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# WAVENUMBER EXPLICIT CONVERGENCE ANALYSIS FOR FINITE ELEMENT DISCRETIZATIONS OF GENERAL WAVE PROPAGATION PROBLEMS

T. CHAUMONT-FRELET AND S. NICAISE

**ABSTRACT.** We analyze the convergence of finite element discretizations of time-harmonic wave propagation problems. We propose a general methodology to derive stability conditions and error estimates that are explicit with respect to the wavenumber. This methodology is formally based on an expansion of the solution in powers of  $k$ , which permits to split the solution into a regular, but oscillating part, and another component that is rough, but behaves nicely when the wavenumber increases. The method is developed in its full generality and is illustrated by two particular cases: the elastic and convected sound waves. Numerical experiments are provided which confirm that the stability conditions and error estimates are sharp.

## 1. INTRODUCTION

The propagation of waves is used in a wide range of applications including resource prospection, radar, and medical imaging [6, 9]. These applications motivate the development of efficient discretization techniques to approximate the solutions to wave propagation problems in an accurate and robust fashion.

In this work, we focus on time-harmonic wave propagation problems. A number of discretization techniques are available to approximate their solutions, but for the sake of simplicity, we only consider finite element methods here. Nevertheless, a great part of the material we present shall be useful in the understanding of other approximation strategies, such as finite difference or integral equation approaches.

The so-called “pollution effect” is a key concept to understand the performance and limitations of finite element discretizations [17, 18]. When the number of wavelength inside the propagation domain is important, the numerical solution is only meaningful under restrictive conditions on the mesh size. This effect is manifested by a gap between the error of the best approximation the finite element scheme is capable of (e.g. the interpolant) and the error of the numerical solution that is actually produced. This gap becomes more important as the frequency increases, unless additional discretization points per wavelength are employed. As a result, finite element methods require a large number

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of degrees of freedom to produce accurate solution of high-frequency problems, in which the number of wavelength inside the domain is important.

As the pollution effect is characterized as a gap between the best approximation error, and the finite element one, it is related to a lack of stability of the finite element scheme. This is typical of wave problems, since the associated sesquilinear forms are not coercive. Because of this lack of coercivity, the quasi-optimality of the finite element solution in the energy norm is not guaranteed for arbitrary meshes. Actually, it is shown in the literature that quasi-optimality of the scheme is achieved only in an ‘‘asymptotic range’’ (i.e. for small enough mesh sizes) that depends on the frequency and the discretization order.

The behaviour of the asymptotic range with respect to the frequency, the mesh size, and the discretization order is the key to understand the efficiency of a finite element method. For the acoustic Helmholtz equation in smooth domains, the asymptotic range for  $hp$ -finite element methods has been characterized in a sequence of papers by J.M. Melenk and collaborators [25, 26, 13]. The main result states that if

$$(1) \quad \frac{kh}{p} \leq C_1, \quad k \left( \frac{kh}{p} \right)^p \leq C_2,$$

then the finite element solution is quasi-optimal. In (1),  $k$  is the wavenumber,  $h$  is the mesh size,  $p$  is the discretization order and  $C_1$  and  $C_2$  are two constant independent of these parameters.

When considering a fixed  $p$ , we deduce from (1) that the condition  $k^{p+1}h^p \leq C$  must be fulfilled to preserve quasi-optimality. In other words, the asymptotic range in which the finite element solution is quasi-optimal is characterized by

$$(2) \quad h \leq h^*(k) = Ck^{-1-1/p}.$$

We also see that the number of element per wavelength  $(kh)^{-1}$  must be increased for increasing frequencies since (1) implies that

$$(3) \quad (kh)^{-1} \geq Ck^{1/p}.$$

In addition, since the constant  $C_1$  and  $C_2$  are independent of  $p$ , stability conditions for varying  $p$  are also possible. Namely, stability is ensured if

$$(4) \quad \frac{kh}{p} = C_3, \quad p \geq C_4 \log k,$$

which means that the number of discretization points per wavelength  $p/(kh)$  is kept constant while the discretization order increases logarithmically with the wavenumber.

The main ingredient to obtain (1) is a splitting of the solution to the Helmholtz equation into a regular but oscillating part, and a rough component that behaves ‘‘nicely’’ for large frequencies. This splitting is based on a filtering of the datum in the Fourier domain, and treating the low and high frequency contents with distinct tools.

While this analysis is sharp and valid for arbitrary order  $p$ , it is currently limited to the acoustic Helmholtz equation in an homogeneous and analytic domain. Our main contribution is to propose another splitting approach to provide (1). Such a splitting is defined using a sequence of solution of elliptic boundary value problems. This sequence

is defined recursively, and the iterates exhibit increasing regularity. The main advantage of this splitting is that it requires less restrictive assumptions on the domain and the wave operator. Indeed, the only important assumption we make is that the elliptic part of the wave operator satisfy a regularity shift (see Definition 1). As a result, as shown in Section 2, our result applies to a variety of wave propagation problems. We illustrate this in Sections 3 and 4 by applying our result to the particular cases of the elastodynamic system and the convected Helmholtz equation. In addition, we required less smoothness on  $\partial\Omega$  and handle more general boundary conditions than [25, 26, 13]. Finally, our proofs are rather elementary and short (as the core of the analysis holds in a few pages).

While our analysis applies to a wider range of problems, it is currently limited in the sense that the constants  $C_1$  and  $C_2$  we obtain in (1) depends on  $p$ . As a result, we recover the same result for a fixed  $p$ : in particular (2) and (3). However, our analysis does not cover varying  $p$ , and thus, a result like (4) is out of the scope of the present article. Nevertheless, we believe that tracking the dependence on  $p$  is not fundamentally impossible.

Our work is organized as follows. The splitting of the solution and the derivation of the stability result (1) are presented for a general wave operator in Section 2. This analysis is illustrated in Sections 3 and 4 by considering the elastodynamic system, and the convected Helmholtz equation. In both cases, numerical experiments are presented, that confirm that the theoretical results are sharp. In the conclusion, we state the main consequences of our results as well as possible extensions.

We finish this section with some notations used in the remainder of the paper. For a bounded domain  $D$ , the usual norm and semi-norm of  $H^s(D)$  ( $s \geq 0$ ) are denoted by  $\|\cdot\|_{s,D}$  and  $|\cdot|_{s,D}$ , respectively. Furthermore, the notation  $A \lesssim B$  (resp.  $A \gtrsim B$ ) means the existence of a positive constant  $C_1$  (resp.  $C_2$ ), which is independent of  $A$  and  $B$  such that  $A \leq C_1 B$  (resp.  $A \geq C_2 B$ ). The notation  $A \sim B$  means that  $A \lesssim B$  and  $A \gtrsim B$  hold simultaneously.

## 2. GENERAL SETTINGS

**2.1. Splitting of the solution.** In the following, we consider a domain  $\Omega \subset \mathbb{R}^N$  ( $N = 2$  or  $3$ ) and a general linear elliptic system  $\mathcal{L}_2$  of order 2 and size  $S \geq 1$  that acts on either scalar (if  $S = 1$ ) or vectorial functions (if  $S \geq 2$ ) of  $\Omega$ . We further suppose fixed two open disjoint parts  $\Gamma_A$  and  $\Gamma_D$  of the boundary  $\partial\Omega$  of  $\Omega$  such that

$$\partial\Omega = \bar{\Gamma}_A \cup \bar{\Gamma}_D.$$

We also suppose fixed two boundary operators  $\mathcal{B}_D$  and  $\mathcal{B}_A$  defined respectively on  $\Gamma_D$  and  $\Gamma_A$ ,  $\mathcal{B}_D$  is supposed to be the natural Dirichlet operator, while  $\mathcal{B}_A$  is supposed to be an operator of order 1 (typically the Neumann boundary operator associated with  $\mathcal{L}_2$  plus possibly a zero order term).

**Definition 1.** *We will say that the triplet  $(\mathcal{L}_2, \mathcal{B}_D, \mathcal{B}_A)$  satisfies the shift property at the order  $\gamma + 2$  for a natural number  $\gamma$  when, for  $0 \leq l \leq \gamma$ , if  $(\tilde{f}, \tilde{g}) \in H^l(\Omega)^S \times H^{l+1/2}(\Gamma_A)^S$ ,*

then there exists a unique solution  $\tilde{u} \in H^{l+2}(\Omega)^S$  of

$$(5) \quad \begin{cases} \mathcal{L}_2 \tilde{u} = \tilde{f} & \text{in } \Omega, \\ \mathcal{B}_A \tilde{u} = \tilde{g} & \text{on } \Gamma_A, \\ \mathcal{B}_D \tilde{u} = 0 & \text{on } \Gamma_D, \end{cases}$$

such that

$$(6) \quad \|\tilde{u}\|_{l+2,\Omega} \lesssim \|\tilde{f}\|_{l,\Omega} + \|\tilde{g}\|_{l+1/2,\Gamma_A}.$$

We consider a time-harmonic wave propagation problem associated with the operator  $\mathcal{L}_2$ . We assume a complex wave number  $\kappa = k - i\delta$  where  $\delta$  is a fixed real number, and  $k \geq 1$  can be arbitrary large. We look for  $u \in H^2(\Omega)^S$  solution to

$$(7) \quad \begin{cases} -\kappa^2 \mathcal{L}_0 u - i\kappa \mathcal{L}_1 u + \mathcal{L}_2 u = f & \text{in } \Omega, \\ \mathcal{B}_A u = i\kappa A u & \text{on } \Gamma_A, \\ \mathcal{B}_D u = 0 & \text{on } \Gamma_D, \end{cases}$$

where  $f \in L^2(\Omega)^S$  is a given source term and  $A$  is a matrix-valued function (or simply a scalar function) with the regularity  $A \in C^{\gamma,1}(\Gamma_A, \mathcal{M}(\mathbb{R}^S))$  and that is independent of  $k$ . In addition,  $\mathcal{L}_0$  and  $\mathcal{L}_1$  are linear operators satisfying

$$(8) \quad \|\mathcal{L}_0 v\|_{l,\Omega} \lesssim \|v\|_{l,\Omega}, \quad \|\mathcal{L}_1 v\|_{l-1,\Omega} \lesssim \|v\|_{l,\Omega},$$

for  $l \in \{0, \dots, \gamma + 1\}$  and  $v \in H^l(\Omega)^S$ .

The operators  $\mathcal{L}_0$  and  $\mathcal{L}_1$  are introduced for the sake of generality, as they permit to recover a number of particular problems. For instance, taking  $\mathcal{L}_0 u = c^{-2}u$ ,  $\mathcal{L}_1 u = 0$ ,  $\mathcal{L}_2 u = -\Delta u$  and  $\delta = 0$ , we recover the standard Helmholtz equation in an heterogeneous medium with a smoothly varying sound speed  $c \in C^\gamma(\bar{\Omega})$ . Other examples are considered and analyzed in the forthcoming sections.

In most applications, system (7) is derived from a time-dependent wave equation by means of Fourier transform. In this case,  $\kappa = k$  is real and  $\delta = 0$ . Though some applications employ a complex wave number (obtained from a Laplace transform), we mostly introduce the case where  $\delta > 0$  as technicality to analyze problems in which  $\Gamma_D$  is empty, as depicted in Section 3.

**Definition 2.** *We will say that system (7) satisfies the  $k$ -stability property if it is well-posed in  $H^2(\Omega)^S$  for all  $k \geq 1$  and if there exists a constant  $\alpha \geq 0$  (independent of  $k$ ) such that for all  $f \in L^2(\Omega)^S$  the solution  $u$  satisfies*

$$(9) \quad k\|u\|_{0,\Omega} + |u|_{1,\Omega} + k^{-1}|u|_{2,\Omega} \lesssim k^\alpha \|f\|_{0,\Omega},$$

for all  $k \geq 1$ .

Before proving our main result, we record a proposition that is very simple, but useful in many places.

