



HAL
open science

A new McKean-Vlasov stochastic interpretation of the parabolic-parabolic Keller-Segel model: The one-dimensional case

Denis Talay, Milica Tomasevic

► **To cite this version:**

Denis Talay, Milica Tomasevic. A new McKean-Vlasov stochastic interpretation of the parabolic-parabolic Keller-Segel model: The one-dimensional case. 2018. hal-01673332v2

HAL Id: hal-01673332

<https://inria.hal.science/hal-01673332v2>

Preprint submitted on 19 Jan 2018 (v2), last revised 25 Jul 2019 (v6)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A new McKean-Vlasov stochastic interpretation of the parabolic-parabolic Keller-Segel model: The one-dimensional case.

Denis Talay* and Milica Tomašević *†

Abstract: In this paper we analyze a stochastic interpretation of the one-dimensional parabolic-parabolic Keller-Segel system without cut-off. It involves an original type of McKean-Vlasov interaction kernel. At the particle level, each particle interacts with all the past of each other particle by means of a time integrated functional involving a singular kernel. At the mean-field level studied here, the McKean-Vlasov limit process interacts with all the past time marginals of its probability distribution in a similarly singular way. We prove that the parabolic-parabolic Keller-Segel system in the whole Euclidean space and the corresponding McKean-Vlasov stochastic differential equation are well-posed for any values of the parameters of the model. **Key words:** Chemotaxis model; Keller–Segel system; Singular McKean-Vlasov non-linear stochastic differential equation.

Classification: 60H30 60H10 60K35.

1 Introduction

The standard d -dimensional parabolic–parabolic Keller–Segel model for chemotaxis describes the time evolution of the density ρ_t of a cell population and of the concentration c_t of a chemical attractant:

$$\begin{cases} \partial_t \rho(t, x) = \nabla \cdot (\frac{1}{2} \nabla \rho - \chi \rho \nabla c), & t > 0, x \in \mathbb{R}^d, \\ \alpha \partial_t c(t, x) = \frac{1}{2} \Delta c - \lambda c + \rho, & t > 0, x \in \mathbb{R}^d. \\ \rho(0, x) = \rho_0(x), c(0, x) = c_0(x), \end{cases} \quad (1)$$

See e.g. Corrias [3], Perthame [12] and references therein for theoretical results on this system of PDEs and applications to Biology.

Recently, stochastic interpretations have been proposed for a simplified version of the model, that is, the parabolic-elliptic model which corresponds to the value $\alpha = 0$. They all rely on the fact that, in the parabolic-elliptic case, the equations for ρ_t and c_t can be decoupled and c_t can be explicitated as the convolution of the initial condition c_0 and the kernel $k(x) = -\frac{x}{2\pi|x|^2}$. Consequently, the stochastic process of McKean–Vlasov type whose ρ_t is the time marginal density involves the singular interaction kernel k . This explains why, so far, only partial results are obtained and heavy techniques are used to get them. In [8], one may find a short review of the works by Haškovec and Schmeiser [5], Fournier and Jourdain [4] and Cattiaux and Pédèches [2].

We here deal with the parabolic–parabolic system ($\alpha > 0$) and study the McKean-Vlasov stochastic representation of the mild formulation of the equation satisfied by ρ_t . This representation involves a singular interaction kernel which is quite different from the one in the above mentioned approaches and does not seem to have been studied in the McKean-Vlasov non-linear SDE literature. The system reads

$$\begin{cases} dX_t = b^\sharp(t, X_t)dt + \left\{ \int_0^t (K_{t-s}^\sharp \star p_s)(X_t) ds \right\} dt + dW_t, & t > 0, \\ p_s(y) dy := \mathcal{L}(X_s), & X_0 \sim \rho_0(x) dx, \end{cases} \quad (2)$$

where $K_t^\sharp(x) := \chi e^{-\lambda t} \nabla \left(\frac{1}{(2\pi t)^{d/2}} e^{-\frac{|x|^2}{2t}} \right)$ and $b^\sharp(t, x) := \chi e^{-\lambda t} \nabla \mathbb{E} c_0(x + W_t)$. Here, $(W_t)_{t \geq 0}$ is a d -dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))$ and X_0 is an \mathbb{R}^d -valued \mathcal{F}_0 -measurable random variable. Notice that the formulation requires that the one dimensional time marginals of the law of the solution are absolutely continuous with respect to Lebesgue’s measure and that the process interacts with all the past time marginals of its probability distribution through a functional involving a singular kernel.

*TOSCA, Inria Sophia Antipolis-Méditerranée, 2004 Route des Lucioles, 06902 Valbonne, France (denis.talay@inria.fr, milica.tomasevic@inria.fr).

†Université Côte d’Azur, Campus Valrose, Batiment M, 28 Avenue de Valrose, 06108 Nice CEDEX 2, France.

The analysis of the well-posedness of this non-linear equation and the proof that $p_s = \rho_s$ for any s are delicate, particularly in the multi-dimensional case when χ is large enough to induce solutions with blow-ups in finite time. This theoretical work is still in progress [14]. As numerical simulations of the related particle system appear to be effective, it seems interesting to validate our approach in the one-dimensional case.

The objective of this paper is to prove general existence and uniqueness results for both the deterministic system (1) and the stochastic dynamics (2) in $d = 1$. In our companion paper Jabir et al. [8] we show the well-posedness and propagation of chaos property of the corresponding particle system where each particle interacts with all the past of the other ones by means of a time integrated singular kernel.

In this one-dimensional framework the PDE (1) was previously studied by [11, 7] in bounded intervals I with periodic boundary conditions while we here deal with the problem posed on the whole space \mathbb{R} . In [11] one assumes $\rho_0 \in L^2(I) \cap L^1(I)$ and $c_0 \in H^1(I)$. In [7] one assumes $\rho_0 \in L^\infty(I) \cap L^1(I)$ and $c_0 \in W^{\sigma,p}(I)$, where p and σ belong to a particular set of parameters. Here, we only suppose that ρ_0 is in $L^1(\mathbb{R})$.

We emphasize that we do not limit ourselves to the specific kernel $K_t^\sharp(x)$ related to the Keller–Segel model. We below show that the mean–field PDE and stochastic differential equation of Keller–Segel type are well-posed for a whole class of time integrated singular kernels. To get this result we use an accurate estimate on densities of one-dimensional diffusions with bounded measurable drifts which is obtained by stochastic methods and allows us to control the L^∞ norms of the time marginal densities of processes constructed in a Picard procedure; thanks to these controls we are able to get uniform bounds on the sequence of drifts, which is essential both to tightness arguments and to get uniqueness of the local solution to the non-linear martingale problem solved by any limit of the Picard procedure. Then, we suitably paste local solutions of a family of non-linear martingale problems in order to exhibit the global solution.

The paper is organized as follows. In Section 2 we state our main results. In Section 3 we prove a preliminary estimate on the probability density of diffusions whose drift is only supposed Borel measurable and bounded. In Section 4 we study a non-linear McKean–Vlasov–Fokker–Planck equation. In Section 5 we prove the local existence and uniqueness of a solution to a non-linear stochastic differential equation more general than (2) (for $d = 1$). In Section 6 we get the global well-posedness of this equation. In Section 7 we apply the preceding result to the specific case of the one-dimensional parabolic–parabolic Keller–Segel model. The appendix section 8 concerns an explicit formula for the transition density of a particular diffusion.

Notation. In all the paper we denote by C_T , $C_T(b_0, p_0)$, etc., any constant which depends on T and the other specified parameters, but is uniform w.r.t. $t \in [0, T]$ and may change from line.

2 Our main results

Our first main result concerns the well-posedness of a non-linear one-dimensional stochastic differential equation (SDE) with a non standard McKean–Vlasov interaction kernel which at each time t involves in a singular way all the time marginals up to time t of the probability distribution of the solution. As our technique of analysis is not limited to the above kernel K^\sharp , we consider the following McKean–Vlasov stochastic equation:

$$\begin{cases} dX_t = b(t, X_t)dt + \left\{ \int_0^t (K_{t-s} \star p_s)(X_t)ds \right\} dt + dW_t, & t \leq T, \\ p_s(y)dy := \mathcal{L}(X_s), & X_0 \sim p_0, \end{cases} \quad (3)$$

and in all the sequel we assume the following conditions on the interaction kernel.

Hypothesis (H). *The function K defined on $\mathbb{R}^+ \times \mathbb{R}$ is such that for any $T > 0$:*

1. $K \in L^1((0, T]; L^1(\mathbb{R})) \cap L^1((0, T]; L^2(\mathbb{R}))$.
2. For any $t > 0$ the function $K_t(x)$ is a bounded continuous function on \mathbb{R} .
3. The set of points $x \in \mathbb{R}$ such that $\lim_{t \rightarrow 0} K_t(x) < \infty$ has Lebesgue measure zero.
4. For any $t > 0$, the functions $f_1(t) := \int_0^t \frac{\|K_{t-s}\|_{L^1(\mathbb{R})}}{\sqrt{s}} ds$ and $f_2(t) := \int_0^t \frac{\|K_{t-s}\|_{L^2(\mathbb{R})}}{s^{1/4}} ds$ are well defined and bounded on $[0, T]$.
5. For any $T > 0$ there exists C_T such that, for any probability density ϕ on \mathbb{R} ,

$$\sup_{(t,x) \in (0,T] \times \mathbb{R}} \int \phi(y) \|K_t(x-y)\|_{L^1(0,t)} dy \leq C_T.$$

6. Finally,

$$\sup_{0 \leq t \leq T} \int_0^T \|K_{T+t-s}\|_{L^1(\mathbb{R})} \frac{1}{\sqrt{s}} ds \leq C_T.$$

As emphasized in the introduction, the well-posedness of the system (3) cannot be obtained by applying known results in the literature.

Given $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$ and a family of densities $(p_t)_{t \leq T}$ we set

$$B(t, x; p) := \int_0^t (K_{t-s} \star p_s)(x) ds. \quad (4)$$

We now define the notion of a weak solution to (3).

Definition 2.1. *The family $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), X, W)$ is said to be a weak solution to the equation (3) up to time $T > 0$ if:*

1. $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))$ is a filtered probability space.
2. The process $X := (X_t)_{t \in [0, T]}$ is real-valued, continuous, and (\mathcal{F}_t) -adapted. In addition, the probability distribution of X_0 has density p_0 .
3. The process $W := (W_t)_{t \in [0, T]}$ is a one-dimensional (\mathcal{F}_t) -Brownian motion.
4. The probability distribution $\mathbb{P} \circ X^{-1}$ has time marginal densities $(p_t, t \in [0, T])$ with respect to Lebesgue measure which satisfy

$$\forall 0 < t \leq T, \quad \|p_t\|_{L^2(\mathbb{R})} \leq \frac{C_T}{t^{1/4}}. \quad (5)$$

5. For all $t \in [0, T]$ and $x \in \mathbb{R}$, one has that $\int_0^t |b(s, x)| ds < \infty$.

6. \mathbb{P} -a.s. the pair (X, W) satisfies (3).

Remark 2.2. *For any $T > 0$ Inequality (5) and Hypothesis (H-4) lead to*

$$\sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}} |B(t, x, p)| \leq C_T.$$

The following theorem provides existence and uniqueness of the weak solution to (3).

