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# A posteriori error estimates and adaptivity for fully mixed finite element discretizations for Biot's consolidation model\*

Elyes Ahmed<sup>†</sup>      Florin Adrian Radu<sup>†</sup>      Jan Martin Nordbotten<sup>†‡</sup>

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## Abstract

This paper is concerned with the analysis of fully mixed finite element method applied to the Biot's consolidation model. We consider two mixed formulations that use the stress tensor and Darcy velocity as primary variables as well as the displacement and pressure. The first formulation is with a symmetric stress tensor while the other enforces the symmetry of the stress weakly through the introduction of a Lagrange multiplier. The well-posedness of the two formulations is shown through a Galerkin's method and suitable a priori estimates. The two formulations are then discretized with the backward Euler scheme in time and with two mixed finite elements in space. We present next a unified a posteriori error analysis for the two formulations. Our estimates are based on  $H^1(\Omega)$ -conforming reconstruction of the pressure and a suitable  $[H^1(\Omega)]^2$ -conforming reconstruction of the displacement; both are continuous and piecewise affine in time. These reconstructions are used to infer a guaranteed and fully computable upper bound on the energy-type error measuring the differences between the exact and the approximate pressure and displacement. The error components resulting from the spatial and the temporal discretization are distinguished. They are then used to design an adaptive space-time algorithm. Numerical experiments illustrate the efficiency of our estimates and the performance of the adaptive algorithm.

**Key words:** Biot's poro-elasticity problem; mixed formulations; well-posedness; a posteriori error estimate; space error; time error; adaptive algorithm.

## 1 Introduction

Biot's consolidation model describes the deformation of an elastic porous medium and the viscous fluid flow inside when the porous medium is saturated by the fluid [9]. The governing equations to describe this interaction between the fluid flow and deforming porous medium are Darcy's law to describe the fluid motion, whereas the deformation of porous media is governed by linear elasticity. This model problem is of interest in a number of applications, such as petroleum engineering, geomechanics, energy storage in the subsurface, CO2 sequestration and understanding of biological tissues.

Many results on this problem and on its numerical approximation have been derived in the past. Results on well-posedness of the classical two-field formulation based on displacement and pressure variables are carried out by Showalter [29]. The corresponding a priori error analysis can be found in [25], and the a posteriori error analysis was derived in [27]. In particular, several numerical schemes have been developed for the three-field formulation [7, 8, 26, 30]. The involved variables in this formulation are the displacements, fluid flux (Darcy velocity), and the pore pressure. This formulation is widely used since it permits flexible discretizations of the problem by using different numerical methods applied separately for the fluid flow and elasticity problems. Among others we cite, in particular, the coupling of continuous and discontinuous Galerkin methods, and mixed finite element method studied in [30]. The coupling of nonconforming and

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<sup>†</sup>Department of Mathematics, University of Bergen, P. O. Box 7800, N-5020 Bergen, Norway. elyes.ahmed@uib.no, florin.radu@math.uib.no, Jan.Nordbotten@uib.no,

<sup>‡</sup>Department of Civil and Environmental Engineering, Princeton University, Princeton, N. J., USA.

mixed finite element methods for the formulation was recently studied in [33]. Particularly, a stabilized discretization for the classical  $\mathbf{P}_1\text{-RT}_0\text{-P}_0$  approach was recently proposed in [28]. One can also see [7, 13, 28] and the references therein for more discretization schemes. More recently, the mixed finite element methods as a numerical approximation of the four-field formulation has received increasing attention. In this formulation, two pairs of mixed finite elements spaces, one for linear elasticity and one for mixed Poisson problem, are used. This formulation uses the stress, displacement, flux, and pressure as primary unknowns. The motivation for using mixed methods for this model problem is manifold. First, these methods are widely used in computational fluid mechanics because they keep conservation law of mass or energy and produces continuous normal fluxes, regardless of mesh quality. Secondly, the stress and the Darcy velocity as variables are of primary interest in many applications. This eliminates the need to use postprocessing techniques to produce the flux and stress from the numerical pressure and displacement. The availability of these variables allows the coupling of the Biot equation with other fluid flow or mechanical systems, where in particular, transmission conditions across physical interfaces, involving the stress and flux are imposed [21]. These variables are also needed to integrate the Biot's consolidation model in fractured media in which the fracture deformation is coupled to that of the surrounding domain through internal boundary conditions [31, 2].

In this work we consider coupled mixed methods for Biot's consolidation model, and we shall mainly be concerned with the a posteriori error analysis for this model problem. To the best of our knowledge, contrarily to the case of a priori error estimates [23, 34], no results are available for a posteriori error estimates for the Biot's consolidation model in mixed form. Residual-based error estimates for the two-field formulation can be found, e.g., in [16]. Recently, a posteriori error analysis of this later formulation has been derived in [27], where the discrete solution is achieved using Taylor–Hood  $H^1$ -conforming finite elements in space (using piecewise polynomials of order  $k \geq 1$  for the pressure and of order  $(k + 1)$  for the displacement) and a backward Euler scheme in time. The estimates are based on flux reconstructions to compute the error indicators that bounds the dual norm of the residual. Therein, the spatial and the temporal discretization are distinguished, then used in an algorithm adapting the mesh and the time so as to equilibrate these error sources.

The main contribution of this article is to derive unified a posteriori error estimates for the Biot's consolidation model given in mixed form. To this aim, first of all to fill the gap of lacking rigorous well-posedness results for the mixed formulation, we analyze the well-posedness of two mixed formulations presented in [23, 34] and show their equivalence in the continuous case. Numerical approximation is then achieved by two mixed finite elements schemes in space (one for the Poisson problem and one for the linear elasticity) and the backward Euler scheme in time. Then a posteriori error estimates are derived which are valid for the aforementioned mixed formulations, and, in a more general sense, for any locally conservative method such as cell-centered finite volume scheme and mimetic finite difference, and others. Precisely, we first show that, using  $H^1(\Omega)$ -conforming reconstruction of the pressure and a  $[H^1(\Omega)]^2$ -conforming reconstruction of the displacement, continuous and piecewise affine in time, an energy-type-norm differences between the exact and the approximate pressure and displacement can be bounded by the dual norm of the residuals [12, 15, 17]. The estimates are then established in this energy-type error by deriving a guaranteed and fully computable upper bound on the dual norm of the residuals (cf. [1, 19, 27]). In the present work, and as in the mentioned works, we will show how a posteriori error estimates can distinguish between the space and time errors as well as between the pressure and displacement errors. This is essential for the development of space–time adaptive marching algorithm as well as adaptive iterative coupling schemes in which we decouple the flow calculation from the deformation calculation and then solve the two equations sequentially using an iterative method; see the fixed-stress split scheme presented in [7, 11].

The paper is organized as follows. In Section 2, we present the physical model and the assumptions on the effective parameters. Section 3 recalls the four-field formulation of the problem and then defines two mixed weak solutions of the problem as well as the relevant functions spaces. We analyze the well-posedness of the aforementioned formulations and shows in particular their equivalence. In Section 4, we present the resulting discrete formulations, by combining in space two mixed finite elements for the flow and mechanical problems, and the backward Euler scheme in time. Section 5 defines the postprocessing of the pressure and displacement as well as the corresponding  $H^1(\Omega)$ -conforming reconstructions. In Section 6, we show how to bound an energy-type error in the pressure and displacement by the dual norm of the residuals. We present then a guaranteed and fully computable error estimate that bounds the energy-type error in the pressure

and displacement. In Section 7, the a posteriori estimate is elaborated by distinguishing its space and time error components. This is next used in Section 8 to propose a space–time adaptive marching algorithm. Finally, we show numerical results in Section 9.

## 2 Presentation of the problem

Let  $\Omega$  be a connected polygonal domain of  $\mathbb{R}^2$ . We denote by  $\partial\Omega$  its boundary (supposed to be Lipschitz-continuous) and by  $\mathbf{n}$  the unit normal to  $\partial\Omega$ , outward to  $\Omega$ . Let a time interval  $(0, T)$  be given with  $T > 0$ . We consider Biot's consolidation equations, modeling flow in deformable porous media. The mathematical form of this problem as it is presented in [9, 29] reads: find  $\mathbf{u}$  and  $p$  such that there holds,

$$-(\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) - \mu\nabla^2 \mathbf{u} + \alpha\nabla p = \mathbf{f}, \quad \text{in } \Omega \times (0, T), \quad (2.1a)$$

$$\partial_t(c_0 p + \alpha\nabla \cdot \mathbf{u}) - \nabla \cdot (\mathbf{K}\nabla p) = g, \quad \text{in } \Omega \times (0, T), \quad (2.1b)$$

where  $\mathbf{f}$  is the body force,  $\mu$  and  $\lambda$  are the Lamé parameters,  $\alpha$  is the Biot–Willis constant,  $\mathbf{K}$  is the permeability tensor divided by fluid viscosity,  $c_0$  is the constrained-specific storage coefficient and  $g$  is the volumetric source term. The vector field  $\mathbf{u}$  is the displacement and  $p$  is the fluid pressure. We denote by  $\boldsymbol{\theta}$  the effective stress tensor;  $\boldsymbol{\theta}(\mathbf{u}) := 2\mu\boldsymbol{\epsilon}(\mathbf{u}) + \lambda\text{tr}(\boldsymbol{\epsilon}(\mathbf{u}))\mathbf{I}$ , where  $\boldsymbol{\epsilon}(\mathbf{u})$  is the linearized strain tensor given by  $\boldsymbol{\epsilon}(\mathbf{u}) := (\nabla\mathbf{u} + \nabla^T\mathbf{u})/2$  and where  $\text{tr}$  denotes the trace of matrices. The total stress tensor  $\boldsymbol{\sigma}(p, \mathbf{u})$  is given by  $\boldsymbol{\sigma}(p, \mathbf{u}) := \boldsymbol{\theta}(\mathbf{u}) - \alpha p\mathbf{I}$ , presenting the internal forces on surface elements. The momentum balance for the fluid is interpreted as the Darcy law for the volumetric fluid flux  $\mathbf{w} := -\mathbf{K}\nabla p$ . To complete the system (2.1a)–(2.1b), we have to prescribe suitable boundary and initial conditions. To this aim, we adopt the following partitions of the boundary;  $\partial\Omega = \Gamma_f \cup \Gamma_p$  and  $\partial\Omega = \Gamma_t \cup \Gamma_d$ , and for the simplicity we consider only homogeneous (mixed) boundary conditions:

$$\mathbf{w} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_f \times (0, T), \quad \text{and} \quad p = 0 \quad \text{on } \Gamma_p \times (0, T), \quad (2.1c)$$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_t \times (0, T), \quad \text{and} \quad \mathbf{u} = 0 \quad \text{on } \Gamma_d \times (0, T). \quad (2.1d)$$

Other types of boundary conditions could be dealt with in the same way as in [10, 11, 23]. We close the model by giving the following initial conditions:

$$\mathbf{u}(0) = 0 \quad \text{and} \quad p(0) = 0, \quad \text{in } \Omega \times \{0\}. \quad (2.1e)$$

For simplicity, we consider only homogeneous initial conditions, but the results are valid for more general case; see Remark 8.1. Let us recall that the well-posedness and regularity of the initial-boundary value problem (2.1) in a slightly different setting for the boundary conditions has been studied by Showalter [29]. In the next section, we present two mixed formulations of (2.1) introduced in [23, 34], where each of which uses two mixed finite element methods, one for linear elasticity of the Hellinger–Reissner formulation and the other for mixed Poisson problems. For the linear elasticity, the first formulation uses symmetric space for the stress unknown, while the second formulation uses mixed finite elements with weakly symmetric stress.

Before transcribing the mixed weak formulations of problem (2.1), we make precise the assumptions on the effective parameters:

**Assumption 2.1** (Data). *1. Let  $c_0$ ,  $\alpha$ ,  $\mu$ , and  $\lambda$  be strictly positive constants.*

*2. Let  $\mathbf{K}$ , be a symmetric and uniformly positive definite tensor which satisfies the following assumption: there exist positive constants  $c_{\mathbf{K}}$  and  $C_{\mathbf{K}}$  such that for or a.e.  $\mathbf{x} \in \Omega$  and for all  $\zeta \in \mathbb{R}^2$*

$$0 < c_{\mathbf{K}}|\zeta|^2 \leq \zeta^T \mathbf{K}^{-1}(\mathbf{x})\zeta, \quad \text{and} \quad |\mathbf{K}(\mathbf{x})\zeta| \leq C_{\mathbf{K}}|\zeta|, \quad (2.2)$$

*and whose terms are for simplicity supposed piecewise constant on the meshes defined below and constant in time.*

### 3 Mixed variational formulation

In this section, we present two mixed formulations of problem (2.1). To do so, we need to introduce some function spaces and their norms that will be used throughout. Firstly, we will use the convention that if  $V$  is a space of functions, then we designate by  $\mathbf{V}$  a space of vector functions having each component in  $V$ , and we designate by  $\mathbb{V}$  the space of tensor functions having each component in  $V$ . The space  $L^2(\Omega)$  is the space of squared integrable functions endowed with its natural inner product written  $(\cdot, \cdot)_{L^2(\Omega)}$  with associated norm denoted by  $\|\cdot\|$ . We designate by  $H^1(\Omega)$  the usual Sobolev space and by  $H_0^1(\Omega)$  for its zero-trace subspace. Its norm and semi-norm are written  $\|\cdot\|_{H_0^1(\Omega)}$  and  $|\cdot|_{H_0^1(\Omega)}$  respectively. In particular,  $H^{-1}(\Omega)$  is the dual of  $H_0^1(\Omega)$ . We will also make use of the subspaces  $H_{0,\Gamma_p}^1(\Omega) = \{\mathbf{v} \in H^1(\Omega) \mid \mathbf{v} = 0 \text{ on } \Gamma_p\}$  and  $\mathbf{H}_{0,\Gamma_d}^1(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega) \mid \mathbf{v} = 0 \text{ on } \Gamma_d\}$ . Further, let  $\mathbf{H}(\text{div}, \Omega)$  be the space of vector-valued functions from  $\mathbf{L}^2(\Omega)$  that admit a weak divergence in  $L^2(\Omega)$ . Its natural norm is

$$\|\mathbf{v}\|_{\text{div},\Omega} := (\|\mathbf{v}\|^2 + \|\nabla \cdot \mathbf{v}\|^2)^{\frac{1}{2}}.$$

We define the subspace

$$\mathbf{H}_{0,\Gamma_f}(\text{div}, \Omega) := \{\mathbf{v} \in \mathbf{H}(\text{div}, \Omega) : \mathbf{v} \cdot \mathbf{n}|_{\Gamma_f} = 0\}.$$

We also define  $\mathbb{H}(\text{div}, \Omega)$  to be the space of tensor-valued functions from  $\mathbb{L}^2(\Omega)$  that admit a weak divergence (by rows) in  $L^2(\Omega)$ . We define its subspace

$$\mathbb{H}_{0,\Gamma_t}(\text{div}, \Omega) := \{\boldsymbol{\tau} \in \mathbb{H}(\text{div}, \Omega) : \boldsymbol{\tau} \cdot \mathbf{n}|_{\Gamma_t} = 0\}.$$

Additionally, we define the symmetric tensor space  $\mathbb{H}^s(\text{div}, \Omega)$

$$\mathbb{H}^s(\text{div}, \Omega) := \{\boldsymbol{\tau} \in \mathbb{H}(\text{div}, \Omega) : \tau_{ij} = \tau_{ji}, 1 \leq i, j \leq 2\},$$

and its subspace

$$\mathbb{H}_{0,\Gamma_t}^s(\text{div}, \Omega) := \{\boldsymbol{\tau} \in \mathbb{H}^s(\text{div}, \Omega) : \boldsymbol{\tau} \cdot \mathbf{n}|_{\Gamma_t} = 0\}.$$

Then, we set

$$Q := L^2(\Omega), \quad \mathbf{W} := \mathbf{H}_{0,\Gamma_f}(\text{div}, \Omega), \quad \mathbb{M} := \mathbb{H}_{0,\Gamma_t}(\text{div}, \Omega), \quad \mathbb{\Sigma} := \mathbb{H}_{0,\Gamma_t}^s(\text{div}, \Omega).$$

#### 3.1 First variational formulation

Following [34] and the references therein, we derive the first mixed formulation for the problem (2.1). To this aim, we first rewrite problem (2.1) as

$$\mathcal{A}\boldsymbol{\sigma} - \boldsymbol{\epsilon}(\mathbf{u}) + \frac{\alpha}{2(\mu + \lambda)}p\mathbf{I} = 0, \quad \text{in } \Omega \times (0, T), \quad (3.1a)$$

$$\partial_t(c_0p + \alpha\nabla \cdot \mathbf{u}) + \nabla \cdot \mathbf{w} = g, \quad \text{in } \Omega \times (0, T), \quad (3.1b)$$

$$\mathbf{K}^{-1}\mathbf{w} + \nabla p = 0, \quad \text{in } \Omega \times (0, T), \quad (3.1c)$$

$$-\nabla \cdot \boldsymbol{\sigma} = \mathbf{f}, \quad \text{in } \Omega \times (0, T), \quad (3.1d)$$

where  $\mathcal{A}$  is the fourth-order compliance tensor given by

$$\mathcal{A}\boldsymbol{\tau} = \frac{1}{2\mu} \left( \boldsymbol{\tau} - \frac{\lambda}{2(\mu + \lambda)} \text{tr}(\boldsymbol{\tau})\mathbf{I} \right), \quad (3.2)$$

and known to be bounded and symmetric definite uniformly with respect to  $\mathbf{x} \in \Omega$ . In addition, we rewrite (3.1b) using the following relationship

$$\nabla \cdot \mathbf{u} = \text{tr}(\boldsymbol{\epsilon}(\mathbf{u})) = \frac{1}{2(\mu + \lambda)} \text{tr}(\boldsymbol{\sigma}) + \frac{\alpha}{\mu + \lambda}p, \quad (3.3)$$

derived by taking the trace-operator on both sides of (3.1a). Let  $c_r = \frac{\alpha^2}{\mu + \lambda}$ . Using (3.3) in (3.1b), and integrating by parts with vanishing boundary conditions, we derive the mixed variational formulation of the

problem (2.1) which reads as follows: for a.e.  $t \in (0, T)$ , find  $(\boldsymbol{\sigma}(t), p(t), \mathbf{w}(t), \mathbf{u}(t)) \in \mathbb{S} \times Q \times \mathbf{W} \times \mathbf{Q}$  such that

$$-(\mathcal{A}\boldsymbol{\sigma}, \boldsymbol{\tau}) - (\mathbf{u}, \nabla \cdot \boldsymbol{\tau}) - \frac{c_r}{2\alpha}(\mathbf{I}p, \boldsymbol{\tau}) = 0, \quad \forall \boldsymbol{\tau} \in \mathbb{S}, \quad (3.4a)$$

$$(c_0 + c_r)(\partial_t p, q) + \frac{c_r}{2\alpha}(\partial_t \boldsymbol{\sigma}, q\mathbf{I}) + (\nabla \cdot \mathbf{w}, q) = (g, q), \quad \forall q \in Q, \quad (3.4b)$$

$$(\mathbf{K}^{-1}\mathbf{w}, \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{W}, \quad (3.4c)$$

$$-(\nabla \cdot \boldsymbol{\sigma}, \mathbf{z}) = (\mathbf{f}, \mathbf{z}), \quad \forall \mathbf{z} \in \mathbf{Q}, \quad (3.4d)$$

together with the initial condition (2.1e). Note that in (3.4), we used that  $(\text{tr}(\boldsymbol{\xi}), q) = (\boldsymbol{\xi}, q\mathbf{I})$  for any matrix  $\boldsymbol{\xi}$  and a scalar  $q$ . In what follows the constrained specific storage coefficient  $c_0$  is assumed to be strictly positive and satisfies:

$$c_0 \geq \frac{1}{2(\lambda + \mu)}. \quad (3.5)$$

We refer to [6] for a similar setting and to [24] for a more detailed discussion of scaling of the Biot system. The well-posedness of problem (3.4), (2.1e) is then given in the following result.

**Theorem 3.1.** *If the source term  $g$  is such that  $g \in L^2(0, T; L^2(\Omega))$  and the body force  $\mathbf{f}$  is in  $H^1(0, T; \mathbf{L}^2(\Omega))$  then the problem (3.4), (2.1e) has a unique solution*

$$(\boldsymbol{\sigma}, \mathbf{u}) \in (L^2(0, T; \mathbb{H}_{0, \Gamma_t}^s(\text{div}, \Omega)) \cap H^1(0, T; L^2(\Omega))) \times H^1(0, T; \mathbf{L}^2(\Omega)), \quad (3.6a)$$

$$(\mathbf{w}, p) \in (L^2(0, T; \mathbf{H}_{0, \Gamma_f}(\text{div}, \Omega)) \cap L^\infty(0, T; \mathbf{L}^2(\Omega))) \times H^1(0, T; L^2(\Omega)). \quad (3.6b)$$

In turn, the well-posedness of problem (3.1), (2.1e) stems from the Lemma given next.

The proof of this result uses a standard approach as in [26] based on Galerkin's method and a priori energy estimates. It is given in Appendix A.

**Lemma 3.2.** *Suppose that  $\mathbf{f} \in H^1(0, T; \mathbf{L}^2(\Omega))$  and  $g \in H^1(0, T; L^2(\Omega))$ . Let  $(\boldsymbol{\sigma}, p, \mathbf{w}, \mathbf{u})$  as given in (3.6) with the improved regularity  $\mathbf{w} \in H^1(0, T; \mathbf{H}_{0, \Gamma_f}(\text{div}, \Omega))$ , be the solution of (3.4), (2.1e). Then,  $(\boldsymbol{\sigma}, p, \mathbf{w}, \mathbf{u})$  solves the initial boundary problem (3.1), (2.1e) in the following sense:*

$$\mathcal{A}\boldsymbol{\sigma}(t) - \boldsymbol{\epsilon}(\mathbf{u})(t) + \frac{c_r}{2\alpha}p(t)\mathbf{I} = 0, \quad \text{in } \Omega, \quad (3.7a)$$

$$(c_0 + c_r)\frac{dp}{dt}(t) + \frac{c_r}{2\alpha}\frac{d(\text{tr}(\boldsymbol{\sigma}))}{dt}(t) + \nabla \cdot \mathbf{w}(t) = g(t), \quad \text{in } \Omega, \quad (3.7b)$$

$$\mathbf{K}^{-1}\mathbf{w}(t) + \nabla p(t) = 0, \quad \text{in } \Omega, \quad (3.7c)$$

$$-\nabla \cdot \boldsymbol{\sigma}(t) = \mathbf{f}(t), \quad \text{in } \Omega, \quad (3.7d)$$

for almost every  $t \in (0, T)$ , and such that (2.1c)–(2.1e) holds true. Conversely, any solution  $(\boldsymbol{\sigma}, p, \mathbf{w}, \mathbf{u})$  of Biot's consolidation problem in the sense given by (3.7) is a solution of (3.4), (2.1e).