**Proposition 1.** *Assume that  $\mathcal{L}_2$  satisfies the shift property introduced in Definition 1 and that  $\tilde{u} \in H^1(\Omega)^S$  solves*

$$\begin{cases} \mathcal{L}_2 \tilde{u} = \mathcal{L}_0 \tilde{f} + i\mathcal{L}_1 \tilde{g} & \text{in } \Omega, \\ \mathcal{B}_N \tilde{u} = iA\tilde{g} & \text{on } \Gamma_A, \\ \mathcal{B}_D \tilde{u} = 0 & \text{on } \Gamma_D, \end{cases}$$

for some  $\tilde{f} \in H^l(\Omega)^S$  and  $\tilde{g} \in H^{l+1}(\Omega)^S$  with  $l \leq \gamma$ . Then  $\tilde{u} \in H^{l+2}(\Omega)^S$  and

$$(10) \quad \|\tilde{u}\|_{l+2,\Omega} \lesssim \|\tilde{f}\|_{l,\Omega} + \|\tilde{g}\|_{l+1,\Omega},$$

where the implicit constant depends on  $\|A\|_{\gamma+1,\infty,\Gamma_A}$ .

*Proof.* The proof simply relies on shift property (6). Indeed, we have

$$\begin{aligned} \|\tilde{u}\|_{l+2,\Omega} &\lesssim \|\mathcal{L}_0 \tilde{f} + i\mathcal{L}_1 \tilde{g}\|_{l,\Omega} + \|iA\tilde{g}\|_{l+1/2,\Gamma_A} \\ &\lesssim \|\mathcal{L}_0 \tilde{f}\|_{l,\Omega} + \|\mathcal{L}_1 \tilde{g}\|_{l,\Omega} + \|A\tilde{g}\|_{l+1/2,\Gamma_A} \\ &\lesssim \|\tilde{f}\|_{l,\Omega} + \|\tilde{g}\|_{l+1,\Omega} + \|A\|_{\gamma+1,\infty,\Gamma_A} \|\tilde{g}\|_{l+1/2,\Gamma_A} \\ &\lesssim \|\tilde{f}\|_{l,\Omega} + \|\tilde{g}\|_{l+1,\Omega} + \|A\|_{\gamma+1,\infty,\Gamma_A} \|\tilde{g}\|_{l+1,\Omega}, \\ &\lesssim \|\tilde{f}\|_{l,\Omega} + \|\tilde{g}\|_{l+1,\Omega}, \end{aligned}$$

where we have used a trace theorem and the fact that  $A$  does not depend on  $k$ .  $\square$

We are now ready to establish the main result of this section.

**Theorem 1.** *Assume that the assumptions of Definitions 1 and 2 are fulfilled. For  $f \in L^2(\Omega)^S$ , define the sequence  $u_j$  by*

$$(11) \quad \begin{cases} \mathcal{L}_2 u_0 = f & \text{in } \Omega, \\ \mathcal{B}_N u_0 = 0 & \text{on } \Gamma_A, \\ \mathcal{B}_D u_0 = 0 & \text{on } \Gamma_D, \end{cases}$$

$$(12) \quad \begin{cases} \mathcal{L}_2 u_1 = i\mathcal{L}_1 u_0 & \text{in } \Omega, \\ \mathcal{B}_N u_1 = iA u_0 & \text{on } \Gamma_A, \\ \mathcal{B}_D u_1 = 0 & \text{on } \Gamma_D, \end{cases}$$

and

$$(13) \quad \begin{cases} \mathcal{L}_2 u_j = \mathcal{L}_0 u_{j-2} + i\mathcal{L}_1 u_{j-1} & \text{in } \Omega, \\ \mathcal{B}_N u_j = iA u_{j-1} & \text{on } \Gamma_A, \\ \mathcal{B}_D u_j = 0 & \text{on } \Gamma_D, \end{cases}$$

for  $j \geq 2$ .

Then, we have  $u_j \in H^{j+2}(\Omega)^S$  with

$$(14) \quad \|u_j\|_{j+2,\Omega} \lesssim \|f\|_{0,\Omega},$$

for  $0 \leq j \leq \gamma$ . Furthermore, for any  $l \leq \gamma$ , if we denote by

$$r_l = u - \sum_{j=0}^{l-1} \kappa^j u_j,$$

where  $u$  is the solution to system (7) associated with  $f$ , then  $r_l \in H^{l+2}(\Omega)^S$  with

$$(15) \quad \|r_l\|_{l+2,\Omega} \lesssim k^{l+1+\alpha} \|f\|_{0,\Omega}.$$

**Remark 1.** The definition of the sequence  $u_j$  is based on the formal expansion

$$(16) \quad u = \sum_{j=0}^{+\infty} \kappa^j u_j,$$

in powers of  $\kappa$ . The recursive definition of the  $u_j$  in terms of  $u_{j-1}$  and  $u_{j-2}$  is obtained by injecting the formal expansion (16) into the boundary value problem (7) and identifying powers of  $\kappa$ .

*Proof.* The fact that each  $u_j \in H^{j+2}(\Omega)^S$  directly follows by applying the shift property (10) recursively in the definition of the sequence. It is also clear that (14) holds, since  $k$  is not involved in the definition of the sequence.

Simple computations show that the sequence  $r_l$ , for  $1 \leq l \leq \gamma$  recursively satisfies

$$(17) \quad \begin{cases} \mathcal{L}_2 r_1 = \kappa^2 \mathcal{L}_0 u + i\kappa \mathcal{L}_1 u & \text{in } \Omega, \\ \mathcal{B}_N r_1 = i\kappa A u & \text{on } \Gamma_A, \\ \mathcal{B}_D r_1 = 0 & \text{on } \Gamma_D, \end{cases}$$

$$(18) \quad \begin{cases} \mathcal{L}_2 r_2 = \kappa^2 \mathcal{L}_0 u + i\kappa \mathcal{L}_1 r_1 & \text{in } \Omega, \\ \mathcal{B}_N r_2 = i\kappa A r_1 & \text{on } \Gamma_A, \\ \mathcal{B}_D r_2 = 0 & \text{on } \Gamma_D, \end{cases}$$

and

$$(19) \quad \begin{cases} \mathcal{L}_2 r_l = \kappa^2 \mathcal{L}_0 r_{l-2} + i\kappa \mathcal{L}_1 r_{l-1} & \text{in } \Omega, \\ \mathcal{B}_N r_l = i\kappa A r_{l-1} & \text{on } \Gamma_A, \\ \mathcal{B}_D r_l = 0 & \text{on } \Gamma_D, \end{cases}$$

for  $l \geq 3$ .

Hence we apply shift property (10) to system (17), so that

$$\|r_1\|_{3,\Omega} \lesssim k^2 \|u\|_{1,\Omega} + k \|u\|_{2,\Omega}.$$

Then, as a result, we obtain

$$(20) \quad \|r_1\|_{3,\Omega} \lesssim k^{2+\alpha} \|f\|_{0,\Omega}$$

by using (9). We thus have proved (15) for  $l = 1$ .

The case  $l = 2$  also follows with the use of (9) and (10). Indeed, since we already proved (20), we have

$$\|r_2\|_{2,\Omega} \lesssim k^2 \|u\|_{2,\Omega} + k \|r_1\|_{3,\Omega} \lesssim k^{3+\alpha} \|f\|_{0,\Omega}.$$

Estimate (15) being established for the case  $l = 1$  and 2. We can obtain the general case by induction. Thus, assume that (15) holds up to some  $n - 1$  with  $3 \leq n \leq \gamma$ . Applying the shift property (10) to (19) and the inductive assumption, we have

$$\|r_n\|_{n+2} \lesssim k^2 \|r_{n-2}\|_{n,\Omega} + k \|r_{n-1}\|_{n+1,\Omega} \lesssim k^{n+1+\alpha} \|f\|_{0,\Omega},$$

which is (15) for  $l = n$ .  $\square$

**2.2. Application to finite element discretizations.** We take advantage of the splitting proposed in Theorem 1 to derive stability conditions and error-estimates for finite element discretizations of system (7).

First we recall the variational setting of problem (7). For that purpose, we introduce the space

$$H_{\Gamma_D}^1(\Omega) = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_D\},$$

equipped with the norm of  $H^1(\Omega)$ . As  $\delta$  is fixed, for each  $k \geq 1$ , we consider a sesquilinear form  $b_k : H_{\Gamma_D}^1(\Omega)^S \times H_{\Gamma_D}^1(\Omega)^S \rightarrow \mathbb{C}$  that represents a variational formulation of (7), so that its solution is characterized as

$$b_k(u, v) = (f, v), \quad \forall v \in H_{\Gamma_D}^1(\Omega)^S.$$

Let us denote by  $\mathcal{P}_k$  and  $\mathcal{P}_k^*$  the linear operators from  $H_{\Gamma_D}^1(\Omega)^S$  to  $(H_{\Gamma_D}^1(\Omega)^S)'$  defined by

$$\langle \mathcal{P}_k u, v \rangle_{-1,1} = b_k(u, v) = \langle u, \mathcal{P}_k^* v \rangle_{1,-1}, \quad \forall u, v \in H_{\Gamma_D}^1(\Omega)^S,$$

where the duality pairing  $\langle \cdot, \cdot \rangle_{-1,1}$  (resp.  $\langle \cdot, \cdot \rangle_{1,-1}$ ) means the one between  $(H_{\Gamma_D}^1(\Omega)^S)'$  and  $H_{\Gamma_D}^1(\Omega)^S$  (resp.  $H_{\Gamma_D}^1(\Omega)^S$  and  $(H_{\Gamma_D}^1(\Omega)^S)'$ ).

Throughout this section, it will be convenient to consider the weighted norm

$$\|v\|^2 = k^2 \|v\|_{0,\Omega}^2 + |v|_{1,\Omega}^2, \quad v \in H^1(\Omega),$$

that is equivalent to the natural  $H^1(\Omega)$  norm. In view of (9), the  $\|\cdot\|$ -norm has the advantage to “balance” the  $L^2$  and  $H^1$  terms of the  $H^1(\Omega)$  norm.

**Assumption 1.** *We will assume that the sesquilinear form  $b_k$  is continuous, in the sense that*

$$(21) \quad |b_k(u, v)| \lesssim \|u\| \|v\|, \quad \forall u, v \in H_{\Gamma_D}^1(\Omega)^S,$$

and satisfies the Gårding inequality

$$(22) \quad \operatorname{Re} b_k(u, u) \gtrsim |u|_{1,\Omega}^2 - k^2 \|u\|_{0,\Omega}^2, \quad \forall u \in H_{\Gamma_D}^1(\Omega)^S.$$

We look for a finite element approximation  $u_{h,p}$  to  $u$ . To this end, we consider a family of meshes  $\{\mathcal{T}_h\}_h$  of  $\Omega$ , where each mesh is made of tetrahedral (or triangular) elements  $K$ . To simplify the analysis, we assume that the boundary of  $\Omega$  is exactly triangulated, and therefore, we consider curved Lagrange finite elements [4]. Also, for each element  $K$ , we denote by  $\mathcal{F}_K$  the mapping taking the reference element  $\hat{K}$  to  $K$ .

Then, for all  $p \leq \gamma + 1$ , the finite element approximation space  $V_{h,p}$  is defined as

$$V_{h,p} = \left\{ v_{h,p} \in H_{\Gamma_D}^1(\Omega)^S \mid v_{h,p}|_K \circ \mathcal{F}_K^{-1} \in \mathbb{P}_p(\hat{K})^S \quad \forall K \in \mathcal{T}_h \right\},$$



where  $\mathbb{P}_p(\hat{K})$  stands for the set of polynomials of total degree less than or equal to  $p$ . At that point, a finite element approximation of  $u$  is obtained by looking for  $u_{h,p} \in V_{h,p}$  such that

$$(23) \quad b(u_{h,p}, v_{h,p}) = (f, v_{h,p}), \quad \forall v_{h,p} \in V_{h,p}.$$

**Remark 2.** *In actual applications, exact triangulations are seldom used, as they are difficult to build (see §6 of [4] in particular). Instead, isoparametric meshes, in which the non-affine mappings  $\mathcal{F}_K$  are assumed to be a polynomial mapping of order  $p$  are usually preferred as they deliver the same orders of convergence. However, the resulting finite element spaces are non-conforming, since the boundary conditions can not be exactly embedded. For the sake of simplicity, we shall focus on the case of exact triangulations with general element mappings, and omit the analysis of the “variational crime” caused by isoparametric elements. The authors believe that similar results as presented here can be obtained for isoparametric elements, at the price of additional complexities in the proofs.*

In the rest of this section, we only consider meshes that have “sufficiently many” elements per wavelength. We thus make the assumption that

$$(24) \quad kh \lesssim 1.$$

This assumption is fairly general, and only rules out meshes that would produce inaccurate solutions. As a result, it is usually satisfied in applications. In addition, the convergence analysis leads to conditions on  $h$  that are actually more restrictive than (24).