Theorem 2.3. *Let $T > 0$. Suppose that $p_0 \in L^1(\mathbb{R})$ is a probability density function and $b \in L^\infty([0, T] \times \mathbb{R})$ is continuous w.r.t. the space variable. Under the hypothesis (H), Eq. (3) admits a weak solution up to T . This solution is unique under the constraint that the one dimensional marginal densities $(p_t)_{t \leq T}$ satisfy the following additional estimate:*

$$\forall 0 < t \leq T, \quad \|p_t\|_{L^\infty(\mathbb{R})} \leq \frac{C_T}{\sqrt{t}}. \quad (6)$$

Our next result concerns the well-posedness of the the one-dimensional parabolic-parabolic Keller-Segel model

$$\begin{cases} \frac{\partial \rho}{\partial t}(t, x) = \frac{\partial}{\partial x} \cdot \left(\frac{1}{2} \frac{\partial \rho}{\partial x} - \chi \rho \frac{\partial c}{\partial x} \right), & t > 0, x \in \mathbb{R}, \end{cases} \quad (7a)$$

$$\begin{cases} \frac{\partial c}{\partial t}(t, x) = \frac{1}{2} \frac{\partial^2 c}{\partial x^2} - \lambda c + \rho, & t > 0, x \in \mathbb{R}, \\ \rho(0, x) = \rho_0(x), \quad c(0, x) = c_0. \end{cases} \quad (7b)$$

The parameters χ and λ are strictly positive. As this system preserves the total mass, that is,

$$\forall t > 0, \quad \int_{\Omega} \rho(t, x) dx = \int_{\Omega} \rho_0(x) dx =: M,$$

the new functions $\tilde{\rho}(t, x) := \frac{\rho(t, x)}{M}$ and $\tilde{c}(t, x) := \frac{c(t, x)}{M}$ satisfy the system (7) with the new parameter $\tilde{\chi} := \chi M$. Therefore, w.l.o.g. we may and do thereafter assume that $M = 1$.

Denote by $g(t, x)$ the density of W_t . We define the notion of solution for the system (7):

Definition 2.4. Given $\chi > 0$ and $T > 0$, the pair (ρ, c) is said to be a solution to K-S model if $\rho \in L^\infty([0, T]; L^1(\mathbb{R}))$, $c \in L^\infty([0, T]; C_b^1(\mathbb{R}))$, and for any $t \in (0, T]$ we have

$$\|\rho_t\|_{L^2(\mathbb{R})} \leq \frac{C}{t^{\frac{1}{4}}}, \quad \|\rho_t\|_{L^\infty(\mathbb{R})} \leq \frac{C_T}{\sqrt{t}},$$

and (ρ_t) satisfies the following equation in the sense of the distributions:

$$\rho_t = g(t, \cdot) * \rho_0 + \chi \int_0^t \frac{\partial g}{\partial x}(t-s, \cdot) * \left(\frac{\partial c}{\partial x}(s, \cdot) \rho_s\right) ds,$$

with

$$c_t = e^{-\lambda t} g(t, \cdot) * c_0 + \int_0^t e^{-\lambda s} \rho_{t-s} * g(s, \cdot) ds.$$

These solutions are known as integral solutions and they have already been studied in PDE literature for the two-dimensional Keller-Segel model for which sub-critical and critical regimes exist depending on the parameters of the model (see [3] and references therein).

In the one-dimensional case there is no critical regime.

Theorem 2.5. Assume that $\rho_0 \in L^1(\mathbb{R})$ and $c_0 \in C_b^1(\mathbb{R})$. Given any $\chi > 0$ and $\lambda \geq 0$, there exists a unique global solution (ρ, c) to (7) in the sense of Definition 2.4.

The condition on c_0 can be replaced by any other condition that ensures that $\frac{\partial}{\partial x} e^{-\lambda t} \mathbb{E}(c_0(x + W_t))$ is bounded by a deterministic constant. In addition, from estimates below we could deduce some additional regularity results which we do not need here: See Remark 3.3. In particular, if $\rho_0 \in L^\infty(\mathbb{R})$, then $\rho \in L^\infty([0, T]; L^1 \cap L^\infty(\mathbb{R}))$. If $\rho_0 \in L^2(\mathbb{R})$, then $\rho \in L^\infty([0, T]; L^1 \cap L^2(\mathbb{R}))$ and $t^{1/4} \|\rho_t\|_{L^\infty(\mathbb{R})} \leq C$; in addition, one can then easily find modifications of the hypotheses on the kernel K allowing to get unique weak solutions with constraints on $\|\rho_t\|_{L^2(\mathbb{R})}$. We prefer to deal with the more general assumption $\rho_0 \in L^1(\mathbb{R})$.

Our last main result shows the relationship between the non-linear SDE (3) and the Keller-Segel system (7).

Corollary 2.6. Assume that $\rho_0 \in L^1(\mathbb{R})$ and $c_0 \in C_b^1(\mathbb{R})$. The kernel $K_t^\sharp(x) := \chi e^{-\lambda t} \frac{-x}{2\pi t^{3/2}} e^{-\frac{x^2}{2t}}$ and the function $b(t, x) := \chi e^{-\lambda t} \mathbb{E} c_0'(x + W_t)$ satisfy the hypothesis of Theorem 2.3. Denote by $\rho_t \equiv p_t$ the time marginals of the probability distribution of the corresponding solution to Eq. (2) with $d = 1$. Set

$$c_t := e^{-\lambda t} g(t, \cdot) * c_0 + \int_0^t e^{-\lambda s} \rho_{t-s} * g(s, \cdot) ds.$$

Then the pair (ρ_t, c_t) is the unique global solution to the Keller-Segel system (7) in the sense of Definition 2.4.

3 Preliminary: A density estimate

Let $X^{(b)}$ be a process defined by

$$X_t^{(b)} = X_0 + \int_0^t b(s, X_s^{(b)}) ds + W_t, \quad t \in [0, T]. \quad (8)$$

To obtain $L^\infty(\mathbb{R})$ estimates for the transition probability density $p^{(b)}(t, x, y)$ of $X^{(b)}$ under the only assumption that the drift $b(t, x)$ is measurable and uniformly bounded we slightly extend the estimate proven in [13] for time homogeneous drift coefficients $b(x)$. We here propose a proof different from the original one in [13]. It avoids the use of densities of pinned diffusions and the claim that $p^{(b)}(t, x, y)$ is continuous w.r.t. all the variables which does not seem obvious to us. We here adapt Makhlof's method [10], the main difference between [10] and our calculation being that instead of the Wiener measure our reference measure is the probability distribution of the particular diffusion process X^β introduced in [13] and defined by

$$X_t^\beta = X_0 + \beta \int_0^t \operatorname{sgn}(y - X_s^\beta) ds + W_t.$$

Theorem 3.1. Let $X^{(b)}$ be the process defined in (8) with $X_0 = x$. Let $p_y^\beta(t, x, z)$ be the transition density of X^β . Assume $\beta := \sup_{t \in [0, T]} \|b(t, \cdot)\|_\infty < \infty$. Then for all $y \in \mathbb{R}$ and $t \in (0, T]$ it holds that

$$p^{(b)}(t, x, y) \leq p_y^\beta(t, x, y) = \frac{1}{\sqrt{2\pi t}} \int_{\frac{|x-y|}{\sqrt{t}}}^{\infty} z e^{-\frac{(z-\beta\sqrt{t})^2}{2}} dz. \quad (9)$$

Proof. Let $f \in C_K^\infty(\mathbb{R})$ and fix $t \in (0, T]$. Consider the parabolic PDE driven by the infinitesimal generator of X^β :

$$\begin{cases} \frac{\partial u}{\partial t}(s, x) + \frac{1}{2} \frac{\partial^2 u}{\partial x^2}(s, x) + \beta \operatorname{sgn}(y - x) \frac{\partial u}{\partial x}(s, x) = 0, & 0 \leq s < t, \quad x \in \mathbb{R}, \\ u(t, x) = f(x), & x \in \mathbb{R}. \end{cases} \quad (10)$$

In view of Veretennikov [15, Thm.1] there exists a solution $u(s, x) \in W_p^{1,2}([0, t] \times \mathbb{R})$. Applying the Itô-Krylov formula to $u(s, X_s^\beta)$ we obtain that

$$u(s, x) = \int f(z) p_y^\beta(t - s, x, z) dz.$$

The formula (30) from our appendix allows us to differentiate under the integral sign:

$$\frac{\partial u}{\partial x}(s, x) = \int f(z) \frac{\partial p_y^\beta}{\partial x}(t - s, x, z) dz, \quad \forall 0 \leq s < t \leq T.$$

Fix $0 < \varepsilon < t$. Now apply the Itô-Krylov formula to $u(s, X_s^{(b)})$ for $0 \leq s \leq t - \varepsilon$ and use the PDE (10). It comes:

$$\mathbb{E}(u(t - \varepsilon, X_{t-\varepsilon}^{(b)})) = u(0, x) + \mathbb{E} \int_0^{t-\varepsilon} (b(s, X_s^{(b)}) - \beta \operatorname{sgn}(y - X_s^{(b)})) \left(\frac{\partial}{\partial x} u \right) (s, X_s^{(b)})^1 ds.$$

In view of Corollary 8.2 in the appendix there exists a function $h \in L^1([0, t] \times \mathbb{R})$ such that

$$\forall 0 < s < t \leq T, \quad \forall y, z \in \mathbb{R}, \quad \mathbb{E} \left| \left(\frac{\partial}{\partial x} p_y^\beta \right) (t - s, X_s^{(b)}, z) \right|^1 \leq C_{T, \beta, x, y} h(s, z). \quad (11)$$

Consequently,

$$\begin{aligned} \mathbb{E}(u(t - \varepsilon, X_{t-\varepsilon}^{(b)})) &= \int f(z) p_y^\beta(t, x, z) dz \\ &+ \int f(z) \int_0^{t-\varepsilon} \mathbb{E} \left\{ (b(s, X_s^{(b)}) - \beta \operatorname{sgn}(y - X_s^{(b)})) \left(\frac{\partial}{\partial x} p_y^\beta \right) (t - s, X_s^{(b)}, z) \right\} ds dz. \end{aligned}$$

Let now ε tend to 0. By Lebesgue's dominated convergence theorem we obtain

$$\begin{aligned} \int f(z) p^{(b)}(t, x, z) dz &= \int f(z) p_y^\beta(t, x, z) dz \\ &+ \int f(z) \int_0^t \mathbb{E} \left\{ (b(s, X_s^{(b)}) - \beta \operatorname{sgn}(y - X_s^{(b)})) \left(\frac{\partial}{\partial x} p_y^\beta \right) (t - s, X_s^{(b)}, z) \right\} ds dz. \end{aligned}$$

Therefore the density $p^{(b)}$ satisfies:

$$p^{(b)}(t, x, z) = p_y^\beta(t, x, z) + \int_0^t \mathbb{E} \left\{ (b(s, X_s^{(b)}) - \beta \operatorname{sgn}(y - X_s^{(b)})) \left(\frac{\partial}{\partial x} p_y^\beta \right) (t - s, X_s^{(b)}, z) \right\} ds.$$

As noticed in [13], in view of Formula (31) from our appendix we have for any $x \in \mathbb{R}$

$$(b(s, x) - \beta \operatorname{sgn}(y - x)) \frac{\partial}{\partial x} p_y^\beta(t - s, x, y) \leq 0.$$

¹ $\left(\frac{\partial}{\partial x} u \right) (s, X_s^{(b)}) = \frac{\partial}{\partial \xi} u(s, \xi) \Big|_{\xi = X_s^{(b)}}$; $\left(\frac{\partial}{\partial x} p_y^\beta \right) (t - s, X_s^{(b)}, z) = \frac{\partial}{\partial \xi} p_y^\beta(t - s, \xi, z) \Big|_{\xi = X_s^{(b)}}$

This notation takes care of the fact that $X_s^{(b)}$ depends on x .

