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# A new McKean-Vlasov stochastic interpretation of the parabolic-parabolic Keller-Segel model: The one-dimensional case.

Denis Talay\* and Milica Tomašević \*†

**Abstract:** In this paper we analyze a stochastic interpretation of the one-dimensional parabolic-parabolic Keller-Segel system without cut-off. It involves an original type of McKean-Vlasov interaction kernel. At the particle level, each particle interacts with all the past of each other particle by means of a time integrated functional involving a singular kernel. At the mean-field level studied here, the McKean-Vlasov limit process interacts with all the past time marginals of its probability distribution in a similarly singular way. We prove that the parabolic-parabolic Keller-Segel system in the whole Euclidean space and the corresponding McKean-Vlasov stochastic differential equation are well-posed for any values of the parameters of the model. **Key words:** Chemotaxis model; Keller–Segel system; Singular McKean-Vlasov non-linear stochastic differential equation.

**Classification:** 60H30 60H10 60K35.

## 1 Introduction

The standard  $d$ -dimensional parabolic–parabolic Keller–Segel model for chemotaxis describes the time evolution of the density  $\rho_t$  of a cell population and of the concentration  $c_t$  of a chemical attractant:

$$\begin{cases} \partial_t \rho(t, x) = \nabla \cdot (\frac{1}{2} \nabla \rho - \chi \rho \nabla c)(t, x), & t > 0, x \in \mathbb{R}^d, \\ \alpha \partial_t c(t, x) = \frac{1}{2} \Delta c(t, x) - \lambda c(t, x) + \rho(t, x), & t > 0, x \in \mathbb{R}^d. \\ \rho(0, x) = \rho_0(x), \quad c(0, x) = c_0(x), \end{cases} \quad (1)$$

See e.g. Corrias [4], Perthame [13] and references therein for theoretical results on this system of PDEs and applications to Biology.

Recently, stochastic interpretations have been proposed for a simplified version of the model, that is, the parabolic-elliptic model which corresponds to the value  $\alpha = 0$ . They all rely on the fact that, in the parabolic-elliptic case, the equations for  $\rho_t$  and  $c_t$  can be decoupled and  $c_t$  can be explicitated as the convolution of the initial condition  $c_0$  and the kernel  $k(x) = -\frac{x}{2\pi|x|^2}$ . Consequently, the stochastic process of McKean–Vlasov type whose  $\rho_t$  is the time marginal density involves the singular interaction kernel  $k$ . This explains why, so far, only partial results are obtained and heavy techniques are used to get them. In Jabir et al. [9], one may find a short review of the works by Haškovec and Schmeiser [6], Fournier and Jourdain [5] and Cattiaux and Pédèches [3].

Budhiraja and Fan [2] have studied a McKean–Vlasov SDE related to a parabolic–parabolic version of the model with cut-off and a forcing potential term. Under a suitable convexity assumption, they obtain uniform in time concentration inequalities for the corresponding particle system and uniform in time error estimates for a numerical approximation of the exact McKean–Vlasov process.

We here deal with the parabolic–parabolic system ( $\alpha > 0$ ) without cut-off and study the McKean-Vlasov stochastic representation of the mild formulation of the equation satisfied by  $\rho_t$ . This representation involves a singular interaction kernel which is different from the one in the above mentioned approaches and does not seem to have been studied in the McKean-Vlasov non-linear SDE literature. The system reads

$$\begin{cases} dX_t = b^\sharp(t, X_t)dt + \left\{ \int_0^t (K_{t-s}^\sharp * p_s)(X_t) ds \right\} dt + dW_t, & t > 0, \\ p_s(y) dy := \mathcal{L}(X_s), \quad X_0 \sim \rho_0(x) dx, \end{cases} \quad (2)$$

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where  $K_t^\sharp(x) := \chi e^{-\lambda t} \nabla \left( \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|x|^2}{2t}} \right)$  and  $b^\sharp(t, x) := \chi e^{-\lambda t} \nabla \mathbb{E} c_0(x + W_t)$ . Here,  $(W_t)_{t \geq 0}$  is a  $d$ -dimensional Brownian motion on a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))$  and  $X_0$  is an  $\mathbb{R}^d$ -valued  $\mathcal{F}_0$ -measurable random variable. Notice that the formulation requires that the one dimensional time marginals of the law of the solution are absolutely continuous with respect to Lebesgue's measure and that the process interacts with all the past time marginals of its probability distribution through a functional involving a singular kernel.

The analysis of the well-posedness of this non-linear equation and the proof that  $p_s = \rho(s, \cdot)$  for any  $s$  are delicate, particularly in the multi-dimensional case when  $\chi$  is large enough to induce solutions with blow-ups in finite time. This theoretical work is still in progress [15]. As numerical simulations of the related particle system appear to be effective, it seems interesting to validate our approach in the one-dimensional case.

The objective of this paper is to prove general existence and uniqueness results for both the deterministic system (1) and the stochastic dynamics (2) in  $d = 1$ . In our companion paper [9] we show the well-posedness and propagation of chaos property of the corresponding particle system where each particle interacts with all the past of the other ones by means of a time integrated singular kernel.

In this one-dimensional framework the PDE (1) was previously studied by [12, 8] in bounded intervals  $I$  with periodic boundary conditions while we here deal with the problem posed on the whole space  $\mathbb{R}$ . In [12] one assumes  $\rho_0 \in L^2(I) \cap L^1(I)$  and  $c_0 \in H^1(I)$ . In [8] one assumes  $\rho_0 \in L^\infty(I) \cap L^1(I)$  and  $c_0 \in W^{\sigma, p}(I)$ , where  $p$  and  $\sigma$  belong to a particular set of parameters. Here, we only suppose that  $\rho_0$  is in  $L^1(\mathbb{R})$ .

We emphasize that we do not limit ourselves to the specific kernel  $K_t^\sharp(x)$  related to the Keller–Segel model. We below show that the mean-field PDE and stochastic differential equation of Keller–Segel type are well-posed for a whole class of time integrated singular kernels. The mean-field SDE cannot be analyzed by means of standard coupling methods or Wasserstein distance contractions. Both to construct local solutions and to go from local to global solutions, an important issue consists in properly defining the set of weak solutions without any assumption on the initial density  $\rho_0$ , which led us to introduce constraints on the time marginal densities. To prove that these constraints are satisfied in the limit of an iterative procedure (where the kernel is not cut off), the norms of the successive time marginal densities cannot be allowed to exponentially depend on the  $L^\infty$ -norm of the successive corresponding drifts. They neither can be allowed to depend on Hölder-norms of the drifts. Therefore, we use an accurate estimate (with explicit constants) on densities of one-dimensional diffusions with bounded measurable drifts which is obtained by a stochastic technique rather than the PDE techniques. This strategy allows us to get uniform bounds on the sequence of drifts, which is essential to get existence and uniqueness of the local solution to the non-linear martingale problem solved by any limit of the Picard procedure, and to suitably paste local solutions when constructing the global solution.

The paper is organized as follows. In Section 2 we state our main results. In Section 3 we prove a preliminary estimate on the probability density of diffusions whose drift is only supposed Borel measurable and bounded. In Section 4 we study a non-linear McKean–Vlasov–Fokker–Planck equation. In Section 5 we prove the local existence and uniqueness of a solution to a non-linear stochastic differential equation more general than (2) (for  $d = 1$ ). In Section 6 we get the global well-posedness of this equation. In Section 7 we apply the preceding result to the specific case of the one-dimensional parabolic–parabolic Keller–Segel model. The appendix section 8 concerns an explicit formula for the transition density of a particular diffusion.

**Notation.** In all the paper we denote by  $C_T$ ,  $C_T(b_0, p_0)$ , etc., any constant which depends on  $T$  and the other specified parameters, but is uniform w.r.t.  $t \in [0, T]$  and may change from line.

## 2 Our main results

Our first main result concerns the well-posedness of a non-linear one-dimensional stochastic differential equation (SDE) with a non standard McKean–Vlasov interaction kernel which at each time  $t$  involves in a singular way all the time marginals up to time  $t$  of the probability distribution of the solution. As our technique of analysis is not limited to the above kernel  $K^\sharp$ , we consider the following McKean–Vlasov stochastic equation:

$$\begin{cases} dX_t = b(t, X_t)dt + \left\{ \int_0^t (K_{t-s} * p_s)(X_t) ds \right\} dt + dW_t, & t \leq T, \\ p_s(y) dy := \mathcal{L}(X_s), & X_0 \sim p_0, \end{cases} \quad (3)$$

and in all the sequel we assume the following conditions on the interaction kernel.

**Hypothesis (H).** *The function  $K$  defined on  $\mathbb{R}^+ \times \mathbb{R}$  is such that for any  $T > 0$ :*

1. For any  $t > 0$ ,  $K_t$  is in  $L^1(\mathbb{R})$ .
2. For any  $t > 0$  the function  $K_t(x)$  is a bounded continuous function on  $\mathbb{R}$ .
3. The set of points  $x \in \mathbb{R}$  such that  $\lim_{t \rightarrow 0} K_t(x) < \infty$  has full Lebesgue measure.
4. For any  $t > 0$ , the function  $f_1(t) := \int_0^t \frac{\|K_{t-s}\|_{L^1(\mathbb{R})}}{\sqrt{s}} ds$  is well defined and bounded on  $[0, T]$ .
5. For any  $T > 0$  there exists  $C_T$  such that, for any probability density  $\phi$  on  $\mathbb{R}$ ,

$$\sup_{(t,x) \in (0,T] \times \mathbb{R}} \int \phi(y) \|K_{\cdot}(x-y)\|_{L^1(0,t)} dy \leq C_T.$$

6. Finally,

$$\sup_{0 \leq t \leq T} \int_0^T \|K_{T+t-s}\|_{L^1(\mathbb{R})} \frac{1}{\sqrt{s}} ds \leq C_T.$$

As emphasized in the introduction, the well-posedness of the system (3) cannot be obtained by applying known results in the literature.

Given  $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$  and a family of densities  $(p_t)_{t \leq T}$  we set

$$B(t, x; p) := \int_0^t (K_{t-s} * p_s)(x) ds. \quad (4)$$

We now define the notion of a weak solution to (3).

**Definition 2.1.** *The family  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), X, W)$  is said to be a weak solution to the equation (3) up to time  $T > 0$  if:*

1.  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))$  is a filtered probability space.
2. The process  $X := (X_t)_{t \in [0, T]}$  is real-valued, continuous, and  $(\mathcal{F}_t)$ -adapted. In addition, the probability distribution of  $X_0$  has density  $p_0$ .
3. The process  $W := (W_t)_{t \in [0, T]}$  is a one-dimensional  $(\mathcal{F}_t)$ -Brownian motion.
4. The probability distribution  $\mathbb{P} \circ X^{-1}$  has time marginal densities  $(p_t, t \in [0, T])$  with respect to Lebesgue measure which satisfy

$$\forall 0 < t \leq T, \quad \|p_t\|_{L^\infty(\mathbb{R})} \leq \frac{C_T}{\sqrt{t}}. \quad (5)$$

5. For all  $t \in [0, T]$  and  $x \in \mathbb{R}$ , one has that  $\int_0^t |b(s, x)| ds < \infty$ .
6.  $\mathbb{P}$ -a.s. the pair  $(X, W)$  satisfies (3).

**Remark 2.2.** *For any  $T > 0$  Inequality (5) and Hypothesis (H-4) lead to*

$$\sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}} |B(t, x, p)| \leq C_T.$$

The following theorem provides existence and uniqueness of the weak solution to (3).

**Theorem 2.3.** *Let  $T > 0$ . Suppose that  $p_0 \in L^1(\mathbb{R})$  is a probability density function and  $b \in L^\infty([0, T] \times \mathbb{R})$  is continuous w.r.t. the space variable. Under the hypothesis (H), Eq. (3) admits a unique weak solution in the sense of Definition 2.1.*

We finally state an easy result which is useful to prove the propagation of chaos in the case of Keller-Segel kernel (see [9]):

**Corollary 2.4.** *In addition to the assumptions of Theorem 2.3 suppose the following hypothesis:*

H-7. *for any  $t > 0$ ,  $K_t$  is in  $L^2(\mathbb{R})$  and the function  $f_2(t) := \int_0^t \frac{\|K_{t-s}\|_{L^2(\mathbb{R})}}{s^{1/4}} ds$  is well defined and bounded on  $[0, T]$ .*

Then, there exists a unique weak solution to (3) in the sense of the Definition 2.1 modified as follows: Instead of (5) one imposes

$$\forall 0 < t \leq T, \quad \|p_t\|_{L^2(\mathbb{R})} \leq \frac{C_T}{t^{1/4}}. \quad (6)$$

Our next result concerns the well-posedness of the the one-dimensional parabolic-parabolic Keller-Segel model

$$\begin{cases} \frac{\partial \rho}{\partial t}(t, x) = \frac{\partial}{\partial x} \cdot \left( \frac{1}{2} \frac{\partial \rho}{\partial x} - \chi \rho \frac{\partial c}{\partial x} \right)(t, x), & t > 0, \quad x \in \mathbb{R}, \\ \frac{\partial c}{\partial t}(t, x) = \frac{1}{2} \frac{\partial^2 c}{\partial x^2}(t, x) - \lambda c(t, x) + \rho(t, x), & t > 0, \quad x \in \mathbb{R}, \\ \rho(0, x) = \rho_0(x), \quad c(0, x) = c_0(x). \end{cases} \quad (7a)$$

$$\begin{cases} \frac{\partial c}{\partial t}(t, x) = \frac{1}{2} \frac{\partial^2 c}{\partial x^2}(t, x) - \lambda c(t, x) + \rho(t, x), & t > 0, \quad x \in \mathbb{R}, \\ \rho(0, x) = \rho_0(x), \quad c(0, x) = c_0(x). \end{cases} \quad (7b)$$

The parameters  $\chi$  and  $\lambda$  are strictly positive. As this system preserves the total mass, that is,

$$\forall t > 0, \quad \int_{\Omega} \rho(t, x) dx = \int_{\Omega} \rho_0(x) dx =: M,$$

the new functions  $\tilde{\rho}(t, x) := \frac{\rho(t, x)}{M}$  and  $\tilde{c}(t, x) := \frac{c(t, x)}{M}$  satisfy the system (7) with the new parameter  $\tilde{\chi} := \chi M$ . Therefore, w.l.o.g. we may and do thereafter assume that  $M = 1$ .