We assume that the considered family of meshes is regular of order  $p$  in the sense given in [4]. Roughly speaking, it consists in assuming that each element  $K$  of each mesh  $\mathcal{T}_h$  is obtained from a single reference element through a mapping  $\mathcal{F}_K$  that is a “small perturbation” of an affine mapping  $\tilde{F}_K$ . In addition, the “perturbations” are assumed to be smooth, and satisfy some bounds on their gradient uniformly in  $h$ .

Under the assumption that the family of mesh is regular, for each  $v \in H^{l+1}(\Omega)^S \cap H_{\Gamma_D}^1(\Omega)^S$  ( $0 \leq l \leq p$ ), there exists an element  $\mathcal{I}_{h,p}v \in V_{h,p}$  such that

$$(25) \quad |v - \mathcal{I}_{h,p}v|_{j,\Omega} \lesssim h^{l+1-j} \|v\|_{l+1,\Omega}, \quad (0 \leq j \leq l).$$

We refer the reader to [4] and in particular to Corollary 5.2 for more details.

**Proposition 2.** *Assume that  $v \in H^{l+1}(\Omega)^S$  with  $1 \leq l \leq p$ , then there exists an element  $\mathcal{I}_{h,p}v \in V_{h,p}$  such that*

$$(26) \quad |||v - \mathcal{I}_{h,p}v||| \lesssim h^l \|v\|_{l+1,\Omega}.$$

*Proof.* The proof is a direct consequence of the assumption that  $kh \lesssim 1$  and estimate (25). Indeed, we have

$$\begin{aligned} |||v - \mathcal{I}_{h,p}v||| &\lesssim k \|v - \mathcal{I}_{h,p}v\|_{0,\Omega} + |v - \mathcal{I}_{h,p}v|_{1,\Omega} \\ &\lesssim kh^{l+1} \|v\|_{l+1,\Omega} + h^l \|v\|_{l+1,\Omega} \\ &\lesssim (1 + kh) h^l \|v\|_{l+1,\Omega}. \end{aligned}$$

□

The main result of this section is an asymptotic error estimate for the finite element solution  $u_{h,p}$  that is explicit in terms of  $h, p$  and  $k$ . The main steps to obtain this estimate rely on slight modifications of the so-called Schatz argument [30], as presented in [25, 26] for the acoustic Helmholtz equation. For the sake of completeness, we briefly reproduce this argument with slight variations for the case of our general settings in Lemmas 1 and 2.

**Lemma 1.** *Assume that the boundary value problem associated with the operator  $\mathcal{P}_k^*$  can be decomposed into system (7) in such a way that the assumptions from Definitions 1 and 2 hold. Then, for  $g \in L^2(\Omega)^S$ , there exists a unique  $s^*(g) \in H_{\Gamma_D}^1(\Omega)^S$  such that*

$$b_k(\phi, s^*(g)) = (\phi, g), \quad \forall \phi \in H_{\Gamma_D}^1(\Omega)^S.$$

Furthermore, if we denote by

$$\eta_{h,p} = \sup_{g \in L^2(\Omega)} \frac{\|s^*(g) - \mathcal{I}_{h,p}s^*(g)\|}{\|g\|_{0,\Omega}},$$

then

$$\eta_{h,p} \lesssim h + k^{p+\alpha}h^p.$$

*Proof.* Let  $g \in L^2(\Omega)^S$ , the existence and uniqueness of  $s^*(g)$  directly follows from the assumption that  $\mathcal{P}_k^*$  satisfies the assumptions from Definition 2. In addition, we see that  $s^*(g)$  can be split up using the technique introduced in the previous section. In other words, we have

$$s^*(g) = \sum_{j=0}^{p-2} \kappa^j s_j + r_{p-1},$$

where  $s_j \in H^{j+2}(\Omega)^S$ ,  $r_{p-1} \in H^{p+1}(\Omega)^S$ , and

$$\|s_j\|_{j+2,\Omega} \lesssim \|g\|_{0,\Omega}, \quad \|r_{p-1}\|_{p+1,\Omega} \lesssim k^p \|g\|_{0,\Omega}.$$

Then, using (26) and recalling the assumption that  $kh \lesssim 1$ , we have

$$\begin{aligned} \|s^*(g) - \mathcal{I}_h s^*(g)\| &\lesssim \sum_{j=0}^{p-2} k^j \|s_j - \mathcal{I}_h s_j\| + \|r_{p-1} - \mathcal{I}_h r_{p-1}\| \\ &\lesssim \sum_{j=0}^{p-2} h^{j+1} k^j \|s_j\|_{j+2,\Omega} + h^p \|r_{p-1}\|_{p+1,\Omega} \\ &\lesssim \sum_{j=0}^{p-2} h(hk)^j \|s_j\|_{j+2,\Omega} + h^p \|r_{p-1}\|_{p+1,\Omega} \\ &\lesssim \sum_{j=0}^{p-2} h \|s_j\|_{j+2,\Omega} + h^p \|r_{p-1}\|_{p+1,\Omega} \\ &\lesssim (h + k^{p+\alpha}h^p) \|g\|_{0,\Omega}, \end{aligned}$$

and the result follows.  $\square$

**Lemma 2.** *Assume that  $b_k$  satisfies Assumption 1. Then, if  $k\eta_{h,p}$  is small enough, there exists a unique finite element solution  $u_{h,p} \in V_{h,p}$  to (23), and we have*

$$|||u - u_{h,p}||| \lesssim \inf_{\phi_{h,p} \in V_{h,p}} |||u - \phi_{h,p}|||.$$

*Proof.* Using (22), there exists a positive constant  $M$  independent of  $k$  and  $h$  such that

$$\begin{aligned} \operatorname{Re} b_k(u - u_{h,p}, u - u_{h,p}) &\geq M(-k^2 \|u - u_{h,p}\|_{0,\Omega}^2 + |u - u_{h,p}|_{1,\Omega}^2) \\ &\geq M |||u - u_{h,p}|||^2 - 2Mk^2 \|u - u_{h,p}\|_{0,\Omega}^2. \end{aligned}$$

If we define  $\xi = 2Mk^2 s^*(u - u_{h,p})$ , we have

$$|||u - u_{h,p}|||^2 \lesssim \operatorname{Re} b_k(u - u_{h,p}, u - u_{h,p} + \xi).$$

Then, by Galerkin orthogonality, we can replace  $u_{h,p}$  by  $\phi_{h,p} + \mu_{h,p}$  for arbitrary  $\phi_{h,p}, \mu_{h,p} \in V_{h,p}$ , and it follows that

$$\begin{aligned} (27) \operatorname{Re} b_k(u - u_{h,p}, u - u_{h,p} + \xi) &= \operatorname{Re} b_k(u - u_{h,p}, u - \phi_{h,p}) + \operatorname{Re} b_k(u - u_{h,p}, \xi - \mu_{h,p}) \\ &\lesssim |||u - u_{h,p}||| (|||u - \phi_{h,p}||| + |||\xi - \mu_{h,p}|||), \end{aligned}$$

where we have used (21). As (27) holds for arbitrary  $\phi_{h,p}, \mu_{h,p} \in V_{h,p}$ , we can take the infimum in the right hand side. In addition, by definition of  $\xi$  and  $\eta_{h,p}$ , we have that

$$\inf_{\mu_{h,p} \in V_{h,p}} |||\xi - \mu_{h,p}||| \lesssim k^2 \eta_{h,p} \|u - u_{h,p}\|_{0,\Omega} \lesssim k\eta_{h,p} |||u - u_{h,p}|||.$$

We thus obtain that

$$|||u - u_{h,p}||| \leq \rho \left( \inf_{\phi_{h,p} \in V_{h,p}} |||u - \phi_{h,p}||| + k\eta_{h,p} |||u - u_{h,p}||| \right),$$

where  $\rho$  is a constant independent of  $h$  and  $k$ , which we can rewrite as

$$(1 - k\eta_{h,p}\rho) |||u - u_{h,p}||| \lesssim \inf_{\phi_{h,p} \in V_{h,p}} |||u - \phi_{h,p}|||.$$

Hence, the main result follows assuming that

$$k\eta_{h,p} \leq \frac{1}{2\rho}.$$

□

We obtain an asymptotic error estimate as a direct consequence of Lemmas 1 and 2.

**Theorem 2.** *Assume that  $\mathcal{P}_k^*$  satisfies the assumptions of Definition 1, that  $\mathcal{P}_k$  and  $\mathcal{P}_k^*$  satisfy the assumptions of Definition 2, and that  $b_k$  satisfies Assumption 1. Let  $f \in L^2(\Omega)^S$ , then, if  $kh$  and  $k^{p+\alpha+1}h^p$  are small enough with  $p \leq \gamma + 1$ , there exists a unique finite element solution  $u_{h,p} \in V_{h,p}$  to (23), and it holds that*

$$(28) \quad |||u - u_{h,p}||| \lesssim \inf_{\phi_{h,p} \in V_{h,p}} |||u - \phi_{h,p}|||.$$

Furthermore, we have that

$$(29) \quad |||u - u_{h,p}||| \lesssim k^{1+\alpha} h \|f\|_{0,\Omega}.$$

**Remark 3.** *Stability estimate (28) is still valid if the assumption on the right-hand-side  $f \in L^2(\Omega)^S$  is weaken to  $f \in (H_{\Gamma_D}^1(\Omega)^S)'$ , however, error estimate (29) does not hold in this case.*

*Proof.* We obtain (28) as a direct consequence of Lemmas 1 and 2. Then, we deduce (29) from (28) by combining Proposition 2 and Definition 2:

$$|||u - u_{h,p}||| \lesssim |||u - \mathcal{I}_{h,p}u||| \lesssim h|u|_{2,\Omega} \lesssim k^{1+\alpha}h|f|_{0,\Omega}.$$

□

### 3. APPLICATION TO THE TIME-HARMONIC ELASTODYNAMIC SYSTEM

We consider time harmonic elastic wave propagation in an homogeneous (possibly) anisotropic medium  $\Omega \subset \mathbb{R}^3$ . We assume that the boundary of  $\Omega$  is sufficiently smooth, namely of class  $C^{\gamma+1,1}$ . For the sake of simplicity we restrict ourselves to the dimension 3, but it is easy to check that the results presented below remain valid in dimension 2.

The elasticity properties of the material are thus represented by a constant second-order tensor  $C = (c_{nmpq})_{n,m,p,q=1}^3$  satisfying the following symmetry and ellipticity properties

$$c_{nmpq} = c_{mnpq} = c_{pqnm}, \quad 1 \leq n, m, p, q \leq 3,$$

and

$$\sum_{n,m,p,q=1}^3 c_{nmpq} \varepsilon_{nm} \varepsilon_{pq} \geq \alpha \sum_{n,m=1}^3 |\varepsilon_{nm}|^2,$$

for all first-order symmetric tensor  $\varepsilon = (\varepsilon_{nm})_{n,m=1}^3$ ,  $\alpha$  being a strictly positive constant.

