delta_1 \varepsilon \text{ and } \Delta'_1 \leq \Delta_1 \varepsilon.\]

This leads to \( \Delta_1 \leq \Delta_1 \varepsilon^2 \). As \( 0 < \varepsilon < 1 \), we have \( \Delta_1 = 0 \), so \( t_A = t_i + 1 \). Then, no processor can be allocated to \( B \) during \([t_i, t_{i+1}]\).

In other words, one task among the first task of \( L_{x_i} \) and the first task of \( L_{x_i} \) is fully processed during interval \([t_i, t_{i+1}]\) and the other one is not processed before \( t_i + 1 \). Because of its critical path length, the chain starting second must be processed at full speed (each task being allocated a number of processors equal to its threshold) and without idle time in the interval \([t_i + 1, M - t_i]\). The last task of the chain starting at time \( t_i \) must then be completed at time \( M - t_i - 1 \) and thus this chain must also be processed at full speed and without idle time. This also implies that all available processors are used in the intervals \([t_i, t_{i+2}]\) and \([M - t_{i+2}, M - t_i]\).

**Inductive step:** Now assume that the lemma holds for \( i - 1 \). With \( t_1 = 0 \) and the inductive property on the last observation we know that no processor is available for chains \( L_{x_i} \) and \( L_{x_i} \) before \( 2(i - 1) \) and after \( M - 2(i - 1) \). The time span available for the remaining chains is thus \( 4n + 2m - 4i + 4 \) while the critical path of chains \( L_{x_2} \) and \( L_{x_3} \) is \( 4n + 2m - 4i + 3 \); these chains cannot be started after \( 2(i - 1) + 1 \) to be completed within the time span. Setting \( t_i = 2(i + 1) \) we reuse the above argument about the scheduling of the two chains \( L_{x_i} \) and \( L_{x_i} \), which proves (i)-(iv).