This leads us to choose $z = y$ in the preceding equality, which gives us

$$p^{(b)}(t, x, y) = p_y^\beta(t, x, y) + \int_0^t \mathbb{E} \left\{ (b(s, X_s^{(b)}) - \beta \operatorname{sgn}(y - X_s^{(b)})) \left(\frac{\partial}{\partial x} p_y^\beta \right) (t - s, X_s^{(b)}, z) \right\} ds,$$

from which

$$\forall t \leq T, \quad p^{(b)}(t, x, y) \leq p_y^\beta(t, x, y).$$

We finally use Qian and Zheng's explicit representation (see [13] and our appendix section 8). □

Corollary 3.2. *Assume $X_0^{(b)}$ is distributed according to the probability density function p_0 on \mathbb{R} . Denote by $p(t, y)$ the probability density of $X_t^{(b)}$. One has*

$$\|p(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{C}{\sqrt{t}} + \beta. \quad (12)$$

Proof. In view of (9) we have

$$\begin{aligned} p(t, y) &\leq \frac{1}{\sqrt{2\pi t}} \int p_0(x) \int_{\frac{|x-y|}{\sqrt{t}}}^{\infty} z e^{-\frac{(z-\beta\sqrt{t})^2}{2}} dz dx \\ &\leq \frac{1}{\sqrt{2\pi t}} \int p_0(x) \int_{\frac{|x-y|}{\sqrt{t}} - \beta\sqrt{t}}^{\infty} (z + \beta\sqrt{t}) e^{-\frac{z^2}{2}} dz dx \\ &= \frac{1}{\sqrt{2\pi t}} \left(\int p_0(x) e^{-\frac{(|x-y|-\beta t)^2}{2t}} dx + \beta\sqrt{t} \int p_0(x) \int_{\frac{|x-y|}{\sqrt{t}} - \beta\sqrt{t}}^{\infty} e^{-\frac{z^2}{2}} dz dx \right) \\ &\leq \frac{1}{\sqrt{2\pi t}} \int p_0(x) e^{-\frac{(|y-x|-\beta t)^2}{2t}} dx + \beta. \end{aligned}$$

□

Remark 3.3. *If $p_0 \in L^\infty(\mathbb{R})$, the above calculation shows that*

$$\|p(t, \cdot)\|_\infty \leq 2\|p_0\|_\infty + \beta.$$

If $p_0 \in L^p(\mathbb{R})$, $p > 1$, Hölder's inequality leads to

$$\frac{1}{\sqrt{2\pi t}} \int p_0(x) e^{-\frac{(|y-x|-\beta t)^2}{2t}} dx \leq \frac{\|p_0\|_{L^p(\mathbb{R})}}{\sqrt{2\pi t}} \left(\int e^{-q\frac{(|y-x|-\beta t)^2}{2t}} dx \right)^{1/q} \leq \frac{C_q t^{\frac{1}{2q}}}{\sqrt{t}} = \frac{C_q}{t^{\frac{1}{2p}}}.$$

4 A non-linear McKean–Vlasov–Fokker–Planck equation

Proposition 4.1. *Let $T > 0$. Assume $p_0 \in L^1(\mathbb{R})$, $b \in L^\infty((0, T) \times \mathbb{R})$ and Hypothesis (H). Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), X, W)$ be a weak solution to (3) until T . Then,*

1. *The marginals $(p_t)_{t \in (0, T]}$ satisfy in the sense of the distributions the mild equation*

$$\forall t \in (0, T], \quad p_t = g(t, \cdot) * p_0 + \int_0^t \frac{\partial}{\partial x} g(t - s, \cdot) * (p_s(b(s, \cdot) + B(s, \cdot; p))) ds. \quad (13)$$

2. *Equation (13) admits at most one solution $(p_t)_{t \in [0, T]}$ which for any $t \in (0, T]$ belongs to $L^1(\mathbb{R})$ and satisfies (5) and (6).*

Proof. We successively prove (13) and the uniqueness of its solution in $L^1(\mathbb{R})$.

1. Now, for $f \in C_b^2(\mathbb{R})$ consider the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial s} G + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} = 0, & 0 \leq s < t, \quad x \in \mathbb{R}, \\ \lim_{s \rightarrow t^-} G(s, x) = f(x). \end{cases} \quad (14)$$

The function

$$G_{t,f}(s, x) = \int f(y)g(t-s, x-y)dy$$

is a smooth solution to (14). Applying Itô's formula we get

$$\begin{aligned} G_{t,f}(t, X_t) - G_{t,f}(0, X_0) &= \int_0^t \frac{\partial G_{t,f}}{\partial s}(s, X_s) ds + \int_0^t \frac{\partial G_{t,f}}{\partial x}(s, X_s)(b(s, X_s) + B(s, X_s; p)) ds \\ &\quad + \int_0^t \frac{\partial}{\partial x} G_{t,f}(s, X_s) dW_s + \frac{1}{2} \int_0^t \frac{\partial^2}{\partial x^2} G_{t,f}(s, X_s) ds. \end{aligned}$$

Using (14) we obtain

$$\mathbb{E}f(X_t) = \mathbb{E}G_{t,f}(0, X_0) + \int_0^t \mathbb{E}\left[\frac{\partial}{\partial x} G_{t,f}(s, X_s)(b(s, X_s) + B(s, X_s; p))\right] ds =: I + II. \quad (15)$$

On the one hand one has

$$I = \int \int f(y)g(t, y-x)dy p_0(x)dx = \int f(y)(g(t, \cdot) * p_0)(y)dy.$$

On the second hand one has

$$\begin{aligned} II &= \int_0^t \int \frac{\partial}{\partial x} \left[\int f(x)g(t-s, y-x)dy \right] (b(s, x) + B(s, x; p)) p_s(x) dx ds \\ &= \int_0^t \int \int f(y) \frac{\partial}{\partial x} g(t-s, y-x) dy (b(s, x) + B(s, x; p)) p_s(x) dx ds \\ &= \int f(y) \int_0^t \left[\frac{\partial}{\partial x} g(t-s, \cdot) * ((b(s, \cdot) + B(s, \cdot; p)) p_s) \right] (y) ds dy. \end{aligned}$$

Thus (15) can be written as

$$\int f(y) p_t(y) dx = \int f(y)(g(t, \cdot) * p_0)(y) dy + \int f(y) \int_0^t \left[\frac{\partial}{\partial x} g(t-s, \cdot) * ((b(s, \cdot) + B(s, \cdot; p)) p_s) \right] (y) ds dy,$$

which is the mild equation (13).

2. Assume p_t^1 and p_t^2 are two mild solutions in the sense of the distributions to (13) which satisfy

$$\exists C > 0, \forall t \in (0, T], \quad \|p_t^1\|_{L^2(\mathbb{R})} + \|p_t^2\|_{L^2(\mathbb{R})} \leq \frac{C_T}{t^{1/4}}.$$

Then,

$$\begin{aligned} \|p_t^1 - p_t^2\|_{L^1(\mathbb{R})} &\leq \int_0^t \left\| \frac{\partial g}{\partial x}(t-s, \cdot) * [B(s, \cdot; p^1) p_s^1 - B(s, \cdot; p^2) p_s^2] \right\|_{L^1(\mathbb{R})} ds \\ &\quad + \int_0^t \left\| \frac{\partial g}{\partial x}(t-s, \cdot) * [b(s, \cdot)(p_s^1 - p_s^2)] \right\|_{L^1(\mathbb{R})} ds \\ &\leq \int_0^t \left\| \frac{\partial g}{\partial x}(t-s, \cdot) * [(B(s, \cdot; p^1) - B(s, \cdot; p^2)) p_s^1] \right\|_{L^1(\mathbb{R})} ds \\ &\quad + \int_0^t \left\| \frac{\partial g}{\partial x}(t-s, \cdot) * [(p_s^1 - p_s^2) B(s, \cdot; p^2)] \right\|_{L^1(\mathbb{R})} ds \\ &\quad + \int_0^t \left\| \frac{\partial}{\partial x} g(t-s, \cdot) * [b(s, \cdot)(p_s^1 - p_s^2)] \right\|_{L^1(\mathbb{R})} ds \\ &=: I + II + III. \end{aligned}$$

As

$$\left\| \frac{\partial}{\partial x} g(t-s, \cdot) \right\|_{L^1(\mathbb{R})} \leq \frac{C_T}{\sqrt{t-s}},$$

the convolution inequality $\|f * h\|_{L^1(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})} \|h\|_{L^1(\mathbb{R})}$ and Remark 2.2 lead to

$$II \leq \int_0^t \left\| \frac{\partial}{\partial x} g(t-s, \cdot) \right\|_{L^1(\mathbb{R})} \|(p_s^1 - p_s^2)B(s, \cdot; p^2)\|_{L^1(\mathbb{R})} ds \leq C_T \int_0^t \frac{\|p_s^1 - p_s^2\|_{L^1(\mathbb{R})}}{\sqrt{t-s}} ds.$$

As b is bounded, we also have

$$|III| \leq C_T \int_0^t \frac{\|p_s^1 - p_s^2\|_{L^1(\mathbb{R})}}{\sqrt{t-s}} ds.$$

We now turn to I . Notice that

$$\|B(s, \cdot; p^1) - B(s, \cdot; p^2)\|_{L^1(\mathbb{R})} \leq \int_0^s \|K_{s-\tau}\|_{L^1(\mathbb{R})} \|p_\tau^1 - p_\tau^2\|_{L^1(\mathbb{R})} d\tau,$$

from which, since by hypothesis (p_t) satisfies (6),

$$\begin{aligned} I &\leq \int_0^t \frac{C_T}{\sqrt{t-s}\sqrt{s}} \int_0^s \|K_{s-\tau}\|_{L^1(\mathbb{R})} \|p_\tau^1 - p_\tau^2\|_{L^1(\mathbb{R})} d\tau ds \\ &\leq \int_0^t \|p_\tau^1 - p_\tau^2\|_{L^1(\mathbb{R})} \int_\tau^t \frac{C_T}{\sqrt{t-s}\sqrt{s}} \|K_{s-\tau}\|_{L^1(\mathbb{R})} ds d\tau. \end{aligned}$$

In addition, using Hypothesis (H.4),

$$\int_\tau^t \frac{1}{\sqrt{t-s}\sqrt{s}} \|K_{s-\tau}\|_{L^1(\mathbb{R})} ds \leq \frac{1}{\sqrt{\tau}} \int_\tau^t \frac{1}{\sqrt{t-s}} \|K_{s-\tau}\|_{L^1(\mathbb{R})} ds = \frac{1}{\sqrt{\tau}} \int_0^{t-\tau} \frac{\|K_s\|_{L^1(\mathbb{R})}}{\sqrt{t-\tau-s}} ds \leq \frac{C_T}{\sqrt{\tau}}.$$

It comes:

$$I \leq C_T \int_0^t \frac{\|p_\tau^1 - p_\tau^2\|_{L^1(\mathbb{R})}}{\sqrt{\tau}} d\tau.$$

Gathering the preceding estimates we obtain

$$\|p_t^1 - p_t^2\|_{L^1(\mathbb{R})} \leq C_T \int_0^t \frac{\|p_s^1 - p_s^2\|_{L^1(\mathbb{R})}}{\sqrt{t-s}} ds + C_T \int_0^t \frac{\|p_s^1 - p_s^2\|_{L^1(\mathbb{R})}}{\sqrt{s}} ds.$$

Applying a Singular Gronwall's Lemma (see Lemma 4.2 below), we conclude

$$\forall t \in (0, T], \quad \|p_t^1 - p_t^2\|_{L^1(\mathbb{R})} = 0,$$

which ends the proof. \square

In the above proof we have used the following result:

Lemma 4.2. *Let $(u(t))_{t \geq 0}$ be a non-negative bounded function such that for a given $T > 0$, there exists a positive constant C_T such that for any $t \in (0, T]$:*

$$u(t) \leq C_T \int_0^t \frac{u(s)}{\sqrt{s}} ds + C_T \int_0^t \frac{u(s)}{\sqrt{t-s}} ds. \quad (16)$$

Then, $u(t) = 0$ for any $t \in (0, T]$.

