

## SHANNON, TURING AND HATS: INFORMATION THEORY INCOMPLETENESS

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WITMSE 2017

SEPTEMBER 13<sup>TH</sup>, 2017

# ROADMAP

1 GETTING RICH WITHOUT A DELOREAN

2 THE SHANNON APPROACH

3 DAYS OF INFINITE PAST

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# BACK TO BACK TO THE FUTURE

## PLOT OF EPISODE 2

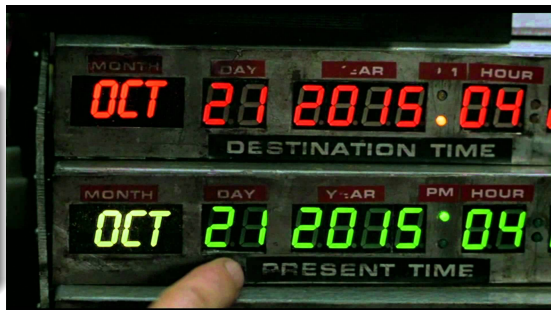
- A POOR GUY



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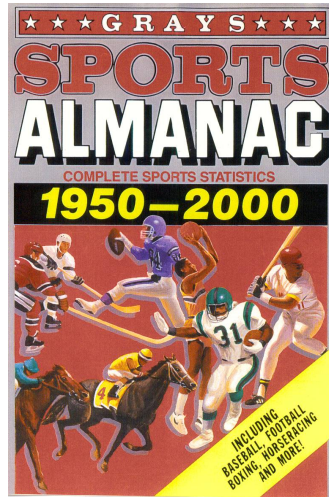
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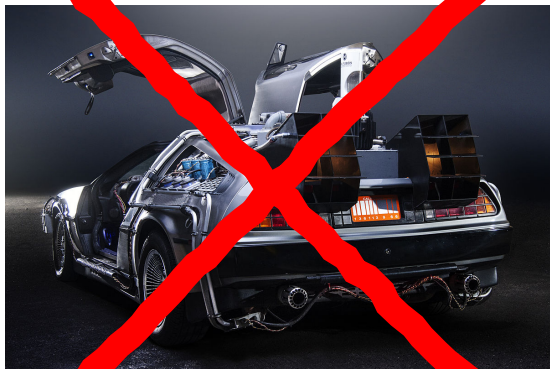


# BACK TO BACK TO THE FUTURE

## PLOT OF EPISODE 2

- A POOR GUY
- A TIME MACHINE
- AN ALMANAC
- A RICH GUY

WHAT IF NO TIME MACHINE?





# YMCA LMCA PROBLEM



BIFF IS A HUGE FAN OF ENGLISH SOCCER

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- CAN HE PREDICT NEXT YEAR CHAMPION?

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## PREDICTION OF A LCMA SEQUENCE

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- WITH A LONG ENOUGH SEQUENCE, BIFF CAN TRY TO PREDICT THINGS.

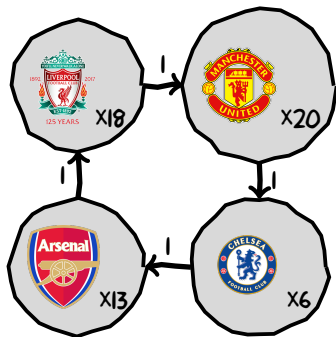
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## FULLY DETERMINISTIC

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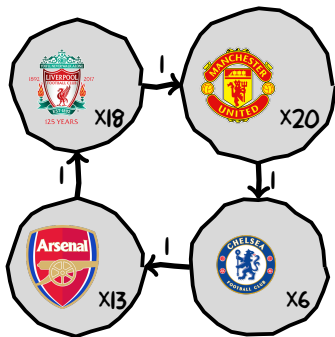
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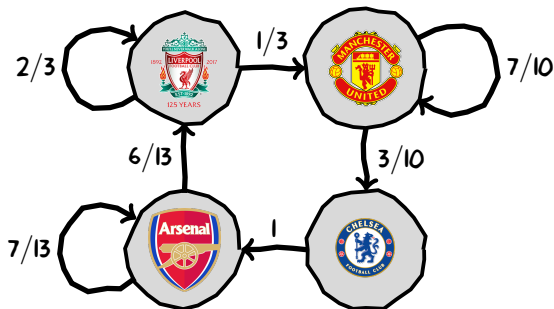
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100/100
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0