*Proof.* It stems from the above considerations that any solution of (3.7) given by the quadruplet  $(\boldsymbol{\sigma}, p, \mathbf{w}, \mathbf{u})$  is such that, for a.e.  $t \in (0, T)$ ,  $(\boldsymbol{\sigma}(t), p(t), \mathbf{w}(t), \mathbf{u}(t)) \in \mathbb{S} \times Q \times \mathbf{W} \times \mathbf{Q}$ , and solves (3.4), (2.1e). Conversely let  $(\boldsymbol{\sigma}, p, \mathbf{w}, \mathbf{u})$  as given in (3.6) with the additional regularity  $\mathbf{w} \in H^1(0, T; \mathbf{H}_{0, \Gamma_f}(\text{div}, \Omega))$  solve (3.4), (2.1e). It follows from the Sobolev embedding theorem for functions of a single variable  $t$  that we have  $(\boldsymbol{\sigma}, p) \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; \mathbb{H}_{0, \Gamma_t}^s(\text{div}, \Omega)) \times C(0, T; L^2(\Omega))$ . Thus, choosing smooth test functions with compact support in  $\Omega$ , we immediately derive that  $(\boldsymbol{\sigma}, p, \mathbf{w}, \mathbf{u})$  satisfies the initial boundary problem (3.7) such that  $\mathbf{u}(0) = 0$  and  $p(0) = 0$ . Furthermore, since  $\mathbf{K}^{-1}\mathbf{w}$  belongs to  $L^2(0, T; \mathbf{L}^2(\Omega))$  (3.7c) implies that  $p \in L^2(0, T; H_{0, \Gamma_p}^1(\Omega))$ . Similarly, since both  $\boldsymbol{\sigma}$  and  $p\mathbf{I}$  are in  $L^2(0, T; L^2(\Omega))$  then it follows from (3.7a) that  $\mathbf{u} \in L^2(0, T; \mathbf{H}_{0, \Gamma_d}^1(\Omega))$ . It remains to recover the regularity of  $(\boldsymbol{\sigma}, p)$  in time. Since both  $\nabla \cdot \mathbf{w}$  and  $g$  are in  $C([0, T]; L^2(\Omega))$ , we infer from (3.7b) that  $(\boldsymbol{\sigma}, p) \in (C^1([0, T]; L^2(\Omega)) \cap L^2(0, T; \mathbb{H}_{0, \Gamma_t}^s(\text{div}, \Omega))) \times (C^1(0, T; L^2(\Omega)) \cap L^2(0, T; H_{0, \Gamma_p}^1(\Omega)))$ . Consequently, Biot's consolidation problem is satisfied in this sense.  $\square$

### 3.2 Second variational formulation

We now present a second mixed formulation of the problem (3.1), (2.1e), in which the symmetry condition for the space of the stress tensor is now relaxed and replaced through the introduction of a Lagrange multiplier. In this formulation as presented in [23], we introduce the new unknown  $\zeta$  as the skew-symmetric part of  $\nabla \mathbf{u}$  and extend the definition of  $\mathcal{A}$  from the symmetric tensor space to all tensors. We denote this extension by  $\mathcal{A}$  as well, since it will also be given by formula (3.2). With these definitions, equation (3.1a) can now be replaced by

$$\mathcal{A}\boldsymbol{\sigma} - \nabla \mathbf{u} + \frac{\alpha}{2(\mu + \lambda)} p \mathbf{I} + \zeta = 0, \quad \text{in } \Omega \times (0, T), \quad (3.8)$$

and then enforcing the symmetry of  $\boldsymbol{\sigma}$  weakly through

$$(\boldsymbol{\sigma}, \boldsymbol{\tau}) = 0, \quad \forall \boldsymbol{\tau} \in \mathbb{Q}_{\text{AS}}, \quad (3.9)$$

where  $\mathbb{Q}_{\text{AS}} = [L^2(\Omega)]_{\text{AS}}^{2 \times 2}$  denotes the subspace of  $[L^2(\Omega)]^{2 \times 2}$  composed of skew symmetric-valued tensors. The mixed formulation for the problem (2.1) reads now: for a.e.  $t \in (0, T)$ , find  $(\boldsymbol{\sigma}(t), p(t), \mathbf{w}(t), \mathbf{u}(t), \zeta(t)) \in \mathbb{M} \times Q \times \mathbf{W} \times \mathbf{Q} \times \mathbb{Q}_{\text{AS}}$  such that

$$-(\mathcal{A}\boldsymbol{\sigma}, \boldsymbol{\tau}) - (\mathbf{u}, \nabla \cdot \boldsymbol{\tau}) - \frac{c_r}{2\alpha} (\mathbf{I}p, \boldsymbol{\tau}) + (\zeta, \boldsymbol{\tau}) = 0, \quad \forall \boldsymbol{\tau} \in \mathbb{M}, \quad (3.10a)$$

$$(c_0 + c_r)(\partial_t p, q) + \frac{c_r}{2\alpha} (\partial_t \boldsymbol{\sigma}, q \mathbf{I}) + (\nabla \cdot \mathbf{w}, q) = (g, q), \quad \forall q \in Q, \quad (3.10b)$$

$$(\mathbf{K}^{-1} \mathbf{w}, \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{W}, \quad (3.10c)$$

$$-(\nabla \cdot \boldsymbol{\sigma}, \boldsymbol{\gamma}) + (\boldsymbol{\sigma}, \boldsymbol{\gamma}) = (\mathbf{f}, \boldsymbol{\gamma}), \quad \forall (\boldsymbol{\gamma}, \boldsymbol{\gamma}) \in \mathbf{Q} \times \mathbb{Q}_{\text{AS}}. \quad (3.10d)$$

Next we prove that problem (3.10), (2.1e) is equivalent to problem (3.4), (2.1e). Since we know from Theorem 3.1 that (3.4), (2.1e) as well as (3.1) has a unique solution, equivalence will also establish that (3.10), (2.1e) is uniquely solvable.

**Lemma 3.3.** *Suppose that  $\mathbf{f} \in H^1(0, T; \mathbf{L}^2(\Omega))$  and  $g \in L^2(0, T; L^2(\Omega))$ , the weak formulation (3.10), (2.1e) admits a unique solution*

$$(\boldsymbol{\sigma}, \mathbf{u}, \zeta) \in (L^2(0, T; \mathbb{H}_{0, \Gamma_t}(\text{div}, \Omega)) \cap H^1(0, T; \mathbb{L}^2(\Omega))) \times H^1(0, T; \mathbf{L}^2(\Omega)) \times H^1(0, T; \mathbb{L}^2(\Omega)), \quad (3.11a)$$

$$(\mathbf{w}, p) \in (L^2(0, T; \mathbf{H}_{0, \Gamma_f}(\text{div}, \Omega)) \cap L^\infty(0, T; \mathbf{L}^2(\Omega))) \times H^1(0, T; L^2(\Omega)). \quad (3.11b)$$

Moreover, under the assumption of Lemma (3.2) the weak formulations (3.4), (2.1e) and (3.10), (2.1e) are equivalent.

*Proof.* The well-posedness of (3.10), (2.1e) can be obtained by mimicking the proof of Theorem 3.1. Let us inspect the equivalence of the two formulations. Let  $(\boldsymbol{\sigma}(t), p(t), \mathbf{w}(t), \mathbf{u}(t), \zeta(t))$  be a solution of (3.10), (2.1e), then  $\boldsymbol{\sigma}$  is symmetric, i.e.,  $\boldsymbol{\sigma}(t) \in \mathbb{H}^s(\text{div}, \Omega)$ , and therefore  $(\boldsymbol{\sigma}(t), p(t), \mathbf{w}(t), \mathbf{u}(t)) \in \Sigma \times Q \times \mathbf{W} \times \mathbf{Q}$  solves the first mixed formulation (3.4), (2.1e). On the other hand, if  $(\boldsymbol{\sigma}, p, \mathbf{w}, \mathbf{u})$  solves (3.7), then  $\mathbf{u}(t) \in \mathbf{H}^1(\Omega)$  and, if we set  $\zeta(t)$  to the skew-symmetric part of  $\nabla \mathbf{u}(t)$ , then  $(\boldsymbol{\sigma}(t), p(t), \mathbf{w}(t), \mathbf{u}(t), \zeta(t))$  solves (3.10), (2.1e). In this respect, the two formulations (3.4), (2.1e) and (3.10), (2.1e) are equivalent under the assumption of Lemma (3.2).  $\square$

## 4 The mixed finite elements method

In this section, after introducing some notations, we present a fully discrete formulation of (3.4), (2.1e), as well as of (3.10), (2.1e). We employ for both formulations discontinuous Galerkin method of order zero in time (DG0), and in space, two mixed finite elements (MFE) for the linear elasticity and flow subproblems. For the first formulation, we use a MFE method with symmetric stress tensors for the elasticity based on the Hellinger–Reissner variational principle [20] and for flow we use standard MFE method with Raviart–Thomas–Nédélec space as used in the second formulation. However, for this later formulation we use mixed finite elements with weakly symmetric stress for the elasticity based on the Arnold–Falk–Winther mixed finite elements [4].

### 4.1 The temporal discretization

For integer values  $N \geq 0$ , let  $(\tau^n)_{0 \leq n \leq N}$  denote a sequence of positive real numbers corresponding to the discrete time steps such that  $T = \sum_{n=1}^N \tau^n$ . Let  $t^0 = 0$ , and  $t^n = \sum_{j=1}^n \tau^j$ ,  $1 \leq n \leq N$  be the discrete times. Let  $I^n = (t^{n-1}, t^n]$ ,  $1 \leq n \leq N$ . For every time step  $0 \leq n \leq N$ , we let  $v_h^n := v_{h\tau}(\cdot, t^n)$  for any sufficiently smooth function  $v_{h\tau}$ .

### 4.2 The spatial meshes

Let  $0 \leq n \leq N$  be fixed. We consider a spatial discretization  $\mathcal{T}_h^n$  of the domain  $\Omega$  at time  $t^n$ , consisting of polygonal elements  $K$ , which are supposed to be closed simplices or closed rectangular parallelepipeds. We assume that the meshes  $\mathcal{T}_h^n$ ,  $0 \leq n \leq N$ , are matching and covers exactly the polygonal domain  $\Omega$ . This means that for  $K, L \in \mathcal{T}_h^n$  with  $K \neq L$ , the Lebesgue measure of  $K \cap L$  is 0, or that there exists a face denoted  $e$  such that  $e = K \cap L$ . We suppose that the initial mesh  $\mathcal{T}_h^0$  is introduced to approximate the initial condition. The meshes are then refined or coarsened as time progresses. For all  $1 \leq n \leq N$ , we denote by  $\mathcal{T}_h^{n-1, n}$ , a common refinement of  $\mathcal{T}_h^{n-1}$  and  $\mathcal{T}_h^n$ .

### 4.3 The discrete functional spaces

Let  $0 \leq n \leq N$  be fixed. We denote by  $\mathbb{P}_l(K)$  the space of polynomials on  $K \in \mathcal{T}_h^n$  of degree less than or equal to  $l$ . We use the notation  $\|\cdot\|_K$  for the norm in  $L^2(K)$ ,  $K \in \mathcal{T}_h^n$ . The corresponding inner product is  $(\cdot, \cdot)_K$ . Let  $|K|$  be the Lebesgue measure of  $K \in \mathcal{T}_h^n$ . We define the broken Sobolev space  $H^1(\mathcal{T}_h^n)$  as the space of all functions  $v \in L^2(\Omega)$  such that  $v|_K \in H^1(K)$ , for all  $K \in \mathcal{T}_h^n$ . The (broken) energy-norm on  $H^1(\mathcal{T}_h^n)$  is given by

$$\|v\|^2 = \sum_{K \in \mathcal{T}_h^n} \|\mathbf{K}^{\frac{1}{2}} \nabla v\|_K^2, \quad \forall v \in H^1(\mathcal{T}_h^n), \quad (4.1)$$

where the sign  $\nabla$  to denote the element-wise gradient, i.e., the gradient of a function restricted to a mesh element  $K \in \mathcal{T}_h^n$ . The (broken) energy-norm in  $\mathbf{L}^2(\Omega)$  is given by

$$\|\mathbf{v}\|_*^2 = \sum_{K \in \mathcal{T}_h^n} \|\mathbf{K}^{-\frac{1}{2}} \mathbf{v}\|_K^2, \quad \forall \mathbf{v} \in \mathbf{L}^2(\mathcal{T}_h^n). \quad (4.2)$$

Let  $E$  be a space of functions defined on  $\Omega$ . We denote  $P_\tau^1(E)$  the vector space of functions continuous in time and with values in  $E$ . We also denote by  $P_\tau^0(E)$  the space of functions piecewise constant in time and with values in  $E$ . We have then if  $v_{h\tau} \in P_\tau^1(E)$ , then  $\partial_t v_{h\tau} \in P_\tau^0(E)$  is such that for all  $0 \leq n \leq N$ ,

$$\partial_t v_h^n := \partial_t v_{h\tau}|_{I^n} = \frac{v_h^n - v_h^{n-1}}{\tau^n}. \quad (4.3)$$

In what follows, we denote respectively by  $c_{\mathbf{K}, K}$  and  $C_{\mathbf{K}, K}$  the smallest and the largest eigenvalue of the tensor  $\mathbf{K}$  in  $K \in \mathcal{T}_h^n$ .

For all  $0 \leq n \leq N$ , let  $Q_h^n \times \mathbf{W}_h^n \subset L^2(\Omega) \times \mathbf{H}(\text{div}, \Omega)$  be the Raviart–Thomas–Nédélec mixed finite element spaces of order zero on the mesh  $\mathcal{T}_h^n$ :

$$\begin{aligned} Q_h^n &:= \{q_h \in L^2(\Omega); \forall K \in \mathcal{T}_h^n, q_h|_K \in \mathbb{P}_0(K)\}, \\ \mathbf{W}_h^n &:= \{\mathbf{v}_h \in \mathbf{H}_{0, \Gamma_f}(\text{div}, \Omega); \forall K \in \mathcal{T}_h^n, \mathbf{v}_h|_K \in \mathbf{RTN}_0(K)\}, \end{aligned}$$

where  $\mathbf{RTN}_0(K)$  denotes the lowest-order Raviart–Thomas–Nédélec finite-dimensional subspace associated with the element  $K \in \mathcal{T}_h^n$ ; any  $\mathbf{v}_h \in \mathbf{RTN}_0(K)$  takes the form  $[\mathbb{P}_0(K)]^2 + \mathbb{P}_0(K)\mathbf{x}$  for the example of simplices. The degrees of freedom of  $\mathbf{v}_h \in \mathbf{RTN}_0(K)$  correspond to the values of the flux of  $\mathbf{v}_h$  across the faces  $e \in \partial K$ .

For all  $0 \leq n \leq N$ , let  $\mathbf{Q}_h^n \times \Sigma_h^n \subset \mathbf{L}^2(\Omega) \times \mathbb{H}^s(\text{div}, \Omega)$  be the Arnold–Winther mixed finite elements for the lowest-order stresses on the mesh  $\mathcal{T}_h^n$ :

$$\begin{aligned} \mathbf{Q}_h^n &:= \{\mathbf{z}_h \in \mathbf{L}^2(\Omega); \forall K \in \mathcal{T}_h^n, \mathbf{z}_h|_K \in [\mathbb{P}_1(K)]^2\}, \\ \Sigma_h^n &:= \{\boldsymbol{\tau}_h \in \mathbb{H}_{0, \Gamma_t}^s(\text{div}, \Omega); \forall K \in \mathcal{T}_h^n, \boldsymbol{\tau}_h|_K \in \mathbf{AW}_1(K)\}, \end{aligned}$$



where  $\mathbf{AW}_1(K)$  denotes the Arnold–Winther Stress space of order 1; any tensor  $\boldsymbol{\tau}_h \in \mathbf{AW}_1(K)$  is such that  $\boldsymbol{\tau}_h \in [\mathbb{P}_3(K)]_S^{2 \times 2}$  with  $\nabla \cdot \boldsymbol{\tau}_h \in [\mathbb{P}_1(K)]^2$ , where  $[\mathbb{P}_3(K)]_S^{2 \times 2}$  denotes the subspace of  $[\mathbb{P}_3(K)]^{2 \times 2}$  composed of symmetric-valued tensors.

Similarly, we will make use of the lowest order Arnold–Falk–Winther mixed finite elements ( $\mathbf{AFW}_1$ ) for the second formulation. For all  $0 \leq n \leq N$ , let  $\mathbf{Q}_h^n \times \mathbb{M}_h^n \times \mathbb{Q}_h^n \subset \mathbf{L}^2(\Omega) \times \mathbb{H}(\operatorname{div}, \Omega) \times \mathbb{L}^2(\Omega)$  be the Arnold–Falk–Winther mixed finite elements for the lowest-order stresses on the mesh  $\mathcal{T}_h^n$ :

$$\begin{aligned} \mathbf{Q}_h^n &:= \{\mathbf{z}_h \in \mathbf{L}^2(\Omega); \forall K \in \mathcal{T}_h^n, \mathbf{z}_h|_K \in [\mathbb{P}_0(K)]^2\}, \\ \mathbb{M}_h^n &:= \{\boldsymbol{\tau}_h \in \mathbb{H}_{0,\Gamma_t}(\operatorname{div}, \Omega); \forall K \in \mathcal{T}_h^n, \boldsymbol{\tau}_h|_K \in \mathbf{AFW}_1(K)\}, \\ \mathbb{Q}_{\text{AS},h}^n &:= \{\boldsymbol{\zeta}_h \in [L^2(\Omega)]_{\text{AS}}^{2 \times 2}; \forall K \in \mathcal{T}_h^n, \boldsymbol{\zeta}_h|_K \in [\mathbb{P}_0(K)]_{\text{AS}}^{2 \times 2}\}, \end{aligned}$$

where  $\mathbf{AFW}_1(K) = [\mathbb{P}_1(K)]^{2 \times 2}$  and  $[\mathbb{P}_0(K)]_{\text{AS}}^{2 \times 2}$  denotes the subspace of  $[\mathbb{P}_0(K)]^{2 \times 2}$  composed of antisymmetric-valued tensors.

#### 4.4 The MFE scheme

The discrete form of the first mixed formulation (3.4), (2.1e) reads as follows: Given  $\boldsymbol{\sigma}_h^0$  and  $p_h^0$ , find  $(\boldsymbol{\sigma}_h^n, p_h^n, \mathbf{w}_h^n, \mathbf{u}_h^n) \in \Sigma_h^n \times Q_h^n \times \mathbf{W}_h^n \times \mathbf{Q}_h^n$ , for all  $1 \leq n \leq N$ , such that

$$-(\mathcal{A}\boldsymbol{\sigma}_h^n, \boldsymbol{\tau}) - (\mathbf{u}_h^n, \nabla \cdot \boldsymbol{\tau}) - \frac{c_r}{2\alpha} (\mathbf{I}p_h^n, \boldsymbol{\tau}) = 0, \quad \forall \boldsymbol{\tau} \in \Sigma_h^n, \quad (4.4a)$$

$$(c_0 + c_r) \left( \frac{p_h^n - p_h^{n-1}}{\tau^n}, q \right) + \frac{c_r}{2\alpha} \left( \frac{\boldsymbol{\sigma}_h^n - \boldsymbol{\sigma}_h^{n-1}}{\tau^n}, q\mathbf{I} \right) + (\nabla \cdot \mathbf{w}_h^n, q) = (g_h^n, q), \quad \forall q \in Q_h^n, \quad (4.4b)$$

$$(\mathbf{K}^{-1}\mathbf{w}_h^n, \mathbf{v}) - (p_h^n, \nabla \cdot \mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{W}_h^n, \quad (4.4c)$$

$$-(\nabla \cdot \boldsymbol{\sigma}_h^n, \mathbf{z}) = (\mathbf{f}_h^n, \mathbf{z}), \quad \forall \mathbf{z} \in \mathbf{Q}_h^n. \quad (4.4d)$$

The well-posedness of this discrete mixed formulation is summarized in [34], where error estimates are derived. The uniqueness is shown by energy arguments. That of the existence is obtained by analyzing the resulting system at each time step.

The discrete form of the second mixed formulation (3.10), (2.1e) reads as follows: Given  $\boldsymbol{\sigma}_h^0$  and  $p_h^0$ , find  $(\boldsymbol{\sigma}_h^n, p_h^n, \mathbf{w}_h^n, \mathbf{u}_h^n, \boldsymbol{\zeta}_h^n) \in \mathbb{M}_h^n \times Q_h^n \times \mathbf{W}_h^n \times \mathbf{Q}_h^n \times \mathbb{Q}_{\text{AS},h}^n$ , for all  $1 \leq n \leq N$ , such that

$$-(\mathcal{A}\boldsymbol{\sigma}_h^n, \boldsymbol{\tau}) - (\mathbf{u}_h^n, \nabla \cdot \boldsymbol{\tau}) - \frac{c_r}{2\alpha} (\mathbf{I}p_h^n, \boldsymbol{\tau}) + (\boldsymbol{\zeta}_h^n, \boldsymbol{\tau}) = 0, \quad \forall \boldsymbol{\tau} \in \Sigma_h^n, \quad (4.5a)$$

$$(c_0 + c_r) \left( \frac{p_h^n - p_h^{n-1}}{\tau^n}, q \right) + \frac{c_r}{2\alpha} \left( \frac{\boldsymbol{\sigma}_h^n - \boldsymbol{\sigma}_h^{n-1}}{\tau^n}, q\mathbf{I} \right) + (\nabla \cdot \mathbf{w}_h^n, q) = (g_h^n, q), \quad \forall q \in Q_h^n, \quad (4.5b)$$

$$(\mathbf{K}^{-1}\mathbf{w}_h^n, \mathbf{v}) - (p_h^n, \nabla \cdot \mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{W}_h^n, \quad (4.5c)$$

$$-(\nabla \cdot \boldsymbol{\sigma}_h^n, \mathbf{z}) + (\boldsymbol{\sigma}_h^n, \boldsymbol{\gamma}) = (\mathbf{f}_h^n, \mathbf{z}), \quad \forall (\mathbf{z}, \boldsymbol{\gamma}) \in \mathbf{Q}_h^n \times \mathbb{Q}_{\text{AS},h}^n. \quad (4.5d)$$

The existence and uniqueness of the solution of this later formulation can be obtained using the same techniques as used for the first one (cf. [34]). A priori error estimates for this formulation were derived in [23]. Let us remark that the two discrete formulations are not equivalent since the discrete stress tensor  $\boldsymbol{\sigma}_h^n$  in this later formulation will not necessarily inherit the symmetry property of  $\boldsymbol{\sigma}_h^n$  from the first formulation. However, the five-fields formulation (4.5) may be advantageous compared to the classical one (4.4), since only mixed finite elements with weakly symmetric stress will be used. This is advantageous because it requires less computational costs and as the hybridization techniques of Fraeijis de Veubeke (cf. [5]) results usually a system with reduced sizes. In what follows, we denote by  $(\boldsymbol{\sigma}_{h\tau}, p_{h\tau}, \mathbf{w}_{h\tau}, \mathbf{u}_{h\tau}, \boldsymbol{\zeta}_{h\tau})$  the discrete space–time functions piecewise constant in time given by  $(\boldsymbol{\sigma}_{h\tau}, p_{h\tau}, \mathbf{w}_{h\tau}, \mathbf{u}_{h\tau}, \boldsymbol{\zeta}_{h\tau})(\cdot, t^n) = (\boldsymbol{\sigma}_h^n, p_h^n, \mathbf{w}_h^n, \mathbf{u}_h^n, \boldsymbol{\zeta}_h^n)$ , for each  $1 \leq n \leq N$ .