Proof. Set $u_t^* := \sup_{s \leq t} u(s)$. The inequality (16) implies that

$$u_t^* \leq 4C_T \sqrt{t} u_t^*.$$

Set $T^* := \frac{1}{32C_T^2}$. If $T \leq T^*$, then for $t \leq T$, we have $u_t^* \leq \frac{u_t^*}{\sqrt{2}}$. Thus, $u_t^* = 0$ for every $t \leq T$ and the lemma is proved. If $T > T^*$, for $T^* < t < T$,

$$u(t) \leq C_T \int_0^{T^*} \left(\frac{1}{\sqrt{t-s}} + \frac{1}{\sqrt{s}} \right) u(s) ds + \int_{T^*}^t \left(\frac{1}{\sqrt{t-s}} + \frac{1}{\sqrt{s}} \right) u(s) ds.$$

The first integral is null, since $u(T^*) = 0$. Thus,

$$u(t) \leq \int_0^{t-T^*} \left(\frac{1}{\sqrt{t-T^*-\theta}} + \frac{1}{\sqrt{T^*+\theta}} \right) u(T^*+\theta) d\theta \leq \int_0^{t-T^*} \left(\frac{1}{\sqrt{t-T^*-\theta}} + \frac{1}{\sqrt{T^*}} \right) u(T^*+\theta) d\theta.$$

For $0 < s \leq T - T^*$, define $v(s) := u(s + T^*)$. The previous inequality becomes:

$$v(t - T^*) \leq C_T \int_0^{t-T^*} \left(\frac{1}{\sqrt{t-T^*-\theta}} + \frac{1}{\sqrt{T^*}} \right) v(\theta) d\theta \leq \frac{\sqrt{T^*} + \sqrt{T - T^*}}{\sqrt{T^*}} \int_0^{t-T^*} \frac{v(\theta)}{\sqrt{t-T^*-\theta}} d\theta.$$

Setting $t - T^* =: \tau$, we thus have

$$v(\tau) \leq \frac{\sqrt{T^*} + \sqrt{T - T^*}}{\sqrt{T^*}} \int_0^\tau \frac{v(\theta)}{\sqrt{\tau-\theta}} d\theta, \quad \forall 0 \leq \tau < T - T^*.$$

Now we are in a position to apply a standard singular Gronwall's lemma (see [6, Lem. 7.1.1]) and conclude that $v(\tau) = 0$ for $0 \leq \tau \leq T - T^*$. Thus, $u(t) = 0$ for $T^* \leq t \leq T$. \square

5 A local existence and uniqueness result for Equation (3)

Set

$$D(T) := \int_0^T \int_{\mathbb{R}} |K_t(x)| dx dt < \infty. \quad (17)$$

The main result in this section is the following theorem.

Theorem 5.1. *Let $T_0 > 0$ be such that $D(T_0) < 1$. Assume $p_0 \in L^1(\mathbb{R})$ and $b \in L^\infty((0, T_0) \times \mathbb{R})$ continuous w.r.t. space variable. Under Hypothesis (H), Equation (3) admits a unique weak solution up to T_0 such that the probability distributions $\mathbb{P} \circ X_t^{-1}$ admit densities which satisfy (6).*

Iterative procedure. Consider the following sequence of SDE's. For $k = 1$

$$\begin{cases} dX_t^1 = b(t, X_t^1) dt + \left\{ \int_0^t (K_{t-s} \star p_0)(X_t^1) ds \right\} dt + dW_t, \\ X_0^1 \sim p_0. \end{cases} \quad (18)$$

Denote the drift of this equation by $b^1(t, x)$. Supposing that, in the step $k - 1$, the one dimensional time marginals of the law of the solution have densities $p^{k-1}(s, y)$, we define the drift in the step k as

$$b^k(t, x, p^{k-1}) = b(t, x) + B(t, x; p^{k-1}).$$

The corresponding SDE is

$$\begin{cases} dX_t^k = b^k(t, X_t^k, p^{k-1}) dt + dW_t, \\ X_0^k \sim p_0. \end{cases} \quad (19)$$

In order to prove the desired local existence and uniqueness result we set up the non-linear martingale problem related to (3).

Definition 5.2. *A probability measure \mathbb{Q} on the canonical space $\mathcal{C}([0, T_0]; \mathbb{R})$ equipped with its canonical filtration and a canonical process (w_t) is a solution to the non-linear martingale problem $(MP(p_0, T_0, b))$ if:*

(i) $\mathbb{Q}_0 = p_0$.

(ii) *For any $t \in (0, T_0]$, the one dimensional time marginals of \mathbb{Q} , denoted by \mathbb{Q}_t , have densities q_t w.r.t. Lebesgue measure on \mathbb{R} . In addition, they satisfy $q_t \in L^2(\mathbb{R})$ and*

$$\forall 0 < t \leq T_0, \quad \|q_t\|_{L^2(\mathbb{R})} \leq \frac{C_{T_0}}{t^{1/4}}. \quad (20)$$

(iii) For any $f \in C_K^2(\mathbb{R})$ the process $(M_t)_{t \leq T_0}$, defined as

$$M_t := f(w_t) - f(w_0) - \int_0^t \left[\frac{1}{2} \frac{\partial^2 f}{\partial x^2}(w_u) + \frac{\partial f}{\partial x}(w_u) (b(u, w_u) + \int_0^u \int K_{u-\tau}(w_u - y) q_\tau(y) dy d\tau) \right] du$$

is a \mathbb{Q} -martingale.

Notice that the arguments in Remark 2.2 justify that all the integrals in the definition of M_t are well defined.

We start with the analysis of Equations (18)-(19).

Proposition 5.3. *Same assumptions as in Theorem (5.1). Then, for any $k \geq 1$, Equations (18)-(19) have a unique weak solution (X, \mathbb{P}^k) up to T_0 . Moreover, for $t \in (0, T_0]$, the time marginals \mathbb{P}_t^k of \mathbb{P}^k have densities p_t^k w.r.t. Lebesgue measure on \mathbb{R} . Setting $\beta^k = \sup_{t \leq T_0} \|b^k(t, \cdot)\|_{L^\infty(\mathbb{R})}$ and $b_0 := \|b\|_{L^\infty(\mathbb{R})}$, one has*

$$\forall 0 < t \leq T_0, \quad \|p_t^k\|_{L^\infty(\mathbb{R})} \leq \frac{C(b_0, T_0)}{\sqrt{t}} \quad \text{and} \quad \beta^k \leq C(b_0, T_0).$$

Proof. We proceed by induction.

Case $k = 1$. In view of (H.5), one has $\beta^1 \leq b_0 + C_{T_0}$. This implies that the equation (18) has a unique weak solution in $[0, T_0]$ with time marginal densities $(p^1(t, y) dy)_{t \leq T_0}$ which in view of (12) satisfy

$$\forall t \in (0, T_0], \quad \|p^1(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{C_{T_0}}{\sqrt{t}} + \beta^1.$$

Case $k > 1$. Assume now that the equation for X^k has a unique weak solution and assume β^k is finite. In addition, suppose that the one dimensional time marginals satisfy

$$\forall t \in (0, T_0], \quad \|p^k(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{C_{T_0}}{\sqrt{t}} + \beta^k.$$

In view of (H.4), the new drift satisfies

$$|b^{k+1}(t, x; p^k)| \leq b_0 + \int_0^t \|p^k(s, \cdot)\|_{L^\infty(\mathbb{R})} \|K_{t-s}\|_{L^1(\mathbb{R})} ds \leq b_0 + \int_0^t \left(\frac{1}{\sqrt{2\pi s}} + \beta^k \right) \|K_{t-s}\|_{L^1(\mathbb{R})} ds \leq b_0 + C_{T_0} + \beta^k D(T_0).$$

Thus, we conclude that $\beta^{k+1} \leq b_0 + C_{T_0} + \beta^k D(T_0)$. Therefore, there exists a unique weak solution to the equation for X^{k+1} . Furthermore, by (12):

$$\forall t \in (0, T_0], \quad \|p^{k+1}(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{C_{T_0}}{\sqrt{t}} + \beta^{k+1}.$$

Notice that

$$\forall k > 1, \quad \beta^{k+1} \leq b_0 + C_{T_0} + \beta^k D(T_0) \quad \text{and} \quad \beta^1 \leq b_0 + C_{T_0}.$$

Thus, as by hypothesis $D(T_0) < 1$, we have

$$\forall k \geq 1, \quad \beta^k \leq \frac{b_0 + C_{T_0}}{1 - D(T_0)} + b_0 + C_{T_0} \tag{21}$$

and

$$\|p^k(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{C_{T_0}}{\sqrt{t}} + \beta^k \leq \frac{C_{T_0}}{\sqrt{t}} + \frac{b_0 + C_{T_0}}{1 - D(T_0)} + b_0 + C_{T_0}. \tag{22}$$

□

Proposition 5.4. *Same assumptions as in Theorem (5.1). Then,*

- 1) *The family of probabilities $(\mathbb{P}^k)_{k \geq 1}$ is tight.*
- 2) *Any weak limit \mathbb{P}^∞ of a convergent subsequence of $(\mathbb{P}^k)_{k \geq 1}$ solves $(MP(p_0, T_0, b))$.*

Proof. In view of (21), we obviously have

$$\exists C_{T_0} > 0, \quad \sup_k |X_t^k - X_s^k|^4 \leq C_{T_0} |t - s|^2, \quad \forall 0 \leq s \leq t \leq T_0.$$

This is a sufficient condition for tightness (see e.g. [9, Chap.2, Pb.4.11]). Therefore, there exists a weakly convergent subsequence of $(\mathbb{P}^k)_{k \geq 1}$ that we will still denote by $(\mathbb{P}^k)_{k \geq 1}$. Denote its limit by \mathbb{P}^∞ . Let us check that \mathbb{P}^∞ solves the martingale problem $(MP(p_0, T_0, b))$.

