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On the scattering of electromagnetic waves by small bodies

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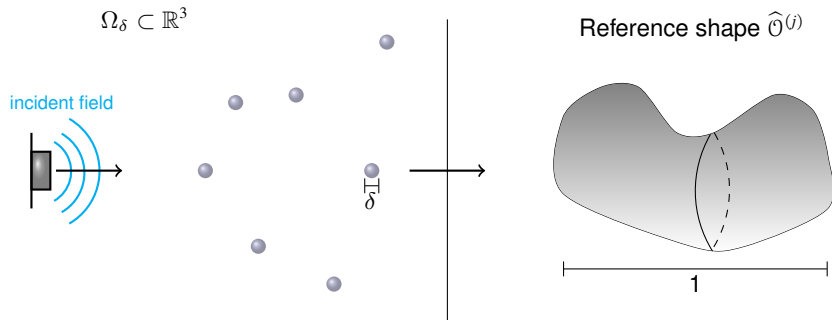
Advisors : Victor PÉRON et Sébastien TORDEUX

Instituto de Matematicas, Pontificia Universidad Católica de Valparaíso, Chile

May 5, 2017



Scattering problems of electromagnetic waves by small inclusions in 3D



δ : small parameter

$\mathcal{O}_\delta^{(1)}, \dots, \mathcal{O}_\delta^{(N)}$: inclusions

$$\Omega_\delta = \mathbb{R}^3 \setminus \bigcup \overline{\mathcal{O}_\delta^{(j)}}$$

$$\Gamma_\delta^{(j)} = \partial \mathcal{O}_\delta^{(j)}$$

$$\Gamma_\delta = \bigcup \Gamma_\delta^{(j)}$$

$$\mathcal{O}_\delta^{(j)} = c_j + \delta \widehat{\mathcal{O}}^{(j)}$$

$\widehat{\mathcal{O}}^{(j)}$ Lipschitz-continuous

Asymptotic assumption

$$\delta \ll \lambda^{\text{inc}}$$

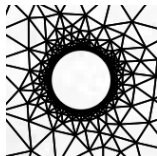
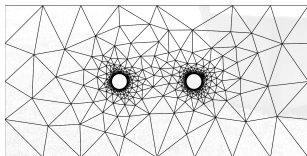
Motivation – Applications and numerical difficulties

► Applications

- Medical imaging: detecting small tumours
- Non-destructive testing civil engineering: detecting impurities into concrete

► Numerical approximation difficulties

- Infinite domain
- Mesh refinement near bodies (pictures from G. Vial)



► Our approach

- Reduced models derived thanks to asymptotic analysis: the radius δ is the small parameter
- Integration in a Boundary Element library (unbounded domain)



Figure:

https://people.eecs.ku.edu/~shontz/nsf_career_project.html

► Electromagnetic waves



H. Ammari, M. S. Vogelius, D. Volkov (2001)

Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of inhomogeneities of small diameter II. The full Maxwell equations



H. Ammari, H. Kang (2004)

Reconstruction of small inhomogeneities from boundary measurements

► Linear elasticity problems



V. Bonnaillie-Noël, D. Brancherie, M. Dambrine, F. Hérau, S. Tordeux, G. Vial (2011)

Multiscale expansion and numerical approximation for surface defects

► Acoustic waves



V. Mattesi (2014)

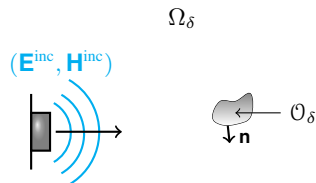
Propagation des ondes dans un domaine comportant des petites hétérogénéités : modélisation asymptotique et calcul numérique



A. Bendali, P-H. Cocquet, S. Tordeux (2014)

Approximation by multipoles of the multiple acoustic scattering by small obstacles and application to the Foldy theory of isotropic scattering

Scattering of electromagnetic waves by **one** small perfect conductor in 3D



$\mathcal{O}_\delta = \delta \widehat{\mathcal{O}}$: perfect conductor

$\Omega_\delta = \mathbb{R}^3 \setminus \overline{\mathcal{O}_\delta}$: exterior domain

$\Gamma_\delta = \partial\Omega_\delta$

κ : wave number

$\mathbf{E}^{\text{inc}}, \mathbf{H}^{\text{inc}}$: incident fields (given)

$\mathbf{E}_\delta^{\text{sca}}, \mathbf{H}_\delta^{\text{sca}}$: scattered fields

Time-harmonic Maxwell equations

$$\left\{ \begin{array}{l} \mathbf{curl} \mathbf{E}_\delta^{\text{sca}} - i\kappa \mathbf{H}_\delta^{\text{sca}} = 0 \quad \text{in } \Omega_\delta \\ \mathbf{curl} \mathbf{H}_\delta^{\text{sca}} + i\kappa \mathbf{E}_\delta^{\text{sca}} = 0 \quad \text{in } \Omega_\delta \\ \mathbf{n} \times \mathbf{E}_\delta^{\text{sca}} = -\mathbf{n} \times \mathbf{E}^{\text{inc}} \quad \text{on } \Gamma_\delta \\ \mathbf{n} \cdot \mathbf{H}_\delta^{\text{sca}} = -\mathbf{n} \cdot \mathbf{H}^{\text{inc}} \quad \text{on } \Gamma_\delta \\ \lim_{|\mathbf{x}| \rightarrow +\infty} |\mathbf{x}| (\mathbf{H}_\delta^{\text{sca}} \times \mathbf{e}_r - \mathbf{E}_\delta^{\text{sca}}) = 0 \end{array} \right.$$

$$\left\{ \begin{array}{l} \kappa^2 = \omega^2 \mu \left(\varepsilon + i \frac{\sigma}{\omega} \right) \\ \Im(\kappa) \geq 0 \end{array} \right.$$

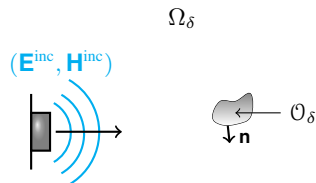
$\omega > 0$: frequency

$\mu > 0$: magnetic permeability

$\varepsilon > 0$: electric permittivity

$\sigma \geq 0$: electric conductivity

Scattering of electromagnetic waves by **one** small perfect conductor in 3D



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$\Omega_\delta = \mathbb{R}^3 \setminus \overline{\mathcal{O}_\delta}$: exterior domain

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$$\left\{ \begin{array}{l} \mathbf{curl} \mathbf{E}^{\text{inc}} - i\kappa \mathbf{H}^{\text{inc}} = 0 \quad \text{in } \mathbb{R}^3 \\ \mathbf{curl} \mathbf{H}^{\text{inc}} + i\kappa \mathbf{E}^{\text{inc}} = \mathbf{F} \quad \text{in } \mathbb{R}^3 \end{array} \right.$$

► **F smooth** source term

► $0 \notin \text{supp } \mathbf{F}$

Approximation method: Matched Asymptotic Expansions

“Formal” part

- ▶ Postulate asymptotic expansions
- ▶ Derive problems independent of δ
- ▶ Define a global approximation \mathbf{E}_δ^N

Analysis part

- ▶ Modal representation of solutions
- ▶ Validation of reduced models

$$\left\| \mathbf{E}_\delta^N - \mathbf{E}_\delta^{\text{sca}} \right\|_X \stackrel{?}{=} O(\delta^N)$$

Numerical resolution method: Equivalent Source Problem

- ▶ Boundary obstacles considered as surface/punctual source terms
- ▶ Implementation of Matched Asymptotic Expansions method
- ▶ Comparison with volumical methods