## 5 Postprocessing and $H^1$ -, and $\mathbf{H}^1$ -conforming reconstructions

The approximate pressure  $p_{h\tau}$  as well as the approximate displacement  $\mathbf{u}_{h\tau}$  inherited from the two methods are only piecewise constant in space and in time. These approximations are not appropriate for an

energy a posteriori error estimate. Following [17, 19], the ingredients for the a posteriori error analysis of problem (3.4), (2.1e) as well as problem (3.10), (2.1e) will be:

- (i) To construct a local postprocessing  $\tilde{p}_{h\tau}$  of the approximate pressure  $p_{h\tau}$  using the known fluxes as well as a local postprocessing  $\tilde{\mathbf{u}}_{h\tau}$  of the approximate displacement  $\mathbf{u}_{h\tau}$  using the known stress. Both postprocessed functions are then discontinuous piecewise quadratic in space and continuous piecewise affine in time.
- (ii) To construct from  $\tilde{p}_{h\tau}$  a scalar function  $\hat{p}_{h\tau}$  and from  $\tilde{\mathbf{u}}_{h\tau}$  a vector function  $\hat{\mathbf{u}}_{h\tau}$ , both are piecewise quadratic in space and continuous in  $\Omega$ , and continuous piecewise affine in time.

### 5.1 The postprocessed scalar variable $\tilde{p}_{h\tau}$ and vector variable $\tilde{\mathbf{u}}_{h\tau}$

We first introduce a postprocessing of the scalar variable  $p_{h\tau}$  as described in [17]. We define the postprocessed approximation  $\tilde{p}_h^n \in \mathbb{P}_2(\mathcal{T}_h^n)$  in each element  $K \in \mathcal{T}_h^n$  as the solution of

$$-\mathbf{K}\nabla\tilde{p}_h^n = \mathbf{w}_h^n, \quad \forall K \in \mathcal{T}_h^n, \quad (5.1a)$$

$$(\tilde{p}_h^n, 1)_K = (p_h^n, 1)_K, \quad \forall K \in \mathcal{T}_h^n. \quad (5.1b)$$

Such a postprocessing is local and its cost is negligible and valid for the two mixed finite schemes described above (cf. [5] for more details). The error analysis of this post-processing is given in [32]. In contrast to the postprocessing of the pressure, we should distinguish that of the displacement inherited from the two schemes. We suppose that the discrete solution is that of the first scheme (4.4). Then, extending the above postprocessing (cf. [19, Section 7.4]) to the vector case, leads to define a postprocessing  $\tilde{\mathbf{u}}_h^n$  of  $\mathbf{u}_h^n$  such that  $\tilde{\mathbf{u}}_h^n \in [\mathbb{P}_2(\mathcal{T}_h^n)]^2$ , and

$$\epsilon(\tilde{\mathbf{u}}_h^n) - \frac{\alpha}{2(\mu + \lambda)} p_h^n \mathbf{I} = \mathcal{A}\boldsymbol{\sigma}_h^n, \quad \forall K \in \mathcal{T}_h^n, \quad (5.2a)$$

$$\frac{(\tilde{\mathbf{u}}_h^n, \mathbf{e}_i)_K}{|K|} = \mathbf{u}_h^{i,n}|_K, \quad i = 1, \dots, d, \quad \forall K \in \mathcal{T}_h^n, \quad (5.2b)$$

where  $\mathbf{e}_i \in \mathbb{R}^2$  denotes the  $i$ -th Euclidean unit vector. For the second scheme (4.5), we modify the postprocessing  $\tilde{\mathbf{u}}_h^n$  of  $\mathbf{u}_h^n$  to be such that  $\tilde{\mathbf{u}}_h^n \in [\mathbb{P}_2(\mathcal{T}_h^n)]^2$ , and satisfies

$$\nabla\tilde{\mathbf{u}}_h^n - \frac{\alpha}{2(\mu + \lambda)} p_h^n \mathbf{I} - \boldsymbol{\zeta}_h^n = \mathcal{A}\boldsymbol{\sigma}_h^n, \quad \forall K \in \mathcal{T}_h^n, \quad (5.3a)$$

$$\frac{(\tilde{\mathbf{u}}_h^n, \mathbf{e}_i)_K}{|K|} = \mathbf{u}_h^{i,n}|_K, \quad i = 1, \dots, d, \quad \forall K \in \mathcal{T}_h^n. \quad (5.3b)$$

It is easy to see that  $\tilde{\mathbf{u}}_h^n$  is well-defined by checking that both reconstructions (5.2) and (5.3) stems from nonsingular linear systems. The error analysis of this post-processing is given in [23]. We define the continuous, piecewise affine in-time functions  $\tilde{p}_{h\tau}$  and  $\tilde{\mathbf{u}}_{h\tau}$  by

$$\tilde{p}_{h\tau}(\cdot, t^n) = \tilde{p}_h^n, \quad \tilde{\mathbf{u}}_{h\tau}(\cdot, t^n) = \tilde{\mathbf{u}}_h^n, \quad 1 \leq n \leq N. \quad (5.4)$$

### 5.2 The reconstructed pressure $\hat{p}_h^n$ and displacement $\hat{\mathbf{u}}_h^n$

We reconstruct from the postprocessed pressure  $\tilde{p}_h^n$  a function  $\hat{p}_h^n$ ,  $H^1(\Omega)$ -conforming, satisfying the mean value constraint (5.1b), and from the postprocessed displacement  $\tilde{\mathbf{u}}_h^n$  a function  $\hat{\mathbf{u}}_h^n$ ,  $\mathbf{H}^1(\Omega)$ -conforming and satisfying the mean value constraint (5.2b). To do so, for all  $1 \leq n \leq N$ , and for a given postprocessed function  $\tilde{p}_h^n \in \mathbb{P}_2(\mathcal{T}_h^n)$  defined by (5.1), we prescribe at the degrees of freedom of  $\mathbb{P}_2(\mathcal{T}_h^n) \cap H^1(\Omega)$  a piecewise continuous polynomial  $\hat{p}_h^n \in \mathbb{P}_2(\mathcal{T}_h^n) \cap H^1(\Omega)$ . If  $\mathbf{x}$  is a Lagrange node situated in the interior of  $\Omega$  or situated at the boundary  $\partial\Omega \setminus \Gamma^p$ , we set

$$\hat{p}_h^n(\mathbf{x}) := \mathcal{I}_{\text{av}}(\tilde{p}_h^n)(\mathbf{x}) + \sum_{K \in \mathcal{T}_h^{n,n+1}} a_K^n b_K(\mathbf{x}),$$

where for all  $K \in \mathcal{T}_h^{n,n+1}$ ,  $b_K$  denotes the standard (time-independent) bubble function supported on  $K$ , defined as the product of the barycentric coordinates of  $K$ , and scaled so that its maximal value is 1, and where  $\mathcal{I}_{\text{av}} : \mathbb{P}_2(\mathcal{T}_h^n) \rightarrow \mathbb{P}_2(\mathcal{T}_h^n) \cap H^1(\Omega)$  is the interpolation operator given by

$$\mathcal{I}_{\text{av}}(\phi_h)(\mathbf{x}) = \frac{1}{|\mathcal{T}_{\mathbf{x}}|} \sum_{K \in \mathcal{T}_{\mathbf{x}}} \phi_h|_K(\mathbf{x}),$$

with  $\mathcal{T}_{\mathbf{x}}$  being the set of all elements of  $\mathcal{T}_h^n$  sharing the node  $\mathbf{x}$ . At the Lagrange nodes  $\mathbf{x}$  situated on the boundary  $\Gamma_p$ , we set  $\hat{p}_h^n(\mathbf{x}) := 0$ . In order to guarantee that the mean value constraint (5.1b) holds true, we choose

$$a_K^n = \frac{1}{(b_K, 1)_K} (\tilde{p}_h^n - \mathcal{I}_{\text{av}}(\tilde{p}_h^n), 1)_K. \quad (5.5)$$

The same procedure applied to the postprocessed displacement  $\tilde{\mathbf{u}}_h^n \in [\mathbb{P}_2(\mathcal{T}_h^n)]^2$ , leads to a  $\mathbf{H}^1(\Omega)$ -conforming vector function as we set

$$\hat{\mathbf{u}}_h^n(\mathbf{x}) := \mathcal{I}_{\text{av}}(\tilde{\mathbf{u}}_h^n)(\mathbf{x}) + \sum_{K \in \mathcal{T}_h^{n,n+1}} \mathbf{a}_K^n \mathbf{b}_K(\mathbf{x}),$$

when  $\mathbf{x}$  is a Lagrange node situated in the interior of  $\Omega$  or situated at the boundary  $\partial\Omega \setminus \Gamma^d$ . If  $\mathbf{x} \in \Gamma^d$ , we set  $\hat{\mathbf{u}}_h^n(\mathbf{x}) = 0$ . Here,  $\mathcal{I}_{\text{av}}$  is given by  $\mathcal{I}_{\text{av}} : [\mathbb{P}_2(\mathcal{T}_h^n)]^2 \rightarrow [\mathbb{P}_2(\mathcal{T}_h^n)]^2 \cap \mathbf{H}^1(\Omega)$ , and where  $\mathbf{b}_K$  denotes the vector bubble function supported on  $K$ . Similarly, in order to obtain a  $\mathbf{H}^1(\Omega)$ -conforming vector function while maintaining (5.2b), we choose

$$\mathbf{a}_K^{i,n} = \frac{1}{(\mathbf{b}_K, \mathbf{e}_i)_K} (\tilde{\mathbf{u}}_h^n - \mathcal{I}_{\text{av}}(\tilde{\mathbf{u}}_h^n), \mathbf{e}_i)_K, \quad i = 1, \dots, d. \quad (5.6)$$

As usually, we define the continuous, piecewise affine in-time functions  $\hat{p}_{h\tau}$  and  $\hat{\mathbf{u}}_{h\tau}$  by

$$\hat{p}_{h\tau}(\cdot, t^n) = \hat{p}_h^n, \quad \hat{\mathbf{u}}_{h\tau}(\cdot, t^n) = \hat{\mathbf{u}}_h^n, \quad 1 \leq n \leq N. \quad (5.7)$$

An important result of the reconstructions above is given by the following Lemma:

**Lemma 5.1.** *Let  $\hat{p}_{h\tau}$  and  $\hat{\mathbf{u}}_{h\tau}$  be the reconstructed functions as obtained in Section 5.2. Define by  $\varphi(p, \mathbf{u}) := c_0 p + \alpha \nabla \cdot \mathbf{u}$  the fluid content. Then there holds*

$$(\partial_t \varphi(\hat{p}_h^n - \tilde{p}_h^n, \hat{\mathbf{u}}_h^n - \tilde{\mathbf{u}}_h^n), 1)_K = 0, \quad \forall K \in \mathcal{T}_h^n, \quad (5.8a)$$

$$(g^n - \partial_t \varphi(\hat{p}_h^n, \hat{\mathbf{u}}_h^n) - \nabla \cdot \mathbf{w}_h^n, 1)_K = 0, \quad \forall K \in \mathcal{T}_h^n. \quad (5.8b)$$

*Proof.* For all  $1 \leq n \leq N$ , and for all  $K \in \mathcal{T}_h^n$ , we have from the definition of  $\varphi$  and (4.3) together with the divergence theorem:

$$\begin{aligned} (\partial_t \varphi(\hat{p}_h^n, \hat{\mathbf{u}}_h^n), 1)_K &= c_0 (\partial_t \hat{p}_h^n, 1)_K + \alpha (\partial_t \nabla \cdot \hat{\mathbf{u}}_h^n, 1)_K, \\ &= \frac{c_0}{\tau^n} (\hat{p}_h^n - \hat{p}_h^{n-1}, 1)_K + \frac{\alpha}{\tau^n} (\nabla \cdot \hat{\mathbf{u}}_h^n - \nabla \cdot \hat{\mathbf{u}}_h^{n-1}, 1)_K, \\ &= \frac{c_0}{\tau^n} (\hat{p}_h^n - \hat{p}_h^{n-1}, 1)_K + \frac{\alpha}{\tau^n} \sum_{e \in \partial K} \sum_{i=1}^2 n_e^i (\hat{\mathbf{u}}_h^n - \hat{\mathbf{u}}_h^{n-1}, \mathbf{e}_i)_K, \end{aligned}$$

where we set  $\mathbf{n}_e = \sum_{i=1}^2 n_e^i \mathbf{e}_i$ . For the first term, we know from the above reconstructions that  $\hat{p}_h^n$  and  $\tilde{p}_h^n$  have the same mean values on all the elements of  $\mathcal{T}_h^n$ . Similarly,  $\hat{p}_h^{n-1}$  and  $\tilde{p}_h^{n-1}$  have the same mean values on all the elements of  $\mathcal{T}_h^{n-1}$ . Thus, for the first term, we infer that  $(\partial_t \hat{p}_h^n, 1)_K = (\partial_t \tilde{p}_h^n, 1)_K$  for all  $K \in \mathcal{T}_h^n$ . For the second term, using the mean value constraint (5.2b) maintained in the reconstruction, we infer that  $(\partial_t \nabla \cdot \hat{\mathbf{u}}_h^n, 1)_K = (\partial_t \nabla \cdot \tilde{\mathbf{u}}_h^n, 1)_K$  for all  $K \in \mathcal{T}_h^n$ . Thus, (5.8a) holds true. The local conservation property of the scheme implies at each time step  $n$ ,  $0 \leq n \leq N$ ,

$$\left( g^n - (c_0 + c_r) \frac{p_h^n - p_h^{n-1}}{\tau^n} - \frac{c_r}{2\alpha} \frac{\boldsymbol{\sigma}_h^n - \boldsymbol{\sigma}_h^{n-1}}{\tau^n} - \nabla \cdot \mathbf{w}_h^n, 1 \right)_K = 0, \quad \forall K \in \mathcal{T}_h^n. \quad (5.9)$$

Taking the trace-operator on both sides of (5.2a) or (5.3a), using the fact that  $\nabla \cdot \tilde{\mathbf{u}}_h^n = \text{tr}(\boldsymbol{\epsilon}(\tilde{\mathbf{u}}_h^n)) = \text{tr}(\nabla \tilde{\mathbf{u}}_h^n)$ , and replacing the result in (5.8a) gives at each time step  $n$ ,  $0 \leq n \leq N$ ,

$$(\partial_t \varphi(\hat{p}_h^n, \hat{\mathbf{u}}_h^n), 1)_K = \left( (c_0 + c_r) \frac{p_h^n - p_h^{n-1}}{\tau^n} + \frac{c_r}{2\alpha} \frac{\boldsymbol{\sigma}_h^n - \boldsymbol{\sigma}_h^{n-1}}{\tau^n}, 1 \right)_K, \quad \forall K \in \mathcal{T}_h^n. \quad (5.10)$$

Replacing the above result in (5.9), we obtain (5.8b).  $\square$

## 6 The a posteriori error estimates

We present in this section our a posteriori error analysis of the two mixed schemes given in the previous section. The idea is to estimate the error between the (exact) weak solution  $(p, \mathbf{u})$  and the approximate solution  $(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})$  presented in Section 5 in an energy norm, for which we shall define first the residuals.

### 6.1 Definition of the residuals

Let  $X_T := L^2(0, T; H_0^1(\Omega))$ , the dual space of  $X_T$  is given by  $X_T' := L^2(0, T; H^{-1}(\Omega))$  and  $\mathbf{Z}_T := H^1(0, T; \mathbf{H}_0^1(\Omega))$ ; for the sake of simplicity, we set  $\Gamma_t = \emptyset$  and  $\Gamma_f = \emptyset$ . To obtain the estimates, we let  $\|\cdot\|_{H_0^1(\Omega)}$  stand for the energy norm

$$\|v\|_{H_0^1(\Omega)} = \left\{ \int_{\Omega} |\mathbf{K}(\mathbf{x})^{\frac{1}{2}} \nabla v(\mathbf{x})|^2 \, d\mathbf{x} \right\}^{\frac{1}{2}}, \quad (6.1)$$

and we equip  $X_T$  with the energy norm;

$$\|v\|_{X_T} = \left\{ \int_0^T \int_{\Omega} |\mathbf{K}(\mathbf{x})^{\frac{1}{2}} \nabla v(\mathbf{x}, t)|^2 \, d\mathbf{x} \, dt \right\}^{\frac{1}{2}}. \quad (6.2)$$

That of  $\mathbf{X}_T$  is left equipped with

$$\|\mathbf{v}\|_{\mathbf{X}_T} = \left\{ \int_0^T \int_{\Omega} |\nabla \mathbf{v}(\mathbf{x}, t)|^2 \, d\mathbf{x} \, dt \right\}^{\frac{1}{2}}. \quad (6.3)$$

Following [29], we introduce the space

$$\mathcal{E}_T := \{(p, \mathbf{u}) \mid p \in X_T, \quad \mathbf{u} \in \mathbf{Z}_T, \text{ such that } \partial_t \varphi(p, \mathbf{u}) \in X_T'\},$$

then define a weak formulation of (2.1) as the following: find  $(p, \mathbf{u}) \in \mathcal{E}_T$  such that  $p(0) = 0$  and  $\mathbf{u}(0) = 0$  and such that

$$\int_0^T \langle \partial_t \varphi(p, \mathbf{u}), q \rangle \, dt + \int_0^T (\mathbf{K} \nabla p, \nabla q) \, dt = \int_0^T (g, q) \, dt, \quad \forall q \in X_T, \quad (6.4a)$$

$$\int_0^T (\boldsymbol{\theta}(\mathbf{u}), \boldsymbol{\epsilon}(\mathbf{v})) \, dt - \alpha \int_0^T (p, \nabla \cdot \mathbf{v}) \, dt = - \int_0^T (\mathbf{f}, \mathbf{v}) \, dt, \quad \forall \mathbf{v} \in \mathbf{X}_T. \quad (6.4b)$$

where  $\langle \cdot, \cdot \rangle$  denotes the duality pairing between  $H^{-1}(\Omega)$  and  $H_0^1(\Omega)$ . Now, recalling that  $(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau}) \in \mathcal{E}_T$  and define the following continuous linear forms  $\mathcal{R}_P(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})$  and  $\mathcal{R}_U(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})$  on  $X_T$  and  $\mathbf{X}_T$  respectively: for all  $q \in X_T$  and  $\mathbf{v} \in \mathbf{X}_T$ ,

$$\langle \mathcal{R}_P(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau}), q \rangle_{X_T', X_T} := \int_0^T (g, q) \, dt - \int_0^T \langle \partial_t \varphi(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau}), q \rangle \, dt - \int_0^T (\mathbf{K} \nabla \hat{p}_{h\tau}, \nabla q) \, dt, \quad (6.5)$$

$$\langle \mathcal{R}_U(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau}), \mathbf{v} \rangle_{\mathbf{X}_T', \mathbf{X}_T} := \int_0^T (\mathbf{f}, \mathbf{v}) \, dt + \int_0^T (\boldsymbol{\theta}(\hat{\mathbf{u}}_{h\tau}), \boldsymbol{\epsilon}(\mathbf{v})) \, dt - \alpha \int_0^T (\hat{p}_{h\tau}, \nabla \cdot \mathbf{v}) \, dt. \quad (6.6)$$

The norm of the residuals are defined in a standard way and given by

$$\|\mathcal{R}_P(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})\|_{X'_T} := \sup_{\substack{q \in X_T \\ \|q\|_{X_T}=1}} \langle \mathcal{R}_P(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau}), q \rangle_{X'_T, X_T}, \quad (6.7)$$

$$\|\mathcal{R}_U(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})\|_{\mathbf{X}'_T} := \sup_{\substack{\mathbf{v} \in \mathbf{X}_T \\ \|\mathbf{v}\|_{\mathbf{X}_T}=1}} \langle \mathcal{R}_U(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau}), \mathbf{v} \rangle_{\mathbf{X}'_T, \mathbf{X}_T}. \quad (6.8)$$

## 6.2 Bounding the error of the displacement and the pressure by the dual norm of the residuals

In this section we prove that the dual norm of the residuals bound an energy-type norm of the differences between the exact and the approximate pressure and displacement. The results are obtained through duality technique [12, 15]. To this aim, and for all times  $t \in (0, T)$ , let

$$Q_t := L^2(0, t; L^2(\Omega)), \quad X_t := L^2(0, t; H_0^1(\Omega)), \quad X'_t := L^2(0, t; H^{-1}(\Omega)), \quad \mathbf{Z}_t = H^1(0, t; \mathbf{H}_0^1(\Omega)),$$

and

$$\mathcal{E}_t := \{(p, \mathbf{u}) \mid p \in X_t, \quad \mathbf{u} \in \mathbf{Z}_t, \text{ such that } \partial_t \varphi(p, \mathbf{u}) \in X'_t\}.$$