- i) Each \mathbb{P}_0^k has density p_0 , and therefore \mathbb{P}_0^∞ also has density p_0 .
- ii) From (22) for any $k \geq 1$ and $0 < t \leq T_0$, we have that

$$p_t^k \in L^2(\mathbb{R}) \quad \text{and} \quad \|p_t^k\|_{L^2(\mathbb{R})} \leq \frac{C(b_0, T_0)}{t^{1/4}}. \quad (23)$$

Define the functional $T_t(g)$ by

$$T_t(g) := \int g(y) \mathbb{P}_t^\infty(dy), \quad g \in C_K(\mathbb{R}).$$

By weak convergence we have

$$T_t(g) = \lim_{k \rightarrow \infty} \int g(y) p_t^k(y) dy,$$

and thus

$$|T_t(g)| \leq \frac{C(b_0, T_0)}{t^{1/4}} \|g\|_{L^2(\mathbb{R})}.$$

Therefore, for each $0 < t \leq T_0$, T_t is a bounded linear functional on a dense subset of $L^2(\mathbb{R})$. Thus, T_t can be extended to a linear functional on $L^2(\mathbb{R})$. By Riesz-representation theorem, there exists a unique $p_t^\infty \in L^2(\mathbb{R})$ such that $\|p_t^\infty\|_{L^2(\mathbb{R})} \leq \frac{C(b_0, T_0)}{t^{1/4}}$ and p_t^∞ is the probability density of $\mathbb{P}_t^\infty(dy)$.

- iii) Set

$$M_t := f(w_t) - f(w_0) - \int_0^t \left[\frac{1}{2} \frac{\partial^2 f}{\partial x^2}(w_u) + \frac{\partial f}{\partial x}(w_u) b(u, w_u) + \int_0^u (K_{u-\tau} \star p_\tau^\infty)(w_u) d\tau \right] du,$$

We have to prove

$$\mathbb{E}_{\mathbb{P}^\infty}[(M_t - M_s)\phi(w_{t_1}, \dots, w_{t_N})] = 0, \quad \forall \phi \in C_b(\mathbb{R}^N) \text{ and } 0 \leq t_1 < \dots < t_N < s \leq t \leq T_0, N \geq 1.$$

The process

$$M_t^k := f(w_t) - f(x(0)) - \int_0^t \left[\frac{1}{2} \frac{\partial^2 f}{\partial x^2}(w_u) + \frac{\partial f}{\partial x}(w_u) b(u, w_u) + \int_0^u (K_{u-\tau} \star p_\tau^{k-1})(w_u) d\tau \right] du$$

is a martingale under \mathbb{P}^k . Therefore, it follows that

$$\begin{aligned} 0 &= \mathbb{E}_{\mathbb{P}^k}[(M_t^k - M_s^k)\phi(w_{t_1}, \dots, w_{t_N})] = \\ &= \mathbb{E}_{\mathbb{P}^k}[\phi(\dots)(f(w_t) - f(w_s))] + \mathbb{E}_{\mathbb{P}^k}[\phi(\dots) \int_s^t \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(w_u) du] \\ &+ \mathbb{E}_{\mathbb{P}^k}[\phi(\dots) \int_s^t \frac{\partial f}{\partial x}(w_u) b(u, w_u) du] + \mathbb{E}_{\mathbb{P}^k}[\phi(\dots) \int_s^t \frac{\partial f}{\partial x}(w_u) \int_0^u (K_{u-\tau} \star p_\tau^{k-1})(w_u) d\tau du]. \end{aligned}$$

Since (\mathbb{P}^k) weakly converges to \mathbb{P}^∞ , the first two terms on the r.h.s. converge. Now, observe that

$$\begin{aligned}
& \mathbb{E}_{\mathbb{P}^k} [\phi(\dots) \int_s^t \frac{\partial f}{\partial x}(w_u) \int_0^u (K_{u-\tau} \star p_\tau^{k-1})(w_u) d\tau du] \\
& - \mathbb{E}_{\mathbb{P}^\infty} [\phi(\dots) \int_s^t \frac{\partial f}{\partial x}(w_u) \int_0^u (K_{u-\tau} \star p_\tau^\infty)(w_u) d\tau du] \\
& = (\mathbb{E}_{\mathbb{P}^k} [\phi(\dots) \int_s^t \frac{\partial f}{\partial x}(w_u) \int_0^u (K_{u-\tau} \star p_\tau^{k-1})(w_u) d\tau du] \\
& - \mathbb{E}_{\mathbb{P}^k} [\phi(\dots) \int_s^t \frac{\partial f}{\partial x}(w_u) \int_0^u (K_{u-\tau} \star p_\tau^\infty)(w_u) d\tau du]) \\
& + (\mathbb{E}_{\mathbb{P}^k} [\phi(\dots) \int_s^t \frac{\partial f}{\partial x}(w_u) \int_0^u (K_{u-\tau} \star p_\tau^\infty)(w_u) d\tau du] \\
& - \mathbb{E}_{\mathbb{P}^\infty} [\phi(\dots) \int_s^t \frac{\partial f}{\partial x}(w_u) \int_0^u (K_{u-\tau} \star p_\tau^\infty)(w_u) d\tau du]) \\
& =: I + II.
\end{aligned}$$

Now,

$$\begin{aligned}
|I| & \leq \|\phi\|_{L^\infty(\mathbb{R})} \int_s^t \int_0^u \int \left| \frac{\partial f}{\partial x}(x) \right| |(K_{u-\tau} \star (p_\tau^{k-1} - p_\tau^\infty))(x)| p_u^k(x) dx d\tau du \\
& \leq \|\phi\|_{L^\infty(\mathbb{R})} \int_s^t \frac{C_T}{u^{1/4}} \int_0^u \sqrt{\int \left(\frac{\partial f}{\partial x}(x) \right)^2 \left(\int K_{u-\tau}(x-y) (p_\tau^{k-1}(y) - p_\tau^\infty(y)) dy \right)^2 dx} d\tau du.
\end{aligned}$$

Then, fix $u \in [s, t]$ and $\tau < u$. By (H), the function $f_1(y) = \mathbb{1}\{\tau < u\} K_{u-\tau}(x-y)$ is a bounded and continuous function of y . By weak convergence

$$\forall \tau < u, \quad \lim_{k \rightarrow \infty} \int K_{u-\tau}(x-y) p_\tau^{k-1}(y) dy = \int K_{u-\tau}(x-y) p_\tau^\infty(y) dy.$$

In view of (23), ii) and (H), one has

$$\left(\frac{\partial f}{\partial x}(x) \right)^2 \left(\int K_{u-\tau}(x-y) (p_\tau^{k-1}(y) - p_\tau^\infty(y)) dy \right)^2 \leq \left(\frac{\partial f}{\partial x}(x) \right)^2 \|K_{u-\tau}\|_{L^2(\mathbb{R})}^2 \frac{C_T}{\sqrt{\tau}}. \quad (24)$$

As $f \in C_K^2(\mathbb{R})$, we apply Lebesgue's dominated convergence to obtain

$$\sqrt{\int \left(\frac{\partial f}{\partial x}(x) \right)^2 \left(\int K_{u-\tau}(x-y) (p_\tau^{k-1}(y) - p_\tau^\infty(y)) dy \right)^2 dx} \xrightarrow{k \rightarrow \infty} 0.$$

Estimate (24) and (H) enable us to apply again Lebesgue's dominated convergence to deduce that $I \rightarrow 0$ as $k \rightarrow \infty$.

Now, to prove that $II \rightarrow 0$, it suffices to prove that the functional $F : \mathcal{C}([0, T_0]; \mathbb{R}) \rightarrow \mathbb{R}$ defined by

$$w. \mapsto \phi(w_{t_1}, \dots, w_{t_N}) \int_s^t \frac{\partial f}{\partial x}(w_u) \int_0^u \int K_{u-\tau}(w_u - y) p_\tau^\infty(y) dy d\tau du$$

is continuous. Let (w^n) a sequence converging in $\mathcal{C}([0, T_0]; \mathbb{R})$ to w . Since ϕ is a continuous function, it suffices to show that

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \int_s^t \frac{\partial f}{\partial x}(w_u^n) \int_0^u \int K_{u-\tau}(w_u^n - y) p_\tau^\infty(y) dy d\tau du \\
& = \int_s^t \frac{\partial f}{\partial x}(w_u) \int_0^u \int K_{u-\tau}(w_u - y) p_\tau^\infty(y) dy d\tau du.
\end{aligned} \quad (25)$$

For $(u, \tau) \in [s, t] \times [0, t]$, set

$$f_3^n(u, \tau) := \mathbb{1}\{\tau < u\} \frac{\partial f}{\partial x}(w_u^n) \int K_{u-\tau}(w_u^n - y) p_\tau^\infty(y) dy.$$

Since w^n converges pointwise to w and K satisfies (H-2), by Lebesgue's dominated convergence theorem,

$$\lim_{n \rightarrow \infty} f_3^n(u, \tau) = \mathbb{1}\{\tau < u\} \frac{\partial f}{\partial x}(w_u) \int K_{u-\tau}(w_u - y) p_\tau^\infty(y) dy.$$

Furthermore,

$$|f_3^n(u, \tau)| \leq C \mathbb{1}\{\tau < u\} \|p_\tau^\infty\|_{L^2(\mathbb{R})} \|K_{u-\tau}\|_{L^2(\mathbb{R})} \leq \frac{C}{\tau^{1/4}} \mathbb{1}\{\tau < u\} \|K_{u-\tau}\|_{L^2(\mathbb{R})}.$$

In view of (H-4),

$$\int_s^t \int_0^t \frac{1}{\tau^{1/4}} \mathbb{1}\{\tau < u\} \|K_{u-\tau}\|_{L^2(\mathbb{R})} d\tau du < \infty.$$

Thus, by Lebesgue's dominated convergence theorem, we get

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_s^t \int_0^t \mathbb{1}\{\tau < u\} \frac{\partial f}{\partial x}(w_u^n) \int K_{u-\tau}(w_u^n - y) p_\tau^\infty(y) dy d\tau du \\ &= \int_s^t \int_0^t \mathbb{1}\{\tau < u\} \frac{\partial f}{\partial x}(w_u) \int K_{u-\tau}(w_u - y) p_\tau^\infty(y) dy d\tau du, \end{aligned}$$

which ends the proof. □

Proof of Theorem 5.1

Proposition 5.4 implies the existence of a weak solution $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), X, W)$ to (3) up to time T_0 . Thus, the marginals $\mathbb{P} \circ X_t^{-1} =: p_t$ satisfy $\|p_t\|_{L^2(\mathbb{R})} \leq \frac{C}{t^{1/4}}$, $t \in (0, T_0]$. In view of Remark 2.2, one has

$$|B(t, x; p)| \leq C(T_0).$$

Therefore the drift coefficient of X is bounded, so that Corollary 3.2 implies that for any $t \in (0, T_0]$

$$\|p_t\|_{L^\infty(\mathbb{R})} \leq \frac{C(T_0, b_0)}{\sqrt{t}}.$$

In addition, $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), X, W)$ is the unique weak solution of the linear SDE

$$d\tilde{X}_t = b(t, \tilde{X}_t) dt + B(t, \tilde{X}_t; p) dt + dW_t, \quad t \leq T_0. \quad (26)$$

Now suppose that there exists another weak solution $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}, (\hat{\mathcal{F}}_t), \hat{X}, \hat{W})$ to (3) up to T_0 such that $\hat{\mathbb{P}} \circ \hat{X}_t^{-1}$ admits a density \hat{p}_t satisfying (6). By Proposition 4.1 we have $\hat{p}_t = p_t$, for $t \leq T_0$. Therefore, $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}, (\hat{\mathcal{F}}_t), \hat{X}, \hat{W})$ is a weak solution to (26), from which $\hat{\mathbb{P}} \circ \hat{X}^{-1} = \mathbb{P} \circ X^{-1}$.