For a given displacement  $v : \Omega \rightarrow \mathbb{C}^3$ , the strain and stress tensor are respectively defined by

$$\varepsilon(u) = \frac{1}{2} (\nabla u + (\nabla u)^\top),$$

and

$$\sigma(u) = C\varepsilon(u).$$

The unknown  $u : \Omega \rightarrow \mathbb{C}^3$  represents the displacement under the action of a load term  $f : \Omega \rightarrow \mathbb{C}^3$  and are related trough the elastodynamic system

$$(30) \quad \begin{cases} -k^2 u - \nabla \cdot \sigma(u) = f & \text{in } \Omega, \\ \sigma(u)n - ikAu = 0 & \text{on } \Gamma_A, \\ u = 0 & \text{on } \Gamma_D, \end{cases}$$

where  $k$  is the (real) wavenumber,  $A : \Gamma_A \rightarrow \mathcal{M}(\mathbb{R}^3)$  is a matrix-valued functions of class  $C^{\gamma,1}$  that is supposed to be symmetric and uniformly positive definite, i.e., there exists a positive real number  $\beta$  such that

$$Y^\top A(x)Y \geq \beta|Y|^2, \quad \forall Y \in \mathbb{R}^3, \quad \forall x \in \Gamma_A.$$

Though our stability results and error estimates hold for arbitrary matrix-valued functions  $A$  satisfying the previous assumptions, it is clear that the efficiency of the absorbing boundary condition heavily depends on  $A$ . As a result, the matrix  $A$  must be carefully

designed in applications. For the special case of tilted transverse isotropic media, we refer the reader to [2, 5]. For the simplest case of an isotropic medium, a possible definition of  $A$  is given in [22, 8, 11], corresponding to the so-called Lysmer-Kuhlemeyer absorbing boundary condition. But in both cases, the symmetry and uniform positive definiteness of  $A$  are respected. In addition, since in the above examples,  $A$  is expressed in terms of normal and tangential vectors to the boundary, it is reasonable to assume that  $A$  has the regularity  $C^{\gamma,1}$ , since the boundary is  $C^{\gamma+1,1}$ .

**3.1. Well-posedness results.** We introduce the variational formulation of problem (30). It reads: find  $u \in H_{\Gamma_D}^1(\Omega)^3$  such that

$$(31) \quad b_k(u, v) = (f, v), \quad \forall v \in H_{\Gamma_D}^1(\Omega)^3.$$

where  $f \in L^2(\Omega)^3$  and

$$b_k(u, v) = -k^2(u, v) - ik\langle Au, v \rangle_{\Gamma_A} + (C\varepsilon(u), \varepsilon(v)).$$

We easily see that  $b_k$  satisfies Assumption 1. Indeed, continuity (21) follows from a trace theorem, and Gårding inequality (22) is a consequence of Korn's inequality.

We now check whether the shift property from Definition 1 is valid for the corresponding system (5), that here takes the form

$$(32) \quad \begin{cases} -\nabla \cdot \sigma(\tilde{u}) &= \tilde{f} & \text{in } \Omega, \\ \sigma(\tilde{u})n &= \tilde{g} & \text{on } \Gamma_A, \\ \tilde{u} &= 0 & \text{on } \Gamma_D. \end{cases}$$

Since this system corresponds to (30) with  $k = 0$ , its variational formulation is clearly (31) with  $k = 0$  and the right-hand-side

$$\langle \psi, v \rangle_{-1,1} = (\tilde{f}, v) + (\tilde{g}, v)_{\Gamma_A}, \quad \forall v \in H_{\Gamma_D}^1(\Omega)^3.$$

As we have just checked that  $b_0$  satisfies Gårding inequality (22), it will be coercive on  $H_{\Gamma_D}^1(\Omega)^3$  if  $\Gamma_D$  is non empty since Poincaré's inequality guarantees that

$$|w|_{1,\Omega}^2 \gtrsim \|w\|_{1,\Omega}^2, \quad \forall w \in H_{\Gamma_D}^1(\Omega).$$

In this case, problem (32) is well-posed in  $H_{\Gamma_D}^1(\Omega)^3$ .

The situation is more delicate if  $\Gamma_D$  is empty, because the well-posedness of (32) is subject to some orthogonality conditions on  $\tilde{f}, \tilde{g}$ . Hence in this case, we perform a change of parameter, namely we set  $\kappa = k - i$  whose real part is exactly  $k$ . Then we transform problem (30) into

$$(33) \quad \begin{cases} -(\kappa^2 + 2i\kappa - 1)u - \nabla \cdot \sigma(u) &= f & \text{in } \Omega, \\ \sigma(u)n - i(\kappa + i)Au &= 0 & \text{on } \partial\Omega, \end{cases}$$

that enters into our abstract setting with  $\delta = 1$ ,

$$\mathcal{L}_2 u = u - \nabla \cdot \sigma(u), \quad \mathcal{L}_1 u = 2u, \quad \mathcal{L}_0 = u,$$

and

$$\mathcal{B}_A = \sigma(u)n + Au.$$

In that situation, the corresponding system (5) is

$$(34) \quad \begin{cases} \tilde{u} - \nabla \cdot \sigma(\tilde{u}) = \tilde{f} & \text{in } \Omega, \\ \sigma(\tilde{u})n + A\tilde{u} = \tilde{g} & \text{on } \partial\Omega, \end{cases}$$

whose variational formulation is

$$b_i(\tilde{u}, v) = (\tilde{f}, v) + (\tilde{g}, v)_{\Gamma_A}, \forall v \in H^1(\Omega)^3,$$

where the sesquilinear form  $b_i$  is given by

$$b_i(u, v) = (u, v) + \langle Au, v \rangle_{\partial\Omega} + (C\varepsilon(u), \varepsilon(v)).$$

Since

$$b_i(u, u) = \|u\|_{0,\Omega}^2 + (C\varepsilon(u), \varepsilon(u)) + \langle Au, u \rangle_{\partial\Omega}$$

by our assumption on  $A$ , we clearly have

$$b_i(u, u) \geq \|u\|_{0,\Omega}^2 + (C\varepsilon(u), \varepsilon(u))$$

and by Korn's inequality we conclude that  $b_i$  is coercive on  $H^1(\Omega)^3$ . Consequently, problem (34) is well-posed in  $H^1(\Omega)^3$ .

Once the well-posedness in  $H^1$  is established, the shift property for both problems (32) and (34) holds owing to standard theory of elliptic systems (the reader will find more details in classical text book such as [27, 21], see also [10, §4.4]) if the boundary of  $\Omega$  is of class  $C^{\gamma+2}$  and if  $\bar{\Gamma}_A \cap \bar{\Gamma}_D = \emptyset$ . The shift property is also valid for less regular boundaries or if  $\bar{\Gamma}_A \cap \bar{\Gamma}_D$  is non empty, see for instance [23].

**3.2. Stability property.** The elasticity system also satisfies the  $k$ -stability property of Definition 2, as we show in the next proposition. Our proof is a simple consequence of a result obtained in [1] for the time-dependent elastodynamic system combined with the next result of functional analysis [29, 16]

**Lemma 3.** *A  $C_0$  semigroup  $(e^{t\mathcal{L}})_{t \geq 0}$  of contractions on a Hilbert space  $H$  is exponentially stable, i.e., satisfies*

$$\|e^{t\mathcal{L}}U_0\| \leq C e^{-\omega t} \|U_0\|_H, \quad \forall U_0 \in H, \quad \forall t \geq 0,$$

for some positive constants  $C$  and  $\omega$  if and only if

$$(35) \quad \rho(\mathcal{L}) \supset \{i\beta \mid \beta \in \mathbb{R}\} \equiv i\mathbb{R},$$

and

$$(36) \quad \sup_{\beta \in \mathbb{R}} \|(i\beta\mathbb{I} - \mathcal{L})^{-1}\| < \infty,$$

where  $\rho(\mathcal{L})$  denotes the resolvent set of the operator  $\mathcal{L}$ .

**Proposition 3.** *Assume that there exists  $x_0 \in \mathbb{R}^3$  such that*

$$(37) \quad (x - x_0) \cdot n(x) \leq 0, \forall x \in \Gamma_D,$$

$$(38) \quad (x - x_0) \cdot n(x) > 0, \forall x \in \bar{\Gamma}_A.$$

Let  $u \in H^1(\Omega)^3$  be a solution of (30) with a right-hand-side  $f \in L^2(\Omega)^3$ . Then  $u \in H^2(\Omega)^3$  and

$$(39) \quad k\|u\|_{0,\Omega} + |u|_{1,\Omega} + k^{-1}|u|_{2,\Omega} \lesssim \|f\|_{0,\Omega},$$

where the hidden constant depends on the domain  $\Omega$ , the elastic tensor  $C$  and the impedance matrix  $A$ , but is independent of  $k$ .

*Proof.* Under our assumptions, the combination of Theorem 1.1 of [20] with Theorem 5.3 of [28] (see also Theorem 1.3 of [1] and section 6.3 of [28]) implies that the time-dependent elastodynamic system

$$\begin{cases} u_{tt} - \nabla \cdot \sigma(u) = 0 & \text{in } \Omega \times (0, \infty), \\ \sigma(u)n - Au_t = 0 & \text{on } \Gamma_A \times (0, \infty), \\ u = 0 & \text{on } \Gamma_D \times (0, \infty), \\ u(\cdot, t=0) = u_0, u_t(\cdot, t=0) = u_1, \end{cases}$$

is exponential stable, in other words, its energy

$$\frac{1}{2} \left( \int_{\Omega} (|u_t(\cdot, t)|^2 + \sigma(u(\cdot, t)) : \varepsilon(u(\cdot, t))) dx \right)$$

decays exponentially (as  $t \rightarrow \infty$ ). In order to make use of Lemma 3, we use the standard reduction of order. Setting  $U = (u, u_t)^\top$ , this system is equivalent to the first order one

$$U_t = \mathcal{A}U \text{ in } \mathcal{H}, t > 0, U(t=0) = (u_0, u_1)^\top,$$

where the Hilbert space  $\mathcal{H}$  and the operator  $\mathcal{A}$  are defined as follows. Let us set

$$\mathcal{H} = H_{\Gamma_D}^1(\Omega)^3 \times L^2(\Omega)^3.$$

If  $\Gamma_D$  is non empty,  $\mathcal{H}$  is equipped with the norm

$$\|(u, v)^\top\|_{\mathcal{H}}^2 := \int_{\Omega} (|v|^2 + \sigma(u) : \varepsilon(\bar{u})) dx, \quad \forall (u, v)^\top \in \mathcal{H},$$

on the contrary  $\mathcal{H}$  is equipped with the norm

$$\|(u, v)^\top\|^2 := \|(u, v)^\top\|_{\mathcal{H}}^2 + \int_{\partial\Omega} Au \cdot \bar{u} d\sigma(x), \quad \forall (u, v)^\top \in \mathcal{H}.$$

In both cases, the norm property is based on Korn's inequality (see [12, Thm 3.3] or [7, Thm 2.1]). Now we define

$$D(\mathcal{A}) = \{(u, v)^\top \in \mathcal{H} : v \in H_{\Gamma_D}^1(\Omega)^3, u \in H^2(\Omega)^3 \text{ satisfying } \sigma(u)n + Av = 0 \text{ on } \Gamma_A\},$$

and set

$$\mathcal{A}(u, v)^\top = (v, \nabla \cdot \sigma(u)), \forall (u, v)^\top \in D(\mathcal{A}).$$

It is well know that  $\mathcal{A}$  generates a  $C_0$  semigroup  $(e^{t\mathcal{A}})_{t \geq 0}$  of contractions on  $\mathcal{H}$ , see for instance [20, p. 972] (as conditions (37)-(38) imply that  $\bar{\Gamma}_A \cap \bar{\Gamma}_D = \emptyset$ ). Hence by the exponential decay of the energy and Lemma 3, when  $\Gamma_D$  is non empty we deduce that

the resolvent of  $\mathcal{A}$  is uniformly bounded in the imaginary axis. In other words for all  $(g_1, f_1)^\top \in \mathcal{H}$  and all  $\xi \in \mathbb{R}$ , there exists a unique solution  $(u, v)^\top \in D(\mathcal{A})$  of

$$(40) \quad (i\xi\mathbb{I} - \mathcal{A})(u, v)^\top = (g_1, f_1)^\top,$$

with

$$(41) \quad \|(u, v)^\top\|_{\mathcal{H}} \lesssim \|(g_1, f_1)^\top\|_{\mathcal{H}}.$$