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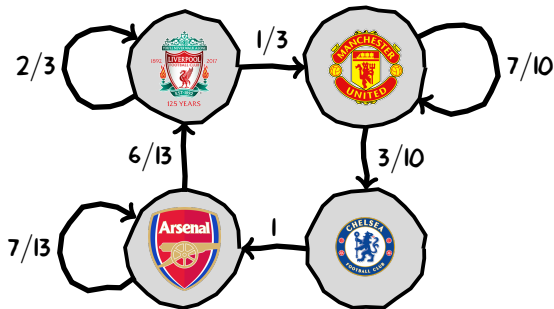
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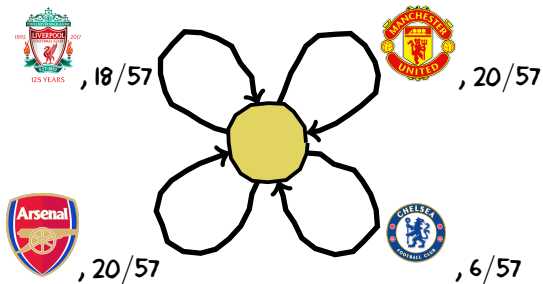
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- BIFF BEST SUCCESS RATE:  
68.4/100
- ENTROPY:  
0.83

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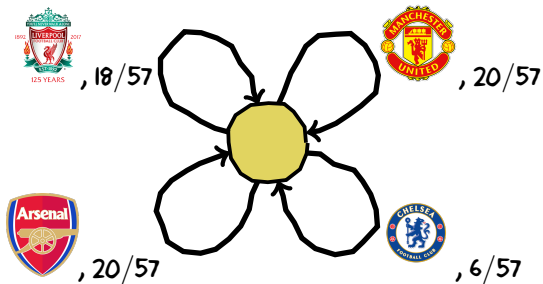
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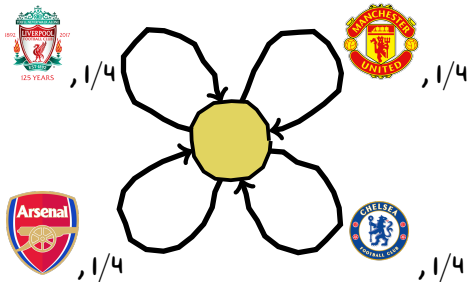
## MEMORYLESS

- BIFF BEST SUCCESS RATE:  
 $20/57 \approx 35/100$
- ENTROPY:  
1.88

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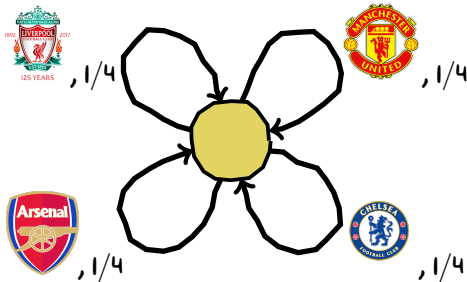
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## UNIFORMLY MEMORYLESS

- BIFF BEST SUCCESS RATE:  
 $1/4$
- ENTROPY:  
 $2$

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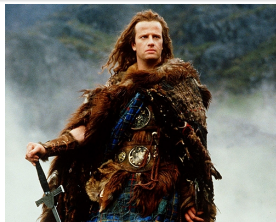
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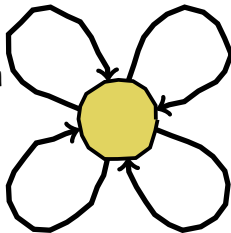
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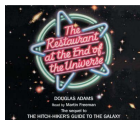
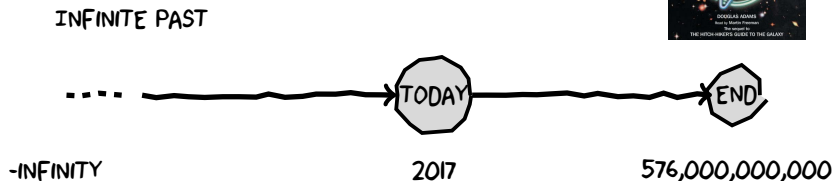




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- UNIVERSE ENDS AT YEAR 576,000,000,000 AD.