Let us define the following two energy-type error measures:

$$\|\mathbf{u} - \hat{\mathbf{u}}_{h\tau}\|_{\Xi_t}^2 := 2\mu \|\boldsymbol{\epsilon}(\mathbf{u} - \hat{\mathbf{u}}_{h\tau})\|_{Q_t}^2 + \lambda \|\nabla \cdot (\mathbf{u} - \hat{\mathbf{u}}_{h\tau})\|_{Q_t}^2, \quad (6.9)$$

and

$$\|(p - \hat{p}_{h\tau}, \mathbf{u} - \hat{\mathbf{u}}_{h\tau})\|_{\mathfrak{a}_t}^2 := c_0 \|p - \hat{p}_{h\tau}\|_{Q_t}^2 + \frac{1}{2} \|\mathbf{u} - \hat{\mathbf{u}}_{h\tau}\|_{\Xi_t}^2 + \frac{1}{2} \|\varphi(p - \hat{p}_{h\tau}, \mathbf{u} - \hat{\mathbf{u}}_{h\tau})(t)\|_{H^{-1}(\Omega)}^2. \quad (6.10)$$

We first state the following bound:

**Lemma 6.1.** *Let  $(p, \mathbf{u})$  be the weak solution of the problem (2.1) given by (6.4). Let  $(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau}) \in \mathcal{E}_t$ . Then, there holds, for a.e.  $t \in (0, T)$ ,*

$$\begin{aligned} \|(p - \hat{p}_{h\tau}, \mathbf{u} - \hat{\mathbf{u}}_{h\tau})\|_{\mathfrak{a}_t}^2 &\leq \frac{1}{2} \|\mathcal{R}_P(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})\|_{X'_t}^2 + \frac{1}{2\mu} \|\mathcal{R}_U(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})\|_{\mathbf{X}'_t}^2 \\ &\quad + \frac{1}{2} \|\varphi(p - \hat{p}_{h\tau}, \mathbf{u} - \hat{\mathbf{u}}_{h\tau})(t)\|_{X'_t}^2. \end{aligned} \quad (6.11)$$

*Proof.* For a.e.  $t \in (0, T)$ , we denote by  $W(\cdot, t) \in H_0^1(\Omega)$  the solution to

$$(\mathbf{K}\nabla W, \nabla \phi) = (\varphi(p - \hat{p}_{h\tau}, \mathbf{u} - \hat{\mathbf{u}}_{h\tau})(t), \phi), \quad \forall \phi \in H_0^1(\Omega). \quad (6.12)$$

For any  $t \in (0, T)$ , the existence and uniqueness of  $W(\cdot, t)$  is guaranteed by standard arguments. Moreover, since both  $\varphi(p, \mathbf{u})(t)$  and  $\varphi(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})(t)$  are in  $X'_t$  there holds  $W \in X_t$ . From (6.12), we infer first that

$$\begin{aligned} \|W(\cdot, t)\|_{H_0^1(\Omega)} &= \sup_{\substack{\phi \in H_0^1(\Omega) \\ \|\phi\|_{H_0^1(\Omega)}=1}} (\mathbf{K}\nabla W, \nabla \phi), \\ &= \sup_{\substack{\phi \in H_0^1(\Omega) \\ \|\phi\|_{H_0^1(\Omega)}=1}} (\varphi(p - \hat{p}_{h\tau}, \mathbf{u} - \hat{\mathbf{u}}_{h\tau})(t), \phi), \\ &= \|\varphi(p - \hat{p}_{h\tau}, \mathbf{u} - \hat{\mathbf{u}}_{h\tau})(t)\|_{H^{-1}(\Omega)}. \end{aligned} \quad (6.13)$$

Now, we choose  $q = W\mathbf{1}_{(0,t)}$  as a test function in (6.5) and  $\mathbf{v} = (\mathbf{u} - \hat{\mathbf{u}}_{h\tau})\mathbf{1}_{(0,t)}$  as a test in (6.6), where  $\mathbf{1}_{(0,t)}$  is the characteristic function of the interval  $(0, t)$ . Summing the resulting equations, we obtain

$$\langle \mathcal{R}_P(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau}), W \rangle_{X'_t, X_t} + \langle \mathcal{R}_U(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau}), \mathbf{u} - \hat{\mathbf{u}}_{h\tau} \rangle_{\mathbf{X}'_t, \mathbf{X}_t} = A_1 + A_2 + A_3 + A_4, \quad (6.14)$$

with

$$\begin{aligned} A_1 &= \int_0^t \langle \partial_t \varphi(p - \hat{p}_{h\tau}, \mathbf{u} - \hat{\mathbf{u}}_{h\tau}), W \rangle ds, \\ A_2 &= \int_0^t (\mathbf{K} \nabla(p - \hat{p}_{h\tau}), \nabla W) ds, \\ A_3 &= \int_0^t (\boldsymbol{\theta}(\mathbf{u} - \hat{\mathbf{u}}_{h\tau}), \boldsymbol{\epsilon}(\mathbf{u} - \hat{\mathbf{u}}_{h\tau})) ds, \\ A_4 &= -\alpha \int_0^t (p - \hat{p}_{h\tau}, \nabla \cdot (\mathbf{u} - \hat{\mathbf{u}}_{h\tau})) ds. \end{aligned}$$

First thanks to (6.12), we obtain

$$A_1 = \int_0^t (\mathbf{K} \partial_t \nabla W, \nabla W) ds = \frac{1}{2} \left( \|W(\cdot, t)\|_{H_0^1(\Omega)}^2 - \|W(\cdot, 0)\|_{H_0^1(\Omega)}^2 \right).$$

Hence from (6.13), we obtain

$$A_1 = \frac{1}{2} \|\varphi(p - \hat{p}_{h\tau}, \mathbf{u} - \hat{\mathbf{u}}_{h\tau})(t)\|_{H^{-1}(\Omega)}^2, \quad (6.15)$$

as  $W(\cdot, 0) = 0$ . Now, from (6.12), we infer that

$$A_2 = c_0 \|p - \hat{p}_{h\tau}\|_{Q_t}^2 + \alpha \int_0^t (\nabla \cdot (\mathbf{u} - \hat{\mathbf{u}}_{h\tau}), p - \hat{p}_{h\tau}) ds. \quad (6.16)$$

By definition, we have

$$A_3 = \|\mathbf{u} - \hat{\mathbf{u}}_{h\tau}\|_{\Xi_t}^2. \quad (6.17)$$

Finally, we infer using Cauchy-Schwarz, young's inequality and then Korn's inequality [22] that

$$\begin{aligned} \langle \mathcal{R}_P(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau}), W \rangle_{X'_t, X_t} + \langle \mathcal{R}_U(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau}), \mathbf{u} - \hat{\mathbf{u}}_{h\tau} \rangle_{\mathbf{X}'_t, \mathbf{X}_t} &\leq \frac{1}{2} \|\mathcal{R}_P(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})\|_{X'_t}^2 + \frac{1}{2\mu} \|\mathcal{R}_U(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})\|_{\mathbf{X}'_t}^2 \\ &+ \frac{1}{2} \|W(\cdot, t)\|_{X_t}^2 + \mu \|\boldsymbol{\epsilon}(\mathbf{u} - \hat{\mathbf{u}}_{h\tau})\|_{Q_t}^2. \end{aligned} \quad (6.18)$$

Employing (6.15), (6.16), (6.17), and (6.18) in (6.14), we obtain (6.11).  $\square$

The next step, is devoted to bound the following energy-type norm

$$\begin{aligned} \|(p - \hat{p}_{h\tau}, \mathbf{u} - \hat{\mathbf{u}}_{h\tau})\|_{\text{en}}^2 &:= \frac{1}{2} \|(p - \hat{p}_{h\tau}, \mathbf{u} - \hat{\mathbf{u}}_{h\tau})\|_{\mathfrak{D}_T}^2 + \frac{1}{2} \|\varphi(p - \hat{p}_{h\tau}, \mathbf{u} - \hat{\mathbf{u}}_{h\tau})\|_{X'_T}^2 \\ &+ 2c_0 \int_0^T \left( \|p - \hat{p}_{h\tau}\|_{Q_t}^2 + \int_0^t \|p - \hat{p}_{h\tau}\|_{Q_s}^2 e^{t-s} ds \right) dt \\ &+ \int_0^T \left( \|\mathbf{u} - \hat{\mathbf{u}}_{h\tau}\|_{\Xi_t}^2 + \int_0^t \|\mathbf{u} - \hat{\mathbf{u}}_{h\tau}\|_{\Xi_s}^2 e^{t-s} ds \right) dt; \end{aligned} \quad (6.19)$$

presenting an energy error of the displacement and the pressure by the dual norm of the residuals.

**Theorem 6.2.** *Under the Assumptions of Lemma 6.1, there holds,*

$$\begin{aligned} \|(p - \hat{p}_{h\tau}, \mathbf{u} - \hat{\mathbf{u}}_{h\tau})\|_{\text{en}}^2 &\leq \frac{1}{2} \|\mathcal{R}_P(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})\|_{X'_T}^2 + \frac{1}{2\mu} \|\mathcal{R}_U(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})\|_{\mathbf{X}'_T}^2 \\ &+ \int_0^T \left( \|\mathcal{R}_P(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})\|_{X'_t}^2 + \int_0^t \|\mathcal{R}_P(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})\|_{X'_s}^2 e^{t-s} ds \right) dt \\ &+ \frac{1}{\mu} \int_0^T \left( \|\mathcal{R}_U(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})\|_{\mathbf{X}'_t}^2 + \int_0^t \|\mathcal{R}_U(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})\|_{\mathbf{X}'_s}^2 e^{t-s} ds \right) dt; \end{aligned} \quad (6.20)$$

*Proof.* First, using Gronwall's Lemma, we infer from (6.11)

$$\begin{aligned}
 & \frac{1}{2} \|\varphi(p - \hat{p}_{h\tau}, \mathbf{u} - \hat{\mathbf{u}}_{h\tau})(t)\|_{H^{-1}(\Omega)}^2 \\
 & \leq \frac{1}{2} \|\mathcal{R}_P(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})\|_{X'_t}^2 + \frac{1}{2\mu} \|\mathcal{R}_U(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})\|_{\mathbf{X}'_t}^2 - \frac{1}{2} (\|\mathbf{u} - \hat{\mathbf{u}}_{h\tau}\|_{\Xi_t}^2 + 2c_0 \|p - \hat{p}_{h\tau}\|_{Q_t}^2) \\
 & + \frac{1}{2} \int_0^t \|\mathcal{R}_P(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})\|_{X'_s}^2 e^{t-s} ds + \frac{1}{2\mu} \int_0^t \|\mathcal{R}_U(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})\|_{\mathbf{X}'_s}^2 e^{t-s} ds \\
 & - \frac{1}{2} \left( \int_0^t \|\mathbf{u} - \hat{\mathbf{u}}_{h\tau}\|_{\Xi_s}^2 e^{t-s} ds + 2c_0 \int_0^t \|p - \hat{p}_{h\tau}\|_{Q_s}^2 e^{t-s} ds \right). \tag{6.21}
 \end{aligned}$$

We integrate the above inequality over the interval  $(0, T)$ , to obtain

$$\begin{aligned}
 \frac{1}{2} \|\varphi(p - \hat{p}_{h\tau}, \mathbf{u} - \hat{\mathbf{u}}_{h\tau})\|_{X'_T}^2 & \leq \frac{1}{2} \int_0^T \left( \|\mathcal{R}_P(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})\|_{X'_t}^2 + \int_0^t \|\mathcal{R}_P(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})\|_{X'_s}^2 e^{t-s} ds \right) dt \\
 & + \frac{1}{2\mu} \int_0^T \left( \|\mathcal{R}_U(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})\|_{\mathbf{X}'_t}^2 + \int_0^t \|\mathcal{R}_U(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})\|_{\mathbf{X}'_s}^2 e^{t-s} ds \right) dt \\
 & - \frac{1}{2} \int_0^T (\|\mathbf{u} - \hat{\mathbf{u}}_{h\tau}\|_{\Xi_t}^2 + 2c_0 \|p - \hat{p}_{h\tau}\|_{Q_t}^2) dt \\
 & - \frac{1}{2} \int_0^T \left( \int_0^t \|\mathbf{u} - \hat{\mathbf{u}}_{h\tau}\|_{\Xi_s}^2 e^{t-s} ds + 2c_0 \int_0^t \|p - \hat{p}_{h\tau}\|_{Q_s}^2 e^{t-s} ds \right) dt. \tag{6.22}
 \end{aligned}$$

Now, adding  $\frac{1}{2} \|\varphi(p - \hat{p}_{h\tau}, \mathbf{u} - \hat{\mathbf{u}}_{h\tau})\|_{X'_T}^2$  in both sides of (6.11) for  $t = T$ , then we use (6.22) to bound the last term, we end up with (6.20)  $\square$

We have also this simple estimate:

**Lemma 6.3** (Simple bounds for the displacement and pressure errors). *Under the Assumptions of Lemma 6.1, there holds,*

$$\|(p - \hat{p}_{h\tau}, \mathbf{u} - \hat{\mathbf{u}}_{h\tau})\|_{\mathfrak{H}_T}^2 \leq \frac{e^T}{2} \left( \|\mathcal{R}_P(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})\|_{X'_T}^2 + \frac{1}{\mu} \|\mathcal{R}_U(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})\|_{\mathbf{X}'_T}^2 \right). \tag{6.23}$$

*Proof.* Using Lemma 6.1, for all  $t \in (0, T)$ , one has

$$\|\varphi(p - \hat{p}_{h\tau}, \mathbf{u} - \hat{\mathbf{u}}_{h\tau})(t)\|_{H^{-1}(\Omega)}^2 \leq \|\mathcal{R}_P(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})\|_{X'_t}^2 + \frac{1}{\mu} \|\mathcal{R}_U(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})\|_{\mathbf{X}'_t}^2 + \|\varphi(p - \hat{p}_{h\tau}, \mathbf{u} - \hat{\mathbf{u}}_{h\tau})(t)\|_{X'_t}^2.$$

The Gronwall lemma yields for all  $t \in (0, T)$ ,

$$\|\varphi(p - \hat{p}_{h\tau}, \mathbf{u} - \hat{\mathbf{u}}_{h\tau})(t)\|_{H^{-1}(\Omega)}^2 \leq e^t \left( \|\mathcal{R}_P(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})\|_{X'_t}^2 + \frac{1}{\mu} \|\mathcal{R}_U(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})\|_{\mathbf{X}'_t}^2 \right). \tag{6.24}$$

Using the above inequality in (6.11) when setting  $t = T$  yields (6.23).  $\square$

**Remark 6.4** (Uniqueness of the weak solution). *If we suppose that  $(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})$  is a weak solution of (2.1). This implies first that  $\mathcal{R}_P(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau}) = \mathcal{R}_U(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau}) = 0$ . Thanks to the estimate (6.20) (or (6.23)), we get  $\|(p - \hat{p}_{h\tau}, \mathbf{u} - \hat{\mathbf{u}}_{h\tau})\|_{\text{en}} = \|(p - \tilde{p}_{h\tau}, \mathbf{u} - \tilde{\mathbf{u}}_{h\tau})\|_{\mathfrak{H}_T} = 0$ , and hence the uniqueness of the weak solution.*

### 6.3 Bounding the dual norm of the residuals by the estimators

We now bound the dual norm of the residuals by fully computable a posteriori error estimators. First, to work with the nonconforming approximation  $(\tilde{\mathbf{u}}_{h\tau}, \tilde{p}_{h\tau})$  of Section 5.1, we introduce the (broken) energy-norm on  $X_T$  where  $\nabla$  is now the broken gradient operator:

$$\|v\|_{X_T}^2 = \sum_{n=1}^N \int_{I_n} \sum_{K \in \mathcal{T}_h^n} \|\mathbf{K}^{\frac{1}{2}} \nabla v\|_K^2 dt, \quad \forall v \in X_T. \tag{6.25}$$

The broken  $\mathbf{X}_T$ -norm is

$$\|\mathbf{v}\|_{\mathbf{X}_T}^2 = \sum_{n=1}^N \int_{I_n} \sum_{K \in \mathcal{T}_h^n} \|\nabla \mathbf{v}\|_K^2 dt, \quad \forall \mathbf{v} \in \mathbf{X}_T. \quad (6.26)$$

Now, let a time step  $1 \leq n \leq N$ , and a mesh element  $K \in \mathcal{T}_h^n$  be given. We define the residual estimators

$$\eta_{\mathbb{R},\mathbb{P},K}^n := C_{\mathbb{P},K} c_{\mathbf{K},K}^{-\frac{1}{2}} h_K \|g^n - \partial_t \varphi(\hat{p}_h^n, \hat{\mathbf{u}}_h^n) - \nabla \cdot \mathbf{w}_h^n\|_K, \quad K \in \mathcal{T}_h^n, \quad (6.27)$$

$$\eta_{\mathbb{R},\mathbb{U},K}^n := C_{\mathbb{P},K} h_K \|\nabla \cdot \boldsymbol{\sigma}_h^n + \mathbf{f}^n\|_K, \quad K \in \mathcal{T}_h^n, \quad (6.28)$$

and the flux estimators

$$\eta_{\mathbb{F},\mathbb{P},K}^n(t) := \| \mathbf{w}_h^n + \mathbf{K} \nabla \hat{p}_{h\tau}(t) \|_{*,K}, \quad K \in \mathcal{T}_h^n, t \in I^n, \quad (6.29)$$

$$\eta_{\mathbb{F},\mathbb{U},K}^n(t) := \| \boldsymbol{\sigma}_h^n - \boldsymbol{\sigma}(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})(t) \|_K, \quad K \in \mathcal{T}_h^n, t \in I^n. \quad (6.30)$$

Herein  $h_K$  denotes the diameter of the mesh element  $K$ , and  $C_{\mathbb{P},K}$  is the constant from Poincaré inequality:

$$\|q - \pi_0 q\| \leq C_{\mathbb{P},K} h_K \|\nabla q\|_K, \quad \forall q \in H^1(K), \quad (6.31)$$

where  $\pi_0 q$  is the mean value of  $q$  on the element  $K$  and  $C_{\mathbb{P},K} = 1/\pi$  whenever the element  $K$  is convex.

We have the following result:

**Theorem 6.5.** *Let  $(p, \mathbf{u})$  be the weak solution of the problem (2.1) given by (6.4). Let  $(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})$  be the reconstructed functions as obtained in Section 5.2. Then there holds*

$$\|\mathcal{R}_{\mathbb{P}}(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})\|_{X_T'} \leq \left\{ \sum_{n=1}^N \int_{I_n} \sum_{K \in \mathcal{T}_h^n} (\eta_{\mathbb{R},\mathbb{P},K}^n + \eta_{\mathbb{F},\mathbb{P},K}^n(t))^2 dt \right\}^{\frac{1}{2}}, \quad (6.32)$$

$$\|\mathcal{R}_{\mathbb{U}}(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})\|_{\mathbf{X}_T'} \leq \left\{ \sum_{n=1}^N \int_{I_n} \sum_{K \in \mathcal{T}_h^n} (\eta_{\mathbb{R},\mathbb{U},K}^n + \eta_{\mathbb{F},\mathbb{U},K}^n(t))^2 dt \right\}^{\frac{1}{2}}. \quad (6.33)$$

*Proof.* Let  $1 \leq n \leq N$  be given and let  $q \in X_T$  with  $\|q\|_{X_T} = 1$ . Adding and subtracting  $(\mathbf{w}_h^n, \nabla q)$  to (6.5) and applying the Green theorem, we obtain

$$\langle \mathcal{R}_{\mathbb{P}}(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau}), q \rangle_{X_T', X_T} := \sum_{n=1}^N \int_{I_n} \{ (g^n - \partial_t \varphi(\hat{p}_h^n, \hat{\mathbf{u}}_h^n) - \nabla \cdot \mathbf{w}_h^n, q) + (\mathbf{K} \nabla(\tilde{p}_h^n - \hat{p}_{h\tau}), \nabla q) \} dt. \quad (6.34)$$

We apply Lemma 5.1, the Poincaré inequality (6.31) and Cauchy-Schwarz inequality, it is inferred that

$$\begin{aligned} & \langle \mathcal{R}_{\mathbb{P}}(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau}), q \rangle_{X_T', X_T} \\ & := \sum_{n=1}^N \int_{I_n} \sum_{K \in \mathcal{T}_h^n} \{ (g^n - \partial_t \varphi(\hat{p}_h^n, \hat{\mathbf{u}}_h^n) - \nabla \cdot \mathbf{w}_h^n, q)_K + (\mathbf{K} \nabla(\tilde{p}_h^n - \hat{p}_{h\tau}), \nabla q)_K \} dt, \\ & := \sum_{n=1}^N \int_{I_n} \sum_{K \in \mathcal{T}_h^n} (g^n - \partial_t \varphi(\hat{p}_h^n, \hat{\mathbf{u}}_h^n) - \nabla \cdot \mathbf{w}_h^n, q - \pi_0 q)_K dt \\ & \quad + \sum_{n=1}^N \int_{I_n} \sum_{K \in \mathcal{T}_h^n} (\mathbf{K} \nabla(\tilde{p}_h^n - \hat{p}_{h\tau}), \nabla q)_K dt, \\ & \leq \sum_{n=1}^N \int_{I_n} \sum_{K \in \mathcal{T}_h^n} \{ (\eta_{\mathbb{R},\mathbb{P},K}^n + \eta_{\mathbb{F},\mathbb{P},K}^n(t)) \|q(t)\|_K \} dt. \end{aligned}$$



Applying the Cauchy-Schwarz inequality yields

$$\|\mathcal{R}_P(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})\|_{X'_T} \leq \left\{ \sum_{n=1}^N \int_{I_n} \sum_{K \in \mathcal{T}_h^n} (\eta_{R,P,K}^n + \eta_{F,P,K}^n(t))^2 dt \right\}^{\frac{1}{2}}. \quad (6.35)$$

Proceeding in a similar way for (6.33) with  $\mathbf{v} \in \mathbf{X}_T$  such that  $\|\mathbf{v}\|_{\mathbf{X}_T} = 1$ ; first recalling that

$$\begin{aligned} \langle \mathcal{R}_U(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau}), \mathbf{v} \rangle_{\mathbf{X}'_T, \mathbf{X}_T} &:= \int_0^T (\boldsymbol{\sigma}(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau}), \boldsymbol{\epsilon}(\mathbf{v})) dt + \int_0^T (\mathbf{f}, \mathbf{v}) dt, \\ &:= \sum_{n=1}^N \int_{I_n} \sum_{K \in \mathcal{T}_h^n} \{(\boldsymbol{\sigma}(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau}), \nabla \mathbf{v})_K + (\mathbf{f}^n, \mathbf{v})_K\} dt, \end{aligned} \quad (6.36)$$

where we used the fact that  $\boldsymbol{\sigma}(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})$  is symmetric so we can replace  $\boldsymbol{\epsilon}(\mathbf{v})$  by  $\nabla \mathbf{v}$  in the first term in the right-hand side of (6.36). Now, adding and subtracting  $(\boldsymbol{\sigma}_h^n, \nabla \mathbf{v})$  to (6.36), applying the Poincaré inequality (6.31) together with the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \langle \mathcal{R}_U(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau}), \mathbf{v} \rangle_{\mathbf{X}'_T, \mathbf{X}_T} &:= \sum_{n=1}^N \int_{I_n} \sum_{K \in \mathcal{T}_h^n} \{(\boldsymbol{\sigma}(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau}) - \boldsymbol{\sigma}_h^n, \nabla \mathbf{v})_K + (\nabla \cdot \hat{\boldsymbol{\sigma}}_h^n + \mathbf{f}^n, \mathbf{v} - \pi_0 \mathbf{v})_K\} dt, \\ &\leq \sum_{n=1}^N \int_{I_n} \sum_{K \in \mathcal{T}_h^n} \{(\eta_{R,U,K}^n + \eta_{F,U,K}^n(t)) \|\nabla \mathbf{v}(t)\|_K\} dt. \end{aligned} \quad (6.37)$$