Denote by  $g_t$  the density of  $W_t$ . We define the notion of solution for the system (7):

**Definition 2.5.** *Given the functions  $\rho_0$  and  $c_0$ , and the constants  $\chi > 0$ ,  $\lambda \geq 0$ ,  $T > 0$ , the pair  $(\rho, c)$  is said to be a solution to (7) if  $\rho(t, \cdot)$  is a probability density function for every  $0 \leq t \leq T$ ,  $c$  is in  $L^\infty([0, T]; C_b^1(\mathbb{R}))$ , one has  $\|\rho(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{C_T}{\sqrt{t}}$  for any  $t \in (0, T]$ , and the following equality*

$$\rho(t, x) = g_t * \rho_0(x) - \chi \int_0^t \frac{\partial g_{t-s}}{\partial x} * \left( \frac{\partial c}{\partial x}(s, \cdot) \rho(s, \cdot) \right)(x) ds \quad (8)$$

is satisfied in the sense of the distributions with

$$c(t, x) = e^{-\lambda t} (g(t, \cdot) * c_0)(x) + \int_0^t e^{-\lambda s} (g_s * \rho(t-s, \cdot))(x) ds. \quad (9)$$

Notice that the function  $c(t, x)$  defined by (9) is a mild solution to (7b). These solutions are known as integral solutions and they have already been studied in PDE literature for the two-dimensional Keller-Segel model for which sub-critical and critical regimes exist depending on the parameters of the model (see [4] and references therein). In the one-dimensional case there is no critical regime as shown by the following theorem.

**Corollary 2.6.** *Assume that  $\rho_0 \in L^1(\mathbb{R})$  and  $c_0 \in C_b^1(\mathbb{R})$ . Given any  $\chi > 0$ ,  $\lambda \geq 0$  and  $T > 0$ , the time marginals  $\rho(t, x) \equiv p_t(x)$  of the probability distribution of the unique solution to Eq. (2) with  $d = 1$  and the corresponding function  $c(t, x)$  provide a global solution to (7) in the sense of Definition 2.5. Any other solution  $(\rho^1, c^1)$  with the same initial condition  $(\rho_0, c_0)$  satisfies  $\|\rho^1(t, \cdot) - \rho(t, \cdot)\|_{L^1(\mathbb{R})} = 0$  and  $\|\frac{\partial c^1}{\partial x}(t, \cdot) - \frac{\partial c}{\partial x}(t, \cdot)\|_{L^1(\mathbb{R})} = 0$  for every  $0 \leq t \leq T$ .*

**Remark 2.7.** *From estimates below we could deduce some additional regularity results which we do not need here: See Remark 3.3. In particular, if  $\rho_0 \in L^\infty(\mathbb{R})$ , then  $\rho \in L^\infty([0, T]; L^1 \cap L^\infty(\mathbb{R}))$ . If  $\rho_0 \in L^2(\mathbb{R})$ , then  $\rho \in L^\infty([0, T]; L^1 \cap L^2(\mathbb{R}))$  and  $t^{1/4} \|\rho_t\|_{L^\infty(\mathbb{R})} \leq C$ . As explained in the introduction, we prefer to only suppose that  $\rho_0 \in L^1(\mathbb{R})$ .*

### 3 Preliminary: A density estimate

In the sequel, we will get local solutions to (3) and extend them to global solutions by means of an iterative procedure. The  $L^\infty$ -norms of the successive drifts are needed to be bounded from above uniformly w.r.t. the iteration step. Standard density estimates obtained by using Girsanov theorem or PDE analysis do not help to this purpose. The reason is that they involve constants which exponentially depend on the  $L^\infty$ -norm (or even Hölder-norm) of the drifts. We therefore proceed by using an accurate pointwise estimate (with explicit constants) on densities of one-dimensional diffusions with bounded measurable drifts. Estimate (11) below is obtained by using a stochastic technique. Its drawback is that the map  $y \mapsto p_y^\beta(t, x, y)$  is not a probability

density function. However, it suffices to nicely bound the successive drifts of the Picard iterations as shown by Proposition 5.3.

Let  $X^{(b)}$  be a process defined by

$$X_t^{(b)} = X_0 + \int_0^t b(s, X_s^{(b)}) ds + W_t, \quad t \in [0, T]. \quad (10)$$

To obtain  $L^\infty(\mathbb{R})$  estimates for the transition probability density  $p^{(b)}(t, x, y)$  of  $X^{(b)}$  under the only assumption that the drift  $b(t, x)$  is measurable and uniformly bounded we slightly extend the estimate proved in Qian and Zheng [14] for time homogeneous drift coefficients  $b(x)$ . We here propose a proof different from the original one. It avoids the use of densities of pinned diffusions and the claim that  $p^{(b)}(t, x, y)$  is continuous w.r.t. all the variables which does not seem obvious to us. In our proof we adapt the method in [11], the main difference being that instead of the Wiener measure our reference measure is the probability distribution of the particular diffusion process  $X^\beta$  considered in [14] and defined by

$$X_t^\beta = X_0 + \beta \int_0^t \operatorname{sgn}(y - X_s^\beta) ds + W_t.$$

**Theorem 3.1.** *Let  $X^{(b)}$  be the process defined in (10) with  $X_0 = x$ . Let  $p_y^\beta(t, x, z)$  be the transition density of  $X^\beta$ . Assume  $\beta := \sup_{t \in [0, T]} \|b(t, \cdot)\|_\infty < \infty$ . Then for all  $y \in \mathbb{R}$  and  $t \in (0, T]$  it holds that*

$$p^{(b)}(t, x, y) \leq p_y^\beta(t, x, y) = \frac{1}{\sqrt{2\pi t}} \int_{\frac{|x-y|}{\sqrt{t}}}^\infty z e^{-\frac{(z-\beta\sqrt{t})^2}{2}} dz. \quad (11)$$

*Proof.* Let  $f \in \mathcal{C}_K^\infty(\mathbb{R})$  and fix  $t \in (0, T]$ . Consider the parabolic PDE driven by the infinitesimal generator of  $X^\beta$ :

$$\begin{cases} \frac{\partial u}{\partial t}(s, x) + \frac{1}{2} \frac{\partial^2 u}{\partial x^2}(s, x) + \beta \operatorname{sgn}(y - x) \frac{\partial u}{\partial x}(s, x) = 0, & 0 \leq s < t, \quad x \in \mathbb{R}, \\ u(t, x) = f(x), & x \in \mathbb{R}. \end{cases} \quad (12)$$

In view of Veretennikov [16, Thm.1] there exists a solution  $u(s, x) \in W_p^{1,2}([0, t] \times \mathbb{R})$ . Applying the Itô-Krylov formula to  $u(s, X_s^\beta)$  we obtain that

$$u(s, x) = \int f(z) p_y^\beta(t - s, x, z) dz.$$

The formula (35) from our appendix allows us to differentiate under the integral sign:

$$\frac{\partial u}{\partial x}(s, x) = \int f(z) \frac{\partial p_y^\beta}{\partial x}(t - s, x, z) dz, \quad \forall 0 \leq s < t \leq T.$$

Fix  $0 < \varepsilon < t$ . Now apply the Itô-Krylov formula to  $u(s, X_s^{(b)})$  for  $0 \leq s \leq t - \varepsilon$  and use the PDE (12). It comes:

$$\mathbb{E}(u(t - \varepsilon, X_{t-\varepsilon}^{(b)})) = u(0, x) + \mathbb{E} \int_0^{t-\varepsilon} (b(s, X_s^{(b)}) - \beta \operatorname{sgn}(y - X_s^{(b)})) \frac{\partial u}{\partial x}(s, X_s^{(b)}) ds.$$

In view of Corollary 8.2 in the appendix there exists a function  $h \in L^1([0, t] \times \mathbb{R})$  such that

$$\forall 0 < s < t \leq T, \quad \forall y, z \in \mathbb{R}, \quad \mathbb{E} \left| \frac{\partial p_y^\beta}{\partial x}(t - s, X_s^{(b)}, z) \right| \leq C_{T, \beta, x, y} h(s, z). \quad (13)$$

Consequently,

$$\begin{aligned} \mathbb{E}(u(t - \varepsilon, X_{t-\varepsilon}^{(b)})) &= \int f(z) p_y^\beta(t, x, z) dz \\ &+ \int f(z) \int_0^{t-\varepsilon} \mathbb{E} \left\{ (b(s, X_s^{(b)}) - \beta \operatorname{sgn}(y - X_s^{(b)})) \frac{\partial p_y^\beta}{\partial x}(t - s, X_s^{(b)}, z) \right\} ds dz. \end{aligned}$$

Let now  $\epsilon$  tend to 0. By Lebesgue's dominated convergence theorem we obtain

$$\begin{aligned} \int f(z)p^{(b)}(t,x,z)dz &= \int f(z)p_y^\beta(t,x,z)dz \\ &+ \int f(z) \int_0^t \mathbb{E} \left\{ (b(s, X_s^{(b)}) - \beta \text{sgn}(y - X_s^{(b)})) \frac{\partial p_y^\beta}{\partial x}(t-s, X_s^{(b)}, z) \right\} ds dz. \end{aligned}$$

Therefore the density  $p^{(b)}$  satisfies:

$$p^{(b)}(t,x,z) = p_y^\beta(t,x,z) + \int_0^t \mathbb{E} \left\{ (b(s, X_s^{(b)}) - \beta \text{sgn}(y - X_s^{(b)})) \frac{\partial p_y^\beta}{\partial x}(t-s, X_s^{(b)}, z) \right\} ds.$$

As noticed in [14], in view of Formula (36) from our appendix we have for any  $x \in \mathbb{R}$

$$(b(s,x) - \beta \text{sgn}(y-x)) \frac{\partial}{\partial x} p_y^\beta(t-s,x,y) \leq 0.$$

This leads us to choose  $z = y$  in the preceding equality, which gives us

$$p^{(b)}(t,x,y) = p_y^\beta(t,x,y) + \int_0^t \mathbb{E} \left\{ (b(s, X_s^{(b)}) - \beta \text{sgn}(y - X_s^{(b)})) \frac{\partial p_y^\beta}{\partial x}(t-s, X_s^{(b)}, y) \right\} ds,$$

from which

$$\forall t \leq T, \quad p^{(b)}(t,x,y) \leq p_y^\beta(t,x,y).$$

We finally use Qian and Zheng's explicit representation (see [14] and our appendix section 8). □

**Corollary 3.2.** *Assume  $X_0$  is distributed according to the probability density function  $p_0$  on  $\mathbb{R}$ . Denote by  $p(t, \cdot)$  the probability density of  $X_t^{(b)}$ . One has*

$$\|p(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2\pi t}} + \beta. \quad (14)$$

*Proof.* In view of (11) we have

$$\begin{aligned} p(t,y) &\leq \frac{1}{\sqrt{2\pi t}} \int p_0(x) \int_{\frac{|x-y|}{\sqrt{t}}}^\infty z e^{-\frac{(z-\beta\sqrt{t})^2}{2}} dz dx \\ &\leq \frac{1}{\sqrt{2\pi t}} \int p_0(x) \int_{\frac{|x-y|}{\sqrt{t}} - \beta\sqrt{t}}^\infty (z + \beta\sqrt{t}) e^{-\frac{z^2}{2}} dz dx \\ &= \frac{1}{\sqrt{2\pi t}} \left( \int p_0(x) e^{-\frac{(|x-y|-\beta t)^2}{2t}} dx + \beta\sqrt{t} \int p_0(x) \int_{\frac{|x-y|}{\sqrt{t}} - \beta\sqrt{t}}^\infty e^{-\frac{z^2}{2}} dz dx \right) \\ &\leq \frac{1}{\sqrt{2\pi t}} \int p_0(x) e^{-\frac{(|y-x|-\beta t)^2}{2t}} dx + \beta. \end{aligned}$$

□

**Remark 3.3.** *If  $p_0 \in L^\infty(\mathbb{R})$ , the above calculation shows that*

$$\|p(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq 2\|p_0\|_{L^\infty(\mathbb{R})} + \beta.$$

*If  $p_0 \in L^p(\mathbb{R})$ ,  $p > 1$ , Hölder's inequality leads to*

$$\frac{1}{\sqrt{2\pi t}} \int p_0(x) e^{-\frac{(|y-x|-\beta t)^2}{2t}} dx \leq \frac{\|p_0\|_{L^p(\mathbb{R})}}{\sqrt{2\pi t}} \left( \int e^{-q\frac{(|y-x|-\beta t)^2}{2t}} dx \right)^{1/q} \leq \frac{C_q t^{\frac{1}{2q}}}{\sqrt{t}} = \frac{C_q}{t^{\frac{1}{2p}}}.$$

## 4 A non-linear McKean–Vlasov–Fokker–Planck equation

**Proposition 4.1.** *Let  $T > 0$ . Assume  $p_0 \in L^1(\mathbb{R})$ ,  $b \in L^\infty([0, T] \times \mathbb{R})$  and Hypothesis (H). Let  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), X, W)$  be a weak solution to (3) until  $T$ . Then,*

1. *The marginals  $(p_t)_{t \in [0, T]}$  satisfy in the sense of the distributions the mild equation*

$$\forall t \in (0, T], \quad p_t = g_t * p_0 - \int_0^t \frac{\partial g_{t-s}}{\partial x} * (p_s(b(s, \cdot) + B(s, \cdot; p))) ds. \quad (15)$$

2. *Equation (15) admits at most one solution  $(p_t)_{t \in [0, T]}$  which for any  $t \in [0, T]$  belongs to  $L^1(\mathbb{R})$  and satisfies (5).*

*Proof.* We successively prove (15) and the uniqueness of its solution in  $L^1(\mathbb{R})$ .