6 Proof of Theorem 2.3: A global existence and uniqueness result for Equation (3)

We now construct a solution for an arbitrary time horizon $T > 0$. We will do it by restarting the equation after the already fixed T_0 . We start with $T = 2T_0$. Then, we will see how to generalize this procedure for an arbitrary $T > 0$.

Throughout this section, we denote by Ω_0 the canonical space $\mathcal{C}([0, T_0]; \mathbb{R})$ and by \mathcal{B}_0 its Borel σ -field. We denote by \mathbb{Q}^1 the probability distribution of the unique weak solution to (3) up to T_0 constructed in the previous section.

6.1 Solution on $[0, 2T_0]$

Proposition 6.1. *Let $T_0 > 0$ be such that $D(T_0) < 1$. Assume $p_0 \in L^1(\mathbb{R})$ and let $b \in L^\infty([0, 2T_0] \times \mathbb{R})$ be continuous w.r.t. the space variable. Under the hypothesis (H), Equation (3) admits a weak solution up to $2T_0$ which is unique in the sense of Theorem 2.3.*

We start with analyzing the dynamics of (3) after T_0 and informally explaining the construction of a solution between T_0 and $2T_0$. Assume, for a while, that Proposition 6.1 holds true. Denote the density of X_t by p_t^1 , for $t \leq T_0$ and by p_t^2 , for $t \in (T_0, 2T_0]$. Let $t \geq 0$. In view of Equation (3), we would have

$$X_{T_0+t} = X_{T_0} + \int_{T_0}^{T_0+t} b(s, X_s) ds + \int_{T_0}^{T_0+t} \int_0^s (K_{s-\theta} \star p_\theta)(X_s) d\theta ds + W_{T_0+t} - W_{T_0}.$$

Observe that

$$\begin{aligned} \int_{T_0}^{T_0+t} \int_0^s (K_{s-\theta} \star p_\theta)(X_s) d\theta ds &= \int_{T_0}^{T_0+t} \int_0^{T_0} (K_{s-\theta} \star p_\theta^1)(X_s) d\theta ds + \int_{T_0}^{T_0+t} \int_{T_0}^s (K_{s-\theta} \star p_\theta^2)(X_s) ds dt \\ &=: B_1 + B_2. \end{aligned}$$

In addition,

$$B_1 = \int_0^t \int_0^{T_0} (K_{T_0+s'-\theta} \star p_\theta^1)(X_{T_0+s'}) d\theta ds',$$

and

$$B_2 = \int_0^t \int_{T_0}^{T_0+s'} (K_{T_0+s'-\theta} \star p_\theta^2)(X_{T_0+s'}) d\theta ds' = \int_0^t \int_0^{s'} (K_{s'-\theta'} \star p_{T_0+\theta'}^2)(X_{T_0+s'}) d\theta' ds'.$$

Now set $Y_t := X_{T_0+t}$ and $\tilde{p}_t(y) := p_{T_0+t}^2(y)$. Consider the new Brownian motion $\bar{W}_t := W_{T_0+t} - W_{T_0}$. It comes:

$$Y_t = Y_0 + \int_0^t b(s + T_0, Y_s) ds + \int_0^t \int_0^{T_0} (K_{T_0+s'-\theta} \star p_\theta^1)(Y_s) d\theta ds + \int_0^t \int_0^{s'} (K_{s'-\theta'} \star \tilde{p}_\theta)(Y_s) d\theta ds + \bar{W}_t,$$

for $t \in [0, T_0]$. Setting

$$b_1(t, x, T_0) := \int_0^{T_0} (K_{T_0+t-s} \star p_s^1)(x) ds \quad \text{and} \quad \tilde{b}(t, x) := b(T_0 + t, x),$$

we have

$$\begin{cases} dY_t = \tilde{b}(t, Y_t) dt + b_1(t, Y_t, T_0) dt + \left\{ \int_0^t (K_{t-s} \star \tilde{p}_s)(Y_t) ds \right\} dt + d\bar{W}_t, & t \leq T_0, \\ Y_0 \sim p_{T_0}^1(y) dy, \quad Y_s \sim \tilde{p}_s(y) dy. \end{cases} \quad (27)$$

To prove Proposition 6.1 we construct a weak solution to (27) on $[0, T_0]$ and suitably paste its probability distribution with \mathbb{Q}^1 . We then prove that the so defined measure solves the desired non-linear martingale problem. Notice that the SDE (27) is of the same type as (3).

Lemma 6.2. *Same assumptions as in Proposition 6.1. Denote by p_t^1 the time marginals of \mathbb{Q}^1 . Set $b_1(t, x, T_0) := \int_0^{T_0} (K_{T_0+t-s} \star p_s^1)(x) ds$ and $\tilde{b}(t, x) := b(T_0 + t, x)$. Then, Equation (27) admits a unique weak solution up to T_0 which is unique in the sense of Theorem 2.3.*

Proof. Let us check that we may apply Theorem 5.1 to (27).

Firstly, by construction the initial law $p_{T_0}^1$ of Y satisfies the assumption of Theorem 5.1. Secondly, the role of the additional drift \tilde{b} is now played by the sum of the two linear drifts, \tilde{b} and b_1 . By hypothesis, \tilde{b} is bounded in $[0, T_0] \times \mathbb{R}$ and continuous in the space variable. Using (6) and (H-6) we conclude that b_1 is bounded uniformly in t and x since

$$|b_1(t, x, T_0)| \leq C_{T_0} \int_0^{T_0} \frac{\|K_{T_0+t-s}\|_{L^1(\mathbb{R})}}{\sqrt{s}} ds < C_{T_0}.$$

To show that $b_1(t, x, T_0)$ is continuous w.r.t. x we use (H-2) and proceed as at the end of the proof of Proposition 5.4.

We now are in a position to apply Theorem 5.1: There exists a unique weak solution to (27) up to T_0 which is unique in the sense of Theorem 2.3. \square

Denote by \mathbb{Q}^2 the probability distribution of the process $(Y_t, t \leq T_0)$. Notice that \mathbb{Q}^2 is the solution to the martingale problem $(MP(p_{T_0}^1, T_0, \tilde{b} + b_1))$.

A new measure on $\mathcal{C}([0, 2T_0]; \mathbb{R})$. Let \mathbb{Q}^1 , \mathbb{Q}^2 and (p_t^1) be as above. Let (p_t^2) denote the time marginal densities of \mathbb{Q}^2 . In particular, $\mathbb{Q}_0^2 = \mathbb{Q}_{T_0}^1$, i.e. $p_0^2(z)dz = p_{T_0}^1(z)dz$. Define the mapping X^0 from Ω_0 to \mathbb{R} as $X^0(w) := w_0$. Using [9, Thm.3.19, Chap.5] to justify the introduction of regular conditional probabilities, for each $y \in \mathbb{R}$ we define the probability measure \mathbb{Q}_y^2 on $(\Omega_0, \mathcal{B}_0)$ by

$$\forall A \in \mathcal{B}_0, \quad \mathbb{Q}_y^2(A) = \mathbb{P}^2(A | X^0 = y).$$

In particular,

$$\mathbb{Q}_y^2(w \in \Omega_0, w_0 = y) = 1.$$

We now set $\Omega := \mathcal{C}([0, 2T_0]; \mathbb{R})$. For $w^1, w^2 \in \Omega_0$ we define the concatenation $w = w^1 \otimes_{T_0} w^2 \in \Omega$ of these two paths as the function in Ω defined by

$$\begin{cases} w_\theta = w_\theta^1, & 0 \leq \theta \leq T_0, \\ w_{\theta+T_0} = w_{T_0}^1 + w_\theta^2 - w_0^2, & 0 \leq \theta \leq t - T_0. \end{cases}$$

On the other hand, for a given path $w \in \Omega$, the two paths $w^1, w^2 \in \Omega_0$ such that $w = w^1 \otimes_{T_0} w^2$ are

$$\begin{cases} w_\theta^1 = w_\theta, & 0 \leq \theta \leq T_0, \\ w_\theta^2 = w_{T_0+\theta}, & 0 \leq \theta \leq T_0. \end{cases}$$

We define the probability distribution \mathbb{Q} on Ω equipped with its Borel σ -field as follows. For any continuous and bounded functional φ on Ω ,

$$\mathbb{E}_{\mathbb{Q}}[\varphi] = \int_{\Omega} \varphi(w) \mathbb{Q}(dw) := \int_{\Omega_0} \int_{\mathbb{R}} \int_{\Omega_0} \varphi(w^1 \otimes_{T_0} w^2) \mathbb{Q}_y^2(dw^2) p_{T_0}^1(y) dy \mathbb{Q}^1(dw^1). \quad (28)$$

Notice that if φ acts only on the part of the path up to $t \leq T_0$ of any $w \in \Omega$, then

$$\mathbb{E}_{\mathbb{Q}}[\varphi((w_\theta)_{\theta \leq t})] = \int_{\Omega_0} \varphi((w_\theta)_{\theta \leq t}) \mathbb{Q}^1(dx) = \mathbb{E}_{\mathbb{Q}^1}[\varphi((w_\theta)_{\theta \leq t})]. \quad (29)$$

Proof of Proposition 6.1. Let us prove that the probability measure \mathbb{Q} solves the non-linear martingale problem $(MP(p_0, 2T_0, b))$ on the canonical space $\mathcal{C}([0, 2T_0]; \mathbb{R})$.

- i) By (29), it is obvious that $\mathbb{Q}_0 = \mathbb{Q}_0^1$. By construction, \mathbb{Q}_0^1 has density p_0 .
- ii) Next, let us characterize the one dimensional time marginals of \mathbb{Q} . For $f \in C_b(\mathbb{R})$ and $t \in [0, 2T_0]$, consider the functional $\varphi(w) = f(w_t)$, for any $x \in \mathcal{C}([0, 2T_0]; \mathbb{R})$. For $t \leq T_0$, by (29),

$$\mathbb{E}_{\mathbb{Q}}[\phi(w)] = \int_{\Omega_0} f(w_t) \mathbb{Q}^1(dx) = \int_{\mathbb{R}} f(z) p_t^1(z) dz.$$

Therefore, $\mathbb{Q}_t(dz) = p_t^1(z) dz$.

For $T_0 \leq t \leq 2T_0$, by (28),

$$\mathbb{E}_{\mathbb{Q}}[\phi(w)] = \int_{\Omega_0} \int_{\mathbb{R}} \int_{\Omega_0} \varphi(w_{t-T_0}^2) \mathbb{Q}_y^2(dw^2) p_{T_0}^2(y) dy \mathbb{Q}^1(dw^1) = \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(z) \mathbb{Q}_{y, t-T_0}^2(dz) p_{T_0}^1(y) dy.$$

By Fubini's theorem:

$$\mathbb{E}_{\mathbb{Q}}[\phi(w)] = \int_{\mathbb{R}} f(z) \int_{\mathbb{R}} \mathbb{Q}_{y, t-T_0}^2(dz) p_{T_0}^1(y) dy.$$

Since $\mathbb{Q}_0^2 = p_{T_0}^1$ we deduce

$$\mathbb{E}_{\mathbb{Q}}[\phi(w)] = \int_{\mathbb{R}} f(z) p_{t-T_0}^2(z) dz,$$

which shows that $\mathbb{Q}_t(dz) = p_{t-T_0}^2(z) dz$.