We infer (6.33) by applying again the Cauchy-Schwarz inequality.  $\square$

## 6.4 The a posteriori error estimate

Now, we combine the upper bound in (6.20) with the results of the previous section to obtain a posteriori estimate for the displacement and pressure errors. To capture the nonconformity from the numerical scheme, we additionally define the nonconformity estimators:

$$\eta_{NC1,U,K}^n(t) := \left\{ 2\mu \|\boldsymbol{\epsilon}(\tilde{\mathbf{u}}_{h\tau} - \hat{\mathbf{u}}_{h\tau})(t)\|_K^2 + \lambda \|\nabla \cdot (\tilde{\mathbf{u}}_{h\tau} - \hat{\mathbf{u}}_{h\tau})(t)\|_K^2 \right\}^{\frac{1}{2}}, \quad K \in \mathcal{T}_h^n, \quad t \in I_n, \quad (6.38)$$

$$\eta_{NC1,P,K}^n(t) := \|(\tilde{p}_{h\tau} - \hat{p}_{h\tau})(t)\|_K, \quad K \in \mathcal{T}_h^n, \quad t \in I_n, \quad (6.39)$$

$$\eta_{NCTF,U,K}^N := \frac{h_K c_{\mathbf{K},K}^{-\frac{1}{2}}}{\pi} \|\nabla \cdot (\tilde{\mathbf{u}}_{h\tau}^N - \hat{\mathbf{u}}_{h\tau}^N)\|_K, \quad K \in \mathcal{T}_h^N, \quad (6.40)$$

$$\eta_{NCTF,P,K}^N := \frac{h_K c_{\mathbf{K},K}^{-\frac{1}{2}}}{\pi} \|(\tilde{p}_{h\tau}^N - \hat{p}_{h\tau}^N)\|_K, \quad K \in \mathcal{T}_h^N, \quad (6.41)$$

$$\eta_{NC2,U,K}^n := \frac{h_K c_{\mathbf{K},K}^{-\frac{1}{2}}}{\pi} (\|\nabla \cdot (\tilde{\mathbf{u}}_h^n - \hat{\mathbf{u}}_h^n)\|_K + \|\nabla \cdot (\tilde{\mathbf{u}}_h^{n-1} - \hat{\mathbf{u}}_h^{n-1})\|_K), \quad K \in \mathcal{T}_h^n, \quad (6.42)$$

$$\eta_{NC2,P,K}^n := \frac{h_K c_{\mathbf{K},K}^{-\frac{1}{2}}}{\pi} (\|\tilde{p}_h^n - \hat{p}_h^n\|_K + \|\tilde{p}_h^{n-1} - \hat{p}_h^{n-1}\|_K), \quad K \in \mathcal{T}_h^n. \quad (6.43)$$

**Theorem 6.6.** *Let  $(p, \mathbf{u})$  be the weak solution of the problem (2.1) given by (6.4). Let  $(\tilde{p}_{h\tau}, \tilde{\mathbf{u}}_{h\tau})$  be the postprocessed pressure and displacement as obtained in Section (5.1). Let  $(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})$  be the reconstructed pressure and displacement as obtained in Section (5.2). Then there holds*

$$\|(p - \tilde{p}_{h\tau}, \mathbf{u} - \tilde{\mathbf{u}}_{h\tau})\|_{\text{en}} \leq \eta_P + \eta_U + \eta_{NC,P} + \eta_{NC,U}, \quad (6.44)$$

where

$$\eta_J := \sqrt{\frac{L_J}{2}} \left\{ \sum_{n=1}^N (\eta_J^n)^2 + 2 \sum_{n=1}^N \tau^n \sum_{l=1}^n (\eta_J^l)^2 + 2 \sum_{n=1}^N \sum_{l=1}^n J_{nl} \left( \sum_{q=1}^l (\eta_J^q)^2 \right) \right\}^{\frac{1}{2}}, \quad J = P, U, \quad (6.45a)$$

$$\begin{aligned} \eta_{\text{NC},J} := & \sqrt{\frac{C_J}{2}} \left\{ \sum_{n=1}^N (\eta_{\text{NC1},J}^n)^2 + 2\delta_J (\eta_{\text{NC2},J}^n)^2 + 4 \sum_{n=1}^N \tau^n \sum_{l=1}^n (\eta_{\text{NC1},J}^l)^2 \right. \\ & \left. + 4 \sum_{n=1}^N \sum_{l=1}^n J_{nl} \left( \sum_{q=1}^l (\eta_{\text{NC1},J}^q)^2 \right) + \delta_J (\eta_{\text{NCTF},J}^N)^2 \right\}^{\frac{1}{2}}, \quad J = P, U, \end{aligned} \quad (6.45b)$$

with

$$\eta_J^n := \left\{ \int_{I^n} \sum_{K \in \mathcal{T}_h^n} (\eta_{\text{R},J,K}^n + \eta_{\text{F},J,K}^n(t))^2 dt \right\}^{\frac{1}{2}}, \quad 1 \leq n \leq N, \quad J = P, U, \quad (6.46)$$

$$\eta_{\text{NC1},J}^n := \left\{ \int_{I^n} \sum_{K \in \mathcal{T}_h^n} (\eta_{\text{NC1},J,K}^n(t))^2 dt \right\}^{\frac{1}{2}}, \quad 1 \leq n \leq N, \quad J = P, U, \quad (6.47)$$

$$\eta_{\text{NC2},J}^n := \left\{ \frac{2}{3} \int_{I^n} \sum_{K \in \mathcal{T}_h^n} (\eta_{\text{NC2},J,K}^n)^2 dt \right\}^{\frac{1}{2}}, \quad 1 \leq n \leq N, \quad J = P, U, \quad (6.48)$$

$$\eta_{\text{NCTF},J}^N := \left\{ \sum_{K \in \mathcal{T}_h^N} (\eta_{\text{NCTF},J,K}^N)^2 \right\}^{\frac{1}{2}}, \quad J = P, U, \quad (6.49)$$

and where we have set  $L_P = 1$  and  $L_U = \frac{1}{\mu}$ ,  $C_P = c_0$  and  $C_U = \frac{1}{2}$ ,  $\delta_P = c_0$  and  $\delta_U = 2\alpha^2$ , and for  $1 \leq n, l \leq N$ ,

$$J_{nl} := \int_{I^n} \int_{I^l} e^{t-s} ds dt.$$

*Proof.* From the triangle inequality, we have

$$\|(p - \tilde{p}_{h\tau}, \mathbf{u} - \tilde{\mathbf{u}}_{h\tau})\|_{\text{en}} \leq \|(p - \hat{p}_{h\tau}, \mathbf{u} - \hat{\mathbf{u}}_{h\tau})\|_{\text{en}} + \|(\hat{p}_{h\tau} - \tilde{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau} - \tilde{\mathbf{u}}_{h\tau})\|_{\text{en}}. \quad (6.50)$$

Then, we decompose  $\|(\hat{p}_{h\tau} - \tilde{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau} - \tilde{\mathbf{u}}_{h\tau})\|_{\text{en}}$  into two parts separating the pressure terms from the displacement terms, using the triangle inequality:

$$\|(\hat{p}_{h\tau} - \tilde{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau} - \tilde{\mathbf{u}}_{h\tau})\|_{\text{en}} \leq \|\hat{p}_{h\tau} - \tilde{p}_{h\tau}\|_P + \|\hat{\mathbf{u}}_{h\tau} - \tilde{\mathbf{u}}_{h\tau}\|_U, \quad (6.51)$$

with

$$\begin{aligned} \|\hat{p}_{h\tau} - \tilde{p}_{h\tau}\|_P^2 := & \frac{c_0}{2} \|\tilde{p}_{h\tau} - \hat{p}_{h\tau}\|_{Q_T}^2 + \frac{c_0^2}{2} \|(\tilde{p}_{h\tau} - \hat{p}_{h\tau})(T)\|_{H^{-1}(\Omega)}^2 + c_0^2 \|\tilde{p}_{h\tau} - \hat{p}_{h\tau}\|_{X'_T}^2 \\ & + 2c_0 \int_0^T \left( \|\tilde{p}_{h\tau} - \hat{p}_{h\tau}\|_{Q_t}^2 + \int_0^t \|\tilde{p}_{h\tau} - \hat{p}_{h\tau}\|_{Q_s}^2 e^{t-s} ds \right) dt, \end{aligned} \quad (6.52)$$

$$\begin{aligned} \|\hat{\mathbf{u}}_{h\tau} - \tilde{\mathbf{u}}_{h\tau}\|_U^2 := & \frac{1}{4} \|\tilde{\mathbf{u}}_{h\tau} - \hat{\mathbf{u}}_{h\tau}\|_{\Xi_T}^2 + \frac{\alpha^2}{2} \|\nabla \cdot (\tilde{\mathbf{u}}_{h\tau} - \hat{\mathbf{u}}_{h\tau})(T)\|_{H^{-1}(\Omega)}^2 + \alpha^2 \|\nabla \cdot (\tilde{\mathbf{u}}_{h\tau} - \hat{\mathbf{u}}_{h\tau})\|_{\mathbf{X}'_T}^2 \\ & + \int_0^T \left( \|\tilde{\mathbf{u}}_{h\tau} - \hat{\mathbf{u}}_{h\tau}\|_{\Xi_t}^2 + \int_0^t \|\tilde{\mathbf{u}}_{h\tau} - \hat{\mathbf{u}}_{h\tau}\|_{\Xi_s}^2 e^{t-s} ds \right) dt. \end{aligned} \quad (6.53)$$

We then apply (6.20), to get

$$\begin{aligned}
& \| (p - \tilde{p}_{h\tau}, \mathbf{u} - \tilde{\mathbf{u}}_{h\tau}) \|_{\text{en}} \\
& \leq \left\{ \frac{1}{2} \| \mathcal{R}_{\text{P}}(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau}) \|_{X'_T}^2 + \int_0^T \left( \| \mathcal{R}_{\text{P}}(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau}) \|_{X'_t}^2 + \int_0^t \| \mathcal{R}_{\text{P}}(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau}) \|_{X'_s}^2 e^{t-s} ds \right) dt \right\}^{\frac{1}{2}} \\
& + \left\{ \frac{1}{2\mu} \| \mathcal{R}_{\text{U}}(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau}) \|_{\mathbf{X}'_T}^2 + \frac{1}{\mu} \int_0^T \left( \| \mathcal{R}_{\text{U}}(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau}) \|_{\mathbf{X}'_t}^2 + \int_0^t \| \mathcal{R}_{\text{U}}(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau}) \|_{\mathbf{X}'_s}^2 e^{t-s} ds \right) dt \right\}^{\frac{1}{2}} \\
& + \| \hat{p}_{h\tau} - \tilde{p}_{h\tau} \|_{\text{P}} + \| \hat{\mathbf{u}}_{h\tau} - \tilde{\mathbf{u}}_{h\tau} \|_{\text{U}}. \tag{6.54}
\end{aligned}$$

We then proceed as in [15, Thm. 5.3] or [1, Thm. 6.1] using the results of Section 6.3 to bound the terms involving  $\| \mathcal{R}_{\text{P}}(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau}) \|_{X'_t}$  and  $\| \mathcal{R}_{\text{U}}(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau}) \|_{\mathbf{X}'_t}$ ,  $t \in (0, T)$ , with the estimators. Thus, there remains to give a computable upper bound to  $\| \hat{p}_{h\tau} - \tilde{p}_{h\tau} \|_{\text{P}}$  as well as to  $\| \hat{\mathbf{u}}_{h\tau} - \tilde{\mathbf{u}}_{h\tau} \|_{\text{U}}$  and then combine these results. For the first term of (6.52), we have

$$\begin{aligned}
\| \tilde{p}_{h\tau} - \hat{p}_{h\tau} \|_{Q_T}^2 &= \sum_{n=1}^N \int_{I_n} \sum_{K \in \mathcal{T}_h^n} \| (\tilde{p}_{h\tau} - \hat{p}_{h\tau})(t) \|_K^2 dt, \\
&= \sum_{n=1}^N \int_{I_n} \sum_{K \in \mathcal{T}_h^n} (\eta_{\text{NC1,P,K}}^n)^2 dt. \tag{6.55}
\end{aligned}$$

Then, since the quantity  $(\tilde{p}_{h\tau} - \hat{p}_{h\tau})$  is piecewise affine and continuous in time, and since for all  $1 \leq n \leq N$  and on each element  $K \in \mathcal{T}_h^n$ ,  $(\tilde{p}_h^n - \hat{p}_h^n)$  has zero mean value by (5.1b):

$$\begin{aligned}
\| (\tilde{p}_{h\tau} - \hat{p}_{h\tau})(T) \|_{H^{-1}(\Omega)}^2 &= \sup_{\substack{q \in H_0^1(\Omega) \\ \| \mathbf{k}^{\frac{1}{2}} \nabla q \| = 1}} ((\tilde{p}_{h\tau} - \hat{p}_{h\tau})(T), q)^2, \\
&= \sup_{\substack{q \in H_0^1(\Omega) \\ \| \mathbf{k}^{\frac{1}{2}} \nabla q \| = 1}} (\tilde{p}_h^N - \hat{p}_h^N, q - \pi_0 q)^2, \\
&\leq \sum_{K \in \mathcal{T}_h^N} \left( \frac{h_K c_{\mathbf{K},K}^{-\frac{1}{2}}}{\pi} \| \tilde{p}_h^N - \hat{p}_h^N \|_K \| q \|_K \right)^2, \\
&\leq \sum_{K \in \mathcal{T}_h^N} \left( \frac{h_K c_{\mathbf{K},K}^{-\frac{1}{2}}}{\pi} \| \tilde{p}_h^N - \hat{p}_h^N \|_K \right)^2, \\
&= \sum_{K \in \mathcal{T}_h^N} (\eta_{\text{NCTF,P,K}}^N)^2. \tag{6.56}
\end{aligned}$$

For the second term, we have first

$$\begin{aligned}
\| \tilde{p}_{h\tau} - \hat{p}_{h\tau} \|_{X'_T}^2 &= \int_0^T \| \tilde{p}_{h\tau} - \hat{p}_{h\tau} \|_{H^{-1}(\Omega)}^2 dt, \\
&= \sum_{n=1}^N \int_{I_n} \| (\tilde{p}_{h\tau} - \hat{p}_{h\tau})(t) \|_{H^{-1}(\Omega)}^2 dt. \tag{6.57}
\end{aligned}$$

Following [17], let  $v_{h\tau} = \tilde{p}_{h\tau} - \hat{p}_{h\tau}$ , and as  $v_{h\tau}$  is piecewise affine and continuous in time, we have

$$\begin{aligned}
\int_{I_n} \| v_{h\tau}(t) \|_{H^{-1}(\Omega)}^2 dt &= \tau^n \int_0^1 \| v_h^{n-1} + \tau(v_h^n - v_h^{n-1}) \|_{H^{-1}(\Omega)}^2 d\tau, \\
&\leq \frac{2}{3} \tau^n \left( \| v_h^{n-1} \|_{H^{-1}(\Omega)}^2 + \| v_h^n \|_{H^{-1}(\Omega)}^2 \right). \tag{6.58}
\end{aligned}$$

Now, for all  $1 \leq n \leq N$  and on each element  $K \in \mathcal{T}_h^n$ ,  $v_h^n$  has zero mean value by (5.1b), so one can proceed as for (6.56), to infer

$$\|\tilde{p}_{h\tau} - \hat{p}_{h\tau}\|_{X'_T}^2 \leq \sum_{n=1}^N \frac{2}{3} \tau^n \sum_{K \in \mathcal{T}_h^n} (\eta_{\text{NC2,P},K}^n)^2. \quad (6.59)$$

We use the same idea to bound  $\|\tilde{\mathbf{u}}_{h\tau} - \hat{\mathbf{u}}_{h\tau}\|_{\mathbf{U}}$ ; the first term of (6.53) is given by

$$\begin{aligned} \|\tilde{\mathbf{u}}_{h\tau} - \hat{\mathbf{u}}_{h\tau}\|_{\Xi_T}^2 &= \sum_{n=1}^N \int_{I_n} \sum_{K \in \mathcal{T}_h^n} \{2\mu \|\epsilon(\tilde{\mathbf{u}}_{h\tau} - \hat{\mathbf{u}}_{h\tau})(t)\|_K^2 + \lambda \|\nabla \cdot (\tilde{\mathbf{u}}_{h\tau} - \hat{\mathbf{u}}_{h\tau})(t)\|_K^2\} dt, \\ &= \sum_{n=1}^N \int_{I_n} \sum_{K \in \mathcal{T}_h^n} (\eta_{\text{NC1,U},K}^n(t))^2 dt. \end{aligned} \quad (6.60)$$

For the second and third terms of (6.53), we have the quantity  $(\tilde{\mathbf{u}}_{h\tau} - \hat{\mathbf{u}}_{h\tau})$  is piecewise affine and continuous in time, and for all  $1 \leq n \leq N$  and on each element  $K \in \mathcal{T}_h^n$ , its discrete-in-time quantity  $(\tilde{\mathbf{u}}_h^n - \hat{\mathbf{u}}_h^n)$  (and also for  $\nabla \cdot (\tilde{\mathbf{u}}_h^n - \hat{\mathbf{u}}_h^n)$ ) has zero mean value by (5.2b). Then, we proceed as for (6.56) and (6.59) to infer

$$\begin{aligned} \|\nabla \cdot (\tilde{\mathbf{u}}_{h\tau} - \hat{\mathbf{u}}_{h\tau})(T)\|_{H^{-1}(\Omega)}^2 &\leq \sum_{K \in \mathcal{T}_h^N} (\eta_{\text{NCTF,U},K}^N)^2, \\ \|\nabla \cdot (\tilde{\mathbf{u}}_{h\tau} - \hat{\mathbf{u}}_{h\tau})\|_{X'_T}^2 &\leq \sum_{n=1}^N \frac{2}{3} \tau^n \sum_{K \in \mathcal{T}_h^n} (\eta_{\text{NC2,U},K}^n)^2. \end{aligned}$$

Gathering the above results in (6.54) to obtain the estimate (6.44).  $\square$

The following simplified version of the a posteriori estimate can be obtained from Lemma 6.3.

**Corollary 6.7** (A simple estimate for the pressure and displacement errors). *Under the Assumptions of Theorem 6.6, there holds,*

$$\|(p - \tilde{p}_{h\tau}, \mathbf{u} - \tilde{\mathbf{u}}_{h\tau})\|_{\mathfrak{H}_T} \leq \eta_P + \eta_U + \eta_{\text{NC,P}} + \eta_{\text{NC,U}}. \quad (6.61)$$

Herein,

$$\eta_J := \sqrt{\frac{L_J}{2}} \left\{ \sum_{n=1}^N (\eta_J^n)^2 \right\}^{\frac{1}{2}}, \quad J = P, U, \quad (6.62a)$$

and  $L_P = e^T$  and  $L_U = \frac{e^T}{\mu}$ .