1. Now, for  $f \in C_b^2(\mathbb{R})$  consider the Cauchy problem

$$\begin{cases} \frac{\partial G}{\partial s} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} = 0, & 0 \leq s < t, \quad x \in \mathbb{R}, \\ \lim_{s \rightarrow t^-} G(s, x) = f(x). \end{cases} \quad (16)$$

The function

$$G_{t,f}(s, x) = \int f(y) g_{t-s}(x - y) dy$$

is a smooth solution to (16). Applying Itô's formula we get

$$\begin{aligned} G_{t,f}(t, X_t) - G_{t,f}(0, X_0) &= \int_0^t \frac{\partial G_{t,f}}{\partial s}(s, X_s) ds + \int_0^t \frac{\partial G_{t,f}}{\partial x}(s, X_s) (b(s, X_s) + B(s, X_s; p)) ds \\ &\quad + \int_0^t \frac{\partial G_{t,f}}{\partial x}(s, X_s) dW_s + \frac{1}{2} \int_0^t \frac{\partial^2 G_{t,f}}{\partial x^2}(s, X_s) ds. \end{aligned}$$

Using (16) we obtain

$$\mathbb{E}f(X_t) = \mathbb{E}G_{t,f}(0, X_0) + \int_0^t \mathbb{E} \left[ \frac{\partial G_{t,f}}{\partial x}(s, X_s) (b(s, X_s) + B(s, X_s; p)) \right] ds =: I + II. \quad (17)$$

On the one hand one has

$$I = \int \int f(y) g_t(y - x) dy p_0(x) dx = \int f(y) (g_t * p_0)(y) dy.$$

On the second hand one has

$$\begin{aligned} II &= \int_0^t \int \frac{\partial}{\partial x} \left[ \int f(y) g_{t-s}(x - y) dy \right] (b(s, x) + B(s, x; p)) p_s(x) dx ds \\ &= \int_0^t \int \int f(y) \frac{\partial g_{t-s}}{\partial x}(x - y) dy (b(s, x) + B(s, x; p)) p_s(x) dx ds \\ &= - \int f(y) \int_0^t \left[ \frac{\partial g_{t-s}}{\partial x} * ((b(s, \cdot) + B(s, \cdot; p)) p_s) \right](y) ds dy. \end{aligned}$$

Thus (17) can be written as

$$\int f(y) p_t(y) dx = \int f(y) (g_t * p_0)(y) dy + \int f(y) \int_0^t \left[ \frac{\partial g_{t-s}}{\partial x} * ((b(s, \cdot) + B(s, \cdot; p)) p_s) \right](y) ds dy,$$

which is the mild equation (15).



2. Assume  $p_t^1$  and  $p_t^2$  are two mild solutions in the sense of the distributions to (15) which satisfy

$$\exists C > 0, \forall t \in (0, T], \quad \|p_t^1\|_{L^\infty(\mathbb{R})} + \|p_t^2\|_{L^\infty(\mathbb{R})} \leq \frac{C_T}{\sqrt{t}}.$$

Then,

$$\begin{aligned} \|p_t^1 - p_t^2\|_{L^1(\mathbb{R})} &\leq \int_0^t \left\| \frac{\partial g_{t-s}}{\partial x} * [B(s, \cdot; p^1)p_s^1 - B(s, \cdot; p^2)p_s^2] \right\|_{L^1(\mathbb{R})} ds \\ &\quad + \int_0^t \left\| \frac{\partial g_{t-s}}{\partial x} * [b(s, \cdot)(p_s^1 - p_s^2)] \right\|_{L^1(\mathbb{R})} ds \\ &\leq \int_0^t \left\| \frac{\partial g_{t-s}}{\partial x} * [(B(s, \cdot; p^1) - B(s, \cdot; p^2))p_s^1] \right\|_{L^1(\mathbb{R})} ds \\ &\quad + \int_0^t \left\| \frac{\partial g_{t-s}}{\partial x} * [(p_s^1 - p_s^2)B(s, \cdot; p^2)] \right\|_{L^1(\mathbb{R})} ds \\ &\quad + \int_0^t \left\| \frac{\partial g_{t-s}}{\partial x} * [b(s, \cdot)(p_s^1 - p_s^2)] \right\|_{L^1(\mathbb{R})} ds \\ &=: I + II + III. \end{aligned}$$

As

$$\left\| \frac{\partial g_{t-s}}{\partial x} \right\|_{L^1(\mathbb{R})} \leq \frac{C_T}{\sqrt{t-s}},$$

the convolution inequality  $\|f * h\|_{L^1(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})} \|h\|_{L^1(\mathbb{R})}$  and Remark 2.2 lead to

$$II \leq \int_0^t \left\| \frac{\partial g_{t-s}}{\partial x} \right\|_{L^1(\mathbb{R})} \|(p_s^1 - p_s^2)B(s, \cdot; p^2)\|_{L^1(\mathbb{R})} ds \leq C_T \int_0^t \frac{\|p_s^1 - p_s^2\|_{L^1(\mathbb{R})}}{\sqrt{t-s}} ds.$$

As  $b$  is bounded, we also have

$$|III| \leq C_T \int_0^t \frac{\|p_s^1 - p_s^2\|_{L^1(\mathbb{R})}}{\sqrt{t-s}} ds.$$

We now turn to  $I$ . Notice that

$$\|B(s, \cdot; p^1) - B(s, \cdot; p^2)\|_{L^1(\mathbb{R})} \leq \int_0^s \|K_{s-\tau}\|_{L^1(\mathbb{R})} \|p_\tau^1 - p_\tau^2\|_{L^1(\mathbb{R})} d\tau,$$

from which, since by hypothesis  $(p_t)$  satisfies (5),

$$\begin{aligned} I &\leq \int_0^t \frac{C_T}{\sqrt{t-s}\sqrt{s}} \int_0^s \|K_{s-\tau}\|_{L^1(\mathbb{R})} \|p_\tau^1 - p_\tau^2\|_{L^1(\mathbb{R})} d\tau ds \\ &= \int_0^t \|p_\tau^1 - p_\tau^2\|_{L^1(\mathbb{R})} \int_\tau^t \frac{C_T}{\sqrt{t-s}\sqrt{s}} \|K_{s-\tau}\|_{L^1(\mathbb{R})} ds d\tau. \end{aligned}$$

In addition, using Hypothesis (H-4),

$$\int_\tau^t \frac{1}{\sqrt{t-s}\sqrt{s}} \|K_{s-\tau}\|_{L^1(\mathbb{R})} ds \leq \frac{1}{\sqrt{\tau}} \int_\tau^t \frac{1}{\sqrt{t-s}} \|K_{s-\tau}\|_{L^1(\mathbb{R})} ds = \frac{1}{\sqrt{\tau}} \int_0^{t-\tau} \frac{\|K_s\|_{L^1(\mathbb{R})}}{\sqrt{t-\tau-s}} ds \leq \frac{C_T}{\sqrt{\tau}}.$$

It comes:

$$I \leq C_T \int_0^t \frac{\|p_\tau^1 - p_\tau^2\|_{L^1(\mathbb{R})}}{\sqrt{\tau}} d\tau.$$

Gathering the preceding estimates we obtain

$$\|p_t^1 - p_t^2\|_{L^1(\mathbb{R})} \leq C_T \int_0^t \frac{\|p_s^1 - p_s^2\|_{L^1(\mathbb{R})}}{\sqrt{t-s}} ds + C_T \int_0^t \frac{\|p_s^1 - p_s^2\|_{L^1(\mathbb{R})}}{\sqrt{s}} ds.$$

Applying a Singular Gronwall Lemma (see Lemma 4.2 below), we conclude

$$\forall t \in (0, T], \quad \|p_t^1 - p_t^2\|_{L^1(\mathbb{R})} = 0,$$

which ends the proof.  $\square$

In the above proof we have used the following result:

**Lemma 4.2.** *Let  $(u(t))_{t \geq 0}$  be a non-negative bounded function such that for a given  $T > 0$ , there exists a positive constant  $C_T$  such that for any  $t \in (0, T]$ :*

$$u(t) \leq C_T \int_0^t \frac{u(s)}{\sqrt{s}} ds + C_T \int_0^t \frac{u(s)}{\sqrt{t-s}} ds. \quad (18)$$

Then,  $u(t) = 0$  for any  $t \in (0, T]$ .

*Proof.* Set  $u_t^* := \sup_{s \leq t} u(s)$ . The inequality (18) implies that

$$u_t^* \leq 4C_T \sqrt{t} u_t^*.$$

Set  $T^* := \frac{1}{32C_T^2}$ . If  $T \leq T^*$ , then for  $t \leq T$ , we have  $u_t^* \leq \frac{u_t^*}{\sqrt{2}}$ . Thus,  $u_t^* = 0$  for every  $t \leq T$  and the lemma is proved. If  $T > T^*$ , for  $T^* < t < T$ ,

$$u(t) \leq C_T \int_0^{T^*} \left( \frac{1}{\sqrt{t-s}} + \frac{1}{\sqrt{s}} \right) u(s) ds + \int_{T^*}^t \left( \frac{1}{\sqrt{t-s}} + \frac{1}{\sqrt{s}} \right) u(s) ds.$$

The first integral is null, since  $u(T^*) = 0$ . Thus,

$$u(t) \leq \int_0^{t-T^*} \left( \frac{1}{\sqrt{t-T^*-\theta}} + \frac{1}{\sqrt{T^*+\theta}} \right) u(T^*+\theta) d\theta \leq \int_0^{t-T^*} \left( \frac{1}{\sqrt{t-T^*-\theta}} + \frac{1}{\sqrt{T^*}} \right) u(T^*+\theta) d\theta.$$

For  $0 < s \leq T - T^*$ , define  $v(s) := u(s + T^*)$ . The previous inequality becomes:

$$v(t - T^*) \leq C_T \int_0^{t-T^*} \left( \frac{1}{\sqrt{t-T^*-\theta}} + \frac{1}{\sqrt{T^*}} \right) v(\theta) d\theta \leq \frac{\sqrt{T^*} + \sqrt{t-T^*}}{\sqrt{T^*}} \int_0^{t-T^*} \frac{v(\theta)}{\sqrt{t-T^*-\theta}} d\theta.$$

Setting  $t - T^* =: \tau$ , we thus have

$$v(\tau) \leq \frac{\sqrt{T^*} + \sqrt{\tau}}{\sqrt{T^*}} \int_0^\tau \frac{v(\theta)}{\sqrt{\tau-\theta}} d\theta, \quad \forall 0 \leq \tau < T - T^*.$$

Now we are in a position to apply a standard singular Gronwall lemma (see [7, Lem. 7.1.1]) and conclude that  $v(\tau) = 0$  for  $0 \leq \tau \leq T - T^*$ . Thus,  $u(t) = 0$  for  $T^* \leq t \leq T$ .  $\square$

## 5 A local existence and uniqueness result for Equation (3)

Set

$$D(T) := \int_0^T \int_{\mathbb{R}} |K_t(x)| dx dt < \infty. \quad (19)$$

The main result in this section is the following theorem.

**Theorem 5.1.** *Let  $T_0 > 0$  be such that  $D(T_0) < 1$ . Assume  $p_0 \in L^1(\mathbb{R})$  and  $b \in L^\infty((0, T_0) \times \mathbb{R})$  continuous w.r.t. space variable. Under Hypothesis (H), Equation (3) admits a unique weak solution up to  $T_0$  such that the probability distributions  $\mathbb{P} \circ X_t^{-1}$  admit densities which satisfy (5).*

**Iterative procedure.** Consider the following sequence of SDE's. For  $k = 1$

$$\begin{cases} dX_t^1 = b(t, X_t^1) dt + \left\{ \int_0^t (K_{t-s} * p_0)(X_t^1) ds \right\} dt + dW_t, \\ X_0^1 \sim p_0. \end{cases} \quad (20)$$

Denote the drift of this equation by  $b^1(t, x)$ . Supposing that, in the step  $k - 1$ , the one dimensional time marginals of the law of the solution have densities  $(p_t^{k-1})_{t \geq 0}$ , we define the drift in the step  $k$  as

$$b^k(t, x, p^{k-1}) = b(t, x) + B(t, x; p^{k-1}).$$

The corresponding SDE is

$$\begin{cases} dX_t^k = b^k(t, X_t^k, p^{k-1})dt + dW_t, \\ X_0^k \sim p_0. \end{cases} \quad (21)$$

In order to prove the desired local existence and uniqueness result we set up the non-linear martingale problem related to (3).