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BIFF CAN CHOOSE ONE REPRESENTATIVE SEQUENCE PER EQUIVALENCE CLASS.

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- TAKE THE REPRESENTATIVE OF THE EQUIVALENCE CLASS.
- ANNOUNCE THE RESULT PREDICTED BY THE REPRESENTATIVE.



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  - TRUE EVEN WITH MAXIMAL ENTROPY 2



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### CONCLUSION

THE EVENT **BIFF IS WRONG** CANNOT BE MEASURED.

# GETTING RID OF THE AXIOM OF CHOICE

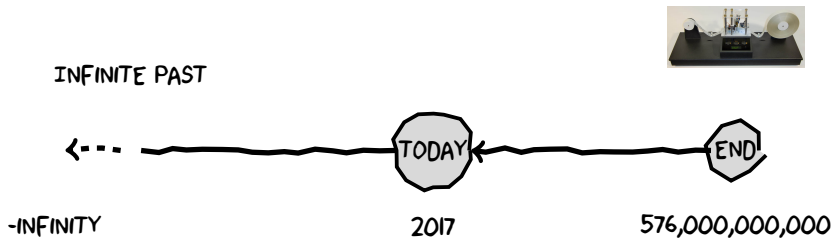
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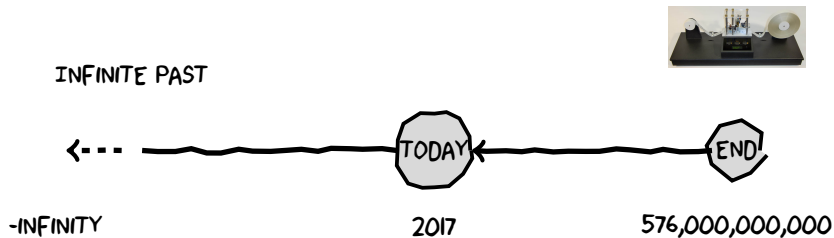
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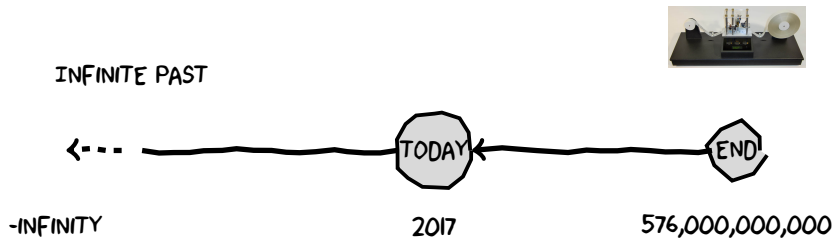
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- AXIOM OF CHOICE IS REMOVED



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- THE SET OF TURING MACHINES IS ENUMERABLE
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  - ▶ BIFF ALWAYS GUESSES THE SAME SEQUENCE
  - ▶ HE IS WRONG ONLY A FINITE NUMBER OF TIMES

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## CONCLUSION

BIFF GETS WRONGER AND WRONGER AS THE END IS NEAR:  
THE LESS BIFF KNOWS, THE BETTER HE PREDICTS.



## CONCLUSION

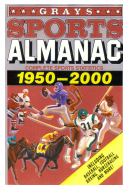
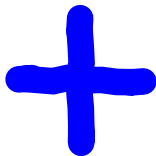
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- IS THERE A PROPER FRAMEWORK?
- VERY EARLY WORK WITH LOT OF OPEN QUESTIONS, E.G. INFINITE FUTURE?

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- BEST WAY TO GET RICH...



THANK YOU!