## 7 An a posteriori error estimate distinguishing the space and time errors

In this section, we elaborate the error estimate (6.44), so as to distinguish the error components resulting from the spatial and the temporal discretization. In particular, separating the space and time errors of the displacement from the estimated space and time errors of the pressure is useful if instead of a monolithic scheme, an iterative coupled solution method is involved for (4.4) and (4.5) (cf. [11, 10, 35]). To this purpose, we define for all  $1 \leq n \leq N$ ,

$$\begin{aligned} (\eta_{\text{sp,P}}^n)^2 &= \int_{I_n} \sum_{K \in \mathcal{T}_h^n} (\eta_{\text{R,P},K}^n + \eta_{\text{DF,P},K}^n)^2 dt, & (\eta_{\text{sp,U}}^n)^2 &= \int_{I_n} \sum_{K \in \mathcal{T}_h^n} (\eta_{\text{R,U},K}^n + \eta_{\text{DF,U},K}^n)^2 dt, \\ (\eta_{\text{tm,P}}^n)^2 &= \int_{I_n} \sum_{K \in \mathcal{T}_h^n} \frac{1}{3} \|\hat{p}_h^n - \hat{p}_h^{n-1}\|_K^2 dt, & (\eta_{\text{tm,U}}^n)^2 &= \int_{I_n} \sum_{K \in \mathcal{T}_h^n} \frac{1}{3} \|\sigma(\hat{p}_h^n, \hat{\mathbf{u}}_h^n) - \sigma(\hat{p}_h^{n-1}, \hat{\mathbf{u}}_h^{n-1})\|_K^2 dt, \end{aligned} \quad (7.1a)$$

where

$$\eta_{\text{DF},\text{P},K}^n := \|\|\|\mathbf{w}_h^n + \mathbf{K}\nabla\hat{p}_h^n\|\|\|_{*,K}, \quad K \in \mathcal{T}_h^n, \quad (7.1b)$$

$$\eta_{\text{DF},\text{U},K}^n := \|\sigma_h^n - \sigma(\hat{p}_h^n, \hat{\mathbf{u}}_h^n)\|_K, \quad K \in \mathcal{T}_h^n. \quad (7.1c)$$

We have this result:

**Theorem 7.1** (A posteriori error estimate distinguishing error components). *Let  $(p, \mathbf{u})$  be the weak solution of the problem (2.1) given by (6.4). Let  $(\tilde{p}_{h\tau}, \tilde{\mathbf{u}}_{h\tau})$  be the postprocessed pressure and displacement as obtained in Section (5.1). Let  $(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})$  be the reconstructed pressure and displacement as obtained in Section (5.2). Then there holds*

$$\|(p - \tilde{p}_{h\tau}, \mathbf{u} - \tilde{\mathbf{u}}_{h\tau})\|_{\text{en}} \leq \eta_{\text{tm},\text{P}} + \eta_{\text{sp},\text{P}} + \eta_{\text{tm},\text{U}} + \eta_{\text{sp},\text{U}}, \quad (7.2)$$

where

$$\begin{aligned} \eta_{\text{tm},\text{J}} &:= \sqrt{\frac{L_{\text{J}}}{2}} \left( \left\{ \sum_{n=1}^N (\eta_{\text{tm},\text{J}}^n)^2 \right\}^{\frac{1}{2}} \right. \\ &\quad \left. + \sqrt{2} \left\{ \sum_{n=1}^N \tau^n \sum_{l=1}^n (\eta_{\text{tm},\text{J}}^l)^2 \right\}^{\frac{1}{2}} + \sqrt{2} \left\{ \sum_{n=1}^N \sum_{l=1}^n J_{nl} \left( \sum_{q=1}^l (\eta_{\text{tm},\text{J}}^q)^2 \right) \right\}^{\frac{1}{2}} \right), \quad \text{J} = \text{P}, \text{U}, \end{aligned} \quad (7.3a)$$

$$\begin{aligned} \eta_{\text{sp},\text{J}} &:= \sqrt{\frac{L_{\text{J}}}{2}} \left( \left\{ \sum_{n=1}^N (\eta_{\text{sp},\text{J}}^n)^2 \right\}^{\frac{1}{2}} \right. \\ &\quad \left. + \sqrt{2} \left\{ \sum_{n=1}^N \tau^n \sum_{l=1}^n (\eta_{\text{sp},\text{J}}^l)^2 \right\}^{\frac{1}{2}} + \sqrt{2} \left\{ \sum_{n=1}^N \sum_{l=1}^n J_{nl} \left( \sum_{q=1}^l (\eta_{\text{sp},\text{J}}^q)^2 \right) \right\}^{\frac{1}{2}} \right) + \eta_{\text{NC},\text{J}}, \quad \text{J} = \text{P}, \text{U}. \end{aligned}$$

*Proof.* The idea is to split the diffusive flux estimator  $\eta_{\text{F},\text{P},K}^n(t)$  as well as the stress estimator  $\eta_{\text{F},\text{U},K}^n(t)$  into two contributions using the triangle inequality (see [17, 27]); for all  $K \in \mathcal{T}_h^n$ ,  $t \in I^n$ ,

$$\|\|\|\mathbf{w}_h^n + \mathbf{K}\nabla\hat{p}_{h\tau}(t)\|\|\|_{*,K} \leq \|\|\|\mathbf{w}_h^n + \mathbf{K}\nabla\hat{p}_h^n\|\|\|_{*,K} + \|\|\|\mathbf{K}(\nabla\hat{p}_{h\tau}(t) - \nabla\hat{p}_h^n)\|\|\|_{*,K}, \quad (7.4)$$

$$\|\sigma_h^n - \sigma(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})(t)\|_K \leq \|\sigma_h^n - \sigma(\hat{p}_h^n, \hat{\mathbf{u}}_h^n)\|_K + \|\sigma(\hat{p}_{h\tau}, \hat{\mathbf{u}}_{h\tau})(t) - \sigma(\hat{p}_h^n, \hat{\mathbf{u}}_h^n)\|_K. \quad (7.5)$$

We then integrate in time the last terms on the right-hand side of above inequalities and use the equality of norms in (7.4), i.e.,  $\|\|\|\mathbf{K}v\|\|\|_{*,K} = \|\|\|\mathbf{K}^{-\frac{1}{2}}(\mathbf{K}v)\|\|\|_K = \|v\|_K$  (cf. [3, Theorem 3.6]). Inserting the resulting inequalities in (6.44) with the use of definition (7.1) leads to the estimate (7.2).  $\square$

In the spirit of Lemma 6.3 the following simplified version of the a posteriori estimate of Theorem 7.1, distinguishing the different error components can immediately be obtained.

**Corollary 7.2** (A simple estimate distinguishing error components). *Under the Assumptions of Lemma 7.1, there holds,*

$$\|(p - \tilde{p}_{h\tau}, \mathbf{u} - \tilde{\mathbf{u}}_{h\tau})\|_{\text{qT}} \leq \eta_{\text{tm},\text{P}} + \eta_{\text{sp},\text{P}} + \eta_{\text{tm},\text{U}} + \eta_{\text{sp},\text{U}}. \quad (7.6)$$

Herein,

$$\begin{aligned} \eta_{\text{tm},\text{J}} &:= \sqrt{\frac{L_{\text{J}}}{2}} \left\{ \sum_{n=1}^N (\eta_{\text{tm},\text{J}}^n)^2 \right\}^{\frac{1}{2}}, \quad \text{J} = \text{P}, \text{U}, \\ \eta_{\text{sp},\text{J}} &:= \sqrt{\frac{L_{\text{J}}}{2}} \left\{ \sum_{n=1}^N (\eta_{\text{sp},\text{J}}^n)^2 \right\}^{\frac{1}{2}} + \eta_{\text{NC},\text{J}}, \quad \text{J} = \text{P}, \text{U}, \end{aligned} \quad (7.7a)$$

and  $L_{\text{P}} = e^T$  and  $L_{\text{U}} = \frac{e^T}{\mu}$ .

## 8 An adaptive algorithm

In this section we investigate the error estimate in Theorem 7.1 to design an adaptive algorithm where the mesh size and time step are locally adapted as in [27, 17, 15, 14] and the references therein. This is based on balancing the space and time errors for an adaptive refinement of the space–time mesh. To do so, first the space and time error components should be equilibrated by selecting the time step  $\tau^n$  and adjusting the spatial meshes  $\mathcal{T}_h^n$  in such a way that

$$\gamma_{\text{tm}}\eta_{\text{sp}}^n \leq \eta_{\text{tm}}^n \leq \Gamma_{\text{tm}}\eta_{\text{sp}}^n, \quad (8.1)$$

where  $\eta_{\text{J}}^n = \eta_{\text{J,P}}^n + \eta_{\text{J,U}}^n$ , for  $\text{J} = \text{sp}, \text{tm}$ , and where  $\gamma_{\text{tm}}, \Gamma_{\text{tm}}$  are two user-given weights, typically close to 1. The goal of this balancing is to have  $\eta_{\text{tm}}^n$  and  $\eta_{\text{sp}}^n$  of comparable size instead of getting  $\eta_{\text{tm}}^n$  much smaller than  $\eta_{\text{sp}}^n$ . This balancing can be defined locally (element-wise) as

$$\gamma_{\text{tm},K}\eta_{\text{sp},K}^n \leq \eta_{\text{tm},K}^n \leq \Gamma_{\text{tm},K}\eta_{\text{sp},K}^n, \quad \forall K \in \mathcal{T}_h^n, \quad (8.2)$$

where  $\gamma_{\text{tm},K}$  and  $\Gamma_{\text{tm},K}$  are the local (element-wise) weights. In the next step, we have to adapt the space mesh; refinement/coarsening so that the local spatial error estimators are distributed equally: for all  $K_1, K_2 \in \mathcal{T}_h^n$ ,

$$\eta_{\text{sp},K_1}^n \approx \eta_{\text{sp},K_2}^n. \quad (8.3)$$

To account for limited computing resources, we fix refinement thresholds  $h_{\min} > 0$  and  $\tau_{\min} > 0$  for both the mesh size and the time step and require, for all  $1 \leq n \leq N$ ,

$$\min_{K \in \mathcal{T}_h^n} h_K \geq h_{\min}, \quad \text{and} \quad \tau^n \geq \tau_{\min}.$$

We also need to fix threshold that the error on the time interval  $I_n$  should not exceed  $e^n$ , for all  $1 \leq n \leq N$ . Based on the above considerations and balancing conditions, we propose the following algorithm:

**Algorithm:** {Adaptive algorithm}  
**Data:** Enter  $T, p_0, \boldsymbol{\sigma}_0$  and  $\Omega, \mathbf{K}, c_0, \mu, \lambda$  and  $\alpha$ .  
**Result:** The quadruplet  $(\boldsymbol{\sigma}_{h\tau}, p_{h\tau}, \mathbf{w}_{h\tau}, \mathbf{u}_{h\tau})$ .  
Choose an initial mesh  $\mathcal{T}_h^0$ , an initial time step  $\tau^0$ , and set  $t^0 = 0$ ;  
Choose the weights  $\gamma_{\text{tm}}$  and  $\Gamma_{\text{tm}}$ , and the refinement thresholds  $\tau_{\min}$  and  $h_{\min}$ ;  
**while**  $t^n \leq T$  **do** {Time loop}  
     $n \leftarrow n + 1$ ;  
     $\mathcal{T}_h^n \leftarrow \mathcal{T}_h^{n-1}$ ;  
     $\tau^n \leftarrow \tau^{n-1}$ ;  
    Calculate the quadruplet  $(\boldsymbol{\sigma}_h^n, p_h^n, \mathbf{w}_h^n, \mathbf{u}_h^n)$ ;  
    Calculate the estimators  $\eta_{\text{sp}}^n$  and  $\eta_{\text{tm}}^n$ ;  
    **repeat** {Space refinement}  
        **repeat** {Space–time balancing}  
            **if**  $\eta_{\text{tm}}^n \leq \gamma_{\text{tm}}\eta_{\text{sp}}^n$  **then**  
                 $\tau^n \leftarrow 2\tau^n$ ;  
            **end**  
            **if**  $\eta_{\text{tm}}^n \geq \Gamma_{\text{tm}}\eta_{\text{sp}}^n$  **then**  
                 $\tau^n \leftarrow \frac{1}{2}\tau^n$ ;  
            **end**  
            **until** the balance(8.1) or (8.2) is satisfied or  $\tau^n \leq \tau_{\min}$ ;  
            Refine or coarsen the cells  $K \in \mathcal{T}_h^n$  until (8.3) is satisfied or  $h_K \leq h_{\min}$ ;  
        **until**  $\eta_{\text{sp}}^n + \eta_{\text{tm}}^n \leq e^n$ ;  
         $(\boldsymbol{\sigma}_h^n, p_h^n, \mathbf{w}_h^n, \mathbf{u}_h^n) \leftarrow (\boldsymbol{\sigma}_{K^n}^n, p_{K^n}^n, \mathbf{w}_{K^n}^n, \mathbf{u}_{K^n}^n)_{K^n \in \mathcal{T}_h^n}$ ;  
         $t^n \leftarrow t^{n-1} + \tau^n$ ;  
    **end**  
**end**

**Algorithm 1:** Complete solution algorithm with adaptive space–time refinement

**Remark 8.1** (The initial data estimator). *In practice, the initial data is inhomogeneous and additional initial condition estimator should be added to the estimate (6.44). The initial condition estimator takes the following form:*

$$\eta_{\text{IC}} := \frac{e^T}{2} \|\varphi(p - \hat{p}_{h\tau}, \mathbf{u} - \hat{\mathbf{u}}_{h\tau})(0)\|_{H^{-1}(\Omega)}. \quad (8.4)$$

*In particular, regarding the proof of Lemma 6.1 and Theorem 6.2 and their proofs implies the appearance of the term  $e^T$ , which can grows exponentially with the final time  $T$ . In practice, this does not affect the global estimate as the initial condition  $\|\varphi(p - \hat{p}_{h\tau}, \mathbf{u} - \hat{\mathbf{u}}_{h\tau})(0)\|_{H^{-1}(\Omega)}$  can be made sufficiently small. Moreover, a new step is added to the above algorithm in which the initial mesh is adapted in such away the local initial condition error estimators  $\eta_{\text{IC},K}$ , for all  $K \in \mathcal{T}_h^0$ , are distributed equally and satisfies  $\eta_{\text{IC}} \leq e^0$ , where  $e^0$  is an additional user-given weight for the initial data (cf. [15, 1] for more details). The initial estimator  $\eta_{\text{IC}}$  can also be decomposed in two components, as one for the initial pressure and one for the initial displacement.*

**Remark 8.2** (Time oscillation of the source terms). *For the sake of simplicity, we assumed the source terms  $\mathbf{f}$  and  $g$  to be piecewise constant in time. When this is not true, additional data time-oscillation estimators should be added to the estimates (7.2), and (6.44);*

$$\eta_{\text{osc},g} := \|\tilde{g} - g\|_{X'_T}, \quad \text{and} \quad \eta_{\text{osc},\mathbf{f}} := \|\tilde{\mathbf{f}} - \mathbf{f}\|_{\mathbf{X}'_T}, \quad (8.5)$$

*where  $\tilde{g} \in P_\tau^0(L^2(\Omega))$  is such that  $\tilde{g}|_{I_n} = \tilde{g}^n$  where we have set  $\tilde{g}^n = \frac{1}{\tau^n} \int_{I_n} g(\cdot, t) dt$ , and accordingly for  $\tilde{\mathbf{f}}$ ;  $\tilde{\mathbf{f}} \in P_\tau^0(\mathbf{L}^2(\Omega))$  such that  $\tilde{\mathbf{f}}|_{I_n} = \tilde{\mathbf{f}}^n$  with  $\tilde{\mathbf{f}}^n = \frac{1}{\tau^n} \int_{I_n} \mathbf{f}(\cdot, t) dt$ . In that case, the residual estimators are given by*

$$\eta_{\text{R},\text{P},K}^n := C_{\text{P},K} c_{\mathbf{K},K}^{-\frac{1}{2}} h_K \|\tilde{g}^n - \partial_t \varphi(\hat{p}_h^n, \hat{\mathbf{u}}_h^n) - \nabla \cdot \mathbf{w}_h^n\|_K, \quad K \in \mathcal{T}_h^n, \quad (8.6)$$

$$\eta_{\text{R},\text{U},K}^n := C_{\text{P},K} h_K \|\nabla \cdot \boldsymbol{\sigma}_h^n + \tilde{\mathbf{f}}^n\|_K, \quad K \in \mathcal{T}_h^n. \quad (8.7)$$

## 9 Numerical results

This section discusses the computational results of applying our a posteriori error estimates when solving (2.1) using the mixed scheme (4.4) or the one given by (4.5). To illustrate numerically our theoretical results, we consider in  $\Omega = [0, 1]^2$  the analytical solution of Biot's consolidation problem (2.1) (see [11]).

$$\mathbf{u}(\mathbf{x}, t) := \begin{pmatrix} tx(1-x)y(1-y) \\ ty(1-y)x(1-x) \end{pmatrix}, \quad \text{and} \quad p(\mathbf{x}, t) = tx(1-x)y(1-y). \quad (9.1)$$

The effective parameters are  $\mathbf{K} = 1$ ,  $c_0 = \alpha = 1$ , and  $\mu = \lambda = 0.6$ . We choose source terms, initial and Dirichlet boundary conditions such that Eq. (9.1) is the solution of problem (2.1). Convergence in space and time are evaluated on a series of space or time uniform refinement. We compare the estimated errors for the pressure and displacement with the analytical errors measured in the norms defined by (6.52) and (6.53). For both methods, we set  $\tau^n = h^3$  in order to compare the convergence rates of the spatial discretization errors ignoring influences of time discretization errors. In table 1, we present the estimated errors of Method 1 with  $\mathbf{AFW}_1$  elements. That of Method 2 are presented in table 2. The last column in both tables displays the corresponding effectivity indices, given by the ratio of the estimates over the global error measured in the energy norm given by (6.19) denoted in the table as  $e_{\text{en}}$ .

We observe that the predicted error overestimated the exact errors—recall that there are no undetermined multiplicative constants in our estimate. One can observe that the convergence rates as well as the effectivity indices (see Figure 3 right) are very satisfactory, and we can see that they are not influenced by the time discretizations. In particular, Method 2 gives a slightly better efficiency. For the temporal refinement, the same convergence behavior with similar effectivity indices was observed (not presented). Figures 1 and 2 depict the distribution of the estimate errors and the analytical errors at the final time step where both methods reflect the distribution of the analytical error in the whole domain.

In the second part of this test case, we compare the previously estimates obtained on uniform and fixed meshes and time steps with the estimates obtained using the Adaptive Algorithm of the previous

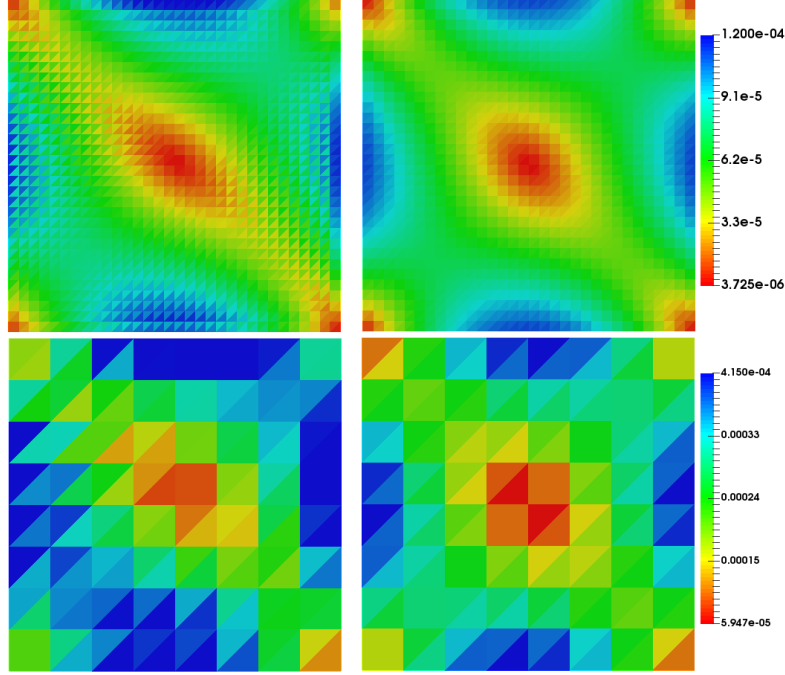


Figure 1: Actual (right) and estimated (left) error distribution on an initial mesh and after two mesh refinements with Method 1.

Method 1						
$1/h$	$\eta_{\text{sp,P}}$	$\eta_{\text{sp,U}}$	$\ p - \tilde{p}_{h\tau}\ _{\text{P}}$	$\ \mathbf{u} - \tilde{\mathbf{u}}_{h\tau}\ _{\text{U}}$	$e_{\text{en}}$	$I_{\text{eff}}$
4	3.15e-2	2.28e-1	1.15e-2	8.05e-2	9.09e-2	2.86
8	1.75e-2	1.35e-1	6.21e-3	4.26e-2	4.86e-2	3.14
16	9.47e-3	7.57e-2	3.31e-3	2.27e-2	2.56e-2	3.32
32	5.01e-3	4.19e-2	1.69e-3	1.34e-2	1.32e-2	3.55

Table 1: Error estimators and analytical errors under space refinement with  $t = T$  and  $\tau^n = h^3$ .

section. We start with an initial mesh of 311 elements. The initial time step  $\tau^0$  was chosen as 0.005 and was not changed by the criterion  $\eta_{\text{IC}} \leq e^0$ . The other parameters used in the Algorithm are  $\gamma_{\text{tm}} = 0.75$  and  $\gamma_{\text{sp}} = 1.4$  and  $e^n = 5 \cdot 10e^{-3}\tau^n$ . The performance of the algorithm is tested for the two methods. In Figure 3 (left and center), we give the initial and final meshes. In Figure 3 (right), we plot the evolution of the error estimates depending on the total number of space–time unknowns of the static algorithm and that of the adaptive algorithm. As shown for both methods, one can reduce the number of unknowns necessary to attain the prescribed precision using the derived estimators and adaptively refined space–time grids. Particularly, with the adaptive algorithm, the two methods attain the prescribed precision with approximately the same number of space–time unknowns. Note that Method 1 require less computational costs compared to Method 2. Figure 4, compares the error estimate at the final time to the exact error for Method 1. Again, the distribution of the estimate error reflects the exact error and one can easily remark that the error is evenly distributed throughout the domain.



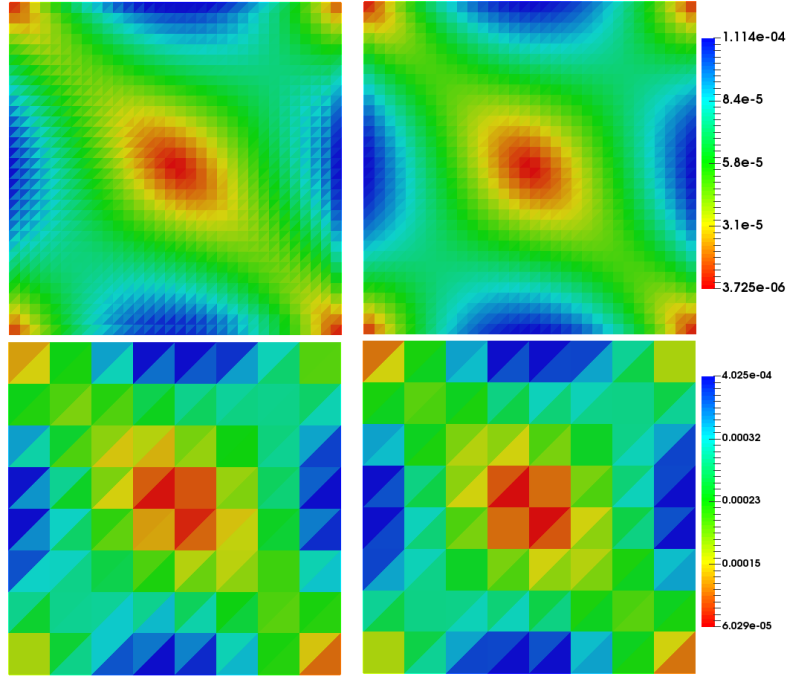


Figure 2: Actual (right) and estimated (left) error distribution on an initial mesh and after two mesh refinements with Method 2.

Method 2						
$1/h$	$\eta_{sp,P}$	$\eta_{sp,U}$	$\ p - \tilde{p}_{h\tau}\ _P$	$\ \mathbf{u} - \tilde{\mathbf{u}}_{h\tau}\ _U$	$e_{en}$	$I_{eff}$
4	2.91e-2	1.61e-1	1.02e-2	5.35e-2	6.37e-2	2.98
8	1.52e-2	8.61e-2	5.25e-3	2.72e-2	3.26e-2	3.10
16	7.88e-3	4.44e-2	2.71e-3	1.40e-2	1.67e-2	3.13
32	4.06e-3	2.30e-2	1.40e-3	7.22e-3	8.58e-3	3.18

Table 2: Error estimators and analytical errors under space refinement with  $t = T$  and  $\tau^n = h^3$ .