**Definition 5.2.** A probability measure  $\mathbb{Q}$  on the canonical space  $\mathcal{C}([0, T_0]; \mathbb{R})$  equipped with its canonical filtration and a canonical process  $(w_t)$  is a solution to the non-linear martingale problem  $(MP(p_0, T_0, b))$  if:

(i)  $\mathbb{Q}_0 = p_0$ .

(ii) For any  $t \in (0, T_0]$ , the one dimensional time marginals of  $\mathbb{Q}$ , denoted by  $\mathbb{Q}_t$ , have densities  $q_t$  w.r.t. Lebesgue measure on  $\mathbb{R}$ . In addition, they satisfy

$$\forall 0 < t \leq T_0, \quad \|q_t\|_{L^\infty(\mathbb{R})} \leq \frac{C_{T_0}}{\sqrt{t}}. \quad (22)$$

(iii) For any  $f \in C_K^2(\mathbb{R})$  the process  $(M_t)_{t \leq T_0}$ , defined as

$$M_t := f(w_t) - f(w_0) - \int_0^t \left[ \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(w_u) + \frac{\partial f}{\partial x}(w_u)(b(u, w_u) + \int_0^u \int K_{u-\tau}(w_u - y)q_\tau(y)dyd\tau) \right] du$$

is a  $\mathbb{Q}$ -martingale.

Notice that the arguments in Remark 2.2 justify that all the integrals in the definition of  $M_t$  are well defined.

We start with the analysis of Equations (20)-(21).

**Proposition 5.3.** Same assumptions as in Theorem 5.1. Then, for any  $k \geq 1$ , Equations (20)-(21) Equations (20)-(21) admit unique weak solutions up to  $T_0$ . For  $k \geq 1$ , denote by  $\mathbb{P}^k$  the law of  $(X_t^k)_{t \leq T_0}$ . Moreover, for  $t \in (0, T_0]$ , the time marginals  $\mathbb{P}_t^k$  of  $\mathbb{P}^k$  have densities  $p_t^k$  w.r.t. Lebesgue measure on  $\mathbb{R}$ . Setting  $\beta^k = \sup_{t \leq T_0} \|b^k(t, \cdot, p^{k-1})\|_{L^\infty(\mathbb{R})}$  and  $b_0 := \|b\|_{L^\infty(\mathbb{R})}$ , one has

$$\forall 0 < t \leq T_0, \quad \|p_t^k\|_{L^\infty(\mathbb{R})} \leq \frac{C(b_0, T_0)}{\sqrt{t}} \quad \text{and} \quad \beta^k \leq C(b_0, T_0).$$

Finally, there exists a function  $p^\infty \in L^\infty([0, T_0]; L^1(\mathbb{R}))$  such that

$$\sup_{t \leq T_0} \|p_t^k - p_t^\infty\|_{L^1(\mathbb{R})} \rightarrow 0, \quad \text{as } k \rightarrow \infty.$$

Moreover,

$$\forall 0 < t \leq T_0, \quad \|p_t^\infty\|_{L^\infty(\mathbb{R})} \leq \frac{C(b_0, T_0)}{\sqrt{t}}. \quad (23)$$

*Proof.* We proceed by induction.

**Case  $k = 1$ .** In view of (H-5), one has  $\beta^1 \leq b_0 + C_{T_0}$ . This implies that the equation (20) has a unique weak solution in  $[0, T_0]$  with time marginal densities  $(p_t^1(y)dy)_{t \leq T_0}$  which in view of (14) satisfy

$$\forall t \in (0, T_0], \quad \|p_t^1\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2\pi t}} + \beta^1.$$

**Case  $k > 1$ .** Assume now that the equation for  $X^k$  has a unique weak solution and assume  $\beta^k$  is finite. In addition, suppose that the one dimensional time marginals satisfy

$$\forall t \in (0, T_0], \quad \|p_t^k\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2\pi t}} + \beta^k.$$

In view of (H-4), the new drift satisfies

$$|b^{k+1}(t, x; p^k)| \leq b_0 + \int_0^t \|p_s^k\|_{L^\infty(\mathbb{R})} \|K_{t-s}\|_{L^1(\mathbb{R})} ds \leq b_0 + \int_0^t \left(\frac{1}{\sqrt{2\pi s}} + \beta^k\right) \|K_{t-s}\|_{L^1(\mathbb{R})} ds \leq b_0 + C_{T_0} + \beta^k D(T_0).$$

Thus, we conclude that  $\beta^{k+1} \leq b_0 + C_{T_0} + \beta^k D(T_0)$ . Therefore, there exists a unique weak solution to the equation for  $X^{k+1}$ . Furthermore, by (14):

$$\forall t \in (0, T_0], \quad \|p_t^{k+1}\|_{L^\infty(\mathbb{R})} \leq \frac{C_{T_0}}{\sqrt{t}} + \beta^{k+1}.$$

Notice that

$$\forall k > 1, \quad \beta^{k+1} \leq b_0 + C_{T_0} + \beta^k D(T_0) \quad \text{and} \quad \beta^1 \leq b_0 + C_{T_0}.$$

Thus, as by hypothesis  $D(T_0) < 1$ , we have

$$\forall k \geq 1, \quad \beta^k \leq \frac{b_0 + C_{T_0}}{1 - D(T_0)} + b_0 + C_{T_0} \quad (24)$$

and

$$\|p_t^k\|_{L^\infty(\mathbb{R})} \leq \frac{C_{T_0}}{\sqrt{t}} + \beta^k \leq \frac{C_{T_0}}{\sqrt{t}} + \frac{b_0 + C_{T_0}}{1 - D(T_0)} + b_0 + C_{T_0}. \quad (25)$$

Finally, it remains to prove that the sequence  $p^k$  converges in  $L^\infty([0, T_0]; L^1(\mathbb{R}))$ . In order to do so, we will prove  $p^k$  is a Cauchy sequence.

Applying the same procedure as in Section 4, one can derive the mild equation for  $(p_t^k)_{t \in [0, T_0]}$ . Thus, for every  $k \geq 1$ , the marginals  $(p_t^k)_{t \in (0, T_0]}$  satisfy the mild equation

$$\forall t \in (0, T], \quad p_t^k = g_t * p_0 - \int_0^t \frac{\partial g_{t-s}}{\partial x} * (p_s^k b^k(s, \cdot, p^{k-1})) ds \quad (26)$$

in the sense of the distributions. Assume for a moment that we have proved that for any  $0 < t \leq T_0$ , one has

$$\|p_t^k - p_t^{k-1}\|_{L^1(\mathbb{R})} \leq C_{T_0} \int_0^t \frac{\|p_s^{k-1} - p_s^{k-2}\|_{L^1(\mathbb{R})}}{\sqrt{s}} ds. \quad (27)$$

Remember that  $\int_0^t f(u_1) \cdots \int_0^{u_{k-1}} f(u_k) du_k \dots du_1 = \frac{1}{k!} \left( \int_0^t f(u) du \right)^k$  for any positive integrable function  $f$ . Then, iterating (27) one gets,

$$\|p_t^k - p_t^{k-1}\|_{L^1(\mathbb{R})} \leq 2 \frac{(C_{T_0} \sqrt{t})^{k-1}}{(k-1)!}.$$

Therefore,  $\sup_{t \leq T_0} \|p_t^k - p_t^{k-1}\|_{L^1(\mathbb{R})} \rightarrow 0$ , as  $k \rightarrow \infty$  as desired.

It remains to prove the inequality (27). In the sequel  $C(T_0) > 0$  will denote a constant that depends on  $T_0$  and may change from line to line. In view of (26), one has

$$\begin{aligned} \|p_t^k - p_t^{k-1}\|_{L^1(\mathbb{R})} &\leq \int_0^t \left\| \frac{\partial g_{t-s}}{\partial x} * (p_s^k b^k(s, \cdot, p^{k-1}) - p_s^{k-1} b^{k-1}(s, \cdot, p^{k-2})) \right\|_{L^1(\mathbb{R})} ds \\ &\leq \int_0^t \frac{1}{\sqrt{t-s}} \|b^{k-1}(s, \cdot, p^{k-2})(p_s^k - p_s^{k-1})\|_{L^1(\mathbb{R})} ds \\ &\quad + \int_0^t \frac{1}{\sqrt{t-s}} \|(b^k(s, \cdot, p^{k-1}) - b^{k-1}(s, \cdot, p^{k-2})) p_s^k\|_{L^1(\mathbb{R})} ds \\ &=: I + II. \end{aligned} \quad (28)$$

According to (24), one has

$$I \leq C(T_0) \int_0^t \frac{\|p_s^k - p_s^{k-1}\|_{L^1(\mathbb{R})}}{\sqrt{t-s}} ds.$$

According to (25), one has

$$II \leq C(T_0) \int_0^t \frac{1}{\sqrt{t-s}\sqrt{s}} \int_0^s \|K_{s-u} * (p_u^{k-1} - p_u^{k-2})\|_{L^1(\mathbb{R})} du ds.$$

Convolution inequality and Fubini-Tonelli's theorem lead to

$$II \leq C(T_0) \int_0^t \|p_u^{k-1} - p_u^{k-2}\|_{L^1(\mathbb{R})} \int_u^t \frac{1}{\sqrt{t-s}\sqrt{s}} \|K_{s-u}\|_{L^1(\mathbb{R})} ds du.$$

Apply the change of variables  $t-s=s'$ . It comes,

$$II \leq C(T_0) \int_0^t \frac{1}{\sqrt{u}} \|p_u^{k-1} - p_u^{k-2}\|_{L^1(\mathbb{R})} \int_0^{t-u} \frac{1}{\sqrt{s'}} \|K_{t-u-s'}\|_{L^1(\mathbb{R})} ds' du.$$

According to (H-4) one has

$$II \leq C(T_0) \int_0^t \frac{1}{\sqrt{u}} \|p_u^{k-1} - p_u^{k-2}\|_{L^1(\mathbb{R})} du.$$

Coming back to (28) and using our above estimates on  $I$  and  $II$ , we obtain

$$\|p_t^k - p_t^{k-1}\|_{L^1(\mathbb{R})} \leq C(T_0) \int_0^t \frac{\|p_s^k - p_s^{k-1}\|_{L^1(\mathbb{R})}}{\sqrt{t-s}} ds + C(T_0) \int_0^t \frac{1}{\sqrt{u}} \|p_u^{k-1} - p_u^{k-2}\|_{L^1(\mathbb{R})} du.$$

We are in the situation

$$\Phi(t) := \|p_t^k - p_t^{k-1}\|_{L^1(\mathbb{R})} \leq A(t) + C \int_0^t \frac{\Phi(s)}{\sqrt{t-s}} ds,$$

where  $A(t) \geq 0$  is a bounded increasing function. Iterate this relation and use the monotonicity of  $A$ . It comes

$$\Phi(t) \leq C_T A(t) + C^2 \int_0^t \frac{1}{\sqrt{t-s}} \int_0^s \frac{\Phi(u)}{\sqrt{s-u}} du ds.$$

Apply Fubini's theorem to get

$$\Phi(t) \leq C_T A(t) + C^2 \int_0^t \Phi(u) \int_u^t \frac{1}{\sqrt{t-s}\sqrt{s-u}} ds du.$$

Notice that  $\int_u^t \frac{1}{\sqrt{t-s}\sqrt{s-u}} ds = \int_0^1 \frac{1}{\sqrt{1-x}\sqrt{x}} dx$ . Now, apply Gronwall's lemma to get (27) and the convergence of  $p^k$  to  $p^\infty$ .

In order to obtain (23), fix  $t \in (0, T]$  and use (25) and the fact that the convergence in  $L^1(\mathbb{R})$  implies the almost sure convergence of a subsequence.  $\square$

The following is an obvious consequence of the preceding proposition:

**Corollary 5.4.** *Same assumptions as in Proposition 5.3. Assume that  $(\mathbb{P}^k)_{k \geq 1}$  admits a weakly convergent subsequence  $(\mathbb{P}^{n_k})_{k \geq 1}$ . Denote its limit by  $\mathbb{Q}$ . Then for any  $t \in (0, T_0]$ , one has that  $\mathbb{Q}_t(dx) = p_t^\infty(x)dx$ , where  $p^\infty$  is constructed in Proposition 5.3.*

**Proposition 5.5.** *Same assumptions as in Theorem (5.1). Then,*

- 1) *The family of probabilities  $(\mathbb{P}^k)_{k > 1}$  is tight.*
- 2) *Any weak limit  $\mathbb{P}^\infty$  of a convergent subsequence of  $(\mathbb{P}^k)_{k \geq 1}$  solves  $(MP(p_0, T_0, b))$ .*

*Proof.* In view of (24), we obviously have

$$\exists C_{T_0} > 0, \quad \sup_k \mathbb{E}|X_t^k - X_s^k|^4 \leq C_{T_0} |t-s|^2, \quad \forall 0 \leq s \leq t \leq T_0.$$

This is a sufficient condition for tightness (see e.g. [10, Chap.2, Pb.4.11]).