1 Method of matched asymptotic expansions

- Asymptotic domain decomposition
- Far field expansion
- Near field expansion
- Matching conditions
- Global approximation

2 Modal decomposition of the asymptotics

- Modal decomposition of the far field
- Modal decomposition of the near field
- Application of the matching conditions
- Case of the sphere: analytical solutions

3 Conclusion and perspectives

1 Method of matched asymptotic expansions

- Asymptotic domain decomposition
- Far field expansion
- Near field expansion
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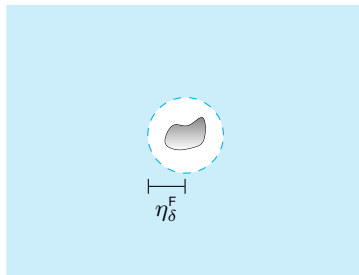
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3 Conclusion and perspectives

Asymptotic domain decomposition

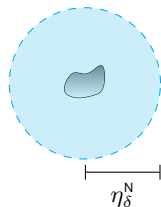
Far field domain



$$\lim_{\delta \rightarrow 0} \eta_{\delta}^F = 0$$

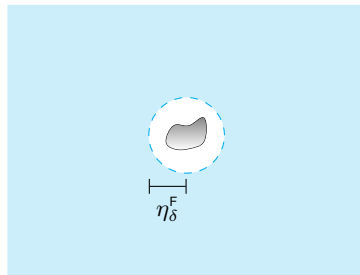
$$\lim_{\delta \rightarrow 0} \frac{\eta_{\delta}^N}{\delta} = +\infty$$

Near field domain



Asymptotic domain decomposition

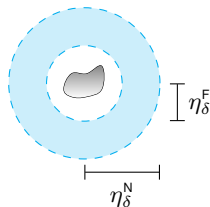
Far field domain



$$\lim_{\delta \rightarrow 0} \eta_{\delta}^F = 0$$

$$\lim_{\delta \rightarrow 0} \frac{\eta_{\delta}^N}{\delta} = +\infty$$

Matching area



Asymptotic assumption

$$\delta \ll \eta_{\delta}^F < \eta_{\delta}^N \ll \lambda^{\text{inc}}$$

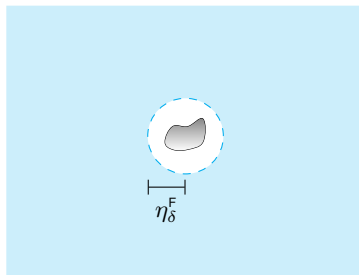


A.M. Il'in (1991)

Matching of Asymptotic Expansions of Solutions of Boundary Value Problems

Asymptotic domain: far field

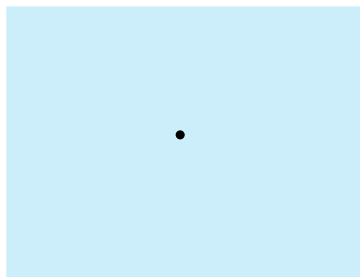
Far field domain



$$\mathbb{R}^3 \setminus \overline{\mathcal{B}(0, \eta_\delta^F)}$$

$$\delta \longrightarrow 0$$

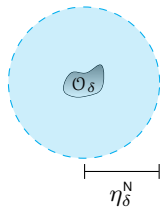
Asymptotic far field domain



$$\Omega^* = \mathbb{R}^3 \setminus \{0\}$$

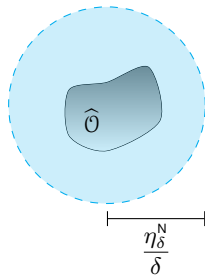
Asymptotic domain: near field

Near field domain

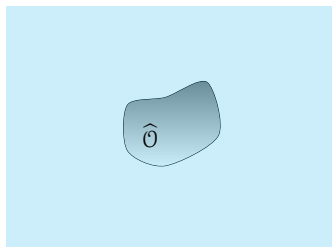


$$\begin{aligned} \xrightarrow{\mathcal{O}_\delta = \delta \widehat{\mathcal{O}}} \\ \widehat{\mathbf{x}} = \frac{\mathbf{x}}{\delta} \end{aligned}$$

Change of coordinates



$$\swarrow \delta \rightarrow 0$$



Asymptotic near field domain

$$\widehat{\Omega} = \mathbb{R}^3 \setminus \overline{\widehat{\mathcal{O}}}$$

Far field expansion in $\Omega^* = \mathbb{R}^3 \setminus \{0\}$

1. Postulate formal series

$$\mathbf{E}_\delta^{\text{sca}}(\mathbf{x}) \underset{\delta \rightarrow 0}{\sim} \sum_{p=0}^{\infty} \delta^p \tilde{\mathbf{E}}_p(\mathbf{x})$$

$$\mathbf{H}_\delta^{\text{sca}}(\mathbf{x}) \underset{\delta \rightarrow 0}{\sim} \sum_{p=0}^{\infty} \delta^p \tilde{\mathbf{H}}_p(\mathbf{x})$$

- ▶ $\tilde{\mathbf{E}}_p, \tilde{\mathbf{H}}_p$ defined in Ω^*
- ▶ $\tilde{\mathbf{E}}_p, \tilde{\mathbf{H}}_p$ independent of δ
- ▶ $\tilde{\mathbf{E}}_p, \tilde{\mathbf{H}}_p$ potentially singular at $\mathbf{x} = 0$

2. Derive formal problems : For $p \geq 0$, find $\tilde{\mathbf{E}}_p, \tilde{\mathbf{H}}_p \in \mathbf{H}_{\text{loc}}(\text{curl}, \Omega^*)$ s.t.

$$\begin{cases} \text{curl } \tilde{\mathbf{E}}_p - i\kappa \tilde{\mathbf{H}}_p = 0 & \text{in } \Omega^* \\ \text{curl } \tilde{\mathbf{H}}_p + i\kappa \tilde{\mathbf{E}}_p = 0 & \text{in } \Omega^* \end{cases}$$

+ Silver-Müller radiation condition

$$\lim_{|\mathbf{x}| \rightarrow +\infty} |\mathbf{x}| \left(\tilde{\mathbf{H}}_p \times \mathbf{e}_r - \tilde{\mathbf{E}}_p \right) = 0$$

- ▶ Ill-posed problems
- ▶ Missing information
 \rightsquigarrow matching conditions
- ▶ Singular behavior at $\mathbf{x} = 0$

Far field expansion in $\Omega^* = \mathbb{R}^3 \setminus \{0\}$

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- ▶ Missing information
 \rightsquigarrow matching conditions
- ▶ Singular behavior at $\mathbf{x} = 0$

Near field expansion in $\widehat{\Omega} = \mathbb{R}^3 \setminus \overline{\widehat{\Omega}}$

1. Postulate **formal series** in fast variable $\widehat{\mathbf{x}} = \frac{\mathbf{x}}{\delta}$

$$\mathbf{E}_\delta^{\text{sca}}(\mathbf{x}) \underset{\delta \rightarrow 0}{\sim} \sum_{p=0}^{\infty} \delta^p \widehat{\mathbf{E}}_p(\widehat{\mathbf{x}})$$

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- ▶ $\widehat{\mathbf{E}}_p, \widehat{\mathbf{H}}_p$ potentially increasing at ∞

$$\mathbf{E}^{\text{inc}}(\mathbf{x}) = \sum_{p=0}^{\infty} \delta^p \underbrace{\sum_{|\alpha|=p} \frac{1}{\alpha!} \mathbf{d}^\alpha \mathbf{E}^{\text{inc}}(0) \cdot \widehat{\mathbf{x}}^\alpha}_{\widehat{\mathbf{E}}_p^{\text{inc}}(\widehat{\mathbf{x}})}$$