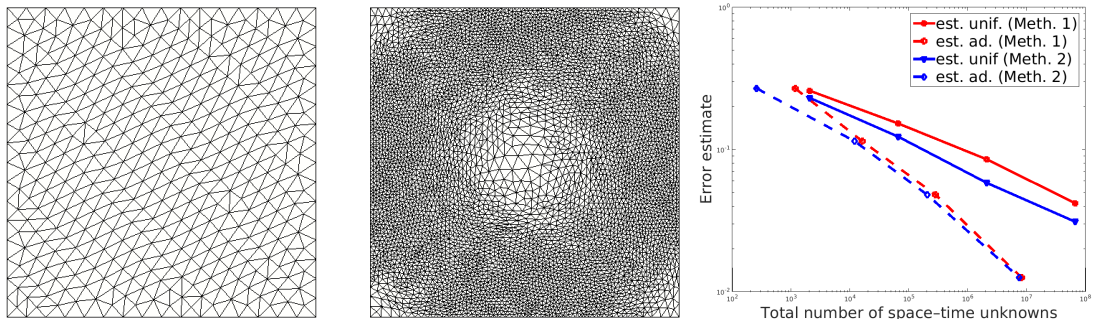


Figure 3: Initial mesh (left) and final mesh (center) and the comparison between a static algorithm (with fixed mesh and time step) and the adaptive algorithm (right).

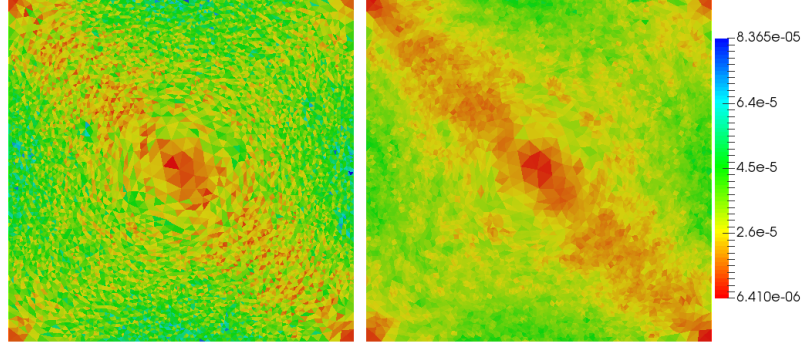


Figure 4: Actual (right) and estimated (left) error distribution at  $t = T$  on the last mesh of an adaptive algorithm with Method 1.

## 10 Conclusions

Two mixed finite element formulations of the Biot's consolidation model are presented and analyzed. Each method utilize a pair of mixed finite element methods, one for the linear elasticity and one for the mixed Poisson problem. The methods uses either symmetric stress space or weakly symmetric stress through the introduction of Lagrange multiplier enforcing the symmetry of the stress in a weak sense. We propose for the two methods a unified and guaranteed a posteriori error estimates based on a conforming reconstructions of the displacement and pressure; both continuous and piecewise affine in time. Precisely, independent of the mixed method, we present a guaranteed and fully computable error estimate that bounds the energy-type error between the unknown exact solution and the approximate solution. This estimate is then decomposed into estimators characterizing the space, and time error components. We also separate the pressure error components from those of displacement errors. This is efficiently used in an adaptive space–time algorithm where the individual error component estimators are used to adjust the time step, and to select the mesh elements to be refined or coarsened. Numerical results with analytical solutions illustrate the effectiveness of our estimates and the performance of the adaptive algorithm. Finally, we shall point out that the developments in this paper also apply to any locally conservative scheme such as cell-centered finite volume scheme and mimetic finite difference.

## A Appendix: Proof of Theorem 3.1

The proof will be broken down in several steps: we first construct solutions of certain finite-dimensional approximations of (3.4), then we derive suitable energy estimates in Lemma A.2 and prove then the Theorem.

We introduce the finite-dimensional approximation of (3.4), (2.1e). To this aim we introduce some notations: Let  $\{\boldsymbol{\tau}_n \mid n \in \mathbb{N}\}$  be a Hilbert basis of  $\boldsymbol{\Sigma}$  and  $\{\mathbf{z}_m \mid m \in \mathbb{N}\}$  a Hilbert basis of  $\mathbf{Q}$ . That of spaces  $\mathbf{W}$  and  $Q$  are,  $\{\mathbf{v}_k \mid k \in \mathbb{N}\}$  and  $\{q_l \mid l \in \mathbb{N}\}$  respectively. For each quadruplet of positive integers  $(n, l, k, m) \in \mathbb{N}^4$ , we define by the following finite dimensional subspaces:

$$\begin{aligned} \boldsymbol{\Sigma}_n &:= \text{span}\{\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \dots, \boldsymbol{\tau}_n\}, & \mathbf{Q}_m &:= \text{span}\{\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_m\}, \\ \mathbf{W}_k &:= \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}, & Q_l &:= \text{span}\{q_1, q_2, \dots, q_l\}. \end{aligned}$$

Therefore, let  $(\boldsymbol{\sigma}_n, p_l, \mathbf{w}_k, \mathbf{u}_m): [0, T]^4 \mapsto \Sigma_n \times Q_l \times \mathbf{W}_k \times \mathbf{Q}_m$  be the solution of the following problem:

$$-(\mathcal{A}\boldsymbol{\sigma}_n, \boldsymbol{\tau}_{i_\sigma}) - (\mathbf{u}_m, \nabla \cdot \boldsymbol{\tau}_{i_\sigma}) - \frac{c_r}{2\alpha}(\mathbf{I}p_l, \boldsymbol{\tau}_{i_\sigma}) = 0, \quad i_\sigma = 1, \dots, n \quad (\text{A.1a})$$

$$(c_0 + c_r)(\partial_t p_l, q_{j_p}) + \frac{c_r}{2\alpha}(\partial_t \boldsymbol{\sigma}_n, q_{j_p} \mathbf{I}) + (\nabla \cdot \mathbf{w}_k, q_{j_p}) = (g(t), q_{j_p}), \quad j_p = 1, \dots, l, \quad (\text{A.1b})$$

$$(\mathbf{K}^{-1} \mathbf{w}_k, \mathbf{v}_{j_w}) - (p_l, \nabla \cdot \mathbf{v}_{j_w}) = 0, \quad j_w = 1, \dots, k, \quad (\text{A.1c})$$

$$(\nabla \cdot \boldsymbol{\sigma}_n, \mathbf{z}_{i_u}) = (-\mathbf{f}(t), \mathbf{z}_{i_u}), \quad i_u = 1, \dots, m, \quad (\text{A.1d})$$

with

$$(\mathbf{u}_m(0), \mathbf{z}_{i_u}) = 0, \quad \forall i_u = 1, \dots, m, \quad \text{and} \quad (p_l(0), q_{j_p}) = 0, \quad \forall j_p = 1, \dots, l. \quad (\text{A.1e})$$

Using the same techniques from [34], one can obtain

**Lemma A.1** (Construction of approximate solutions). *For each quadruplet  $(n, l, k, m) \in \mathbb{N}^4$ , there exists a unique solution  $(\boldsymbol{\sigma}_n, p_l, \mathbf{w}_k, \mathbf{u}_m)$  to problem (A.1).*

*Proof.* Let us introduce the following notations:

$$(G_l(t))_{i_p} = (g(t), q_{i_p}), \quad (\mathbf{C}_l)_{i_p j_p} = (c_0 + c_r)(q_{j_p}, q_{i_p}), \quad \forall 0 \leq i_p, j_p \leq l,$$

$$(\mathbf{E}_n)_{i_\sigma j_\sigma} = (\mathcal{A}\boldsymbol{\tau}_{j_\sigma}, \boldsymbol{\tau}_{i_\sigma}), \quad \forall 0 \leq i_\sigma, j_\sigma \leq n, \quad (\mathbf{A}_{nl})_{j_p j_\sigma} = \frac{c_r}{2\alpha}(\mathbf{I}q_{j_p}, \boldsymbol{\tau}_{i_\sigma}), \quad \forall 0 \leq j_p \leq l, 0 \leq i_\sigma \leq n,$$

$$(\mathbf{F}_m(t))_{i_u} = (\mathbf{f}(t), \mathbf{z}_{i_u}), \quad \forall 0 \leq i_u \leq m, \quad (\mathbf{B}_{mn})_{i_u j_\sigma} = (\nabla \cdot \boldsymbol{\tau}_{j_\sigma}, \mathbf{z}_{i_u}), \quad \forall 0 \leq i_u \leq m, 0 \leq j_\sigma \leq n,$$

$$(\mathbf{D}_{kl})_{i_w j_p} = (\nabla \cdot \mathbf{v}_{i_w}, q_{j_p}), \quad \forall 0 \leq i_w \leq k, \forall 0 \leq j_p \leq l, \quad (\mathbf{V}_k)_{i_w j_w} = (\mathbf{K}^{-1} \mathbf{v}_{i_w}, \mathbf{v}_{j_w}), \quad \forall 0 \leq i_w, j_w \leq k.$$

We also use the following notations;  $\mathbf{P}_l$  to denote the vector of degrees of freedom of  $p_l(t)$  with respect to the basis  $\{q_{i_p}\}_0^l$ ,  $\mathbf{S}_n$  to denote the vector of degrees of freedom of  $\boldsymbol{\sigma}_n(t)$  with respect to the basis  $\{\boldsymbol{\tau}_{i_\sigma}\}_0^n$ ,  $\mathbf{U}_m$  to denote the vector of degrees of freedom of  $\mathbf{u}_m(t)$  with respect to the basis  $\{\mathbf{v}_{i_u}\}_0^m$  and  $\mathbf{W}_k$  to denote the vector of degrees of freedom of  $\mathbf{w}_k(t)$  with respect to the basis  $\{\mathbf{z}_{i_w}\}_0^k$ . Now, using the above notations, and for all  $(n, l, k, m) \in \mathbb{N}^4$ , problem (A.1) may be rewritten as

$$-\mathbf{E}_n \mathbf{S}_n(t) - \mathbf{B}_{mn}^T \mathbf{U}_m(t) - \mathbf{A}_{nl} \mathbf{P}_l(t) = 0, \quad (\text{A.2})$$

$$\mathbf{C}_l \frac{d\mathbf{P}_l}{dt}(t) + \mathbf{A}_{nl}^T \frac{d\mathbf{S}_n}{dt}(t) + \mathbf{D}_{lk} \mathbf{W}_k(t) = G_l(t), \quad (\text{A.3})$$

$$\mathbf{V}_k \mathbf{W}_k(t) - \mathbf{D}_{lk}^T \mathbf{P}_l(t) = 0, \quad (\text{A.4})$$

$$\mathbf{B}_{mn} \mathbf{S}_n(t) = -\mathbf{F}_m(t), \quad (\text{A.5})$$

$$\mathbf{P}_n(0) = 0, \quad \mathbf{U}_m(0) = 0. \quad (\text{A.6})$$

Regarding the assumptions concerning  $\mathbf{K}$  as well as the property of  $\mathcal{A}$ , we obtain that  $\mathbf{V}_k$  and  $\mathbf{E}_n$  are symmetric and positive definite square matrices of size  $k$  and  $n$  respectively, and we have

$$(\mathbf{E}_n^{-1})_{i_\sigma j_\sigma} = 2(\lambda + \mu)(\boldsymbol{\tau}_{j_\sigma}, \boldsymbol{\tau}_{i_\sigma}), \quad \forall 0 \leq i_\sigma, j_\sigma \leq n, \quad (\text{A.7})$$

$$(\mathbf{V}_k^{-1})_{i_w j_w} = (\mathbf{K} \mathbf{v}_{i_w}, \mathbf{v}_{j_w}), \quad \forall 0 \leq i_w, j_w \leq k. \quad (\text{A.8})$$

We have then

$$\mathbf{S}_n(t) = -\mathbf{E}_n^{-1} \mathbf{B}_{mn}^T \mathbf{U}_m(t) - \mathbf{E}_n^{-1} \mathbf{A}_{nl} \mathbf{P}_l(t), \quad (\text{A.9})$$

$$\mathbf{W}_k(t) = \mathbf{V}_k^{-1} \mathbf{D}_{lk}^T \mathbf{P}_l(t). \quad (\text{A.10})$$

Substituting (A.9) and (A.10) into (A.3) and (A.5), we obtain

$$(\mathbf{C}_l + \mathbf{J}_l) \frac{d\mathbf{P}_l}{dt}(t) - \mathbf{H}_{lm} \frac{d\mathbf{U}_m}{dt}(t) + \mathbf{L}_l \mathbf{P}_l(t) = G_l(t), \quad (\text{A.11})$$

$$\mathbf{M}_m \mathbf{U}_m(t) + \mathbf{H}_{lm}^T \mathbf{P}_l(t) = -\mathbf{F}_m(t), \quad (\text{A.12})$$

where  $\mathbf{J}_l = -\mathbf{A}_{nl}^T \mathbf{E}_n^{-1} \mathbf{A}_{nl}$ ,  $\mathbf{H}_{lm} = \mathbf{A}_{nl}^T \mathbf{E}_n^{-1} \mathbf{B}_{mn}^T$ ,  $\mathbf{L}_l = \mathbf{D}_{lk} \mathbf{V}_k^{-1} \mathbf{D}_{lk}^T$  and  $\mathbf{M}_m = \mathbf{B}_{mn} \mathbf{E}_n^{-1} \mathbf{B}_{mn}^T$ . Now, we differentiate (A.12) with respect to time and plugging the result into (A.11), we obtain

$$(\mathbf{C}_l + \mathbf{J}_l + \mathbf{H}_{lm} \mathbf{M}_m^{-1} \mathbf{H}_{lm}^T) \frac{d\mathbf{P}_l}{dt}(t) + \mathbf{L}_l \mathbf{p}_l(t) = \mathbf{G}_l(t) + \mathbf{M}_m^{-1} \mathbf{F}_m(t). \quad (\text{A.13})$$

Now, it is inferred from (A.7) that  $(\mathbf{D}_l)_{i_p j_p} = (\mathbf{C}_l + \mathbf{J}_l)_{i_p j_p} = c_0(q_{j_p}, q_{i_p})$ . Thus,  $(\mathbf{D}_l + \mathbf{H}_{lm} \mathbf{M}_m^{-1} \mathbf{H}_{lm}^T)$  is the sum of a symmetric, positive-definite matrix  $\mathbf{D}_l$  and a symmetric, nonnegative definite matrix  $\mathbf{H}_{lm} \mathbf{M}_m^{-1} \mathbf{H}_{lm}^T$ . Therefore, their sum is symmetric and positive definite, and thus can be inverted in (A.13), leading to a system of  $l$  linear ODEs of order 1 with initial condition (A.6). Hence, there exists a unique function  $\mathbf{P}_l \in (C[0, T])^l$ , with  $\frac{d\mathbf{P}_l}{dt} \in (L^2(0, T))^l$  which satisfies (A.13) together with (A.6). From (A.12), we obtain  $\mathbf{U}_m \in (C[0, T])^m$  such that  $\frac{d\mathbf{U}_m}{dt} \in (L^2(0, T))^l$ . Similarly, from (A.9)–(A.10), one obtain  $\mathbf{W}_k \in (C[0, T])^k$  such that  $\frac{d\mathbf{W}_k}{dt} \in (L^2(0, T))^k$  as well as  $\mathbf{S}_n \in (C[0, T])^n$  such that  $\frac{d\mathbf{S}_n}{dt} \in (L^2(0, T))^n$ . As a result  $(\boldsymbol{\sigma}_n, p_l, \mathbf{w}_k, \mathbf{u}_m)$  is the unique solution to problem (A.1).  $\square$

The next step is to derive a priori estimates with the aim to pass to the limit in a subsequence of the approximated solution.

**Lemma A.2** (A priori estimates). *There exists a constant  $C$  independent of  $(n, l, k, m)$  such that*

$$\begin{aligned} & \|p_l\|_{H^1(0, T; L^2(\Omega))} + \|\mathbf{w}_k\|_{L^\infty(0, T; L^2(\Omega))} + \|\mathbf{w}_k\|_{L^2(0, T; \mathbf{H}(\text{div}, \Omega))} \\ & + \|\partial_t \boldsymbol{\sigma}_n\|_{L^2(0, T; L^2(\Omega))} + \|\boldsymbol{\sigma}_n\|_{L^2(0, T; \mathbb{H}^s(\text{div}, \Omega))} + \|\mathbf{u}_l\|_{H^1(0, T; L^2(\Omega))} \\ & \leq C (\|g\|_{L^2(0, T; L^2(\Omega))} + \|\mathbf{f}\|_{H^1(0, T; L^2(\Omega))}), \quad \forall (n, l, k, m) \geq 1. \end{aligned}$$

For the proof of this Lemma, we will make use of this intermediate result

**Lemma A.3.** *There exist the constant  $C_1$  and  $C_2$  independent of  $(n, l, k, m)$  such that*

$$\|\boldsymbol{\sigma}_n\|_{\mathcal{A}}^2 \leq \left( c_r + \frac{1}{2(\lambda + \mu)} \right) \|p_l\|^2 + C_1 \|\mathbf{f}\|^2, \quad \forall n, l, k, m \geq 1, \quad (\text{A.14})$$

$$\|\mathbf{u}_m\|^2 \leq C_2 \left( c_r + \frac{1}{2(\lambda + \mu)} \right) \|p_l\|^2 + \frac{1}{2(\lambda + \mu)} \|\mathbf{f}\|^2, \quad \forall n, l, k, m \geq 1, \quad (\text{A.15})$$

where  $\|\cdot\|_{\mathcal{A}}$  is the energy-norm given by

$$\|\boldsymbol{\tau}\|_{\mathcal{A}} = \int_{\Omega} \mathcal{A} \boldsymbol{\tau} : \boldsymbol{\tau} \, dx.$$

*Proof.* For any given  $t \in (0, T)$ , we know that there exist  $\tilde{\boldsymbol{\sigma}}(\cdot, t) \in \mathbb{H}^1(\Omega) \cap \Sigma_n$  such that  $-\nabla \cdot \tilde{\boldsymbol{\sigma}}(\cdot, t) = \mathbf{u}_m(\cdot, t)$  and  $\|\tilde{\boldsymbol{\sigma}}\|_{H_0^1(\Omega)} \leq C \|\mathbf{u}_m\|$ . Then, using equation (A.1a), we obtain

$$\begin{aligned} \|\mathbf{u}_m\|^2 &= -(\mathbf{u}_m, \nabla \cdot \tilde{\boldsymbol{\sigma}}) = (\mathcal{A} \boldsymbol{\sigma}_n, \tilde{\boldsymbol{\sigma}}) + \frac{c_r}{2\alpha} (\mathbf{I} p_l, \tilde{\boldsymbol{\sigma}}), \\ &\leq C (\|\boldsymbol{\sigma}_n\| + \|p_l\|) \|\tilde{\boldsymbol{\sigma}}\|, \\ &\leq \tilde{C} (\|\boldsymbol{\sigma}_n\| + \|p_l\|) \|\mathbf{u}_m\|. \end{aligned}$$

Therefore,

$$\|\mathbf{u}_m\| \leq \tilde{C} (\|\boldsymbol{\sigma}_n\| + \|p_l\|). \quad (\text{A.16})$$

Now, take  $\boldsymbol{\tau}_{i_\sigma} = \boldsymbol{\sigma}_n$  in (A.1a) and  $\mathbf{z}_{i_u} = \mathbf{u}_m$  in (A.1d) and summing, we obtain:

$$(\mathcal{A} \boldsymbol{\sigma}_n, \boldsymbol{\sigma}_n) = -\frac{c_r}{2\alpha} (\mathbf{I} p_l, \boldsymbol{\sigma}_n) + (\mathbf{f}(t), \mathbf{u}_m). \quad (\text{A.17})$$

Using the definition of  $\mathcal{A}$  and applying the Cauchy-Schwarz and Young's inequality leads to

$$\|\boldsymbol{\sigma}_n\|_{\mathcal{A}}^2 \leq \frac{\alpha^2}{2(\alpha + \mu)} \|p_l\|^2 + \frac{1}{8(\lambda + \mu)} \|\boldsymbol{\sigma}_n\|^2 + \frac{1}{2\epsilon} \|\mathbf{f}\|^2 + \frac{\epsilon}{2} \|\mathbf{u}_m\|^2, \quad (\text{A.18})$$

for any  $\epsilon > 0$ . Now, using (A.16) and setting the suitable  $\epsilon$ , say  $\epsilon = 1/(8\tilde{C}^2(\lambda + \mu))$ , we get

$$\|\boldsymbol{\sigma}_n\|_{\mathcal{A}}^2 \leq \frac{2\alpha^2 + 1}{2(\lambda + \mu)} \|p_l\|^2 + \frac{1}{4\tilde{C}^2(\lambda + \mu)} \|\mathbf{f}\|^2 + \frac{1}{4(\lambda + \mu)} \|\boldsymbol{\sigma}_n\|^2. \quad (\text{A.19})$$

Now we use the fact that  $\|\cdot\|_{\mathcal{A}}$  is equivalent to the  $L^2$ -norm, i.e.,

$$\frac{1}{2(\lambda + \mu)} \|\boldsymbol{\tau}\|^2 \leq \|\boldsymbol{\tau}\|_{\mathcal{A}}^2 \leq \frac{1}{2\mu} \|\boldsymbol{\tau}\|^2, \quad (\text{A.20})$$

to deduce from (A.19)

$$\|\boldsymbol{\sigma}_n\|_{\mathcal{A}}^2 \leq \left( c_r + \frac{1}{2(\lambda + \mu)} \right) \|p_l\|^2 + C_1 \|\mathbf{f}\|^2, \quad (\text{A.21})$$

with  $C_1 = \frac{1}{4\tilde{C}^2(\lambda + \mu)}$ . This result together with (A.16) implies (A.15).  $\square$

We prove now Lemma A.2 by deriving successively the estimates on  $p_l$ ,  $\partial_t p_l$  and  $\boldsymbol{\sigma}_m$ ,  $\partial_t \boldsymbol{\sigma}_m$ , and finally estimates for  $\nabla \cdot \mathbf{w}_k$  for the  $\mathbf{H}(\text{div}, \Omega)$ -norm and  $\nabla \cdot \boldsymbol{\sigma}_n$  for the  $\mathbb{H}(\text{div}, \Omega)$ -norm. That of the estimate for  $\mathbf{u}_m$  will be straightforwardly deduced from the previous Lemma.