This equivalently means that  $(u, v)^\top \in D(\mathcal{A})$  satisfies

$$\begin{aligned} i\xi u - v &= g_1 \text{ in } \Omega, \\ i\xi v - \nabla \cdot \sigma(u) &= f_1 \text{ in } \Omega. \end{aligned}$$

The first identity means that

$$(42) \quad v = i\xi u - g_1 \text{ in } \Omega,$$

and eliminating  $v$  in the second equation we find that  $u$  is solution of

$$(43) \quad -\xi^2 u - \nabla \cdot \sigma(u) = f_1 + i\xi g_1 \text{ in } \Omega.$$

Reminding the boundary conditions satisfied by  $(u, v)^\top \in D(\mathcal{A})$ , namely  $u = 0$  on  $\Gamma_D$  and

$$(44) \quad \sigma(u)n + Av = 0 \text{ on } \Gamma_A,$$

and using (42) we find that

$$(45) \quad \sigma(u)n + i\xi Au = Ag_1 \text{ on } \Gamma_A.$$

Taking  $g_1 = 0$  and  $f_1 = f$ , we see that  $u$  is solution of (30) if we chose  $\xi = -k$ . Coming back to (41), using Korn's inequality and again using (42) with  $g_1 = 0$ , one deduces the estimate

$$(46) \quad \|u\|_{1,\Omega} + k\|u\|_{0,\Omega} \lesssim \|f\|_{0,\Omega}.$$

As  $u$  can be seen as the solution of

$$\begin{cases} -\nabla \cdot \sigma(u) &= f + k^2 u & \text{in } \Omega, \\ \sigma(u)n &= ikAu & \text{on } \Gamma_A, \\ u &= 0 & \text{on } \Gamma_D, \end{cases}$$

by the elliptic regularity we find that

$$\|u\|_{2,\Omega} \lesssim \|f\|_{0,\Omega} + k^2\|u\|_{0,\Omega} + k\|u\|_{1/2,\Gamma_A}.$$

By a trace theorem and the estimate (46), we conclude that

$$\|u\|_{2,\Omega} \lesssim k\|f\|_{0,\Omega}.$$

This estimate and (46) lead to (39).

The situation is more delicate in the case  $\Gamma_D = \emptyset$ . Indeed the problem is that the quantity  $\|\cdot\|_{\mathcal{H}}$  is no more a norm in  $\mathcal{H}$  (hence the exponential decay of  $\|(u(\cdot, t), v(\cdot, t))\|_{\mathcal{H}}$  does not allow to apply directly Lemma 3). Indeed in such a case

$$\|(u, v)^\top\|_{\mathcal{H}} = 0$$



if and only if  $v = 0$  and  $u \in \mathcal{R}$ , where  $\mathcal{R}$  is the set of rigid displacement given by

$$\mathcal{R} = \{\mathbf{a} \times x + \mathbf{b} : \mathbf{a}, \mathbf{b} \in \mathbb{C}^3\},$$

and is of dimension 6.

Hence we introduce the subspace

$$\mathcal{H}_0 = \{(u, v)^\top \in \mathcal{H} : \int_{\Omega} v \bar{r} dx + \int_{\Gamma} Au \bar{r} d\sigma(x) = 0, \forall r \in \mathcal{R}\}.$$

The introduction of this subspace has two main features. First

$$(\mathcal{R} \times \{0\}) \cap \mathcal{H}_0 = \{0\},$$

and hence the semi-norm  $\|\cdot\|_{\mathcal{H}}$  is a norm on  $\mathcal{H}_0$ . Indeed if  $(u, v)^\top \in (\mathcal{R} \times \{0\}) \cap \mathcal{H}_0$ , then  $v = 0$  and  $u \in H^1(\Omega)^3 \cap \mathcal{R}$  satisfies

$$\int_{\Gamma} Au \bar{r} d\sigma(x) = 0, \forall r \in \mathcal{R}.$$

In particular, it yields

$$\int_{\Gamma} Au \bar{u} d\sigma(x) = 0,$$

and consequently  $u = 0$ , since it is not difficult to check that the quantity  $(\int_{\Gamma} Au \bar{u} d\sigma(x))^{\frac{1}{2}}$  is a norm on  $\mathcal{R}$ . Second we see that

$$(47) \quad \mathcal{A}(u, v)^\top \in \mathcal{H}_0, \forall (u, v)^\top \in D(\mathcal{A}).$$

Indeed for  $(u, v)^\top \in D(\mathcal{A})$ , we have  $\mathcal{A}(u, v)^\top = (v, \nabla \cdot \sigma(u))$ , hence (47) holds if and only if

$$(48) \quad \int_{\Omega} \nabla \cdot \sigma(u) \bar{r} dx + \int_{\Gamma} Av \bar{r} d\sigma(x) = 0.$$

But according to Green's formula, we have

$$\int_{\Omega} \nabla \cdot \sigma(u) \bar{r} dx = \int_{\Gamma} \sigma(u) n \bar{r} d\sigma(x)$$

and using the boundary condition (44), we find that

$$\int_{\Omega} \nabla \cdot \sigma(u) \bar{r} dx = - \int_{\Gamma} Av \bar{r} d\sigma(x),$$

which proves (48).

This justifies the introduction of the operator  $\mathcal{A}_0$  that is the restriction of  $\mathcal{A}$  to  $\mathcal{H}_0$ . Trivially this restriction generates a  $C_0$  semigroup of contractions on  $\mathcal{H}_0$ . Hence by the exponential decay of the energy and Lemma 3 (applied to  $\mathcal{A}_0$  in  $\mathcal{H}_0$ ), we deduce that the resolvent of  $\mathcal{A}_0$  is uniformly bounded on the imaginary axis, in other words, for any  $(g_1, f_1)^\top \in \mathcal{H}_0$  and all  $\xi \in \mathbb{R}$ , there exists a unique solution  $(u, v)^\top \in D(\mathcal{A}_0)$  of (40) with

the estimate (41). As before, we deduce that  $v$  is given by (42) and that  $u$  is solution of (43) and (45). The problem is that the pair  $(g_1, f_1)$  fulfils the condition

$$(49) \quad \int_{\Omega} f_1 \bar{r} \, dx + \int_{\Gamma} A g_1 \bar{r} \, d\sigma(x) = 0, \forall r \in \mathcal{R},$$

and that we can no more choose  $g_1 = 0$ , since we want to take arbitrary right-hand side  $f$  in (30). The solution is to take

$$(50) \quad g_1 = \varphi \sum_{j=1}^6 \alpha_j r_j, f_1 = f - i\xi g_1,$$

with  $\alpha_j \in \mathbb{C}$  determined below and a fixed function  $\varphi \in \mathcal{D}(\Omega)$  such that  $\varphi = 1$  in an open non empty subset  $O$  of  $\Omega$  and  $\{r_j\}_{j=1}^6$  is a basis of  $\mathcal{R}$  such that

$$(51) \quad \int_{\Omega} \varphi r_j \bar{r}_k \, dx = \delta_{j,k}, \forall j, k = 1, \dots, 6.$$

This is possible because

$$\int_{\Omega} \varphi r \bar{s} \, dx$$

is an inner product on  $\mathcal{R}$ .

With these choices, if  $\xi \neq 0$ , we see that condition (49) holds if and only if (since  $g_1 = 0$  on  $\Gamma$ )

$$\int_{\Omega} (f - i\xi g_1) \bar{r}_j \, dx = 0, \forall j = 1, \dots, 6,$$

or equivalently using (51),

$$(52) \quad \alpha_j = -\frac{1}{i\xi} \int_{\Omega} f \bar{r}_j \, dx, \forall j = 1, \dots, 6.$$

By choosing  $\xi = -k$ , we then deduce that  $u$  is solution of (30). It then remains to prove (46). But reminding the estimate (41) and the choice (50), we have

$$\|(u, v)^{\top}\|_{\mathcal{H}} \lesssim (1+k) \sum_{j=1}^6 |\alpha_j| + \|f\|_{0,\Omega}.$$

But (52) and Cauchy-Schwarz's inequality yield

$$(53) \quad |\alpha_j| \lesssim \frac{1}{k} \|f\|_{0,\Omega}, \forall j = 1, \dots, 6,$$

and therefore the previous estimate becomes

$$\|(u, v)^{\top}\|_{\mathcal{H}} \lesssim \|f\|_{0,\Omega}.$$

This yields the estimates

$$(54) \quad \left( \int_{\Omega} \sigma(u) : \varepsilon(u) \, dx \right)^{\frac{1}{2}} \lesssim \|f\|_{0,\Omega},$$

$$(55) \quad \|v\|_{0,\Omega} \leq C_3 \|f\|_{0,\Omega},$$

and to catch the  $L^2$ -norm of  $u$ , we exploit the second estimate. Indeed (42) yields

$$\|u\|_{0,\Omega} \leq \frac{1}{k}(\|v\|_{0,\Omega} + \|g_1\|_{0,\Omega}),$$

hence by the previous estimate (55), (50) and (53), one gets

$$\|u\|_{0,\Omega} \lesssim \frac{1}{k}\|f\|_{0,\Omega}.$$

This estimate and (54) lead to (46). The estimate of the  $H^2$ -norm of  $u$  being the same than before, the proof is complete.  $\square$

**Remark 4.** *If  $\Gamma_D$  is non empty, the previous result directly furnishes the stability property for system (30). On the contrary if  $\Gamma_D$  is empty, it indirectly furnishes the stability property for system (33), since this last system is equivalent to (30).*

**3.3. Finite element discretization.** In order to use Theorem 2 for conforming finite element discretizations of the elastodynamic system described in Subsection 2.2, it remains to check that  $\mathcal{P}_k^*$  satisfies the assumptions from Definitions 1 and 2.

For that purpose, we characterize the boundary value problem associated with  $\mathcal{P}_k^*$ . Consider an arbitrary  $v \in H_{\Gamma_D}^1(\Omega)^3$  and let  $g = \mathcal{P}_k v \in (H_{\Gamma_D}^1(\Omega)^3)'$ . Then, for all  $\phi \in H_{\Gamma_D}^1(\Omega)^3$ , we have

$$\langle \phi, \mathcal{P}_k^* \bar{v} \rangle_{1,-1} = b_k(\phi, \bar{v}) = b_k(v, \bar{\phi}) = \langle \mathcal{P}_k v, \bar{\phi} \rangle_{-1,1} = \langle g, \bar{\phi} \rangle_{-1,1} = \langle \bar{g}, \phi \rangle_{-1,1},$$

so that  $\mathcal{P}_k \bar{v} = \bar{g}$ . Hence, since  $\mathcal{P}_k$  satisfies the assumptions from Definitions 1 and 2, so does  $\mathcal{P}_k^*$ .

As a result, Theorem 2 holds for the conforming finite element discretizations of the elastodynamic system.

**3.4. Numerical experiments.** We consider a toy experiment in the unit square  $\Omega = (0, 1)^2$  to illustrate our results. For additional experiments, we refer to the recent paper [11], where the performance of the Lysmer-Kuhlemeyer absorbing boundary condition is compared with a higher order absorbing boundary condition (that is not covered by our framework). We consider an isotropic elastic medium with Lamé parameter  $\lambda = \mu = 1$  and density  $\rho = 1$ . We set very simple boundary conditions corresponding to the case  $A = I$  and  $\Gamma_D = \emptyset$ . We consider an analytic solution that is the superposition of a P wave and a S wave travelling in different directions:

$$\begin{aligned} u(x) &= (\cos(\theta_p), \sin(\theta_p)) \frac{1}{k} \exp\left(\frac{ik}{\sqrt{\lambda + 2\mu}}(\cos(\theta_p)x_1 + \sin(\theta_p)x_2)\right) \\ &+ (-\sin(\theta_s), \cos(\theta_s)) \frac{1}{k} \exp\left(\frac{ik}{\sqrt{\mu}}(\cos(\theta_s)x_1 + \sin(\theta_s)x_2)\right), \end{aligned}$$

with  $\theta_p = \pi/3$  and  $\theta_s = -\pi/3$ .