Let  $(\mathbb{P}^{n_k})$  be a weakly convergent subsequence of  $(\mathbb{P}^k)_{k \geq 1}$  and let  $\mathbb{P}^\infty$  denote its limit. Let us check that  $\mathbb{P}^\infty$  solves the martingale problem  $(MP(p_0, T_0, b))$ . To simplify the notation, we below write  $\mathbb{P}^k$  instead of  $\mathbb{P}^{n_k}$  and  $\bar{p}^{k-1}$  instead of  $p^{n_{k-1}}$ .

- i) Each  $\mathbb{P}_0^k$  has density  $p_0$ , and therefore  $\mathbb{P}_0^\infty$  also has density  $p_0$ .
- ii) Corollary 5.4 implies that the time marginals of  $\mathbb{P}^\infty$  are absolutely continuous with respect to Lebesgue's measure and satisfy (22).

iii) Set

$$M_t := f(w_t) - f(w_0) - \int_0^t \left[ \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(w_u) + \frac{\partial f}{\partial x}(w_u)(b(u, w_u) + \int_0^u (K_{u-\tau} * p_\tau^\infty)(w_u) d\tau) \right] du,$$

We have to prove

$$\mathbb{E}_{\mathbb{P}^\infty}[(M_t - M_s)\phi(w_{t_1}, \dots, w_{t_N})] = 0, \quad \forall \phi \in C_b(\mathbb{R}^N) \text{ and } 0 \leq t_1 < \dots < t_N < s \leq t \leq T_0, N \geq 1.$$

The process

$$M_t^k := f(w_t) - f(x(0)) - \int_0^t \left[ \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(w_u) + \frac{\partial f}{\partial x}(w_u)(b(u, w_u) + \int_0^u (K_{u-\tau} * \bar{p}_\tau^{k-1})(w_u) d\tau) \right] du$$

is a martingale under  $\mathbb{P}^k$ . Therefore, it follows that

$$\begin{aligned} 0 &= \mathbb{E}_{\mathbb{P}^k}[(M_t^k - M_s^k)\phi(w_{t_1}, \dots, w_{t_N})] \\ &= \mathbb{E}_{\mathbb{P}^k}[\phi(\dots)(f(w_t) - f(w_s))] + \mathbb{E}_{\mathbb{P}^k}[\phi(\dots) \int_s^t \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(w_u) du] \\ &\quad + \mathbb{E}_{\mathbb{P}^k}[\phi(\dots) \int_s^t \frac{\partial f}{\partial x}(w_u) b(u, w_u) du] + \mathbb{E}_{\mathbb{P}^k}[\phi(\dots) \int_s^t \frac{\partial f}{\partial x}(w_u) \int_0^u (K_{u-\tau} * \bar{p}_\tau^{k-1})(w_u) d\tau du]. \end{aligned}$$

Since  $(\mathbb{P}^k)$  weakly converges to  $\mathbb{P}^\infty$ , the first two terms on the r.h.s. obviously converge. Now, observe that

$$\begin{aligned} &\mathbb{E}_{\mathbb{P}^k}[\phi(\dots) \int_s^t \frac{\partial f}{\partial x}(w_u) \int_0^u (K_{u-\tau} * \bar{p}_\tau^{k-1})(w_u) d\tau du] \\ &\quad - \mathbb{E}_{\mathbb{P}^\infty}[\phi(\dots) \int_s^t \frac{\partial f}{\partial x}(w_u) \int_0^u (K_{u-\tau} * p_\tau^\infty)(w_u) d\tau du] \\ &= (\mathbb{E}_{\mathbb{P}^k}[\phi(\dots) \int_s^t \frac{\partial f}{\partial x}(w_u) \int_0^u (K_{u-\tau} * \bar{p}_\tau^{k-1})(w_u) d\tau du] \\ &\quad - \mathbb{E}_{\mathbb{P}^k}[\phi(\dots) \int_s^t \frac{\partial f}{\partial x}(w_u) \int_0^u (K_{u-\tau} * p_\tau^\infty)(w_u) d\tau du]) \\ &\quad + (\mathbb{E}_{\mathbb{P}^k}[\phi(\dots) \int_s^t \frac{\partial f}{\partial x}(w_u) \int_0^u (K_{u-\tau} * p_\tau^\infty)(w_u) d\tau du] \\ &\quad - \mathbb{E}_{\mathbb{P}^\infty}[\phi(\dots) \int_s^t \frac{\partial f}{\partial x}(w_u) \int_0^u (K_{u-\tau} * p_\tau^\infty)(w_u) d\tau du]) \\ &=: I + II. \end{aligned}$$

Now, in view of (25) and the definition of  $D(T)$  as in (19), one has

$$\begin{aligned} |I| &\leq \|\phi\|_{L^\infty(\mathbb{R})} \int_s^t \int_0^u \int \left| \frac{\partial f}{\partial x}(x) \right| |(K_{u-\tau} * (\bar{p}_\tau^{k-1} - p_\tau^\infty))(x)| p_u^k(x) dx d\tau du \\ &\leq \|\phi\|_{L^\infty(\mathbb{R})} \left\| \frac{\partial f}{\partial x} \right\|_{L^\infty(\mathbb{R})} \int_s^t \frac{C_{T_0}}{\sqrt{u}} \int_0^u \|K_{u-\tau}\|_{L^1(\mathbb{R})} \|\bar{p}_\tau^{k-1} - p_\tau^\infty\|_{L^1(\mathbb{R})} d\tau du \\ &\leq C_{T_0} D(T_0) \|\phi\|_{L^\infty(\mathbb{R})} \left\| \frac{\partial f}{\partial x} \right\|_{L^\infty(\mathbb{R})} \sup_{r \leq T_0} \|\bar{p}_r^{k-1} - p_r^\infty\|_{L^1(\mathbb{R})}. \end{aligned}$$

Proposition 5.3 implies that  $I \rightarrow 0$  as  $k \rightarrow \infty$ .

Now, to prove that  $II \rightarrow 0$ , it suffices to prove that the functional  $F : \mathcal{C}([0, T_0]; \mathbb{R}) \rightarrow \mathbb{R}$  defined by

$$w. \mapsto \phi(w_{t_1}, \dots, w_{t_N}) \int_s^t \frac{\partial f}{\partial x}(w_u) \int_0^u \int K_{u-\tau}(w_u - y) p_\tau^\infty(y) dy d\tau du$$

is continuous. Let  $(w^n)$  a sequence converging in  $\mathcal{C}([0, T_0]; \mathbb{R})$  to  $w$ . Since  $\phi$  is a continuous function, it suffices to show that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_s^t \frac{\partial f}{\partial x}(w_u^n) \int_0^u \int K_{u-\tau}(w_u^n - y) p_\tau^\infty(y) dy d\tau du \\ &= \int_s^t \frac{\partial f}{\partial x}(w_u) \int_0^u \int K_{u-\tau}(w_u - y) p_\tau^\infty(y) dy d\tau du. \end{aligned} \quad (29)$$

For  $(u, \tau) \in [s, t] \times [0, t]$ , set

$$h_{u,\tau}(x) := \mathbb{1}\{\tau < u\} \frac{\partial f}{\partial x}(x_u) \int K_{u-\tau}(x - y) p_\tau^\infty(y) dy.$$

The hypothesis (H-2) implies the continuity of  $h_{u,\tau}$  on  $\mathbb{R}$ . Furthermore,

$$|h_{u,\tau}(x)| \leq C \mathbb{1}\{\tau < u\} \|p_\tau^\infty\|_{L^\infty(\mathbb{R})} \|K_{u-\tau}\|_{L^1(\mathbb{R})} \leq \frac{C}{\sqrt{\tau}} \mathbb{1}\{\tau < u\} \|K_{u-\tau}\|_{L^1(\mathbb{R})}.$$

In view of (H-4), we apply the theorem of dominated convergence to conclude (29). This ends the proof.  $\square$

### Proof of Theorem 5.1

Proposition 5.5 implies the existence of a weak solution  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), X, W)$  to (3) up to time  $T_0$ . Thus, the marginals  $\mathbb{P} \circ X_t^{-1} =: p_t$  satisfy  $\|p_t\|_{L^\infty(\mathbb{R})} \leq \frac{C}{\sqrt{t}}$ ,  $t \in (0, T_0]$ . In addition, as  $|B(t, x; p)| \leq C(T_0)$ , one has that  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), X, W)$  is the unique weak solution of the linear SDE

$$d\tilde{X}_t = b(t, \tilde{X}_t)dt + B(t, \tilde{X}_t; p)dt + dW_t, \quad t \leq T_0. \quad (30)$$

Now suppose that there exists another weak solution  $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}, (\hat{\mathcal{F}}_t), \hat{X}, \hat{W})$  to (3) up to  $T_0$  and denote  $\hat{\mathbb{P}} \circ \hat{X}_t^{-1}(dx) = \hat{p}_t(x)dx$ . By Proposition 4.1 we have  $\hat{p}_t = p_t$ , for  $t \leq T_0$ . Therefore,  $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}, (\hat{\mathcal{F}}_t), \hat{X}, \hat{W})$  is a weak solution to (30), from which  $\hat{\mathbb{P}} \circ \hat{X}^{-1} = \mathbb{P} \circ X^{-1}$ .

## 6 Proofs of Theorem 2.3 and Corollary 2.4: A global existence and uniqueness result for Equation (3)

We now construct a solution for an arbitrary time horizon  $T > 0$ . We will do it by restarting the equation after the already fixed  $T_0$ . We start with  $T = 2T_0$ . Then, we will see how to generalize this procedure for an arbitrary  $T > 0$ .

Throughout this section, we denote by  $\Omega_0$  the canonical space  $\mathcal{C}([0, T_0]; \mathbb{R})$  and by  $\mathcal{B}_0$  its Borel  $\sigma$ -field. We denote by  $\mathbb{Q}^1$  the probability distribution of the unique weak solution to (3) up to  $T_0$  constructed in the previous section.

### 6.1 Solution on $[0, 2T_0]$

**Proposition 6.1.** *Let  $T_0 > 0$  be such that  $D(T_0) < 1$ . Assume  $p_0 \in L^1(\mathbb{R})$  and let  $b \in L^\infty([0, 2T_0] \times \mathbb{R})$  be continuous w.r.t. the space variable. Under the hypothesis (H), Equation (3) admits a unique weak solution up to  $2T_0$ .*

We start with analyzing the dynamics of (3) after  $T_0$  and informally explaining the construction of a solution between  $T_0$  and  $2T_0$ . Assume, for a while, that Proposition 6.1 holds true. Denote the density of  $X_t$  by  $p_t^1$ , for  $t \leq T_0$  and by  $p_t^2$ , for  $t \in (T_0, 2T_0]$ . Let  $t \geq 0$ . In view of Equation (3), we would have

$$X_{T_0+t} = X_{T_0} + \int_{T_0}^{T_0+t} b(s, X_s)ds + \int_{T_0}^{T_0+t} \int_0^s (K_{s-\theta} * p_\theta)(X_s) d\theta ds + W_{T_0+t} - W_{T_0}.$$

Observe that

$$\begin{aligned} \int_{T_0}^{T_0+t} \int_0^s (K_{s-\theta} * p_\theta)(X_s) d\theta ds &= \int_{T_0}^{T_0+t} \int_0^{T_0} (K_{s-\theta} * p_\theta^1)(X_s) d\theta ds + \int_{T_0}^{T_0+t} \int_{T_0}^s (K_{s-\theta} * p_\theta^2)(X_s) ds dt \\ &=: B_1 + B_2. \end{aligned}$$

In addition,

$$B_1 = \int_0^t \int_0^{T_0} (K_{T_0+s'-\theta} * p_\theta^1)(X_{T_0+s'}) d\theta ds',$$

and

$$B_2 = \int_0^t \int_{T_0}^{T_0+s'} (K_{T_0+s'-\theta} * p_\theta^2)(X_{T_0+s'}) d\theta ds' = \int_0^t \int_0^{s'} (K_{s'-\theta'} * p_{T_0+\theta'}^2)(X_{T_0+s'}) d\theta' ds'.$$

Now set  $Y_t := X_{T_0+t}$  and  $\tilde{p}_t(y) := p_{T_0+t}^2(y)$ . Consider the new Brownian motion  $\overline{W}_t := W_{T_0+t} - W_{T_0}$ . It comes:

$$Y_t = Y_0 + \int_0^t b(s+T_0, Y_s) ds + \int_0^t \int_0^{T_0} (K_{T_0+s'-\theta} * p_\theta^1)(Y_s) d\theta ds + \int_0^t \int_0^{s'} (K_{s'-\theta'} * \tilde{p}_\theta)(Y_s) d\theta ds + \overline{W}_t,$$

for  $t \in [0, T_0]$ . Setting

$$b_1(t, x, T_0) := \int_0^{T_0} (K_{T_0+t-s} * p_s^1)(x) ds \quad \text{and} \quad \tilde{b}(t, x) := b(T_0 + t, x),$$

we have

$$\begin{cases} dY_t = \tilde{b}(t, Y_t) dt + b_1(t, Y_t, T_0) dt + \left\{ \int_0^t (K_{t-s} * \tilde{p}_s)(Y_t) ds \right\} dt + d\overline{W}_t, & t \leq T_0, \\ Y_0 \sim p_{T_0}^1(y) dy, \quad Y_s \sim \tilde{p}_s(y) dy. \end{cases} \quad (31)$$

To prove Proposition 6.1 we construct a weak solution to (31) on  $[0, T_0]$  and suitably paste its probability distribution with  $\mathbb{Q}^1$ . We then prove that the so defined measure solves the desired non-linear martingale problem. Notice that the SDE (31) is of the same type as (3).