*Proof.* Let  $(n, l, k, m) \geq 1$  and take  $\partial_t \boldsymbol{\sigma}_n$  as test function in (A.1a), then derivate (A.1d) with respect to  $t$  and take  $\mathbf{u}_m$  as test function. We take then  $p_l$  as test function (A.1b) and  $\mathbf{w}_k$  as test function in (A.1c). Summing the four equations, we obtain

$$(c_0 + c_r)(\partial_t p_l, p_l) + (\mathbf{K}^{-1} \mathbf{w}_k, \mathbf{w}_k) = (\mathcal{A} \boldsymbol{\sigma}_n, \partial_t \boldsymbol{\sigma}_n) + (g(t), p_l) + (-\partial_t \mathbf{f}(t), \mathbf{u}_m). \quad (\text{A.22})$$

Using the properties of  $\mathbf{K}$ ,  $(c_0 + c_r)$  and applying the Cauchy-Schwarz inequality and Young's inequality, we get

$$(c_0 + c_r)(\partial_t p_l, p_l) = \frac{(c_0 + c_r)}{2} \frac{d}{dt} \|p_l(t)\|^2, \\ (\mathbf{K}^{-1} \mathbf{w}_k, \mathbf{w}_k) \geq c_{\mathbf{K}} \|\mathbf{w}_k\|^2,$$

$$(\mathcal{A} \boldsymbol{\sigma}_n, \partial_t \boldsymbol{\sigma}_n) + (g(t), p_l) + (-\partial_t \mathbf{f}(t), \mathbf{u}_m) \leq \frac{1}{2} \frac{d}{dt} \|\boldsymbol{\sigma}_n\|_{\mathcal{A}}^2 + \frac{1}{2\epsilon_1} \|p_l\|^2 + \frac{\epsilon_1}{2} \|g\|^2 + \frac{\epsilon_2}{2} \|\partial_t \mathbf{f}\|^2 + \frac{1}{2\epsilon_2} \|\mathbf{u}_m\|^2,$$

for any  $\epsilon_1, \epsilon_2 > 0$ . We deduce from these inequalities

$$\frac{(c_0 + c_r)}{2} \frac{d}{dt} \|p_l(t)\| + c_{\mathbf{K}} \|\mathbf{w}_k\|^2 \leq \frac{1}{2} \frac{d}{dt} \|\boldsymbol{\sigma}_n\|_{\mathcal{A}}^2 + \frac{1}{2\epsilon_1} \|p_l\|^2 + \frac{\epsilon_1}{2} \|g\|^2 + \frac{\epsilon_2}{2} \|\partial_t \mathbf{f}\|^2 + \frac{1}{2\epsilon_2} \|\mathbf{u}_m\|^2,$$

We integrate this inequality over  $(0, t)$ , for  $t \in (0, T)$ , to get

$$(c_0 + c_r) \|p_l(t)\|^2 + 2c_{\mathbf{K}} \int_0^t \|\mathbf{w}_k(s)\|^2 ds \leq \|\boldsymbol{\sigma}_n(t)\|_{\mathcal{A}}^2 + \int_0^t \left( \frac{1}{\epsilon_1} \|p_l\|^2 + \frac{1}{\epsilon_2} \|\mathbf{u}_m\|^2 \right) ds \\ + \int_0^t (\epsilon_1 \|g\|^2 + \epsilon_2 \|\partial_t \mathbf{f}\|^2) ds.$$

We now replace (A.14) and (A.15) in the above inequality, letting  $\gamma_0 = c_0 - \frac{1}{2(\lambda + \mu)} > 0$  using (3.5), and adjusting  $\epsilon_1$  and  $\epsilon_2$ , we get

$$\|p_l(t)\|^2 + \frac{2c_{\mathbf{K}}}{\gamma_0} \int_0^t \|\mathbf{w}_k(s)\|^2 ds \leq C(\|g\|_{L^2(0,T;L^2(\Omega))}^2 + \|\mathbf{f}\|_{H^1(0,T;L^2(\Omega))}^2) + C_1 \|\mathbf{f}(t)\|_{L^2(\Omega)} + \int_0^t \|p_l(s)\|^2 ds.$$

It is inferred from Gronwall's lemma that

$$\|p_l\|_{L^2(0,T;L^2(\Omega))}^2 \leq C(\|g\|_{L^2(0,T;L^2(\Omega))}^2 + \|\mathbf{f}\|_{H^1(0,T;L^2(\Omega))}^2).$$

To derive the estimate for  $\partial_t p_l$ , we take  $\partial_t p_l$  as test function in (A.1b) and derivate (A.1c) with respect to  $t$  and taking  $\mathbf{v}_j = \mathbf{w}_k$ . Therefore, we derivate (A.1a) with respect to  $t$  and take  $\partial_t \boldsymbol{\sigma}_n$  as test function, then derivate (A.1d) with respect to  $t$  and take  $\partial_t \mathbf{u}_m$  as test function. Summing the four equations, we get

$$(c_0 + c_r)(\partial_t p_l, \partial_t p_l) + (\mathbf{K}^{-1} \partial_t \mathbf{w}_k, \mathbf{w}_k) = (\mathcal{A} \partial_t \boldsymbol{\sigma}_n, \partial_t \boldsymbol{\sigma}_n) + (g, \partial_t p_l) + (-\partial_t \mathbf{f}, \partial_t \mathbf{u}_m). \quad (\text{A.23})$$

We recall that  $\partial_t \tilde{\boldsymbol{\sigma}} \in \mathbb{H}^1(\Omega) \cap \mathbb{H}^s(\text{div}, \Omega)$  and satisfies  $-\nabla \cdot (\partial_t \tilde{\boldsymbol{\sigma}}) = \partial_t \mathbf{u}_m$  such that  $\|\partial_t \tilde{\boldsymbol{\sigma}}\| \leq C \|\partial_t \mathbf{u}_m\|$ . We use the same techniques as as used to derive (A.16), it is inferred that

$$\|\partial_t \mathbf{u}_m\| \leq C (\|\partial_t \boldsymbol{\sigma}_n\| + \|\partial_t p_l\|). \quad (\text{A.24})$$

We now differentiate (A.1a) and (A.1d) with respect to  $t$ , and take  $\boldsymbol{\tau}_{i_\sigma} = \partial_t \boldsymbol{\sigma}_n$  in (A.1a) and  $\mathbf{z}_{i_u} = \partial_t \mathbf{u}_m$  in (A.1d) then we sum the resulting equations, we infer that

$$(\mathcal{A} \partial_t \boldsymbol{\sigma}_n, \partial_t \boldsymbol{\sigma}_n) = -\frac{c_r}{2\alpha} (\mathbf{I} \partial_t p_l, \partial_t \boldsymbol{\sigma}_n) + (-\partial_t \mathbf{f}(t), \partial_t \mathbf{u}_m). \quad (\text{A.25})$$

We then follow the same steps as used to derive (A.21): Applying the Cauchy-Schwarz and Young's inequality to obtain

$$\|\partial_t \boldsymbol{\sigma}_n\|_{\mathcal{A}}^2 \leq \left( c_r + \frac{1}{2(\lambda + \mu)} \right) \|\partial_t p_l\|^2 + C_1 \|\partial_t \mathbf{f}\|^2. \quad (\text{A.26})$$

Using the property of  $\mathbf{K}$  and Cauchy-Schwarz inequality as well as the inequalities (A.24) and (A.26), it is inferred from (A.23) that there exist  $C$  such that

$$\|\partial_t p_l\|^2 + \frac{2}{\gamma_0} \frac{d}{dt} \|\sqrt{\mathbf{K}^{-1}} \mathbf{w}_k\|^2 \leq C(\|g\|^2 + \|\partial_t \mathbf{f}\|^2),$$

We integrate this inequality over  $(0, t)$  for  $t \in (0, T)$ , we find

$$\int_0^t \|\partial_t p_l(s)\|^2 ds + \frac{2}{\gamma_0} \|\sqrt{\mathbf{K}^{-1}} \mathbf{w}_k\|^2 \leq C \int_0^t (\|g\|^2 + \|\partial_t \mathbf{f}\|^2) ds. \quad (\text{A.27})$$

since  $\mathbf{w}_k(0) = 0$ . This result yields

$$\|\partial_t p_l\|_{L^2(0,T;L^2(\Omega))}^2 + \|\mathbf{w}_k\|_{L^\infty(0,T;L^2(\Omega))}^2 \leq C(\|g\|_{L^2(0,T;L^2(\Omega))}^2 + \|\mathbf{f}\|_{H^1(0,T;L^2(\Omega))}^2). \quad (\text{A.28})$$

Now, it remains to bound  $\|\nabla \cdot \mathbf{w}_k\|_{L^2(0,T;L^2(\Omega))}$  and  $\|\nabla \cdot \boldsymbol{\sigma}_n\|_{L^2(0,T;L^2(\Omega))}$ . We fix  $k$  and let  $\nabla \cdot \mathbf{w}_k = \sum_{j_p}^{\infty} \xi_k^{j_p} q_{j_p}$  as  $\nabla \cdot \mathbf{w}_k \in L^2(\Omega)$ , for a.e.  $t \in (0, T)$ , where we set  $\xi_k^{j_p} = (\nabla \cdot \mathbf{w}_k(t), q_{j_p})$ . Now we fix  $l$  and multiply (A.1b) by  $\xi_k^{j_p}$  and summing over  $j_p = 1, \dots, l$ , and using (A.26), and (A.28) we find out

$$(\nabla \cdot \mathbf{w}_k, \sum_{j_p}^l \xi_k^{j_p} q_{j_p}) \leq \frac{1}{2} \left\| \sum_{j_p}^l \xi_k^{j_p} q_{j_p} \right\|^2 + C (\|\partial_t \mathbf{f}\|^2 + \|g\|^2). \quad (\text{A.29})$$

Integrating over the time we find

$$\int_0^T (\nabla \cdot \mathbf{w}_k, \sum_{j_p}^l \xi_k^{j_p} q_{j_p}) dt \leq C \left( \|\mathbf{f}\|_{H^1(0,T;L^2(\Omega))}^2 + \|g\|_{L^2(0,T;L^2(\Omega))}^2 \right) + \frac{1}{2} \int_0^T \left\| \sum_{j_p}^l \xi_k^{j_p} q_{j_p} \right\|^2 dt. \quad (\text{A.30})$$

We tend  $s$  to  $+\infty$ , then use the fact that  $\nabla \cdot \mathbf{w}_k = \sum_{j_p}^{\infty} \xi_k^{j_p} q_{j_p}$ , we obtain

$$\int_0^T \|\nabla \cdot \mathbf{w}_k\|^2 dt \leq C \left( \|\mathbf{f}\|_{H^1(0,T;L^2(\Omega))}^2 + \|g\|_{L^2(0,T;L^2(\Omega))}^2 \right) + \frac{1}{2} \int_0^T \|\nabla \cdot \mathbf{w}_k\|_{L^2(\Omega)}^2 dt. \quad (\text{A.31})$$

Thus,

$$\|\nabla \cdot \mathbf{w}_k\|_{L^2(0,T;L^2(\Omega))}^2 \leq C \left( \|\mathbf{f}\|_{H^1(0,T;L^2(\Omega))}^2 + \|g\|_{L^2(0,T;L^2(\Omega))}^2 \right). \quad (\text{A.32})$$

We use the same techniques to obtain

$$\|\nabla \cdot \boldsymbol{\sigma}_n\|_{L^2(0,T;L^2(\Omega))}^2 \leq C \|\mathbf{f}\|_{L^2(0,T;L^2(\Omega))}^2. \quad (\text{A.33})$$

We now prove the first part of Theorem 3.1: there exist a unique solution  $(\boldsymbol{\sigma}, \mathbf{u}, \mathbf{w}, p)$  to the problem (3.4), (2.1e).

- Lemma A.2 implies that the sequences  $\{p_l\}_0^\infty$  and  $\{\partial p_l\}_0^\infty$  are bounded in  $L^2(0,T;L^2(\Omega))$ . That of  $\{\boldsymbol{\sigma}_n\}_0^\infty$  is bounded in  $L^2(0,T;\mathbb{H}^s(\text{div},\Omega))$ , such that  $\{\partial_t \boldsymbol{\sigma}_n\}_0^\infty$  is bounded in  $L^2(0,T;L^2(\Omega))$ . The sequences  $\{\mathbf{u}_m\}_0^\infty$  and  $\{\partial_t \mathbf{u}_m\}_0^\infty$  are bounded in  $L^2(0,T;L^2(\Omega))$ . Finally, the sequence  $\{\mathbf{w}_n\}_0^\infty$  is bounded in  $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;\mathbf{H}(\text{div},\Omega))$ . Therefore, there exist subsequences, still denoted by  $\{p_l\}_0^\infty$ ,  $\{\boldsymbol{\sigma}_n\}_0^\infty$ ,  $\{\mathbf{w}_n\}_0^\infty$  and  $\{\mathbf{u}_m\}_0^\infty$  and functions  $p \in L^2(0,T;L^2(\Omega))$  with  $\partial_t p \in L^2(0,T;L^2(\Omega))$ ,  $\boldsymbol{\sigma} \in L^2(0,T;\mathbb{H}^s(\text{div},\Omega))$  with  $\partial_t \boldsymbol{\sigma} \in L^2(0,T;L^2(\Omega))$ ,  $\mathbf{w} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;\mathbf{H}(\text{div},\Omega))$  and  $\mathbf{u} \in L^2(0,T;L^2(\Omega))$  with  $\partial_t \mathbf{u} \in L^2(0,T;L^2(\Omega))$  such that

$$\begin{aligned} p_l &\rightharpoonup p \text{ in } L^2(0,T;L^2(\Omega)), \\ \partial_t p_l &\rightharpoonup \partial_t p \text{ in } L^2(0,T;L^2(\Omega)), \\ \mathbf{w}_k &\rightharpoonup \mathbf{w} \text{ in } L^2(0,T;\mathbf{H}(\text{div},\Omega)), \\ \mathbf{u}_m &\rightharpoonup \mathbf{u} \text{ in } L^2(0,T;L^2(\Omega)), \\ \partial_t \mathbf{u}_m &\rightharpoonup \partial_t \mathbf{u} \text{ in } L^2(0,T;L^2(\Omega)), \\ \boldsymbol{\sigma}_n &\rightharpoonup \boldsymbol{\sigma} \text{ in } L^2(0,T;\mathbb{H}^s(\text{div},\Omega)), \\ \partial_t \boldsymbol{\sigma}_n &\rightharpoonup \partial_t \boldsymbol{\sigma} \text{ in } L^2(0,T;L^2(\Omega)). \end{aligned} \quad (\text{A.34})$$

For fixed set  $(n_0, l_0, k_0, m_0) \geq 1$ , we let  $(\boldsymbol{\tau}, q, \mathbf{v}, \mathbf{z}) \in \times C^1(0,T;\Sigma_{n_0} \times Q_{l_0} \times \mathbf{W}_{k_0} \times \mathbf{Q}_{m_0})$  as test functions in (A.1) and then integrate each equation with respect to time, for  $(n, l, k, m) \geq (n_0, l_0, k_0, m_0)$ ,

$$-\int_0^T \{(\mathcal{A}\boldsymbol{\sigma}_n, \boldsymbol{\tau}) - (\mathbf{u}_m, \nabla \cdot \boldsymbol{\tau}) - \frac{c_r}{2\alpha} (\mathbf{I}p_l, \boldsymbol{\tau})\} dt = 0, \quad (\text{A.35})$$

$$(c_0 + c_r) \int_0^T \{(\partial_t p_l, q) + \frac{c_r}{2\alpha} (\partial_t \boldsymbol{\sigma}_n, q\mathbf{I}) + (\nabla \cdot \mathbf{w}_k, q)\} dt = \int_0^T (g(t), q) dt, \quad (\text{A.36})$$

$$\int_0^T \{(\mathbf{K}^{-1}\mathbf{w}_k, \mathbf{v}) - (p_l, \nabla \cdot \mathbf{v})\} dt = 0, \quad (\text{A.37})$$

$$\int_0^T (\nabla \cdot \boldsymbol{\sigma}_n, \mathbf{z}) dt = \int_0^T (-\mathbf{f}(t), \mathbf{z}) dt, \quad (\text{A.38})$$

and passing to the limit, we obtain

$$-\int_0^T \{(\mathcal{A}\boldsymbol{\sigma}, \boldsymbol{\tau}) - (\mathbf{u}, \nabla \cdot \boldsymbol{\tau}) - \frac{c_r}{2\alpha} (\mathbf{I}p, \boldsymbol{\tau})\} dt = 0, \quad (\text{A.39})$$

$$(c_0 + c_r) \int_0^T \{(\partial_t p, q) + \frac{c_r}{2\alpha} (\partial_t \boldsymbol{\sigma}, q\mathbf{I}) + (\nabla \cdot \mathbf{w}, q)\} dt = \int_0^T (g(t), q) dt, \quad (\text{A.40})$$

$$\int_0^T \{(\mathbf{K}^{-1}\mathbf{w}, \mathbf{v}) - (p, \nabla \cdot \mathbf{v})\} dt = 0, \quad (\text{A.41})$$

$$\int_0^T (\nabla \cdot \boldsymbol{\sigma}, \mathbf{z}) dt = \int_0^T (-\mathbf{f}(t), \mathbf{z}) dt. \quad (\text{A.42})$$

We then use the density of the spaces for the test functions in  $L^2(0,T;\Sigma) \times L^2(0,T;Q) \times L^2(0,T;\mathbf{W}) \times L^2(0,T;\mathbf{Q})$  to show that equations (3.4), (2.1e) holds true for a.e.  $t \in (0,T)$  (see [18]).

- To end this part of the theorem, we need to see that  $p(0) = 0$  as well as  $\boldsymbol{\sigma}(0) = 0$ . We take  $q \in C^1(0, T; Q)$  with  $q(T) = 0$  and integrate (A.36) by parts in time, we obtain

$$-(c_0 + c_r) \int_0^T \{(\partial_t q, p_l) - \frac{c_r}{2\alpha} (\partial_t q, \text{tr}(\boldsymbol{\sigma}_n)) + (\nabla \cdot \mathbf{w}_k, q)\} dt = \int_0^T (g(t), q) dt + (p_l(0), q(0)) + (\boldsymbol{\sigma}_n(0), q(0)\mathbf{I}). \quad (\text{A.43})$$

From (A.40), we get

$$-(c_0 + c_r) \int_0^T \{(\partial_t q, p_l) - \frac{c_r}{2\alpha} (\partial_t q, \text{tr}(\boldsymbol{\sigma}_n)) + (\nabla \cdot \mathbf{w}, q)\} dt = \int_0^T (g(t), q) dt + (p(0), q(0)) + (\boldsymbol{\sigma}(0), q(0)\mathbf{I}). \quad (\text{A.44})$$

Thus, when passing to the limit in (A.43), we obtain that  $p(0) = 0$  as well as  $\boldsymbol{\sigma}(0) = 0$  since  $p_l(0) \rightarrow 0$  in  $L^2(\Omega)$ ,  $\boldsymbol{\sigma}_n(0) \rightarrow 0$  in  $\mathbb{L}^2(\Omega)$  and as  $q(0)$  is chosen arbitrarily.

- To end the proof we analyze the uniqueness of the solution. As the equations are linear, it suffices to check that  $(\boldsymbol{\sigma}, p, \mathbf{w}, \mathbf{u}) := 0$ , for  $g = 0$  and  $\mathbf{f} = 0$ . We take  $\partial_t \boldsymbol{\sigma}$  as test function in (3.4a), then derivate (3.4d) with respect to  $t$  and take  $\mathbf{u}$  as test function. We take then  $p$  as test function in (3.4b) and  $\mathbf{w}$  as test function in (3.4c). We obtain

$$\frac{(c_0 + c_r)}{2} \frac{d}{dt} \|p\|^2 + (\mathbf{K}^{-1} \mathbf{w}, \mathbf{w}) = \frac{1}{2} \frac{d}{dt} \|\boldsymbol{\sigma}\|_{\mathcal{A}}^2. \quad (\text{A.45})$$

Integrating with respect to  $t$ , we see that

$$\frac{(c_0 + c_r)}{2} \|p(t)\|^2 + \int_0^t (\mathbf{K}^{-1} \mathbf{w}, \mathbf{w}) dt = \frac{1}{2} \|\boldsymbol{\sigma}(t)\|_{\mathcal{A}}^2. \quad (\text{A.46})$$

Now, take  $\boldsymbol{\tau} = \boldsymbol{\sigma}$  in (3.4a) and  $\mathbf{z} = \mathbf{u}$  in (3.4d) and summing, we obtain:

$$(\mathcal{A}\boldsymbol{\sigma}, \boldsymbol{\sigma}) = -\frac{c_r}{2\alpha} (\mathbf{I}p, \boldsymbol{\sigma}). \quad (\text{A.47})$$

Applying the Cauchy-Schwarz and Young's inequality leads to

$$\|\boldsymbol{\sigma}\|_{\mathcal{A}}^2 \leq C_r \|p\|^2. \quad (\text{A.48})$$

Combine (A.46) and (A.48) and using the properties of  $\mathbf{K}$  to see

$$\frac{c_0}{2} \|p(t)\|^2 + \int_0^t (\mathbf{K}^{-1} \mathbf{w}, \mathbf{w}) dt \leq 0, \quad (\text{A.49})$$

and then that  $p = 0$  and  $\mathbf{w} = 0$ . From (A.48), we deduce that  $\boldsymbol{\sigma} = 0$ . It remains to see that  $\mathbf{u} = 0$ . We know that for any given  $t \in (0, T)$ , there exist  $\tilde{\mathbf{e}}_{\boldsymbol{\sigma}}(\cdot, t) \in \mathbb{H}^1(\Omega) \cap \Sigma_n$  such that  $-\nabla \cdot \tilde{\mathbf{e}}_{\boldsymbol{\sigma}}(\cdot, t) = \mathbf{u}(\cdot, t)$  and  $\|\tilde{\mathbf{e}}_{\boldsymbol{\sigma}}\|_{H_0^1(\Omega)} \leq C \|\mathbf{u}\|$ . From (3.4a), we get

$$\begin{aligned} \|\mathbf{u}\|^2 &= -(\mathbf{u}, \nabla \cdot \tilde{\mathbf{e}}_{\boldsymbol{\sigma}}) = (\mathcal{A}\boldsymbol{\sigma}, \tilde{\mathbf{e}}_{\boldsymbol{\sigma}}) + \frac{c_r}{2\alpha} (\mathbf{I}p, \tilde{\mathbf{e}}_{\boldsymbol{\sigma}}), \\ &\leq C (\|\boldsymbol{\sigma}\| + \|p\|) \|\tilde{\mathbf{e}}_{\boldsymbol{\sigma}}\|, \\ &\leq C (\|\boldsymbol{\sigma}\| + \|p\|) \|\mathbf{u}\|. \end{aligned}$$

Therefore,

$$\|\mathbf{u}\| \leq C (\|\boldsymbol{\sigma}\| + \|p\|), \quad (\text{A.50})$$

and therefore  $\mathbf{u} = 0$  for a.e.  $t \in (0, T)$ .

□



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