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$(\widehat{\mathbf{E}}_{-1} = \widehat{\mathbf{H}}_{-1} = 0)$ + Boundary condition on $\widehat{\Gamma} = \partial\widehat{\Omega}$

$$\begin{cases} \mathbf{n} \times \widehat{\mathbf{E}}_p = -\mathbf{n} \times \widehat{\mathbf{E}}_p^{\text{inc}} & \text{on } \widehat{\Gamma} \\ \mathbf{n} \cdot \widehat{\mathbf{H}}_p = -\mathbf{n} \cdot \widehat{\mathbf{H}}_p^{\text{inc}} & \text{on } \widehat{\Gamma} \end{cases}$$

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▶ Ill-posed problems

▶ Missing information
 \rightsquigarrow matching conditions

▶ Increasing behavior at ∞

Some notations

Far field decomposition

$$\tilde{\mathbf{E}}_p(\mathbf{x}) = \tilde{\mathbf{E}}_p^{\text{reg}}(\mathbf{x}) + \tilde{\mathbf{E}}_p^{\text{sing}}(\mathbf{x}) \quad \mathbf{x} \in \Omega^*$$

$$\tilde{\mathbf{H}}_p(\mathbf{x}) = \tilde{\mathbf{H}}_p^{\text{reg}}(\mathbf{x}) + \tilde{\mathbf{H}}_p^{\text{sing}}(\mathbf{x}) \quad \mathbf{x} \in \Omega^*$$

- ▶ $\tilde{\mathbf{E}}_p^{\text{reg}}, \tilde{\mathbf{H}}_p^{\text{reg}}$: regular behavior at $\mathbf{x} = 0$ (**regular part** of the far field)
- ▶ $\tilde{\mathbf{E}}_p^{\text{sing}}, \tilde{\mathbf{H}}_p^{\text{sing}}$: singular behavior at $\mathbf{x} = 0$ (**singular part** of the far field)

Near field decomposition

$$\hat{\mathbf{E}}_p(\hat{\mathbf{x}}) = \hat{\mathbf{E}}_p^{\text{reg}}(\hat{\mathbf{x}}) + \hat{\mathbf{E}}_p^{\text{sing}}(\hat{\mathbf{x}}) \quad \hat{\mathbf{x}} \in \hat{\Omega}$$

$$\hat{\mathbf{H}}_p(\hat{\mathbf{x}}) = \hat{\mathbf{H}}_p^{\text{reg}}(\hat{\mathbf{x}}) + \hat{\mathbf{H}}_p^{\text{sing}}(\hat{\mathbf{x}}) \quad \hat{\mathbf{x}} \in \hat{\Omega}$$

- ▶ $\hat{\mathbf{E}}_p^{\text{reg}}, \hat{\mathbf{H}}_p^{\text{reg}}$: “**decreasing**” behavior at ∞ (**decreasing part** of the near field)
- ▶ $\hat{\mathbf{E}}_p^{\text{sing}}, \hat{\mathbf{H}}_p^{\text{sing}}$: increasing behavior at ∞ (**increasing part** of the near field)

Be careful. Weighted spaces



X. Claeys, PhD thesis (2008)

Analyse asymptotique et numérique de la diffraction d'ondes par des fils minces

Matching conditions

Find missing information

- ▶ Singular part of the far field ← Decreasing part of the near field
- ▶ Increasing part of the near field ← Regular part of the far field

Idea : develop $\tilde{\mathbf{E}}_p$ close to $\mathbf{x} = 0$ and $\hat{\mathbf{E}}_p$ near ∞

$$\tilde{\mathbf{E}}_p(\mathbf{x}) \underset{r \rightarrow 0}{\sim} \sum_{\ell=-\infty}^{-1} r^\ell \tilde{\mathbf{E}}_\ell^{(p)}(\theta, \varphi) + \sum_{\ell=0}^{+\infty} r^\ell \tilde{\mathbf{E}}_\ell^{(p)}(\theta, \varphi) \quad \mathbf{x} = (r, \theta, \varphi)$$

$$\hat{\mathbf{E}}_p(\hat{\mathbf{x}}) \underset{R \rightarrow \infty}{\sim} \sum_{\ell=-\infty}^{-1} R^\ell \hat{\mathbf{E}}_\ell^{(p)}(\theta, \varphi) + \sum_{\ell=0}^{+\infty} R^\ell \hat{\mathbf{E}}_\ell^{(p)}(\theta, \varphi) \quad \hat{\mathbf{x}} = (R, \theta, \varphi) = \left(\frac{r}{\delta}, \theta, \varphi \right)$$

Matching conditions

Find missing information

- ▶ Singular part of the far field ← Decreasing part of the near field
- ▶ Increasing part of the near field ← Regular part of the far field

Idea

- ▶ Far field expansion in $\mathcal{V}(0)$

$$\mathbf{E}_{\delta}^{\text{sca}}(\mathbf{x}) \underset{\delta \rightarrow 0}{\sim} \sum_{p=-\infty}^{+\infty} \delta^p \left(\sum_{\ell=-\infty}^{-1} r^{\ell} \tilde{\mathbf{E}}_{\ell}^{(p)}(\theta, \varphi) + \sum_{\ell=0}^{+\infty} r^{\ell} \tilde{\mathbf{E}}_{\ell}^{(p)}(\theta, \varphi) \right)$$

- ▶ Near field expansion in $\mathcal{V}(\infty)$ with $R = \frac{r}{\delta}$

$$\mathbf{E}_{\delta}^{\text{sca}}(\mathbf{x}) \underset{\delta \rightarrow 0}{\sim} \sum_{p=-\infty}^{+\infty} \delta^p \left(\sum_{\ell=-\infty}^{-1} R^{\ell} \hat{\mathbf{E}}_{\ell}^{(p)}(\theta, \varphi) + \sum_{\ell=0}^{+\infty} R^{\ell} \hat{\mathbf{E}}_{\ell}^{(p)}(\theta, \varphi) \right)$$

Matching conditions

Find missing information

- ▶ Singular part of the far field ← Decreasing part of the near field
- ▶ Increasing part of the near field ← Regular part of the far field

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- ▶ Far field expansion in $\mathcal{V}(0)$

$$\mathbf{E}_\delta^{\text{sca}}(\mathbf{x}) \underset{\delta \rightarrow 0}{\sim} \sum_{p=-\infty}^{+\infty} \delta^p \left(\sum_{\ell=-\infty}^{+\infty} r^\ell \tilde{\mathbf{E}}_\ell^{(p)}(\theta, \varphi) \right)$$

- ▶ Near field expansion in $\mathcal{V}(\infty)$ with $R = \frac{r}{\delta}$

$$\mathbf{E}_\delta^{\text{sca}}(\mathbf{x}) \underset{\delta \rightarrow 0}{\sim} \sum_{p=-\infty}^{+\infty} \delta^p \left(\sum_{\ell=-\infty}^{+\infty} R^\ell \hat{\mathbf{E}}_\ell^{(p)}(\theta, \varphi) \right)$$

- ▶ **Identification** of the two series

$$\begin{cases} \tilde{\mathbf{E}}_\ell^{(p)} = \hat{\mathbf{E}}_\ell^{(p+\ell)} & p \geq 0, \ell = -p, \dots, -1 \\ \hat{\mathbf{E}}_\ell^{(p)} = \tilde{\mathbf{E}}_\ell^{(p-\ell)} & p \geq 0, \ell = 0, \dots, p \end{cases} \quad \begin{cases} \tilde{\mathbf{E}}_\ell^{(p)} = 0 & \ell < -p \\ \hat{\mathbf{E}}_\ell^{(p)} = 0 & \ell > p \end{cases}$$