Since these plane waves satisfy the homogeneous equation inside  $\Omega$ , we use inhomogeneous boundary condition to impose them as the solution. Hence, the problem read:

$$(56) \quad \begin{cases} -k^2 u - \nabla \cdot \sigma(u) = 0 & \text{in } \Omega, \\ \sigma(u)n - iku = g, \end{cases}$$

where

$$\sigma(u) = \lambda(\nabla \cdot u)I + \mu\varepsilon(u).$$

and  $g = \sigma(u)n - iku$ .

Even if the boundary of  $\Omega$  is not of class  $C^3$ , the shift property from Definition 1 holds with  $\gamma = 1$ . Indeed in such a situation, according to Theorem 4.1.1 from [14], the weak solution  $\tilde{u} \in H^1(\Omega)^2$  of

$$\begin{cases} -\nabla \cdot \sigma(\tilde{u}) = \tilde{f} & \text{in } \Omega, \\ \sigma(\tilde{u})n = \tilde{g}, \end{cases}$$

with  $\tilde{f} \in H^l(\Omega)^2$  and  $\tilde{g} \in H^{l+1/2}(\partial\Omega)^2$  with  $l = 0$  or  $1$  belongs to  $H^{l+2}(\Omega)^2$  if the transcendental equation

$$\sin^2\left(\frac{\alpha\pi}{2}\right) = \alpha^2.$$

has no root  $\alpha \in \mathbb{C}$  in the strip  $\text{Re } \alpha \in [0, l+1]$ . But according to Theorems 4.2.1 and 4.2.2 of [19], the strip  $\text{Re } \alpha \in [0, 2]$  is free of such a root and consequently the weak solution belongs to  $H^{l+2}(\Omega)^2$ .

We need to quantify the best approximation of the scheme. To this end, we will also solve the auxiliary problem to find  $w_{h,p} \in V_{h,p}$

$$k^2(w_{h,p}, \phi_{h,p}) + (\nabla w_{h,p}, \nabla \phi_{h,p}) = k^2(u, \phi_{h,p}) + (\nabla u, \nabla \phi_{h,p})$$

for all  $\phi_{h,p} \in V_{h,p}$ . One observes that this  $w_{h,p}$  satisfies

$$\| \|u - w_{h,p}\| \| = \min_{\phi_{h,p} \in V_{h,p}} \| \|u - \phi_{h,p}\| \|.$$

In essence, the main result of our theoretical analysis states that if  $k^{p+1}h^p \lesssim 1$  and  $p = 1$  or  $2$ , then

$$(57) \quad \| \|u - u_{h,p}\| \| \lesssim \| \|u - w_{h,p}\| \|.$$

With this experiment, we illustrate that this bound is sharp. To do so, we compute  $u_{h,p}$  and  $w_{h,p}$  for different values of  $h$  and  $k$ . More precisely, we first fix a value of  $k$ , and then compute the finite element solution and projection for decreasing values of  $h$ , as depicted on Figure 1 and 2. These figures illustrate that as expected, the projection  $w_{h,p}$  converges as  $h$  and  $h^4$  to the solution  $u$  for  $\mathbb{P}_1$  and  $\mathbb{P}_4$  elements respectively. In addition, the finite element solution  $u_{h,p}$  is almost as accurate as the projection  $w_{h,p}$  for small values of  $h$ . However, a gap is present for largest values of  $h$ . One also sees that this gap increases as  $k$  increases.

We proceed as follow to determine if (57) is sharp. For each  $k$ , we denote by  $h^*(k)$  the greatest value  $h_0$  such that

$$(58) \quad \| \|u - u_{h,p}\| \| \leq 2 \| \|u - w_{h,p}\| \|, \quad \forall h \leq h_0.$$

The value of  $h^*(k)$  for a given  $k$  is obtained by inspecting the corresponding convergence curve (see Figures 1 and 2). Condition (58) states that the finite element solution must be quasi optimal in the  $|||\cdot|||$  norm, uniformly in  $k$  (with the arbitrary constant 2).

The graph of  $h^*(k)$  is represented on Figure 3 for  $\mathbb{P}_1$  and  $\mathbb{P}_4$  elements. We observe that in both cases  $h^*(k) \sim k^{-1-1/p}$ , which is in accordance with (57). Indeed, it means that quasi-optimality in the sense of (58) is achieved under the condition that  $h \leq h^*(k) \sim k^{-1-1/p}$ , which is equivalent to  $k^{p+1}h^p \leq k^{p+1}[h^*(k)]^p \lesssim 1$ . We thus conclude that the stability condition presented in Theorem 2 is sharp.

Note that our experiments indicate that the first statement from Theorem 2 may remain valid for values of  $p$  larger than the theoretical one, that is here equal to 2.

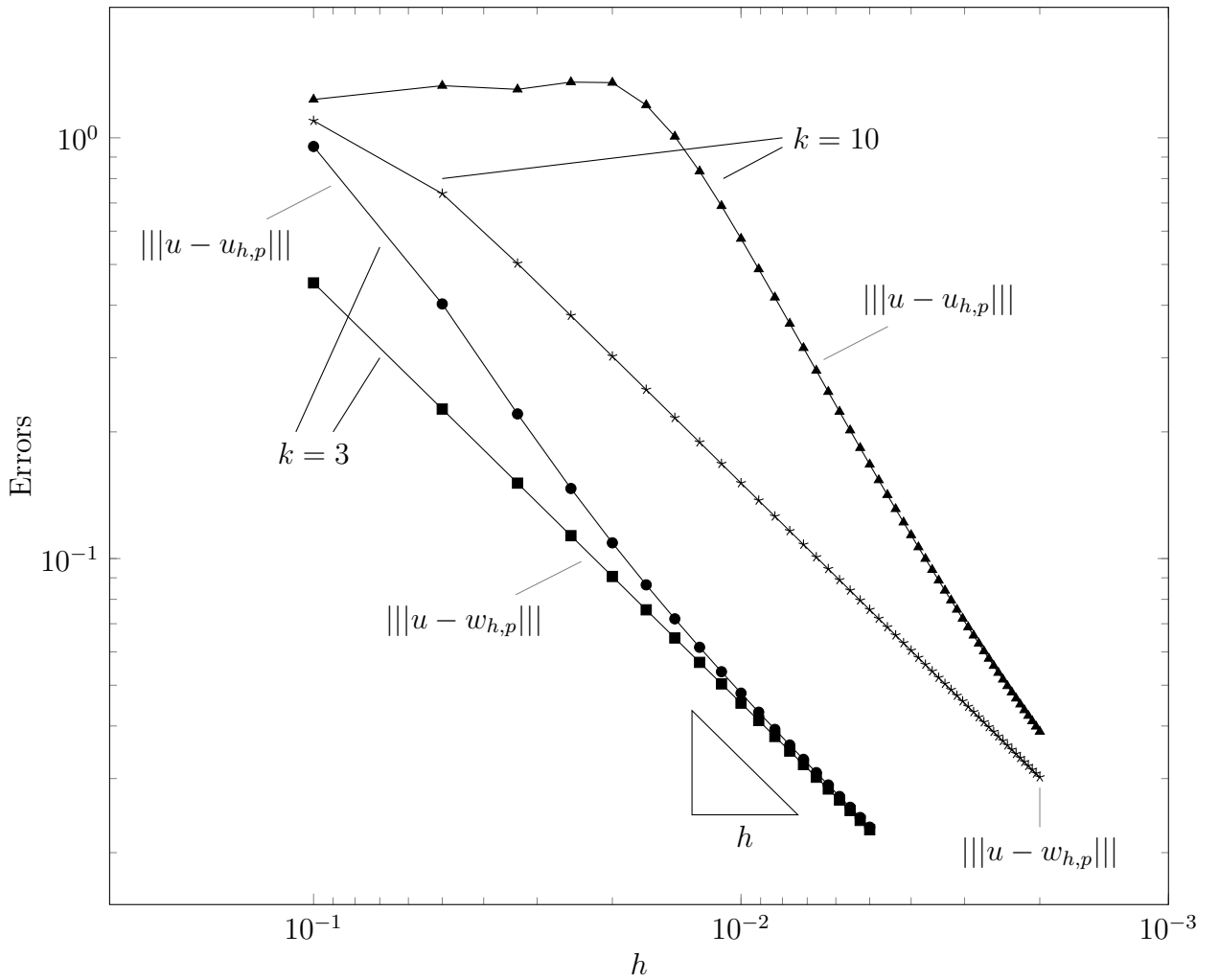
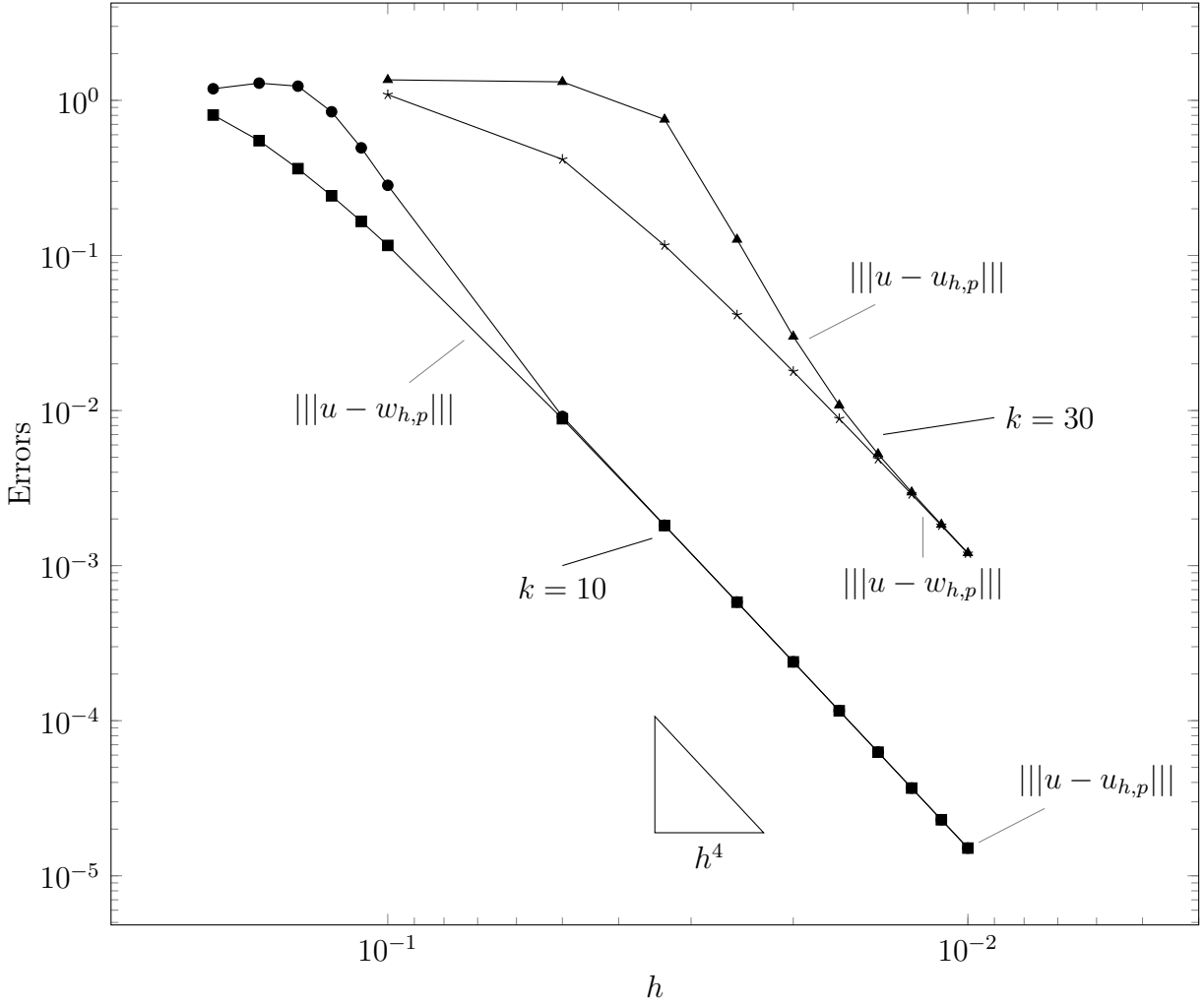