Therefore, the one dimensional marginals of \mathbb{Q} have densities q_t which, by construction, belong to $L^2(\mathbb{R})$ and satisfy $\|q_t\|_{L^2(\mathbb{R})} \leq \frac{C}{t^{1/4}}$.

iii) It remains to show that M_t defined as

$$M_t := f(w_t) - f(w_0) - \int_0^t \left[\frac{1}{2} \frac{\partial^2 f}{\partial x^2}(w_u) + \frac{\partial f}{\partial x}(w_u) (b(u, w_u) + \int_0^u \int K_{u-\tau}(w_u - y) q_\tau(y) dy d\tau) \right] du$$

is a \mathbb{Q} -martingale, i.e. $\mathbb{E}_{\mathbb{Q}}(M_t | \mathcal{B}_s) = M_s$.

(a) Let $s \leq t \leq T_0$:

For any $n \in \mathbb{N}$, any continuous bounded functional ϕ on \mathbb{R}^n , and any $t_1 \leq \dots \leq t_n < s \leq t \leq T_0$, by (29):

$$\mathbb{E}_{\mathbb{Q}}(\phi(w_{t_1}, \dots, w_{t_n})(M_t - M_s)) = \mathbb{E}_{\mathbb{Q}^1}(\phi(w_{t_1}, \dots, w_{t_n})(M_t - M_s)).$$

As \mathbb{Q}^1 solves the $(MP(p_0, T_0, b))$ up to T_0 ,

$$\mathbb{E}_{\mathbb{Q}}(\phi(w_{t_1}, \dots, w_{t_n})(M_t - M_s)) = 0.$$

(b) For $s \leq T_0 \leq t \leq 2T_0$,

$$\mathbb{E}_{\mathbb{Q}}(M_t | \mathcal{B}_s) = \mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}(M_t | \mathcal{B}_{T_0}) | \mathcal{B}_s].$$

Let us prove that $\mathbb{E}_{\mathbb{Q}}(M_t | \mathcal{B}_{T_0}) = M_{T_0}$. Notice that

$$\begin{aligned} M_t - M_{T_0} &= f(w_t) - f(w_{T_0}) - \int_{T_0}^t \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(w_u) du - \int_{T_0}^t \frac{\partial f}{\partial x}(w_u) b(u, w_u) du \\ &\quad - \int_{T_0}^t \frac{\partial f}{\partial x}(w_u) \int_0^u \int K_{u-\tau}(w_u - y) q_\tau(y) dy d\tau du. \end{aligned}$$

Write the last integral as

$$\begin{aligned} &\int_{T_0}^t \frac{\partial f}{\partial x}(w_u) \int_0^u \int K_{u-\tau}(w_u - y) q_\tau(y) dy d\tau du = \int_{T_0}^t \frac{\partial f}{\partial x}(w_u) \int_0^{T_0} \int K_{u-\tau}(w_u - y) p_\tau^1(y) dy d\tau du \\ &+ \int_{T_0}^t \frac{\partial f}{\partial x}(w_u) \int_{T_0}^u \int K_{u-\tau}(w_u - y) p_{\tau-T_0}^2(y) dy d\tau du \\ &=: I_1 + I_2. \end{aligned}$$

Now,

$$I_1 = \int_0^{t-T_0} \frac{\partial f}{\partial x}(w_{u+T_0}) \int_0^{T_0} \int K_{u+T_0-\tau}(w_{u+T_0} - y) p_\tau^1(y) dy d\tau du.$$

For $w \in \Omega$ identify $w^1, w^2 \in \Omega_0$ such that $w = w^1 \otimes_{T_0} w^2$. Then,

$$I_1 = \int_0^{t-T_0} \frac{\partial f}{\partial x}(w_u^2) \int_0^{T_0} (K_{u+T_0-\tau} \star p_\tau^1)(w_u^2) d\tau du = \int_0^{t-T_0} \frac{\partial f}{\partial x}(w_u^2) b_1(u, w_u^2, T_0) du.$$

Proceeding as above,

$$\begin{aligned} I_2 &= \int_0^{t-T_0} \frac{\partial f}{\partial x}(w_{u+T_0}) \int_0^u \int K_{u-\tau}(w_{u+T_0} - y) p_\tau^2(y) dy d\tau du \\ &= \int_0^{t-T_0} \frac{\partial f}{\partial x}(w_u^2) \int_0^u \int K_{u-\tau}(w_u^2 - y) p_\tau^2(y) dy d\tau du. \end{aligned}$$

Similarly

$$\begin{aligned} &\int_{T_0}^t \frac{\partial f}{\partial x}(w_u) b(u, w_u) du = \int_0^{t-T_0} \frac{\partial f}{\partial x}(w_{u+T_0}) b(u+T_0, w_{u+T_0}) du \\ &= \int_0^{t-T_0} \frac{\partial f}{\partial x}(w_u^2) b(u+T_0, w_u^2) du = \int_0^{t-T_0} \frac{\partial f}{\partial x}(w_u^2) \tilde{b}(u, w_u^2) du. \end{aligned}$$

It comes:

$$\begin{aligned} M_t - M_{T_0} &= f(w_{t-T_0}^2) - f(w_0^2) - \int_0^{t-T_0} \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(w_u^2) du - \int_0^{t-T_0} \frac{\partial f}{\partial x}(w_u^2) \tilde{b}(u, w_u^2) du \\ &\quad - \int_0^{t-T_0} \frac{\partial f}{\partial x}(w_u^2) [b_1(u, w_u^2, T_0) + \int_0^u \int K_{u-\tau}(w_u^2 - y) p_\tau^2(y) dy d\tau] du =: F(w^2). \end{aligned}$$

By definition of the measure \mathbb{Q} ,

$$\mathbb{E}^{\mathbb{Q}}(\phi(w_{t_1}, \dots, w_{t_n})(M_t - M_{T_0})) = \int_{\Omega_0} \phi(w_{t_1}^1, \dots, w_{t_n}^1) \int_{\mathbb{R}} \int_{\Omega_0} F(w^2) \mathbb{Q}_y^2(dw^2) p_{T_0}^1(y) dy \mathbb{Q}^1(dw^1).$$

By the definition of \mathbb{Q}^2 :

$$\mathbb{E}^{\mathbb{Q}}(\phi(w_{t_1}, \dots, w_{t_n})(M_t - M_{T_0})) = \int_{\Omega_0} \phi(w_{t_1}^1, \dots, w_{t_n}^1) \int_{\Omega_0} F(w^2) \mathbb{Q}^2(dw^2) \mathbb{Q}^1(dw^1).$$

As \mathbb{Q}^2 solves $(MP(p_{T_0}^1, T_0, \tilde{b} + b_1))$, one has

$$\mathbb{E}_{\mathbb{Q}^2}(\varphi) = \int_{\Omega_0} F(w^2) \mathbb{Q}^2(dw^2) = 0.$$

Finally, we conclude that $\mathbb{E}_{\mathbb{Q}}(M_t | \mathcal{B}_{T_0}) = M_{T_0}$ and therefore $\mathbb{E}_{\mathbb{Q}}(M_t | \mathcal{B}_s) = M_s$ for all $s \leq T_0 \leq t \leq 2T_0$.

(c) For $T_0 \leq s \leq t \leq 2T_0$: we may rewrite the difference $M_t - M_s$ in the same manner:

$$\begin{aligned} M_t - M_s &= f(w_{t-T_0}^2) - f(w_{s-T_0}^2) - \int_{s-T_0}^{t-T_0} \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(w_u^2) du \\ &\quad - \int_{s-T_0}^{t-T_0} \frac{\partial f}{\partial x}(w_u^2) [b(u, w_u^2) + b_1(u, w_u^2, T_0) + \int_0^u \int K_{u-\tau}(w_{u+T_0} - y) p_\tau^2(y) dy d\tau] du \\ &=: F(w^2). \end{aligned}$$

Now, take $t_1 \leq \dots \leq t_n < s$. Let us suppose that the first m are before T_0 and others after. We have that

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}(\phi(w_{t_1}, \dots, w_{t_n})(M_t - M_s)) &= \\ &= \int_{\Omega_0} \int_{\mathbb{R}} \int_{\Omega_0} F(w^2) \phi(w_{t_1}^1, \dots, w_{t_m}^1, w_{t_{m+1}-T_0}^2, \dots, w_{t_n-T_0}^2) \varphi(w^2) d\mathbb{Q}_y^2(dw^2) p_{T_0}^1(y) dy \mathbb{Q}^1(dw^1). \end{aligned}$$

Since \mathbb{Q}^2 solves the $(MP(p_{T_0}^1, T_0, \tilde{b} + b_1))$, one has that $\mathbb{E}_{\mathbb{Q}^2}(\phi'(w_{t'_1}^2, \dots, w_{t'_n}^2)F) = 0$ for any continuous bounded functional ϕ' on \mathbb{R}^n , any $n \in N$ and any $t'_1 \leq \dots \leq t'_n < s - T_0$. Taking $\phi'(w_{t'_1}^2, \dots, w_{t'_n}^2) = \phi(w_{t_1}^1, \dots, w_{t_m}^1, w_{t_{m+1}-T_0}^2, \dots, w_{t_n-T_0}^2)$ for a fixed x^1 , we conclude that

$$\int_{\mathbb{R}} \int_{\Omega_0} \phi(w_{t_1}^1, \dots, w_{t_m}^1, w_{t_{m+1}-T_0}^2, \dots, w_{t_n-T_0}^2) \varphi(w^2) \mathbb{Q}_y^2(dw^2) p_{T_0}^1(y) dy = 0.$$

Therefore,

$$\mathbb{E}^{\mathbb{Q}}(\phi(w_{t_1}, \dots, w_{t_n})(M_t - M_s)) = 0.$$

Thus, $\mathbb{E}^{\mathbb{Q}}(M_t | \mathcal{B}_s) = M_s$ for $T_0 \leq s \leq t \leq 2T_0$.

To summarize the preceding, we have just shown the well-posedness of $(MP(p_0, 2T_0, b))$. Finally, we proceed as in the proof of Theorem 5.1 to deduce the existence and uniqueness of a weak solution to (3) up to $2T_0$ with the desired property on its time marginals.

6.2 End of the proof of Theorem 2.3: construction of the global solution

Given any finite time horizon $T > 0$, split the interval $[0, T]$ into $n = \lfloor \frac{T}{T_0} \rfloor + 1$ intervals of length T_0 and repeat n times the procedure used in the preceding subsection. By construction, the time marginals of this solution to $(MP(p_0, T, b))$ has probability densities which satisfy (5).

Remark 6.3. *Using similar arguments as above one can construct a solution to (3) when the initial condition p_0 is in $L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ or, respectively, $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. In these cases we use Remark 3.3 in the iterative procedure. Consequently, the weak solution is unique under the constraint that the one dimensional marginal densities $(p_t)_{t \leq T}$ belong to $L^\infty((0, T); L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}))$ or, respectively, satisfy*

$$\|p_t\|_{L^2(\mathbb{R})} \leq \frac{C_T}{t^{1/4}}.$$

7 Application to the one-dimensional Keller–Segel model

In this section we prove Corollary 2.6. We start with checking that K^\sharp satisfies Hypothesis (H).