Matching conditions

- **Identification** of the two series

$$\begin{cases} \tilde{\mathbf{E}}_\ell^{(p)} = \hat{\mathbf{E}}_\ell^{(p+\ell)} & p \geq 0, \ell = -p, \dots, -1 \\ \hat{\mathbf{E}}_\ell^{(p)} = \tilde{\mathbf{E}}_\ell^{(p-\ell)} & p \geq 0, \ell = 0, \dots, p \end{cases} \quad \begin{cases} \tilde{\mathbf{E}}_\ell^{(p)} = 0 & \ell < -p \\ \hat{\mathbf{E}}_\ell^{(p)} = 0 & \ell > p \end{cases}$$

- **Term by term**

$$\tilde{\mathbf{E}}_p(\mathbf{x}) \underset{r \rightarrow 0}{\sim} \sum_{\ell=-p}^{-1} r^\ell \hat{\mathbf{E}}_\ell^{(p+\ell)}(\theta, \varphi) + \sum_{\ell=0}^{+\infty} r^\ell \tilde{\mathbf{E}}_\ell^{(p)}(\theta, \varphi) \quad (\text{far field})$$

$$\hat{\mathbf{E}}_p(\hat{\mathbf{x}}) \underset{R \rightarrow \infty}{\sim} \sum_{\ell=-\infty}^{-1} R^\ell \hat{\mathbf{E}}_\ell^{(p)}(\theta, \varphi) + \sum_{\ell=0}^p R^\ell \tilde{\mathbf{E}}_\ell^{(p-\ell)}(\theta, \varphi) \quad (\text{near field})$$

- For $p = 0$, $\tilde{\mathbf{E}}_0$ is **only** regular

Global approximation

Truncated series

$$\tilde{\mathbf{E}}_\delta^N(\mathbf{x}) = \sum_{p=0}^N \delta^p \tilde{\mathbf{E}}_p(\mathbf{x})$$

$$\tilde{\mathbf{H}}_\delta^P(\mathbf{x}) = \sum_{p=0}^N \delta^p \tilde{\mathbf{H}}_p(\mathbf{x}) \quad (\text{far fields})$$

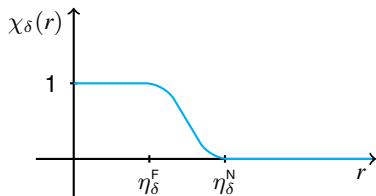
$$\hat{\mathbf{E}}_\delta^N(\mathbf{x}) = \sum_{p=0}^N \delta^p \hat{\mathbf{E}}_p(\hat{\mathbf{x}})$$

$$\hat{\mathbf{H}}_\delta^N(\mathbf{x}) = \sum_{p=0}^N \delta^p \hat{\mathbf{H}}_p(\hat{\mathbf{x}}) \quad (\text{near fields})$$

Global approximation

$$\mathbf{E}_\delta^P(\mathbf{x}) = \chi_\delta(|\mathbf{x}|) \hat{\mathbf{E}}_\delta^P(\mathbf{x}) + (1 - \chi_\delta(|\mathbf{x}|)) \tilde{\mathbf{E}}_\delta^P(\mathbf{x}) \quad \mathbf{x} \in \Omega_\delta$$

$$\mathbf{H}_\delta^P(\mathbf{x}) = \chi_\delta(|\mathbf{x}|) \hat{\mathbf{H}}_\delta^P(\mathbf{x}) + (1 - \chi_\delta(|\mathbf{x}|)) \tilde{\mathbf{H}}_\delta^P(\mathbf{x}) \quad \mathbf{x} \in \Omega_\delta$$



Consistence and stability Determine ϕ s.t.
 $\phi(N) \xrightarrow{N \rightarrow \infty} +\infty$ and for any $N > 0, \rho > \rho_0$

$$\left\| \mathbf{E}_\delta^{\text{sca}} - \mathbf{E}_\delta^N \right\|_{\mathbf{H}(\text{curl}, \Omega_\delta \cap \mathcal{B}(0, \rho))} \leq C_{N, \rho}^E \delta^{\phi(N)}$$

$$\left\| \mathbf{H}_\delta^{\text{sca}} - \mathbf{H}_\delta^N \right\|_{\mathbf{H}(\text{curl}, \Omega_\delta \cap \mathcal{B}(0, \rho))} \leq C_{N, \rho}^H \delta^{\phi(N)}$$

Global approximation

Truncated series

$$\tilde{\mathbf{E}}_\delta^N(\mathbf{x}) = \sum_{p=0}^N \delta^p \tilde{\mathbf{E}}_p(\mathbf{x})$$

$$\tilde{\mathbf{H}}_\delta^P(\mathbf{x}) = \sum_{p=0}^N \delta^p \tilde{\mathbf{H}}_p(\mathbf{x}) \quad (\text{far fields})$$

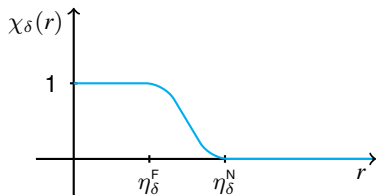
$$\hat{\mathbf{E}}_\delta^N(\mathbf{x}) = \sum_{p=0}^N \delta^p \hat{\mathbf{E}}_p(\hat{\mathbf{x}})$$

$$\hat{\mathbf{H}}_\delta^N(\mathbf{x}) = \sum_{p=0}^N \delta^p \hat{\mathbf{H}}_p(\hat{\mathbf{x}}) \quad (\text{near fields})$$

Global approximation

$$\mathbf{E}_\delta^P(\mathbf{x}) = \chi_\delta(|\mathbf{x}|) \hat{\mathbf{E}}_\delta^P(\mathbf{x}) + (1 - \chi_\delta(|\mathbf{x}|)) \tilde{\mathbf{E}}_\delta^P(\mathbf{x}) \quad \mathbf{x} \in \Omega_\delta$$

$$\mathbf{H}_\delta^P(\mathbf{x}) = \chi_\delta(|\mathbf{x}|) \hat{\mathbf{H}}_\delta^P(\mathbf{x}) + (1 - \chi_\delta(|\mathbf{x}|)) \tilde{\mathbf{H}}_\delta^P(\mathbf{x}) \quad \mathbf{x} \in \Omega_\delta$$



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$$\left\| \mathbf{H}_\delta^{\text{sca}} - \mathbf{H}_\delta^N \right\|_{\mathbf{H}(\text{curl}, \Omega_\delta \cap \mathcal{B}(0, \rho))} \leq C_{N, \rho}^H \delta^{\phi(N)}$$

Outline

1 Method of matched asymptotic expansions

- Asymptotic domain decomposition
- Far field expansion
- Near field expansion
- Matching conditions
- Global approximation

2 Modal decomposition of the asymptotics

- Modal decomposition of the far field
- Modal decomposition of the near field
- Application of the matching conditions
- Case of the sphere: analytical solutions

3 Conclusion and perspectives

Modal decomposition of the far field of order 0

Find $\tilde{\mathbf{E}}_0$ and $\tilde{\mathbf{H}}_0 \in \mathbf{H}_{\text{loc}}(\text{curl}, \mathbb{R}^3)$ s.t.