FIGURE 1. Convergence curves for  $\mathcal{P}_1$  elements


 FIGURE 2. Convergence curves for  $\mathcal{P}_4$  elements

#### 4. APPLICATION TO THE CONVECTED HELMHOLTZ EQUATION

We now considered the convected Helmholtz equation in a fluid with uniform horizontal motion [3]. The equation reads:

$$(59) \quad \begin{cases} -k^2 u - 2ikM \frac{\partial u}{\partial \mathbf{x}_1} - (1 - M^2) \frac{\partial^2 u}{\partial \mathbf{x}_1^2} - \frac{\partial^2 u}{\partial \mathbf{x}_2^2} = f & \text{in } \Omega, \\ (1 - M^2) \frac{\partial u}{\partial \mathbf{x}_1} \mathbf{n}_1 + \frac{\partial u}{\partial \mathbf{x}_2} \mathbf{n}_2 - ik(1 - M \mathbf{n}_1)u = 0 & \text{on } \Gamma_A, \\ u = 0 & \text{on } \Gamma_D, \end{cases}$$

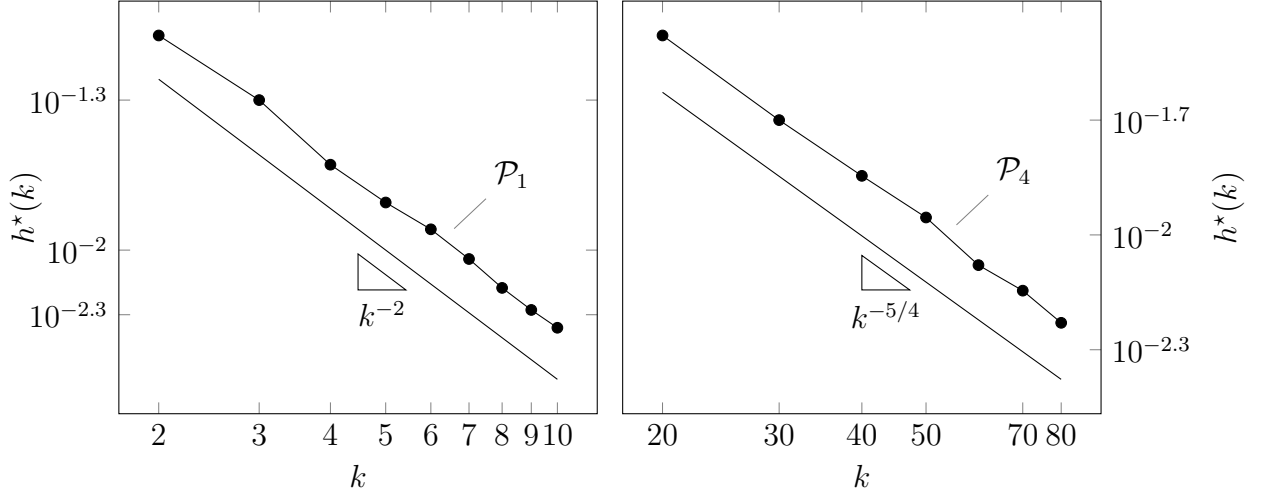


FIGURE 3. Elastic wave equation: asymptotic ranges  $h^*(k)$  for  $\mathcal{P}_1$  (left) and  $\mathcal{P}_4$  (right) elements

where  $M \in (-1, 1)$  is the Mach number,  $\Omega = (0, 1)^2$  is the unit square and  $\Gamma_D = (0, 1) \times \{0\}$  denotes its lower edge. The absorbing boundary condition on  $\Gamma_A = \partial\Omega \setminus \overline{\Gamma_D}$  is designed to perfectly filter incoming “plane wave” travelling normally to the boundary.

Problem (59) fit in our general framework with  $S = 1$ ,

$$\mathcal{L}_0 u = u, \quad \mathcal{L}_1 u = 2M \frac{\partial u}{\partial \mathbf{x}_1}, \quad \mathcal{L}_2 u = -(1 - M^2) \frac{\partial^2 u}{\partial \mathbf{x}_1} - \frac{\partial^2 u}{\partial \mathbf{x}_2},$$

and

$$\mathcal{B}_D u = u, \quad \mathcal{B}_A u = (1 - M^2) \frac{\partial u}{\partial \mathbf{x}_1} \mathbf{n}_1 + \frac{\partial u}{\partial \mathbf{x}_2} \mathbf{n}_2, \quad Au = (1 - M \mathbf{n}_1)u.$$

Since  $|M| < 1$ , it is clear that the triplet  $(\mathcal{L}_2, \mathcal{B}_A, \mathcal{B}_D)$  satisfies the shift property introduced in Definition 1. In addition, it is easily seen that  $\mathcal{L}_0$  and  $\mathcal{L}_1$  are consistent with (8). Thus, we only need to verify that the assumptions from Definition 2 hold. For that purpose, we perform a change of unknown in order to get back to a standard Helmholtz problem.

**Lemma 4.** *Assume that  $u \in H^1(\Omega)$  solves (59). Then the function  $\tilde{u} \in H^1(\Omega)$  defined as*

$$\tilde{u}(\mathbf{x}_1, \mathbf{x}_2) = u(\mathbf{x}_1, \frac{\mathbf{x}_2}{\beta}) e^{-\rho \mathbf{x}_1}$$

is solution to

$$\begin{cases} -\tilde{k}^2 \tilde{u} - \tilde{\Delta} \tilde{u} = \tilde{f} & \text{in } \tilde{\Omega} \\ \tilde{\nabla} u \cdot \tilde{\mathbf{n}} - i\tilde{k} \tilde{u} = 0 & \text{on } \tilde{\Gamma}_A \\ \tilde{u} = 0 & \text{on } \tilde{\Gamma}_D \end{cases}$$

where

$$\tilde{\Omega} = (0, 1) \times (0, \frac{1}{\beta}), \quad \tilde{\Gamma}_D = (0, 1) \times \{0\}, \quad \tilde{\Gamma}_A = \tilde{\Omega} \setminus \overline{\tilde{\Gamma}_D},$$

$$\tilde{f}(\mathbf{x}_1, \mathbf{x}_2) = f(\mathbf{x}_1, \frac{\mathbf{x}_2}{\beta})e^{-\rho\mathbf{x}_1}, \quad \tilde{k} = \frac{k}{1-M^2}$$

and

$$\rho = -\frac{ikM}{1-M^2}, \quad \beta = \sqrt{1-M^2}.$$

*Proof.* The proof relies on simple but tedious computations which simply amounts to compute the derivatives of  $\tilde{u}$  up to order 2 and arrange the results to obtain an Helmholtz equation. These computations are reported in Appendix A.  $\square$

**Remark 5.** *In practice, it is more convenient to solve directly problem (59) instead of its equivalent Helmholtz problem (in  $\tilde{u}$ ) since in this last one spurious oscillations might occur.*

**Proposition 4.** *Problem (59) admits a unique solution  $u \in H^1(\Omega)$  for all  $k \geq 1$ . In addition, shift property 1 is satisfied with  $\alpha = 0$ .*

*Proof.* Let us define  $\tilde{u} \in H^1(\tilde{\Omega})$  as in Lemma 4. Since  $\tilde{u}$  is defined as the solution of a standard Helmholtz problem, it is uniquely defined (see [15] or [24] for instance). In addition,  $\tilde{u}$  satisfies

$$(60) \quad k\|\tilde{u}\|_{0,\tilde{\Omega}} + |\tilde{u}|_{1,\tilde{\Omega}} + k^{-1}|\tilde{u}|_{2,\tilde{\Omega}} \lesssim \|\tilde{f}\|_{0,\tilde{\Omega}}.$$

Then, one easily checks that

$$\|u\|_{0,\Omega} \lesssim \|\tilde{u}\|_{0,\tilde{\Omega}}, \quad \|\tilde{f}\|_{0,\tilde{\Omega}} \lesssim \|f\|_{0,\Omega} \lesssim \|\tilde{f}\|_{0,\tilde{\Omega}},$$

so that

$$k\|u\|_{0,\Omega} \lesssim \|f\|_{0,\Omega}.$$

Furthermore, we remark that

$$|u|_{1,\Omega} \lesssim |\tilde{u}|_{1,\tilde{\Omega}} + k\|\tilde{u}\|_{0,\tilde{\Omega}},$$

hence using (60), we get

$$|u|_{1,\Omega} \lesssim \|f\|_{0,\Omega}.$$

Similarly, for the estimate of the  $H^2(\Omega)$  semi-norm, we notice that

$$|u|_{2,\Omega} \lesssim |\tilde{u}|_{2,\tilde{\Omega}} + k|\tilde{u}|_{1,\tilde{\Omega}} + k^2\|\tilde{u}\|_{0,\tilde{\Omega}},$$

and we conclude that

$$|u|_{2,\Omega} \lesssim k\|f\|_{0,\Omega},$$

again using (60).  $\square$



**4.1. Finite element discretization.** The sesquilinear form associated with the convected Helmholtz equation (59) reads:

$$\begin{aligned} b_k(u, v) &= -k^2(u, v) + (1 - M^2) \left( \frac{\partial u}{\partial \mathbf{x}_1}, \frac{\partial v}{\partial \mathbf{x}_1} \right) + \left( \frac{\partial u}{\partial \mathbf{x}_2}, \frac{\partial v}{\partial \mathbf{x}_2} \right) \\ &\quad - 2ikM \left( \frac{\partial u}{\partial \mathbf{x}_1}, v \right) - \langle ik(1 - M\mathbf{n}_1)u, v \rangle_{\partial\Omega} \end{aligned}$$

for all  $u, v \in H^1(\Omega)$ .

In order to apply Theorem 2, we need to identify the adjoint operator. To do so, we consider  $g \in L^2(\Omega)$  and assume that  $v \in H^1(\Omega)$  satisfies  $b(\phi, v) = (\phi, g)$  for all  $\phi \in H^1(\Omega)$ . We perform integration by parts on the left and side, and we observe that

$$\begin{cases} -k^2v - 2ikM \frac{\partial v}{\partial \mathbf{x}_1} - (1 - M^2) \frac{\partial^2 v}{\partial \mathbf{x}_1^2} - \frac{\partial^2 v}{\partial \mathbf{x}_2^2} = g & \text{in } \Omega, \\ (1 - M^2) \frac{\partial v}{\partial \mathbf{x}_1} \mathbf{n}_1 + \frac{\partial v}{\partial \mathbf{x}_1} \mathbf{n}_2 + ik(1 + M\mathbf{n}_1)v = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, we take the complex conjugate to get

$$\begin{cases} -k^2\bar{v} + 2ikM \frac{\partial \bar{v}}{\partial \mathbf{x}_1} - (1 - M^2) \frac{\partial^2 \bar{v}}{\partial \mathbf{x}_1^2} - \frac{\partial^2 \bar{v}}{\partial \mathbf{x}_2^2} = \bar{g} & \text{in } \Omega, \\ (1 - M^2) \frac{\partial \bar{v}}{\partial \mathbf{x}_1} \mathbf{n}_1 + \frac{\partial \bar{v}}{\partial \mathbf{x}_1} \mathbf{n}_2 - ik(1 + M\mathbf{n}_1)\bar{v} = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence,  $\bar{v}$  is solution to the direct problem with an opposite Mach number and  $\bar{g}$  as a right-hand-side. We thus see that both  $\mathcal{P}_k$  and  $\mathcal{P}_k^*$  satisfy the assumptions from Definitions 1 and 2 and therefore, Theorem 2 applies.