**Lemma 6.2.** *Same assumptions as in Proposition 6.1. Denote by  $p_t^1$  the time marginals of  $\mathbb{Q}^1$ . Set  $b_1(t, x, T_0) := \int_0^{T_0} (K_{T_0+t-s} * p_s^1)(x) ds$  and  $\tilde{b}(t, x) := b(T_0 + t, x)$ . Then, Equation (31) admits a unique weak solution up to  $T_0$ .*

*Proof.* Let us check that we may apply Theorem 5.1 to (31).

Firstly, by construction the initial law  $p_{T_0}^1$  of  $Y$  satisfies the assumption of Theorem 5.1. Secondly, the role of the additional drift  $b$  is now played by the sum of the two linear drifts,  $\tilde{b}$  and  $b_1$ . By hypothesis,  $\tilde{b}$  is bounded in  $[0, T_0] \times \mathbb{R}$  and continuous in the space variable. Using (5) and (H-6) we conclude that  $b_1$  is bounded uniformly in  $t$  and  $x$  since

$$|b_1(t, x, T_0)| \leq C_{T_0} \int_0^{T_0} \frac{\|K_{T_0+t-s}\|_{L^1(\mathbb{R})}}{\sqrt{s}} ds < C_{T_0}.$$

To show that  $b_1(t, x, T_0)$  is continuous w.r.t.  $x$  we use (H-2) and proceed as at the end of the proof of Proposition 5.5.

We now are in a position to apply Theorem 5.1: There exists a unique weak solution to (31) up to  $T_0$ .  $\square$

Denote by  $\mathbb{Q}^2$  the probability distribution of the process  $(Y_t, t \leq T_0)$ . Notice that  $\mathbb{Q}^2$  is the solution to the martingale problem  $(MP(p_{T_0}^1, T_0, \tilde{b} + b_1))$ .

**A new measure on  $\mathcal{C}([0, 2T_0]; \mathbb{R})$ .** Let  $\mathbb{Q}^1, \mathbb{Q}^2$  and  $(p_t^1)$  be as above. Let  $(p_t^2)$  denote the time marginal densities of  $\mathbb{Q}^2$ . In particular,  $\mathbb{Q}_0^2 = \mathbb{Q}_{T_0}^1$ , i.e.  $p_0^2(z) dz = p_{T_0}^1(z) dz$ . Define the mapping  $X^0$  from  $\Omega_0$  to  $\mathbb{R}$  as  $X^0(w) := w_0$ . Using [10, Thm.3.19, Chap.5] to justify the introduction of regular conditional probabilities, for each  $y \in \mathbb{R}$  we define the probability measure  $\mathbb{Q}_y^2$  on  $(\Omega_0, \mathcal{B}_0)$  by

$$\forall A \in \mathcal{B}_0, \quad \mathbb{Q}_y^2(A) = \mathbb{P}^2(A | X^0 = y).$$



In particular,

$$\mathbb{Q}_y^2(w \in \Omega_0, w_0 = y) = 1.$$

We now set  $\Omega := \mathcal{C}([0, 2T_0]; \mathbb{R})$ . For  $w^1, w^2 \in \Omega_0$  we define the concatenation  $w = w^1 \otimes_{T_0} w^2 \in \Omega$  of these two paths as the function in  $\Omega$  defined by

$$\begin{cases} w_\theta = w_\theta^1, & 0 \leq \theta \leq T_0, \\ w_{\theta+T_0} = w_{T_0}^1 + w_\theta^2 - w_0^2, & 0 \leq \theta \leq t - T_0. \end{cases}$$

On the other hand, for a given path  $w \in \Omega$ , the two paths  $w^1, w^2 \in \Omega_0$  such that  $w = w^1 \otimes_{T_0} w^2$  are

$$\begin{cases} w_\theta^1 = w_\theta, & 0 \leq \theta \leq T_0, \\ w_\theta^2 = w_{T_0+\theta}, & 0 \leq \theta \leq T_0. \end{cases}$$

We define the probability distribution  $\mathbb{Q}$  on  $\Omega$  equipped with its Borel  $\sigma$ -field as follows. For any continuous and bounded functional  $\varphi$  on  $\Omega$ ,

$$\mathbb{E}_{\mathbb{Q}}[\varphi] = \int_{\Omega} \varphi(w) \mathbb{Q}(dw) := \int_{\Omega_0} \int_{\mathbb{R}} \int_{\Omega_0} \varphi(w^1 \otimes_{T_0} w^2) \mathbb{Q}_y^2(dw^2) p_{T_0}^1(y) dy \mathbb{Q}^1(dw^1). \quad (32)$$

Notice that if  $\varphi$  acts only on the part of the path up to  $t \leq T_0$  of any  $w \in \Omega$ , then

$$\mathbb{E}_{\mathbb{Q}}[\varphi((w_\theta)_{\theta \leq t})] = \int_{\Omega_0} \varphi((w_\theta)_{\theta \leq t}) \mathbb{Q}^1(dx) = \mathbb{E}_{\mathbb{Q}^1}[\varphi((w_\theta)_{\theta \leq t})]. \quad (33)$$

**Proof of Proposition 6.1.** Let us prove that the probability measure  $\mathbb{Q}$  solves the non-linear martingale problem  $(MP(p_0, 2T_0, b))$  on the canonical space  $\mathcal{C}([0, 2T_0]; \mathbb{R})$ .

- i) By (33), it is obvious that  $\mathbb{Q}_0 = \mathbb{Q}_0^1$ . By construction,  $\mathbb{Q}_0^1$  has density  $p_0$ .
- ii) Next, let us characterize the one dimensional time marginals of  $\mathbb{Q}$ . For  $f \in C_b(\mathbb{R})$  and  $t \in [0, 2T_0]$ , consider the functional  $\varphi(w) = f(w_t)$ , for any  $x \in \mathcal{C}([0, 2T_0]; \mathbb{R})$ . For  $t \leq T_0$ , by (33),

$$\mathbb{E}_{\mathbb{Q}}[\varphi(w)] = \int_{\Omega_0} f(w_t) \mathbb{Q}^1(dx) = \int_{\mathbb{R}} f(z) p_t^1(z) dz.$$

Therefore,  $\mathbb{Q}_t(dz) = p_t^1(z) dz$ .

For  $T_0 \leq t \leq 2T_0$ , by (32),

$$\mathbb{E}_{\mathbb{Q}}[\varphi(w)] = \int_{\Omega_0} \int_{\mathbb{R}} \int_{\Omega_0} f(w_{t-T_0}^2) \mathbb{Q}_y^2(dw^2) p_{T_0}^2(y) dy \mathbb{Q}^1(dw^1) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(z) \mathbb{Q}_{y, t-T_0}^2(dz) p_{T_0}^1(y) dy.$$

By Fubini's theorem:

$$\mathbb{E}_{\mathbb{Q}}[\varphi(w)] = \int_{\mathbb{R}} f(z) \int_{\mathbb{R}} \mathbb{Q}_{y, t-T_0}^2(dz) p_{T_0}^1(y) dy.$$

Since  $\mathbb{Q}_0^2 = p_{T_0}^1$  we deduce

$$\mathbb{E}_{\mathbb{Q}}[\varphi(w)] = \int_{\mathbb{R}} f(z) p_{t-T_0}^2(z) dz,$$

which shows that  $\mathbb{Q}_t(dz) = p_{t-T_0}^2(z) dz$ .

Therefore, the one dimensional marginals of  $\mathbb{Q}$  have densities  $q_t$  which, by construction, satisfy  $\|q_t\|_{L^\infty(\mathbb{R})} \leq \frac{C}{\sqrt{t}}$ .

- iii) It remains to show that  $(M_t)_{t \leq 2T_0}$  defined as

$$M_t := f(w_t) - f(w_0) - \int_0^t \left[ \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(w_u) + \frac{\partial f}{\partial x}(w_u) (b(u, w_u) + \int_0^u \int K_{u-\tau}(w_u - y) q_\tau(y) dy d\tau) \right] du$$

is a  $\mathbb{Q}$ -martingale, i.e.  $\mathbb{E}_{\mathbb{Q}}(M_t | \mathcal{B}_s) = M_s$ .

(a) Let  $s \leq t \leq T_0$  :

For any  $n \in N$ , any continuous bounded functional  $\phi$  on  $\mathbb{R}^n$ , and any  $t_1 \leq \dots \leq t_n < s \leq t \leq T_0$ , by (33):

$$\mathbb{E}_{\mathbb{Q}}(\phi(w_{t_1}, \dots, w_{t_n})(M_t - M_s)) = \mathbb{E}_{\mathbb{Q}^1}(\phi(w_{t_1}, \dots, w_{t_n})(M_t - M_s)).$$

As  $\mathbb{Q}^1$  solves the  $(MP(p_0, T_0, b))$  up to  $T_0$ ,

$$\mathbb{E}_{\mathbb{Q}}(\phi(w_{t_1}, \dots, w_{t_n})(M_t - M_s)) = 0.$$

(b) For  $s \leq T_0 \leq t \leq 2T_0$ ,

$$\mathbb{E}_{\mathbb{Q}}(M_t | \mathcal{B}_s) = \mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}(M_t | \mathcal{B}_{T_0}) | \mathcal{B}_s].$$

Let us prove that  $\mathbb{E}_{\mathbb{Q}}(M_t | \mathcal{B}_{T_0}) = M_{T_0}$ . Notice that

$$\begin{aligned} M_t - M_{T_0} &= f(w_t) - f(w_{T_0}) - \int_{T_0}^t \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(w_u) du - \int_{T_0}^t \frac{\partial f}{\partial x}(w_u) b(u, w_u) du \\ &\quad - \int_{T_0}^t \frac{\partial f}{\partial x}(w_u) \int_0^u \int K_{u-\tau}(w_u - y) q_{\tau}(y) dy d\tau du. \end{aligned}$$

Write the last integral as

$$\begin{aligned} &\int_{T_0}^t \frac{\partial f}{\partial x}(w_u) \int_0^u \int K_{u-\tau}(w_u - y) q_{\tau}(y) dy d\tau du = \int_{T_0}^t \frac{\partial f}{\partial x}(w_u) \int_0^{T_0} \int K_{u-\tau}(w_u - y) p_{\tau}^1(y) dy d\tau du \\ &\quad + \int_{T_0}^t \frac{\partial f}{\partial x}(w_u) \int_{T_0}^u \int K_{u-\tau}(w_u - y) p_{\tau-T_0}^2(y) dy d\tau du \\ &=: I_1 + I_2. \end{aligned}$$

Now,

$$I_1 = \int_0^{t-T_0} \frac{\partial f}{\partial x}(w_{u+T_0}) \int_0^{T_0} \int K_{u+T_0-\tau}(w_{u+T_0} - y) p_{\tau}^1(y) dy d\tau du.$$

For  $w \in \Omega$  identify  $w^1, w^2 \in \Omega_0$  such that  $w = w^1 \otimes_{T_0} w^2$ . Then,

$$I_1 = \int_0^{t-T_0} \frac{\partial f}{\partial x}(w_u^2) \int_0^{T_0} (K_{u+T_0-\tau} * p_{\tau}^1)(w_u^2) d\tau du = \int_0^{t-T_0} \frac{\partial f}{\partial x}(w_u^2) b_1(u, w_u^2, T_0) du.$$

Proceeding as above,

$$\begin{aligned} I_2 &= \int_0^{t-T_0} \frac{\partial f}{\partial x}(w_{u+T_0}) \int_0^u \int K_{u-\tau}(w_{u+T_0} - y) p_{\tau}^2(y) dy d\tau du \\ &= \int_0^{t-T_0} \frac{\partial f}{\partial x}(w_u^2) \int_0^u \int K_{u-\tau}(w_u^2 - y) p_{\tau}^2(y) dy d\tau du. \end{aligned}$$

Similarly

$$\begin{aligned} &\int_{T_0}^t \frac{\partial f}{\partial x}(w_u) b(u, w_u) du = \int_0^{t-T_0} \frac{\partial f}{\partial x}(w_{u+T_0}) b(u+T_0, w_{u+T_0}) du \\ &= \int_0^{t-T_0} \frac{\partial f}{\partial x}(w_u^2) b(u+T_0, w_u^2) du = \int_0^{t-T_0} \frac{\partial f}{\partial x}(w_u^2) \tilde{b}(u, w_u^2) du. \end{aligned}$$

It comes:

$$\begin{aligned} M_t - M_{T_0} &= f(w_{t-T_0}^2) - f(w_0^2) - \int_0^{t-T_0} \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(w_u^2) du - \int_0^{t-T_0} \frac{\partial f}{\partial x}(w_u^2) \tilde{b}(u, w_u^2) du \\ &\quad - \int_0^{t-T_0} \frac{\partial f}{\partial x}(w_u^2) [b_1(u, w_u^2, T_0) + \int_0^u \int K_{u-\tau}(w_u^2 - y) p_{\tau}^2(y) dy d\tau] du =: F(w^2). \end{aligned}$$

By definition of the measure  $\mathbb{Q}$ ,

$$\mathbb{E}^{\mathbb{Q}}(\phi(w_{t_1}, \dots, w_{t_n})(M_t - M_{T_0})) = \int_{\Omega_0} \phi(w_{t_1}^1, \dots, w_{t_n}^1) \int_{\mathbb{R}} \int_{\Omega_0} F(w^2) \mathbb{Q}_y^2(dw^2) p_{T_0}^1(y) dy \mathbb{Q}^1(dw^1).$$

By the definition of  $\mathbb{Q}^2$ :

$$\mathbb{E}^{\mathbb{Q}}(\phi(w_{t_1}, \dots, w_{t_n})(M_t - M_{T_0})) = \int_{\Omega_0} \phi(w_{t_1}^1, \dots, w_{t_n}^1) \int_{\Omega_0} F(w^2) \mathbb{Q}^2(dw^2) \mathbb{Q}^1(dw^1).$$

As  $\mathbb{Q}^2$  solves  $(MP(p_{T_0}^1, T_0, \tilde{b} + b_1))$ , one has

$$\mathbb{E}_{\mathbb{Q}^2}(\varphi) = \int_{\Omega_0} F(w^2) \mathbb{Q}^2(dw^2) = 0.$$

Finally, we conclude that  $\mathbb{E}_{\mathbb{Q}}(M_t | \mathcal{B}_{T_0}) = M_{T_0}$  and therefore  $\mathbb{E}_{\mathbb{Q}}(M_t | \mathcal{B}_s) = M_s$  for all  $s \leq T_0 \leq t \leq 2T_0$ .