$$\begin{cases} \text{curl } \tilde{\mathbf{E}}_0 - i\kappa \tilde{\mathbf{H}}_0 = 0 & \text{in } \mathbb{R}^3 \\ \text{curl } \tilde{\mathbf{H}}_0 + i\kappa \tilde{\mathbf{E}}_0 = 0 & \text{in } \mathbb{R}^3 \\ \lim_{|\mathbf{x}| \rightarrow \infty} |\mathbf{x}| \left(\tilde{\mathbf{H}}_0 \times \mathbf{e}_r - \tilde{\mathbf{E}}_0 \right) = 0 \end{cases}$$

- ▶ $\tilde{\mathbf{E}}_0, \tilde{\mathbf{H}}_0$ are **regular** (no singularity at $\mathbf{x} = 0$)
- ▶ $\tilde{\mathbf{E}}_0, \tilde{\mathbf{H}}_0$ satisfy S.M. $\implies \tilde{\mathbf{E}}_0 = 0$ and $\tilde{\mathbf{H}}_0 = 0$
- ▶ Far field expansion

$$\mathbf{E}_\delta^{\text{sca}}(\mathbf{x}) \sim \sum_{p=1}^{\infty} \delta^p \tilde{\mathbf{E}}_p(\mathbf{x})$$

Modal decomposition of the far field of order $p > 0$

Find $\tilde{\mathbf{E}}_p$ and $\tilde{\mathbf{H}}_p \in \mathbf{H}_{\text{loc}}(\mathbf{curl}, \Omega^*)$ s.t.

$$\left\{ \begin{array}{ll} \mathbf{curl} \tilde{\mathbf{E}}_p - i\kappa \tilde{\mathbf{H}}_p = 0 & \text{in } \Omega^* \\ \mathbf{curl} \tilde{\mathbf{H}}_p + i\kappa \tilde{\mathbf{E}}_p = 0 & \text{in } \Omega^* \\ \lim_{|\mathbf{x}| \rightarrow \infty} |\mathbf{x}| \left(\tilde{\mathbf{H}}_p \times \mathbf{e}_r - \tilde{\mathbf{E}}_p \right) = 0 & \end{array} \right.$$

► $\tilde{\mathbf{E}}_p, \tilde{\mathbf{H}}_p$ are **singular** at $\mathbf{x} = 0$

Modal decomposition of the far field of order $p > 0$

Find $\tilde{\mathbf{E}}_p$ and $\tilde{\mathbf{H}}_p \in \mathbf{H}_{\text{loc}}(\mathbf{curl}, \Omega^*)$ s.t.

$$\begin{cases} \mathbf{curl} \tilde{\mathbf{E}}_p - i\kappa \tilde{\mathbf{H}}_p = 0 & \text{in } \Omega^* \\ \mathbf{curl} \tilde{\mathbf{H}}_p + i\kappa \tilde{\mathbf{E}}_p = 0 & \text{in } \Omega^* \\ \lim_{|\mathbf{x}| \rightarrow \infty} |\mathbf{x}| \left(\tilde{\mathbf{H}}_p \times \mathbf{e}_r - \tilde{\mathbf{E}}_p \right) = 0 \end{cases}$$

- ▶ $\tilde{\mathbf{E}}_p, \tilde{\mathbf{H}}_p$ are **singular** at $\mathbf{x} = 0$
- ▶ Modal decomposition (singular at 0)

$$\tilde{\mathbf{E}}_p(\mathbf{x}) = \sum_{n=1}^{\infty} \sum_{m=-n}^n \left\{ \tilde{\beta}_{n,m}^{(p),E} \mathbf{curl} \left[h_n^{(1)}(\kappa|\mathbf{x}|) Y_{n,m}(\hat{\mathbf{x}}) \mathbf{x} \right] + \frac{\tilde{\beta}_{n,m}^{(p),H}}{i\kappa} \mathbf{curl} \mathbf{curl} \left[h_n^{(1)}(\kappa|\mathbf{x}|) Y_{n,m}(\hat{\mathbf{x}}) \mathbf{x} \right] \right\}$$

- ▶ **Finite number** of $\tilde{\beta}_{n,m}^{(p),E}, \tilde{\beta}_{n,m}^{(p),H}$ non-equal to 0
- ▶ $\tilde{\beta}_{n,m}^{(p),E}, \tilde{\beta}_{n,m}^{(p),H}$ are given by **matching conditions**
- ▶ $h_n^{(1)}$: spherical Hankel function of first kind

Modal decomposition of the far field of order $p > 0$

Find $\tilde{\mathbf{E}}_p$ and $\tilde{\mathbf{H}}_p \in \mathbf{H}_{\text{loc}}(\mathbf{curl}, \Omega^*)$ s.t.

$$\begin{cases} \mathbf{curl} \tilde{\mathbf{E}}_p - i\kappa \tilde{\mathbf{H}}_p = 0 & \text{in } \Omega^* \\ \mathbf{curl} \tilde{\mathbf{H}}_p + i\kappa \tilde{\mathbf{E}}_p = 0 & \text{in } \Omega^* \\ \lim_{|\mathbf{x}| \rightarrow \infty} |\mathbf{x}| \left(\tilde{\mathbf{H}}_p \times \mathbf{e}_r - \tilde{\mathbf{E}}_p \right) = 0 \end{cases}$$

- ▶ $\tilde{\mathbf{E}}_p, \tilde{\mathbf{H}}_p$ are **singular** at $\mathbf{x} = 0$
- ▶ Modal decomposition (singular at 0)

$$\begin{aligned} \tilde{\mathbf{E}}_p(\mathbf{x}) = & \sum_{n=1}^{\infty} \sum_{m=-n}^n \left\{ \tilde{\beta}_{n,m}^{(p),E} h_n^{(1)}(\kappa r) \mathbf{curl}_{S^2} Y_{n,m}(\theta, \varphi) \right. \\ & \left. + \frac{\tilde{\beta}_{n,m}^{(p),H}}{i\kappa r} \left([h_n^{(1)}(\kappa r) + \kappa r h_n^{(1)'}(\kappa r)] \nabla_{S^2} Y_{n,m}(\theta, \varphi) + n(n+1) h_n^{(1)}(\kappa r) Y_{n,m}(\theta, \varphi) \mathbf{e}_r \right) \right\} \end{aligned}$$

- ▶ **Finite number** of $\tilde{\beta}_{n,m}^{(p),E}, \tilde{\beta}_{n,m}^{(p),H}$ non-equal to 0
- ▶ $\tilde{\beta}_{n,m}^{(p),E}, \tilde{\beta}_{n,m}^{(p),H}$ are given by **matching conditions**

→ **Behavior at 0 ?**

Behavior of the far field in a neighbourhood of 0

Expansion of spherical Hankel function (1st)

$$h_n^{(1)}(z) = \sum_{\ell \geq -n-1} h_{n,\ell} z^\ell \quad | \quad h_{n,\ell} = 0 \text{ iff } (\ell - n \text{ even and } \ell < n) \text{ or } (\ell < -n - 1)$$

Expansion of the far field close to 0

$$\begin{aligned} \tilde{\mathbf{E}}_{p>0}(\mathbf{x}) = & \sum_{n=1}^{\infty} \sum_{m=-n}^n \left\{ \tilde{\beta}_{n,m}^{(p),E} \sum_{\ell \geq -n-1} \left\{ (\kappa r)^\ell h_{n,\ell} \mathbf{curl}_{S^2} Y_{n,m}(\theta, \varphi) \right\} + \right. \\ & \left. \tilde{\beta}_{n,m}^{(p),H} \sum_{\ell \geq -n-2} \left\{ (\kappa r)^\ell \frac{h_{n,\ell+1}}{i} [(\ell + 2)\nabla_{S^2} Y_{n,m}(\theta, \varphi) + n(n + 1)Y_{n,m}(\theta, \varphi)\mathbf{e}_r] \right\} \right\} \end{aligned}$$