**4.2. Numerical experiments.** Here, we consider problem (59) with a Mach number  $M = 0.4$  and  $f = \mathbf{1}_{(1/2-1/16, 1/2+1/16)^2}$ . The problem is solved for various wavenumbers  $k$ .

We proceed as in Section 3.4 to assess the results of our theoretical analysis. As in this example the analytical solution to the problem is unknown, an ‘‘overkilling’’ finite element solution, computed on a very fine mesh, is used as reference instead. This accurate solution is also used to compute the best approximation  $w_{hp}$ .

The value  $h^*(k)$  designates the beginning of the asymptotic range for a given  $k$ , and is defined as in Section 3.4. Figure 5 shows the behaviour of  $h^*(k)$  with respect to  $k$  for  $\mathbb{P}_1$  and  $\mathbb{P}_2$  elements. In both case, we observe that  $h^*(k) \sim k^{-1-1/p}$ , which is accordance with the condition provided in Theorem 2 and confirm that this condition is sharp.

## 5. CONCLUSION

We have presented a general approach to split the solution of a wave propagation problem into a regular oscillating part and a  $H^2$  component whose norm can be controlled explicitly with respect to the wavenumber. For the case of the acoustic Helmholtz equation, this type of splitting is available in the literature [25, 26, 13]. However, our approach to build the splitting is different. The main advantage of our construction is that it is valid for a large

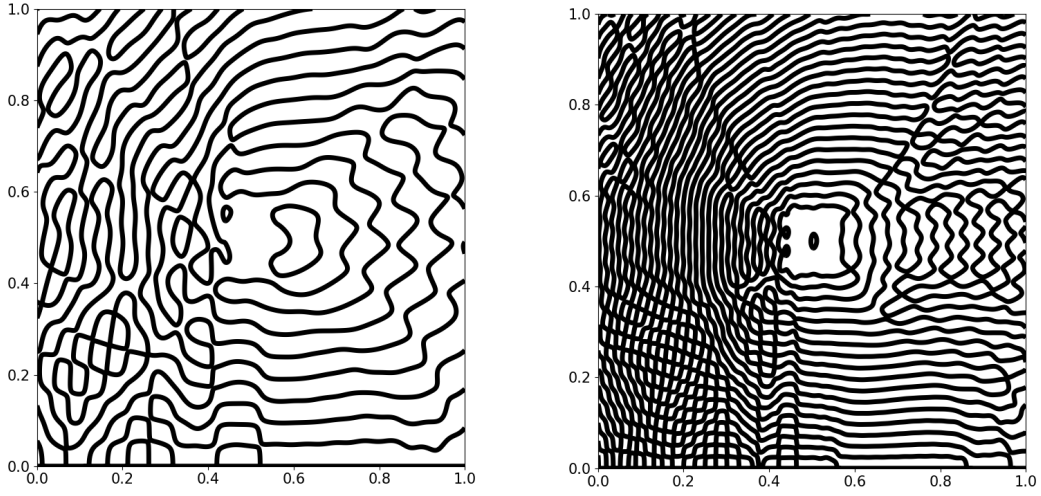


FIGURE 4. Zero-levelset curves of the real parts of solutions to the convected Helmholtz equation for  $k = 10$  (Left) and  $20$  (Right)

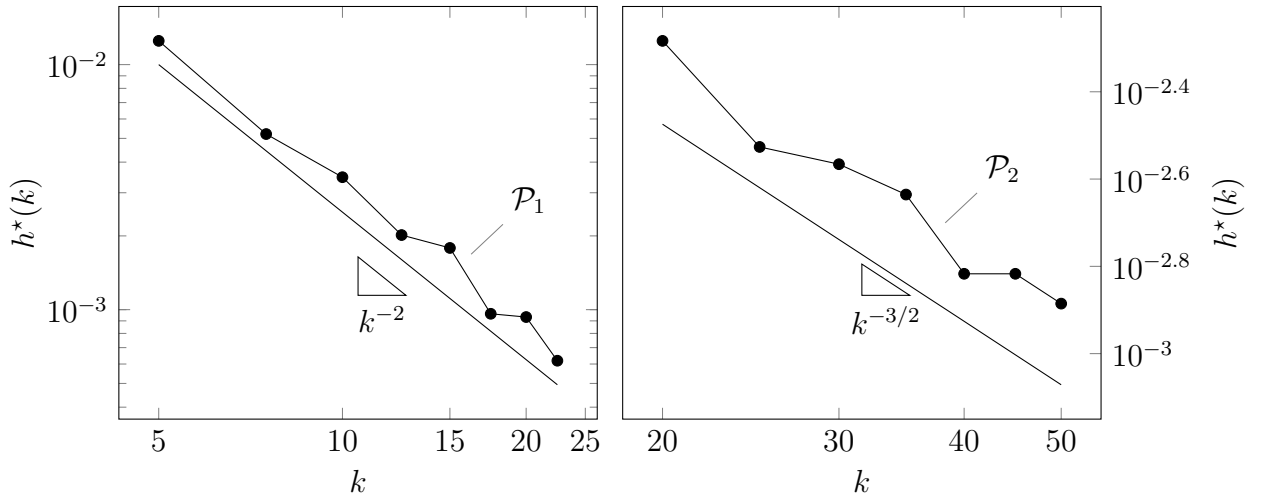


FIGURE 5. Convected Helmholtz equation: asymptotic ranges  $h^*(k)$  for  $\mathcal{P}_1$  (left) and  $\mathcal{P}_2$  (right) elements

class of wave propagation operators and for inhomogeneous and non analytic domains. We highlight this by considering the elastodynamic system and the convected Helmholtz equation.

We have taken advantage of the splitting to derive asymptotic error estimates for finite element discretizations. In this analysis, we consider finite element spaces of arbitrary high, but fixed, discretization order  $p$ . The main conclusion is that the finite element solution is

quasi-optimal under the condition that  $kh$  and  $k^{p+1}h^p$  are small enough, where  $k$  and  $h$  are the wavenumber and the mesh size. Such a result was already established in [25, 26, 13] for the case of the acoustic Helmholtz equation in homogeneous media, so that our result is a generalization to more complex wave propagation problems.

We performed numerical experiments for the case of the elastodynamic system and the convected Helmholtz equation. These examples indicate that our theoretical analysis is sharp.

The analysis presented in this paper focuses on finite element discretization of fixed discretization order  $p$ . In future works, we would like to explicitly track the  $p$  dependence in the constants to consider fully  $hp$  finite element methods. Furthermore, though we have focused on smooth domains for the case of simplicity, the method is general and can apply in more general settings. Possible extensions thus include domains with corners, or wave operators with discontinuous coefficients.

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#### APPENDIX A. PROOF OF LEMMA 4

Let  $v(\mathbf{x}_1, \mathbf{x}_2) = ue^{-\rho\mathbf{x}_1}$ , so that  $u(\mathbf{x}_1, \mathbf{x}_2) = v(\mathbf{x}_1, \mathbf{x}_2)e^{\rho\mathbf{x}_1}$ . By elementary calculations, we show that

$$\begin{aligned} -k^2u - 2ikM\frac{\partial u}{\partial \mathbf{x}_1} - (1 - M^2)\frac{\partial^2 u}{\partial \mathbf{x}_1^2} - \frac{\partial^2 u}{\partial \mathbf{x}_2^2} &= - (k^2 + 2ikM\rho + (1 - M^2)\rho^2)ve^{\rho\mathbf{x}_1} \\ &\quad - (2ikM + 2(1 - M)^2\rho)\frac{\partial v}{\partial \mathbf{x}_1}e^{\rho\mathbf{x}_1} \\ &\quad - (1 - M^2)\frac{\partial^2 v}{\partial \mathbf{x}_1^2}e^{\rho\mathbf{x}_1} - \frac{\partial^2 v}{\partial \mathbf{x}_2^2}e^{\rho\mathbf{x}_1}, \end{aligned}$$

so that

$$\begin{aligned} - (k^2 + 2ikM\rho + (1 - M^2)\rho^2)v - 2(ikM + (1 - M)^2\rho)\frac{\partial v}{\partial \mathbf{x}_1} - (1 - M^2)\frac{\partial^2 v}{\partial \mathbf{x}_1^2} - \frac{\partial^2 v}{\partial \mathbf{x}_2^2} \\ = fe^{-\rho\mathbf{x}_1}. \end{aligned}$$

We now select the value of  $\rho$  to guarantee that

$$ikM + (1 - M^2)\rho = 0,$$

hence we take  $\rho = -ikM/(1 - M^2)$ . With this choice, we get and

$$\begin{aligned} k^2 + 2ikM\rho + (1 - M^2)\rho^2 &= k^2 - 2\frac{(ikM)^2}{1 - M^2} + \frac{(-ikM)^2}{1 - M^2} \\ &= k^2 + \frac{M^2k^2}{1 - M^2} \\ &= \frac{k^2}{1 - M^2}. \end{aligned}$$

As a result,  $v$  satisfies

$$-\frac{k^2}{1 - M^2}v - (1 - M^2)\frac{\partial^2 v}{\partial \mathbf{x}_1^2} - \frac{\partial^2 v}{\partial \mathbf{x}_2^2} = fe^{-\rho\mathbf{x}_1}.$$

On the boundary, it holds that

$$\begin{aligned} 0 &= (1 - M^2) \frac{\partial u}{\partial \mathbf{x}_1} \mathbf{n}_1 + \frac{\partial u}{\partial \mathbf{x}_2} \mathbf{n}_2 - ik(1 - M\mathbf{n}_1)u \\ &= (1 - M^2) \frac{\partial v}{\partial \mathbf{x}_1} e^{\rho \mathbf{x}_1} \mathbf{n}_1 + \frac{\partial v}{\partial \mathbf{x}_2} e^{\rho \mathbf{x}_1} \mathbf{n}_2 + ((1 - M^2)\rho - ik(1 - M\mathbf{n}_1)) v e^{\rho \mathbf{x}_1}. \end{aligned}$$

Hence

$$(1 - M^2) \frac{\partial v}{\partial \mathbf{x}_1} \mathbf{n}_1 + \frac{\partial v}{\partial \mathbf{x}_2} \mathbf{n}_2 + ((1 - M^2)\mathbf{n}_1 \rho - ik(1 - M\mathbf{n}_1)) v = 0.$$

As simple calculations yield

$$(1 - M^2)\rho - ik(1 - M\mathbf{n}_1) = -ikM\mathbf{n}_1 - ik + ikM\mathbf{n}_1 = -ik,$$

the previous boundary condition is equivalent to

$$(1 - M^2) \frac{\partial v}{\partial \mathbf{x}_1} \mathbf{n}_1 + \frac{\partial v}{\partial \mathbf{x}_2} \mathbf{n}_2 - ikv = 0.$$

Now, we set  $v(\mathbf{x}_1, \mathbf{x}_2) = \tilde{u}(\mathbf{x}_1, \beta \mathbf{x}_2)$ , so that

$$\frac{\partial^2 v}{\partial \mathbf{x}_2^2} = \beta^2 \frac{\partial^2 \tilde{u}}{\partial \tilde{\mathbf{x}}_2^2}(\mathbf{x}_1, \beta \mathbf{x}_2)$$

and

$$\frac{\partial v}{\partial \mathbf{x}_2} \mathbf{n}_2 = \beta^2 \frac{\partial \tilde{u}}{\partial \tilde{\mathbf{x}}_2}(\mathbf{x}_1, \beta \mathbf{x}_2) \tilde{\mathbf{n}}_2.$$

Setting  $\beta = \sqrt{1 - M^2}$  and dividing both equations by  $1 - M^2$ , we obtain that

$$-\left(\frac{k}{1 - M^2}\right)^2 \tilde{u} - \tilde{\Delta} \tilde{u} = \frac{1}{1 - M^2} f e^{-\rho \mathbf{x}_1}$$

in  $\tilde{\Omega}$  and

$$\tilde{\nabla} \tilde{u} \cdot \tilde{\mathbf{n}} - \frac{k}{1 - M^2} \tilde{u} = 0$$

on  $\partial \tilde{\Omega}$ .

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