(c) For  $T_0 \leq s \leq t \leq 2T_0$ : we may rewrite the difference  $M_t - M_s$  in the same manner:

$$\begin{aligned} M_t - M_s &= f(w_{t-T_0}^2) - f(w_{s-T_0}^2) - \int_{s-T_0}^{t-T_0} \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(w_u^2) du \\ &\quad - \int_{s-T_0}^{t-T_0} \frac{\partial f}{\partial x}(w_u^2) [b(u, w_u^2) + b_1(u, w_u^2, T_0) + \int_0^u \int K_{u-\tau}(w_{u+T_0} - y) p_{\tau}^2(y) dy d\tau] du \\ &=: F(w^2). \end{aligned}$$

Now, take  $t_1 \leq \dots \leq t_n < s$ . Let us suppose that the first  $m$  are before  $T_0$  and others after. We have that

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}(\phi(w_{t_1}, \dots, w_{t_n})(M_t - M_s)) &= \\ &= \int_{\Omega_0} \int_{\mathbb{R}} \int_{\Omega_0} F(w^2) \mathbb{Q}_y^2(dw^2) p_{T_0}^1(y) dy \mathbb{Q}^1(dw^1). \end{aligned}$$

Since  $\mathbb{Q}^2$  solves the  $(MP(p_{T_0}^1, T_0, \tilde{b} + b_1))$ , one has that  $\mathbb{E}_{\mathbb{Q}^2}(\phi'(w_{t'_1}^2, \dots, w_{t'_n}^2) F) = 0$  for any continuous bounded functional  $\phi'$  on  $\mathbb{R}^n$ , any  $n \in \mathbb{N}$  and any  $t'_1 \leq \dots \leq t'_n < s - T_0$ . Taking  $\phi'(w_{t'_1}^2, \dots, w_{t'_n}^2) = \phi(w_{t_1}^1, \dots, w_{t_m}^1, w_{t_{m+1}-T_0}^2, \dots, w_{t_n-T_0}^2)$  for a fixed  $x^1$ , we conclude that

$$\int_{\mathbb{R}} \int_{\Omega_0} \phi(w_{t_1}^1, \dots, w_{t_m}^1, w_{t_{m+1}-T_0}^2, \dots, w_{t_n-T_0}^2) \varphi(w^2) \mathbb{Q}_y^2(dw^2) p_{T_0}^1(y) dy = 0.$$

Therefore,

$$\mathbb{E}_{\mathbb{Q}}(\phi(w_{t_1}, \dots, w_{t_n})(M_t - M_s)) = 0.$$

Thus,  $\mathbb{E}_{\mathbb{Q}}(M_t | \mathcal{B}_s) = M_s$  for  $T_0 \leq s \leq t \leq 2T_0$ .

To summarize the preceding, we have just shown the existence of a solution to  $(MP(p_0, 2T_0, b))$ . Finally, we proceed as in the proof of Theorem 5.1 to deduce the existence and uniqueness of a weak solution to (3) up to  $2T_0$ .

## 6.2 End of the proof of Theorem 2.3: construction of the global solution

Given any finite time horizon  $T > 0$ , split the interval  $[0, T]$  into  $n = \lceil \frac{T}{T_0} \rceil + 1$  intervals of length not exceeding  $T_0$  and repeat  $n$  times the procedure used in the preceding subsection. By construction, the time marginals of this solution to  $(MP(p_0, T, b))$  has probability densities which satisfy (6).

**Remark 6.3.** *Using similar arguments as above one can construct a solution to (3) when the initial condition  $p_0$  is in  $L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$  or, respectively,  $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ . In these cases we use Remark 3.3 in the iterative procedure. Consequently, the weak solution is unique under the constraint that the one dimensional marginal densities  $(p_t)_{t \leq T}$  belong to  $L^\infty((0, T); L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}))$  or, respectively, satisfy*

$$\|p_t\|_{L^\infty(\mathbb{R})} \leq \frac{C_T}{t^{1/4}}.$$

### 6.3 Proof of Corollary 2.4

As (5) implies (6), Theorem (2.3) implies the existence of a solution to (3) in the modified sense.

Let  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), X, W)$  be any solution to (3) in this new sense. Denote by  $(p_t)_{t \leq T}$  the time marginal densities of  $\mathbb{P} \circ X^{-1}$ . Then, (6) and (H-7) imply that  $B(t, x; p)$  is a bounded function on  $[0, T] \times \mathbb{R}$ . In view of Corollary 3.2,  $(p_t)_{t \leq T}$  satisfies (5) and as such it is a solution to (3) in the sense of Definition 2.1. Theorem 2.3 implies the uniqueness of this solution in the modified sense.

## 7 Application to the one-dimensional Keller–Segel model

In this section we prove Corollary 2.6. We start with checking that  $K^\sharp$  satisfies Hypothesis (H). The condition (H-1) is satisfied since for  $t > 0$  one has

$$\|K_t^\sharp\|_{L^1(\mathbb{R})} = \frac{C}{\sqrt{t}} \int |z| e^{-\frac{z^2}{2}} dz.$$

From the definition of  $K^\sharp$  it is clear that for  $t > 0$ ,  $K_t^\sharp$  is a bounded and continuous function on  $\mathbb{R}$ . The condition (H-3) is also obviously satisfied. As already noticed,

$$\|K_{t-s}^\sharp\|_{L^1(\mathbb{R})} = \frac{C}{\sqrt{t-s}},$$

from which,

$$f_1(t) := \int_0^t \frac{\|K_{t-s}^\sharp\|_{L^1(\mathbb{R})}}{\sqrt{s}} ds = C \int_0^t \frac{1}{\sqrt{s}\sqrt{t-s}} ds = C \int_0^1 \frac{1}{\sqrt{x}\sqrt{1-x}} dx = C,$$

where  $C$  is a universal constant. Now let  $\phi$  be a probability density on  $\mathbb{R}$ . For  $(t, x) \in (0, T] \times \mathbb{R}$ , one has

$$\begin{aligned} \int \phi(y) \|K^\sharp(x-y)\|_{L^1(0,t)} dy &\leq C \int \phi(y) |x-y| \int_0^t \frac{1}{s^{3/2}} e^{-\frac{|x-y|^2}{2s}} ds dy \\ &= \int \phi(y) |x-y| \int_{\frac{|x-y|}{\sqrt{t}}}^\infty \frac{z^3}{|x-y|^3} e^{-\frac{z^2}{2}} \frac{|x-y|^2}{z^3} dz dy \\ &= \int \phi(y) \int_{\frac{|x-y|}{\sqrt{t}}}^\infty e^{-\frac{z^2}{2}} dz dy. \end{aligned}$$

This shows that (H-5) is satisfied. Finally, to prove (H-6) we notice that for every  $t \in [0, T]$

$$\int_0^T \|K_{T+t-s}^\sharp\|_{L^1(\mathbb{R})} \frac{1}{\sqrt{s}} ds \leq \int_0^T \frac{C}{\sqrt{T+t-s}\sqrt{s}} ds \leq C \int_0^T \frac{1}{\sqrt{T-s}\sqrt{s}} ds = C.$$

Therefore, in view of Theorem 2.3, Equation (2) with  $d = 1$  is well-posed.<sup>1</sup>

Denote by  $\rho(t, x) \equiv p_t(x)$  the time marginals of the constructed probability distribution. Now, define the function  $c$  as in (9). In view of Inequality (5), for any  $t \in (0, T]$  the function  $c(t, \cdot)$  is well defined and bounded continuous. Let us show that  $c \in L^\infty([0, T]; C_b^1(\mathbb{R}))$ .

We have

$$\frac{\partial c}{\partial x}(t, x) = \frac{\partial}{\partial x} (e^{-\lambda t} \mathbb{E}(c_0(x + W_t))) + \frac{\partial}{\partial x} \left( \mathbb{E} \int_0^t e^{-\lambda s} \rho(t-s, x + W_s) ds \right).$$

Then observe that

$$\begin{aligned} \frac{\partial}{\partial x} \mathbb{E} \int_0^t e^{-\lambda s} \rho(t-s, x + W_s) ds &= \frac{\partial}{\partial x} \int_0^t e^{-\lambda s} \int \rho(t-s, x+y) \frac{1}{\sqrt{2\pi s}} e^{-\frac{y^2}{4s}} dy ds \\ &= \frac{\partial}{\partial x} \int_0^t e^{-\lambda(t-s)} \int \rho(s, y) \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(y-x)^2}{4(t-s)}} dy ds \\ &=: \frac{\partial}{\partial x} \int_0^t f(s, x) ds. \end{aligned}$$

<sup>1</sup>With similar calculations as for  $f_1$ , one easily checks that the function  $f_2$  is bounded on any compact time interval. Thus, Corollary 2.4 applies as well as Theorem 2.3.

As for any  $0 < s < t$

$$\left| \frac{\partial}{\partial x} \frac{1}{\sqrt{t-s}} e^{-\frac{(y-x)^2}{2(t-s)}} \right| \leq \frac{|y-x|}{2(t-s)^{3/2}} e^{-\frac{(y-x)^2}{2(t-s)}} \leq \frac{C}{t-s},$$

we have

$$\frac{\partial f}{\partial x}(s, x) = e^{-\lambda(t-s)} \int \rho(s, y) \frac{y-x}{2\sqrt{2\pi}(t-s)^{3/2}} e^{-\frac{(y-x)^2}{2(t-s)}} dy.$$

Now, we repeat the same argument for  $\frac{\partial}{\partial x} \int_0^t f(s, x) ds$ . In order to justify the differentiation under the integral sign we notice that

$$\left| \frac{\partial f}{\partial x}(s, x) \right| \leq \frac{C_T}{\sqrt{(t-s)s}}.$$

Gathering the preceding calculations we have obtained

$$\frac{\partial c}{\partial x}(t, x) = e^{-\lambda t} \mathbb{E} c'_0(x + W_t) + \int_0^t e^{-\lambda(t-s)} \int \rho_s(y) \frac{y-x}{\sqrt{2\pi}(t-s)^{3/2}} e^{-\frac{(y-x)^2}{2(t-s)}} dy ds.$$

Using the assumption on  $c_0$  and Inequality (5), for any  $t \in (0, T]$  one has

$$\left\| \frac{\partial c}{\partial x}(t, \cdot) \right\|_{L^\infty(\mathbb{R})} \leq \|c'_0\|_{L^\infty(\mathbb{R})} + C_T.$$

In addition, the preceding calculation and Lebesgue's Dominated Convergence Theorem show that  $\frac{\partial c}{\partial x}(t, \cdot)$  is continuous on  $\mathbb{R}$ . We thus have obtained the desired property.

The above discussion shows that we are in a position to apply Proposition 4.1 with  $b(t, x) \equiv \chi e^{-\lambda t} \mathbb{E} c'_0(x + W_t)$  and  $B(t, x; \rho)$  defined as in (4) with  $K \equiv K^\sharp$ : the function  $\rho(t, x)$  satisfies (8) in the sense of the distributions. Therefore, it is a solution to the Keller Segel system (7) in the sense of Definition 2.5. We now check the uniqueness of this solution.

Assume there exists another solution  $\rho^1$  satisfying Definition 2.5 with the same initial condition as  $\rho$ . For notation convenience, in the calculation below we set  $c_t(x) := c(t, x)$ ,  $c_t^1(x) := c^1(t, x)$ ,  $\rho_t(x) := \rho(t, x)$ , and  $\rho_t^1(x) := \rho^1(t, x)$ .