No **regular** part for the far field \implies No **increasing** part for the near field

Behavior of the far field in a neighbourhood of 0

Expansion of spherical Hankel function (1st)

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Expansion of the far field close to 0

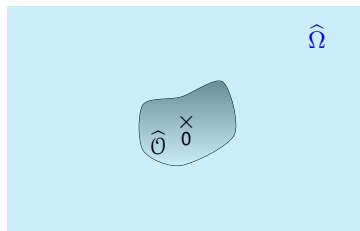
$$\begin{aligned} \tilde{\mathbf{E}}_{p>0}(\mathbf{x}) = & \sum_{n=1}^{\infty} \sum_{m=-n}^n \left\{ \tilde{\beta}_{n,m}^{(p),E} \sum_{\ell \geq -n-1} \left\{ (\kappa r)^\ell h_{n,\ell} \mathbf{curl}_{S^2} Y_{n,m}(\theta, \varphi) \right\} + \right. \\ & \left. \tilde{\beta}_{n,m}^{(p),H} \sum_{\ell \geq -n-2} \left\{ (\kappa r)^\ell \frac{h_{n,\ell+1}}{i} [(\ell + 2) \nabla_{S^2} Y_{n,m}(\theta, \varphi) + n(n + 1) Y_{n,m}(\theta, \varphi) \mathbf{e}_r] \right\} \right\} \end{aligned}$$

No **regular** part for the far field \implies No **increasing** part for the near field

Modal decomposition of the near field

Find $(\widehat{\mathbf{E}}_p, \widehat{\mathbf{H}}_p)_{p \geq 0} \in \mathbf{H}_{\text{loc}}(\text{curl}, \widehat{\Omega})$ s.t.

$$\begin{cases} \text{curl } \widehat{\mathbf{E}}_p = i\kappa \widehat{\mathbf{H}}_{p-1} & \text{in } \widehat{\Omega} \\ \text{curl } \widehat{\mathbf{H}}_p = -i\kappa \widehat{\mathbf{E}}_{p-1} & \text{in } \widehat{\Omega} \\ \mathbf{n} \times \widehat{\mathbf{E}}_p = -\mathbf{n} \times \widehat{\mathbf{E}}_p^{\text{inc}} (*) & \text{on } \widehat{\Gamma} \\ \mathbf{n} \cdot \widehat{\mathbf{H}}_p = -\mathbf{n} \cdot \widehat{\mathbf{H}}_p^{\text{inc}} & \text{on } \widehat{\Gamma} \end{cases}$$



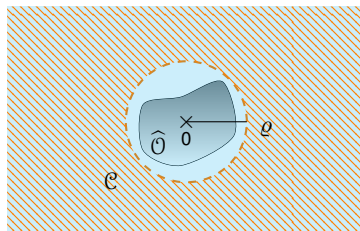
$$(*) \widehat{\mathbf{E}}_p^{\text{inc}}(\widehat{\mathbf{x}}) = \sum_{|\alpha|=p} \frac{1}{\alpha!} \mathbf{d}^\alpha \mathbf{E}^{\text{inc}}(0) \cdot \widehat{\mathbf{x}}^\alpha$$

- ▶ No **increasing** part for the near field
- ▶ Matching conditions – **separation of variables**
- ▶ **“Decreasing”** part – by induction with **integral equations**

Modal decomposition of the near field

$$\begin{cases} \mathbf{curl} \hat{\mathbf{E}}_{p,0} = 0 & \text{in } \mathcal{C} \\ \mathbf{curl} \hat{\mathbf{H}}_{p,0} = 0 & \text{in } \mathcal{C} \\ \mathbf{div} \hat{\mathbf{E}}_{p,0} = 0 & \text{in } \mathcal{C} \\ \mathbf{div} \hat{\mathbf{H}}_{p,0} = 0 & \text{in } \mathcal{C} \end{cases}$$

$$\begin{cases} \mathbf{curl} \hat{\mathbf{E}}_{p,\ell} = i\kappa \hat{\mathbf{H}}_{p-1,\ell-1} & \text{in } \mathcal{C} \\ \mathbf{curl} \hat{\mathbf{H}}_{p,\ell} = -i\kappa \hat{\mathbf{E}}_{p-1,\ell-1} & \text{in } \mathcal{C} \end{cases}$$



$$\mathcal{C} = \{\hat{\mathbf{x}} \in \mathbb{R}^3, |\hat{\mathbf{x}}| > \varrho\}$$

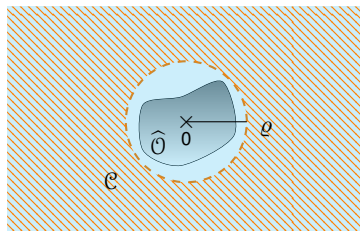
Near field expansions into \mathcal{C}

$$\hat{\mathbf{E}}_p(\hat{\mathbf{x}}) = \underbrace{\hat{\mathbf{E}}_{p,0}(\hat{\mathbf{x}})}_{\text{HEAD}} + \underbrace{\sum_{\ell=1}^p \hat{\mathbf{E}}_{p,\ell}(\hat{\mathbf{x}})}_{\text{SHADOWS}} \quad \hat{\mathbf{H}}_p(\hat{\mathbf{x}}) = \underbrace{\hat{\mathbf{H}}_{p,0}(\hat{\mathbf{x}})}_{\text{HEAD}} + \underbrace{\sum_{\ell=1}^p \hat{\mathbf{H}}_{p,\ell}(\hat{\mathbf{x}})}_{\text{SHADOWS}}$$

- ▶ **HEADs** : solution of homogeneous Maxwell **static** equations
- ▶ **SHADOWS** : particular solution of **nested** equations

Modal decomposition of the near field

$$\left\{ \begin{array}{ll} \mathbf{curl} \hat{\mathbf{E}}_{p,0} = 0 & \text{in } \mathcal{C} \\ \mathbf{curl} \hat{\mathbf{H}}_{p,0} = 0 & \text{in } \mathcal{C} \\ \operatorname{div} \hat{\mathbf{E}}_{p,0} = 0 & \text{in } \mathcal{C} \\ \operatorname{div} \hat{\mathbf{H}}_{p,0} = 0 & \text{in } \mathcal{C} \end{array} \right.$$



$$\mathcal{C} = \{\hat{\mathbf{x}} \in \mathbb{R}^3, |\hat{\mathbf{x}}| > \varrho\}$$

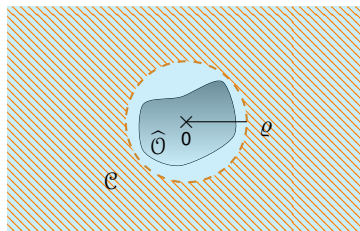
Head expansions into \mathcal{C}

$$\hat{\mathbf{E}}_{p,0} = \sum_{n=1}^{\infty} \sum_{m=-n}^n (\kappa R)^{-n-2} \hat{\alpha}_{n,m}^{(p),H} \frac{h_{n,-n-1}}{i} \left[-n \nabla_{S^2} Y_{n,m} + n(n+1) Y_{n,m} \mathbf{e}_r \right]$$

$$\hat{\mathbf{H}}_{p,0} = \sum_{n=1}^{\infty} \sum_{m=-n}^n (\kappa R)^{-n-2} \hat{\alpha}_{n,m}^{(p),E} \frac{h_{n,-n-1}}{i} \left[-n \nabla_{S^2} Y_{n,m} + n(n+1) Y_{n,m} \mathbf{e}_r \right]$$

Modal decomposition of the near field

$$\begin{cases} \operatorname{curl} \widehat{\mathbf{E}}_{p,\ell} = i\kappa \widehat{\mathbf{H}}_{p-1,\ell-1} & \text{in } \mathcal{C} \\ \operatorname{curl} \widehat{\mathbf{H}}_{p,\ell} = -i\kappa \widehat{\mathbf{E}}_{p-1,\ell-1} & \text{in } \mathcal{C} \end{cases}$$



Shadow expansions into \mathcal{C}

$$\mathcal{C} = \{\widehat{\mathbf{x}} \in \mathbb{R}^3, |\widehat{\mathbf{x}}| > \rho\}$$

$$\widehat{\mathbf{E}}_{p,\ell} = \sum_{n=1}^{\infty} \sum_{m=-n}^n (\kappa R)^{\ell-n-2} \left\{ \widehat{\alpha}_{n,m}^{(p-\ell),E} h_{n,\ell-n-2} \operatorname{curl}_{S^2} Y_{n,m} + \widehat{\alpha}_{n,m}^{(p-\ell),H} \frac{h_{n,\ell-n-1}}{i} [(\ell-n)\nabla_{S^2} Y_{n,m} + n(n+1)Y_{n,m}\mathbf{e}_r] \right\}$$

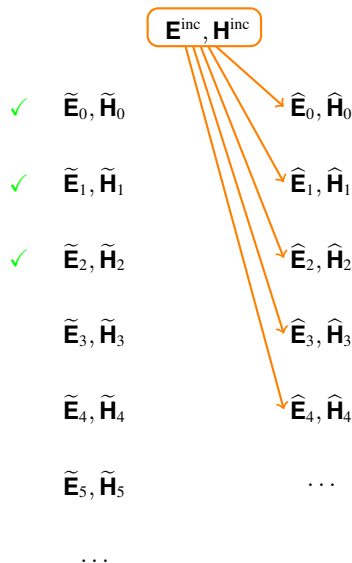
$$\widehat{\mathbf{H}}_{p,\ell} = \sum_{n=1}^{\infty} \sum_{m=-n}^n (\kappa R)^{\ell-n-2} \left\{ -\widehat{\alpha}_{n,m}^{(p-\ell),H} h_{n,\ell-n-2} \operatorname{curl}_{S^2} Y_{n,m} + \widehat{\alpha}_{n,m}^{(p-\ell),E} \frac{h_{n,\ell-n-1}}{i} [(\ell-n)\nabla_{S^2} Y_{n,m} + n(n+1)Y_{n,m}\mathbf{e}_r] \right\}$$

Application of the matching conditions

► Far field expansion

$$\begin{aligned} \tilde{\mathbf{E}}_p = & \sum_{n=1}^{p-2} \sum_{m=-n}^n \left\{ \hat{\alpha}_{n,m}^{(p-n-2),E} h_n^{(1)}(\kappa r) \mathbf{curl}_{S^2} Y_{n,m} \right. \\ & + \frac{\hat{\alpha}_{n,m}^{(p-n-2),H}}{i\kappa r} \left([h_n^{(1)}(\kappa r) + \kappa r h_n^{(1)'}(\kappa r)] \nabla_{S^2} Y_{n,m} \right. \\ & \left. \left. + n(n+1) h_n^{(1)}(\kappa r) Y_{n,m} \mathbf{e}_r \right) \right\} \end{aligned}$$

► Resolution of the near field problem?



Application of the matching conditions

► Far field expansion

$$\begin{aligned} \tilde{\mathbf{E}}_p = & \sum_{n=1}^{p-2} \sum_{m=-n}^n \left\{ \hat{\alpha}_{n,m}^{(p-n-2),E} h_n^{(1)}(\kappa r) \mathbf{curl}_{S^2} Y_{n,m} \right. \\ & + \frac{\hat{\alpha}_{n,m}^{(p-n-2),H}}{i\kappa r} \left([h_n^{(1)}(\kappa r) + \kappa r h_n^{(1)'}(\kappa r)] \nabla_{S^2} Y_{n,m} \right. \\ & \left. \left. + n(n+1) h_n^{(1)}(\kappa r) Y_{n,m} \mathbf{e}_r \right) \right\} \end{aligned}$$

► Resolution of the near field problem?

$\mathbf{E}^{\text{inc}}, \mathbf{H}^{\text{inc}}$

✓	$\tilde{\mathbf{E}}_0, \tilde{\mathbf{H}}_0$	$\hat{\mathbf{E}}_0, \hat{\mathbf{H}}_0$	✓
✓	$\tilde{\mathbf{E}}_1, \tilde{\mathbf{H}}_1$	$\hat{\mathbf{E}}_1, \hat{\mathbf{H}}_1$	✓
✓	$\tilde{\mathbf{E}}_2, \tilde{\mathbf{H}}_2$	$\hat{\mathbf{E}}_2, \hat{\mathbf{H}}_2$	✓
	$\tilde{\mathbf{E}}_3, \tilde{\mathbf{H}}_3$	$\hat{\mathbf{E}}_3, \hat{\mathbf{H}}_3$	✓
	$\tilde{\mathbf{E}}_4, \tilde{\mathbf{H}}_4$	$\hat{\mathbf{E}}_4, \hat{\mathbf{H}}_4$	✓
	$\tilde{\mathbf{E}}_5, \tilde{\mathbf{H}}_5$...	✓

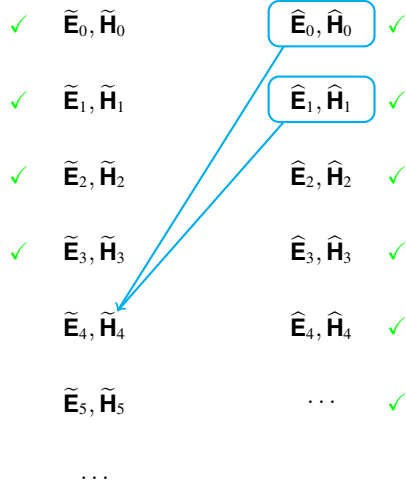
Application of the matching conditions

► Far field expansion

$$\begin{aligned} \tilde{\mathbf{E}}_p = & \sum_{n=1}^{p-2} \sum_{m=-n}^n \left\{ \hat{\alpha}_{n,m}^{(p-n-2),E} h_n^{(1)}(\kappa r) \mathbf{curl}_{S^2} Y_{n,m} \right. \\ & + \frac{\hat{\alpha}_{n,m}^{(p-n-2),H}}{i\kappa r} \left([h_n^{(1)}(\kappa r) + \kappa r h_n^{(1)'}(\kappa r)] \nabla_{S^2} Y_{n,m} \right. \\ & \left. \left. + n(n+1) h_n^{(1)}(\kappa r) Y_{n,m} \mathbf{e}_r \right) \right\} \end{aligned}$$

► Resolution of the near field problem?

$\mathbf{E}^{\text{inc}}, \mathbf{H}^{\text{inc}}$



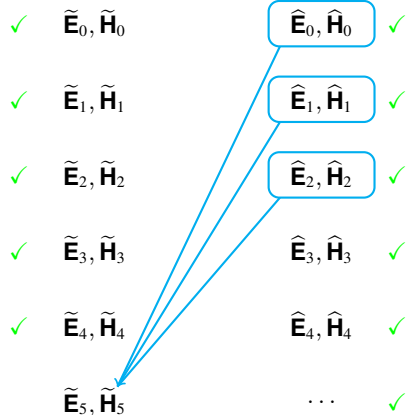
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► Resolution of the near field problem?