Using Definition 2.5,

$$\begin{aligned} \|\rho_t^1 - \rho_t\|_{L^1(\mathbb{R})} &\leq \int_0^t \left\| \frac{\partial g_{t-s}}{\partial x} * \left( \frac{\partial c_s^1}{\partial x} \rho_s^1 - \frac{\partial c_s}{\partial x} \rho_s \right) \right\|_{L^1(\mathbb{R})} ds \\ &\leq \int_0^t \left\| \frac{\partial g_{t-s}}{\partial x} * \left( \frac{\partial c^1}{\partial x} (\rho_s^1 - \rho_s) \right) \right\|_{L^1(\mathbb{R})} ds + \int_0^t \left\| \frac{\partial g_{t-s}}{\partial x} * \left( \rho_s \left( \frac{\partial c^1}{\partial x} - \frac{\partial c_s}{\partial x} \right) \right) \right\|_{L^1(\mathbb{R})} ds \\ &=: I + II. \end{aligned}$$

Using standard convolution inequalities and  $\left\| \frac{\partial g_{t-s}}{\partial x} \right\|_{L^1(\mathbb{R})} \leq \frac{C}{\sqrt{t-s}}$  we deduce:

$$I \leq C \int_0^t \frac{\|\rho_s^1 - \rho_s\|_{L^1(\mathbb{R})}}{\sqrt{t-s}} ds \quad \text{and} \quad II \leq C \int_0^t \frac{\left\| \frac{\partial c_s^1}{\partial x} - \frac{\partial c_s}{\partial x} \right\|_{L^1(\mathbb{R})}}{\sqrt{t-s}\sqrt{s}} ds.$$

Therefore

$$\left\| \frac{\partial c_s^1}{\partial x} - \frac{\partial c_s}{\partial x} \right\|_{L^1(\mathbb{R})} \leq \int_0^s \left\| (\rho_u^1 - \rho_u) * \frac{\partial g_{s-u}}{\partial x} \right\|_{L^1(\mathbb{R})} du \leq C \int_0^s \frac{\|\rho_u^1 - \rho_u\|_{L^1(\mathbb{R})}}{\sqrt{s-u}} du, \quad (34)$$

from which

$$\begin{aligned} II &\leq C \int_0^t \frac{1}{\sqrt{s}\sqrt{t-s}} \int_0^s \frac{\|\rho_u^1 - \rho_u\|_{L^1(\mathbb{R})}}{\sqrt{s-u}} du ds \\ &\leq C \int_0^t \|\rho_u^1 - \rho_u\|_{L^1(\mathbb{R})} \int_u^t \frac{1}{\sqrt{s}\sqrt{s-u}\sqrt{t-s}} ds du \leq C_T \int_0^t \frac{\|\rho_u^1 - \rho_u\|_{L^1(\mathbb{R})}}{\sqrt{u}} du. \end{aligned}$$

Gathering the preceding bounds for  $I$  and  $II$  we get

$$\|p_t^1 - p_t\|_{L^1(\mathbb{R})} \leq C_T \int_0^t \frac{\|p_s^1 - p_s\|_{L^1(\mathbb{R})}}{\sqrt{t-s}} ds + C_T \int_0^t \frac{\|p_s^1 - p_s\|_{L^1(\mathbb{R})}}{\sqrt{s}} ds.$$

Lemma 4.2 implies that  $\|\rho_t^1 - \rho_t\|_{L^1(\mathbb{R})} = 0$  for every  $t \leq T$ . In view of (34) we also have  $\left\| \frac{\partial c_t^1}{\partial x} - \frac{\partial c_t}{\partial x} \right\|_{L^1(\mathbb{R})} = 0$ . This completes the proof of Corollary 2.6.

## 8 Appendix

We here propose a light simplification of the calculations in [14].

**Proposition 8.1.** *Let  $y \in \mathbb{R}$  and let  $\beta$  be a constant. Denote by  $p_y^\beta(t, x, z)$  the transition probability density (with respect to the Lebesgue measure) of the unique weak solution to*

$$X_t = x + \beta \int_0^t \operatorname{sgn}(y - X_s) ds + W_t.$$

Then

$$\begin{aligned} p_y^\beta(t, x, z) &= \frac{1}{\sqrt{2\pi t^{3/2}}} \int_0^\infty e^{\beta(|y-x|+\bar{y}-|z-y|)-\frac{\beta^2}{2}t} (\bar{y} + |z-y| + |y-x|) e^{-\frac{(\bar{y}+|z-y|+|y-x|)^2}{2t}} d\bar{y} \\ &\quad + \frac{1}{\sqrt{2\pi t}} e^{\beta(|y-x|-|z-y|)-\frac{\beta^2}{2}t} (e^{-\frac{(z-x)^2}{2t}} - e^{-\frac{(|z-y|+|y-x|)^2}{2t}}). \end{aligned} \quad (35)$$

In particular,

$$p_y^\beta(t, x, y) = \frac{1}{\sqrt{2\pi t}} \int_{\frac{|x-y|}{\sqrt{t}}}^\infty z e^{-\frac{(z-\beta\sqrt{t})^2}{2}} dz. \quad (36)$$

*Proof.* Let  $f$  be a bounded continuous function. The Girsanov transform leads to

$$\mathbb{E}(f(X_t)) = \mathbb{E}(f(x + W_t) e^{\beta \int_0^t \operatorname{sgn}(y-x-W_s) dW_s - \frac{\beta^2}{2}t}).$$

Let  $L_t^a$  be the Brownian local time. By Tanaka's formula ([10], p. 205):

$$|W_t - a| = |a| + \int_0^t \operatorname{sgn}(W_s - a) dW_s + L_t^a.$$

Therefore

$$\int_0^t \operatorname{sgn}(y-x-W_s) dW_s = |y-x| + L_t^a - |W_t - (y-x)|,$$

from which

$$\mathbb{E}(f(X_t)) = \mathbb{E}(f(x + W_t) e^{\beta(|y-x|+L_t^{y-x}-|W_t-(y-x)|)-\frac{\beta^2}{2}t}).$$

Recall that  $(W_t, L_t^a)$  has the following joint distribution (see [1, p.200, Eq.(1.3.8)]):

$$\begin{cases} \bar{y} > 0 : & \mathbb{P}(W_t \in dz, L_t^a \in d\bar{y}) = \frac{1}{\sqrt{2\pi t^{3/2}}} (\bar{y} + |z-a| + |a|) e^{-\frac{(\bar{y}+|z-a|+|a|)^2}{2t}} d\bar{y} dz. \\ & \mathbb{P}(W_t \in dz, L_t^a = 0) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{z^2}{2t}} dz - \frac{1}{\sqrt{2\pi t}} e^{-\frac{(|z-a|+|a|)^2}{2t}} dz. \end{cases}$$

It comes:

$$\begin{aligned} \mathbb{E}(f(X_t)) &= \frac{1}{\sqrt{2\pi t^{3/2}}} \int_{\mathbb{R}} \int_0^\infty f(x+z) e^{\beta(|y-x|+\bar{y}-|z-(y-x)|)-\frac{\beta^2}{2}t} (\bar{y} + |z-(y-x)| + |y-x|) \\ &\quad e^{-\frac{(\bar{y}+|z-(y-x)|+|y-x|)^2}{2t}} d\bar{y} dz \\ &\quad + \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} f(x+z) e^{\beta(|y-x|-|z-(y-x)|)-\frac{\beta^2}{2}t} (e^{-\frac{z^2}{2t}} - e^{-\frac{(|z-(y-x)|+|y-x|)^2}{2t}}) dz. \end{aligned}$$

The change of variables  $x+z = z'$  leads to

$$\begin{aligned} \mathbb{E}(f(X_t)) &= \frac{1}{\sqrt{2\pi t^{3/2}}} \int_{\mathbb{R}} f(z') \int_0^\infty e^{\beta(|y-x|+\bar{y}-|z'-y|)-\frac{\beta^2}{2}t} (\bar{y} + |z'-y| + |y-x|) e^{-\frac{(\bar{y}+|z'-y|+|y-x|)^2}{2t}} d\bar{y} dz' \\ &\quad + \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} f(z') e^{\beta(|y-x|-|z'-y|)-\frac{\beta^2}{2}t} (e^{-\frac{(z'-x)^2}{2t}} - e^{-\frac{(|z'-y|+|y-x|)^2}{2t}}) dz', \end{aligned}$$

from which the desired result follows.  $\square$

In the next Corollary we use the same notation as in the proof of Theorem 3.1.

**Corollary 8.2.** *Let  $0 < s < t \leq T$ . Then for any  $z, y \in \mathbb{R}$ , there exists  $C_{T,\beta,x,y}$  such that*

$$\mathbb{E} \left| \left( \frac{\partial}{\partial x} p_y^\beta \right) (t-s, X_s^{(b)}, z) \right| \leq C_{T,\beta,x,y} h(s, z),$$

where  $h$  belongs to  $L^1([0, t] \times \mathbb{R})$ .

*Proof.* By Girsanov's theorem, for some constant  $C_{T,\beta}$  we have

$$\mathbb{E} \left| \left( \frac{\partial}{\partial x} p_y^\beta \right) (t-s, X_s^{(b)}, z) \right| \leq C_{T,\beta} \sqrt{\mathbb{E} \left| \left( \frac{\partial}{\partial x} p_y^\beta \right) (t-s, W_s^x, z) \right|^2}.$$

Observe that

$$\begin{aligned} \frac{\partial}{\partial x} p_y^\beta(t-s, \bar{x}, z) &= \frac{\beta}{\sqrt{2\pi(t-s)}} e^{-2\beta|z-y|} e^{-\frac{(|z-y|+|y-\bar{x}|-\beta(t-s))^2}{2(t-s)}} \operatorname{sgn}(\bar{x}-y) \\ &\quad + \frac{\beta}{\sqrt{2\pi(t-s)}} e^{-\beta|z-y|-\frac{\beta^2}{2}(t-s)} e^{\beta|y-\bar{x}|-\frac{(z-\bar{x})^2}{2(t-s)}} \operatorname{sgn}(\bar{x}-y) \\ &\quad + \frac{z-\bar{x}}{2\pi(t-s)^{3/2}} e^{-\beta|z-y|-\frac{\beta^2}{2}(t-s)} e^{\beta|y-\bar{x}|-\frac{(z-\bar{x})^2}{2(t-s)}}. \end{aligned}$$

The sum of the absolute values of the first two terms in the right-hand side is bounded from above by

$$\frac{\beta}{\sqrt{2\pi(t-s)}} e^{-2\beta|z-y|+\beta|y-\bar{x}|}.$$

Thus,

$$\mathbb{E} \left| \left( \frac{\partial}{\partial x} p_y^\beta \right) (t-s, X_s^{(b)}, z) \right| \leq \frac{C_{T,\beta}}{\sqrt{2\pi(t-s)}} \sqrt{\mathbb{E} e^{2\beta|y-W_s^x|}} + \frac{C_{T,\beta}}{(t-s)^{3/2}} \sqrt{\mathbb{E} (|z-W_s^x|^2 e^{2\beta|y-W_s^x|-\frac{(z-W_s^x)^2}{t-s}})} =: B+A.$$

Notice that

$$A \leq \frac{C_{T,\beta}}{(t-s)^{3/2}} (\mathbb{E} [|z-W_s^x|^4 e^{-2\frac{(z-W_s^x)^2}{t-s}}] \mathbb{E} [e^{4\beta|y-W_s^x|}])^{1/4} =: \frac{C_{T,\beta}}{(t-s)^{3/2}} (A_1 A_2)^{1/4}.$$

Firstly, as there exists an  $\alpha > 0$  such that  $|a|^4 e^{-a^2} \leq C e^{-\alpha a^2}$ , one has

$$A_1 \leq C(t-s)^2 \int e^{-\alpha \frac{(z-u)^2}{t-s}} g_s(u-x) du \leq \frac{(t-s)^{2+1/2}}{\sqrt{s+(t-s)/(2\alpha)}} e^{-\frac{(z-x)^2}{2(s+(t-s)/(2\alpha))}}.$$

Secondly,

$$\begin{aligned} A_2 &= \int e^{4\beta|y-u|} g_s(u-x) du = e^{-4\beta y} \int_y^\infty e^{4\beta u} \frac{1}{\sqrt{s}} e^{-\frac{(u-x)^2}{2s}} du + e^{4\beta y} \int_{-\infty}^y e^{-4\beta u} \frac{1}{\sqrt{s}} e^{-\frac{(u-x)^2}{2s}} du \\ &= e^{4\beta(x-y)} e^{8\beta^2 s} \int_y^\infty \frac{1}{\sqrt{s}} e^{-\frac{(u-x-4\beta s)^2}{2s}} du + e^{4\beta(y-x)} e^{8\beta^2 s} \int_{-\infty}^y \frac{1}{\sqrt{s}} e^{-\frac{(u-x+4\beta s)^2}{2s}} du \leq e^{8\beta^2 s} C_{\beta,x,y}. \end{aligned}$$

Therefore,

$$A \leq C_{T,\beta,x,y} \frac{1}{(t-s)^{7/8}} g_{s+(t-s)/(2\alpha)}(z-x).$$

The term  $B$  is treated in the similar way as  $A_2$ . □

# References