✓	$\tilde{\mathbf{E}}_0, \tilde{\mathbf{H}}_0$	$\hat{\mathbf{E}}_0, \hat{\mathbf{H}}_0$	✓
✓	$\tilde{\mathbf{E}}_1, \tilde{\mathbf{H}}_1$	$\hat{\mathbf{E}}_1, \hat{\mathbf{H}}_1$	✓
✓	$\tilde{\mathbf{E}}_2, \tilde{\mathbf{H}}_2$	$\hat{\mathbf{E}}_2, \hat{\mathbf{H}}_2$	✓
✓	$\tilde{\mathbf{E}}_3, \tilde{\mathbf{H}}_3$	$\hat{\mathbf{E}}_3, \hat{\mathbf{H}}_3$	✓
✓	$\tilde{\mathbf{E}}_4, \tilde{\mathbf{H}}_4$	$\hat{\mathbf{E}}_4, \hat{\mathbf{H}}_4$	✓
✓	$\tilde{\mathbf{E}}_5, \tilde{\mathbf{H}}_5$...	✓

Etcetera ...

Explicit computation in the case of the sphere $\mathcal{O}_\delta = \mathcal{B}(0, \delta)$

► Local approximations

$$\mathbf{E}_\delta^{\text{sca}}(\mathbf{x}) \underset{\delta \rightarrow 0}{\sim} \delta^3 \tilde{\mathbf{E}}_3(\mathbf{x}) + \delta^4 \tilde{\mathbf{E}}_4(\mathbf{x}) + \underset{\delta \rightarrow 0}{O}(\delta^5)$$

$$\mathbf{E}_\delta^{\text{sca}}(\mathbf{x}) \underset{\delta \rightarrow 0}{\sim} \hat{\mathbf{E}}_0(\hat{\mathbf{x}}) + \delta \hat{\mathbf{E}}_1(\hat{\mathbf{x}}) + \underset{\delta \rightarrow 0}{O}(\delta^2)$$

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► Expression of $\tilde{\mathbf{E}}_3$ (matching conditions)

$$\tilde{\mathbf{E}}_3 = \sum_{m=-1}^1 \left\{ \hat{\alpha}_{1,m}^{(0),E} h_1^{(1)}(\kappa r) \mathbf{curl}_{S^2} Y_{1,m} + \frac{\hat{\alpha}_{1,m}^{(0),H}}{i\kappa r} \left([h_1^{(1)}(\kappa r) + \kappa r h_1^{(1)'}(\kappa r)] \nabla_{S^2} Y_{1,m} + 2 h_1^{(1)}(\kappa r) Y_{1,m} \mathbf{e}_r \right) \right\}$$

Explicit computation in the case of the sphere $\mathcal{O}_\delta = \mathcal{B}(0, \delta)$

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$$\mathbf{E}_\delta^{\text{sca}}(\mathbf{x}) \underset{\delta \rightarrow 0}{\sim} \delta^3 \tilde{\mathbf{E}}_3(\mathbf{x}) + \delta^4 \tilde{\mathbf{E}}_4(\mathbf{x}) + \underset{\delta \rightarrow 0}{O}(\delta^5)$$

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► Modal representation of $(\hat{\mathbf{E}}_0, \hat{\mathbf{H}}_0)$

$$\hat{\mathbf{E}}_0(\mathbf{X}) = (\kappa R)^{-3} \frac{h_{1,-2}}{i} \sum_{m=-1}^1 \hat{\alpha}_{1,m}^{(0),\text{H}} \left[-\nabla_{S^2} Y_{1,m} + 2 Y_{1,m} \mathbf{e}_r \right] + \underset{R \rightarrow \infty}{O}(R^{-4})$$

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Boundary conditions on $\hat{\Gamma} = S^2 \implies$ Unicity of $\hat{\alpha}_{1,m}^{(0),\text{H}}$ and $\hat{\alpha}_{1,m}^{(0),\text{E}}$

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► Expression of $\tilde{\mathbf{E}}_4$ (matching conditions)

$$\begin{aligned} \tilde{\mathbf{E}}_4 = & \sum_{m=-1}^1 \left\{ \hat{\alpha}_{1,m}^{(1),E} h_1^{(1)}(\kappa r) \mathbf{curl}_{\mathbb{S}^2} Y_{1,m} \right. \\ & \left. + \frac{\hat{\alpha}_{1,m}^{(1),H}}{i\kappa r} \left([h_1^{(1)}(\kappa r) + \kappa r h_1^{(1)'}(\kappa r)] \nabla_{\mathbb{S}^2} Y_{1,m} + 2 h_1^{(1)}(\kappa r) Y_{1,m} \mathbf{e}_r \right) \right\} \\ & + \sum_{m=-2}^2 \left\{ \hat{\alpha}_{2,m}^{(0),E} h_2^{(1)}(\kappa r) \mathbf{curl}_{\mathbb{S}^2} Y_{2,m} \right. \\ & \left. + \frac{\hat{\alpha}_{2,m}^{(0),H}}{i\kappa r} \left([h_2^{(1)}(\kappa r) + \kappa r h_2^{(1)'}(\kappa r)] \nabla_{\mathbb{S}^2} Y_{2,m} + 6 h_2^{(1)}(\kappa r) Y_{2,m} \mathbf{e}_r \right) \right\} \end{aligned}$$

Explicit computation in the case of the sphere $\mathcal{O}_\delta = \mathcal{B}(0, \delta)$

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$$\hat{\mathbf{E}}_1(\mathbf{X}) = (\kappa R)^{-3} \frac{h_{1,-2}}{i} \sum_{m=-1}^1 \hat{\alpha}_{1,m}^{(1),\text{H}} [-\nabla_{\mathbb{S}^2} Y_{1,m} + 2 Y_{1,m} \mathbf{e}_r] + \underset{R \rightarrow \infty}{O}(R^{-4})$$

$$\hat{\mathbf{H}}_1(\mathbf{X}) = (\kappa R)^{-3} \frac{h_{1,-2}}{i} \sum_{m=-1}^1 \hat{\alpha}_{1,m}^{(1),\text{E}} [-\nabla_{\mathbb{S}^2} Y_{1,m} + 2 Y_{1,m} \mathbf{e}_r] + \underset{R \rightarrow \infty}{O}(R^{-4})$$

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Boundary conditions on $\hat{\Gamma} = S^2 \implies$ Unicity of $\hat{\alpha}_{2,m}^{(0),H}$ and $\hat{\alpha}_{2,m}^{(0),E}$

Outline

1 Method of matched asymptotic expansions

- Asymptotic domain decomposition
- Far field expansion
- Near field expansion
- Matching conditions
- Global approximation

2 Modal decomposition of the asymptotics

- Modal decomposition of the far field
- Modal decomposition of the near field
- Application of the matching conditions
- Case of the sphere: analytical solutions

3 Conclusion and perspectives

Conclusion and perspectives

- ▶ These tools are still valid for
 - ▶ Extension to **transmission problem**
 - ▶ Extension to **multi-scattering problem**
 - Superposition principle
- ▶ To finalize
 - ▶ **Stability** and **consistance**
 - Far field error
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 - ▶ **Boundary integral equations** for near field problems
- ▶ To do
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;; Thank you for your attention!

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