1. Notice that

$$\|K_t^\sharp\|_{L^1(\mathbb{R})} \leq \frac{C_T \chi}{\sqrt{t}} \int |z| e^{-\frac{z^2}{2}} dz$$

and

$$\|K_t^\sharp\|_{L^2(\mathbb{R})} \leq \frac{C_T}{t^{3/4}}.$$

The condition (H.1) is thus satisfied.

2. From the definition of K^\sharp it is clear that for $t > 0$, K_t^\sharp is a bounded and continuous function on \mathbb{R} .
3. The condition (H.3) is obviously satisfied.
4. As already noticed,

$$\|K_{t-s}^\sharp\|_{L^1(\mathbb{R})} = \frac{C}{\sqrt{t-s}},$$

from which, by definition of the β -function,

$$f_1(t) = \int_0^t \frac{\|K_{t-s}^\sharp\|_{L^1(\mathbb{R})}}{\sqrt{s}} ds = C \int_0^t \frac{1}{\sqrt{s}\sqrt{t-s}} ds = C\beta(1/2, 1/2),$$

where C is a universal constant. With similar arguments one easily checks that the function f_2 is bounded on any compact time interval.

5. Let ϕ be a probability density on \mathbb{R} . For $(t, x) \in (0, T] \times \mathbb{R}$, one has

$$\begin{aligned} \int \phi(y) \|K^\sharp(x-y)\|_{L^1(0,t)} dy &\leq C \int \phi(y) |x-y| \int_0^t \frac{1}{s^{3/2}} e^{-\frac{|x-y|^2}{2s}} ds dy \\ &= \int \phi(y) |x-y| \int_{\frac{|x-y|}{\sqrt{t}}}^\infty \frac{z^3}{|x-y|^3} e^{-\frac{z^2}{2}} \frac{|x-y|^2}{z^3} dz dy \\ &= \int \phi(y) \int_{\frac{|x-y|}{\sqrt{t}}}^\infty e^{-\frac{z^2}{2}} dz dy. \end{aligned}$$

This shows that (H.5) is satisfied.

6. Finally, to prove (H.6) we notice that for $t \in [0, T]$, one has

$$\int_0^T \|K_{T+t-s}^\sharp\|_{L^1(\mathbb{R})} \frac{1}{\sqrt{s}} ds \leq \int_0^T \frac{C}{\sqrt{T+t-s}\sqrt{s}} ds \leq C\beta(1/2, 1/2).$$

Now, since $c_0 \in C_b^1(\mathbb{R})$, we have

$$|\chi e^{-\lambda t} \mathbb{E} c_0'(x + W_t)| \leq \chi \|c_0'\|_{L^\infty(\mathbb{R})}.$$

In addition, it is continuous w.r.t. the space variable. Thus, in view of Theorem 2.3, Equation (2) with $d = 1$ is well-posed. Denote by $\rho_t \equiv p_t$ the time marginals of the constructed probability distribution. Set

$$c_t := e^{-\lambda t} g(t, \cdot) * c_0 + \int_0^t e^{-\lambda s} \rho_{t-s} * g(s, \cdot) ds.$$

For any $t \in (0, T]$, by Inequality (5), c_t is well defined, continuous, and

$$\|c_t\|_{L^\infty(\mathbb{R})} \leq \|c_0\|_{L^\infty(\mathbb{R})} + C_T.$$

In addition,

$$\frac{\partial}{\partial x} c(t, x) = \frac{\partial}{\partial x} (e^{-\lambda t} \mathbb{E}(c_0(x + W_t))) + \frac{\partial}{\partial x} \left(\mathbb{E} \int_0^t e^{-\lambda s} \rho_{t-s}(x + W_s) ds \right).$$

Then observe that

$$\begin{aligned} \frac{\partial}{\partial x} \mathbb{E} \int_0^t e^{-\lambda s} \rho_{t-s}(x + W_s) ds &= \frac{\partial}{\partial x} \int_0^t e^{-\lambda s} \int \rho_{t-s}(x + y) \frac{1}{\sqrt{2\pi s}} e^{-\frac{y^2}{4s}} dy ds \\ &= \frac{\partial}{\partial x} \int_0^t e^{-\lambda(t-s)} \int \rho_s(y) \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(y-x)^2}{4(t-s)}} dy ds =: \frac{\partial}{\partial x} \int_0^t f(s, x) ds. \end{aligned}$$

As

$$\left| \frac{\partial}{\partial x} \left(\rho_s(y) \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(y-x)^2}{2(t-s)}} \right) \right| \leq \rho_s(y) \frac{|y-x|}{2\sqrt{2\pi}(t-s)^{3/2}} e^{-\frac{(y-x)^2}{2(t-s)}} \leq \frac{C_T \rho_s(y)}{t-s},$$

we have

$$\frac{\partial}{\partial x} f(s, x) = e^{-\lambda(t-s)} \int \rho_s(y) \frac{y-x}{2\sqrt{2\pi}(t-s)^{3/2}} e^{-\frac{(y-x)^2}{2(t-s)}} dy.$$

Now, we repeat the same argument for $\frac{\partial}{\partial x} \int_0^t f(s, x) ds$. In order to justify the differentiation under the integral sign we notice that

$$\left| \frac{\partial}{\partial x} f(s, x) \right| \leq \frac{C_T}{\sqrt{(t-s)s}}.$$

Gathering the preceding calculations we have obtained

$$\frac{\partial}{\partial x} c(t, x) = e^{-\lambda t} \mathbb{E}(c_0'(x + W_t)) + \int_0^t e^{-\lambda(t-s)} \int \rho_s(y) \frac{y-x}{\sqrt{2\pi}(t-s)^{3/2}} e^{-\frac{(y-x)^2}{2(t-s)}} dy ds.$$

Using the assumption on c_0 and Inequality (5), for any $t \in (0, T]$ one has

$$\left\| \frac{\partial}{\partial x} c(t, \cdot) \right\|_{L^\infty(\mathbb{R})} \leq \|c_0'\|_{L^\infty(\mathbb{R})} + C_T.$$

In addition, by Lebesgue's Dominated Convergence Theorem, $\frac{\partial}{\partial x} c(t, x)$ is continuous on \mathbb{R} . Thus $c_t \in C_b^1(\mathbb{R})$ for all $t \in [0, T]$.

As $\chi \frac{\partial}{\partial x} c(t, x)$ is the drift coefficient in the SDE (3), in view of Proposition 4.1 the function ρ_t satisfies

$$\rho_t = g(t, \cdot) * \rho_0 + \chi \int_0^t \frac{\partial}{\partial x} g(t-s, \cdot) * \left(\frac{\partial}{\partial x} c(s, \cdot) \rho_s \right) ds$$

in the sense of the distributions. Therefore, the pair (p, c) is a solution to the Keller Segel system (7) in the sense of Definition 2.4. We now check the uniqueness of this solution.

Assume there exists another pair (ρ^1, c^1) satisfying Definition (2.4) with the same initial condition as (ρ, c) . Let us prove $(\rho^1, c^1) = (\rho, c)$ a.e.

Using Definition 2.4,

$$\begin{aligned}
\|\rho_t^1 - \rho_t\|_{L^1(\mathbb{R})} &\leq \int_0^t \left\| \frac{\partial}{\partial x} g(t-s, \cdot) * \left(\frac{\partial}{\partial x} c^1(s, \cdot) \rho_s^1 - \frac{\partial}{\partial x} c(s, \cdot) \rho_s \right) \right\|_{L^1(\mathbb{R})} ds \\
&\leq \int_0^t \left\| \frac{\partial}{\partial x} g(t-s, \cdot) * \left(\frac{\partial}{\partial x} c^1(s, \cdot) (\rho_s^1 - \rho_s) \right) \right\|_{L^1(\mathbb{R})} ds + \int_0^t \left\| \frac{\partial}{\partial x} g_{t-s} * \left(\rho_s \left(\frac{\partial}{\partial x} c^1(s, \cdot) - \frac{\partial}{\partial x} c(s, \cdot) \right) \right) \right\|_{L^1(\mathbb{R})} ds \\
&=: I + II.
\end{aligned}$$

Using standard convolution inequalities and $\left\| \frac{\partial}{\partial x} g(t-s, \cdot) \right\|_{L^1(\mathbb{R})} \leq \frac{C}{\sqrt{t-s}}$ we deduce:

$$I \leq C \int_0^t \frac{\|\rho_s^1 - \rho_s\|_{L^1(\mathbb{R})}}{\sqrt{t-s}} ds \quad \text{and} \quad II \leq C \int_0^t \frac{\left\| \frac{\partial}{\partial x} c^1(s, \cdot) - \frac{\partial}{\partial x} c(s, \cdot) \right\|_{L^1(\mathbb{R})}}{\sqrt{t-s}\sqrt{s}} ds.$$

Therefore

$$\left\| \frac{\partial}{\partial x} c^1(s, \cdot) - \frac{\partial}{\partial x} c(s, \cdot) \right\|_{L^1(\mathbb{R})} \leq \int_0^s \left\| (\rho_u^1 - \rho_u) * \frac{\partial}{\partial x} g_{s-u} \right\|_{L^1(\mathbb{R})} du \leq C \int_0^s \frac{\|\rho_u^1 - \rho_u\|_{L^1(\mathbb{R})}}{\sqrt{s-u}} du,$$

from which

$$\begin{aligned}
II &\leq C \int_0^t \frac{1}{\sqrt{s}\sqrt{t-s}} \int_0^s \frac{\|\rho_u^1 - \rho_u\|_{L^1(\mathbb{R})}}{\sqrt{s-u}} du ds \\
&\leq C \int_0^t \|\rho_u^1 - \rho_u\|_{L^1(\mathbb{R})} \int_u^t \frac{1}{\sqrt{s}\sqrt{s-u}\sqrt{t-s}} ds du \leq C_T \int_0^t \frac{\|\rho_u^1 - \rho_u\|_{L^1(\mathbb{R})}}{\sqrt{u}} du.
\end{aligned}$$

Gathering the preceding bounds for I and II we get

$$\|p_t^1 - p_t\|_{L^1(\mathbb{R})} \leq C_T \int_0^t \frac{\|p_s^1 - p_s\|_{L^1(\mathbb{R})}}{\sqrt{t-s}} ds + C_T \int_0^t \frac{\|p_s^1 - p_s\|_{L^1(\mathbb{R})}}{\sqrt{s}} ds.$$

In view of Lemma 4.2, for all $t \leq T$

$$\|\rho_t^1 - \rho_t\|_{L^1(\mathbb{R})} = 0.$$

We deduce that c_t^1 and c_t are equal almost everywhere for all $t \leq T$. This completes the proof of Corollary 2.6.

8 Appendix

We here propose a light simplification of the calculations in [13].

Proposition 8.1. *Let $y \in \mathbb{R}$ and let β be a constant. Denote by $p_y^\beta(t, x, z)$ the transition probability density (with respect to the Lebesgue measure) of the unique weak solution to*

$$X_t = x + \beta \int_0^t \text{sgn}(y - X_s) ds + W_t.$$

Then

$$\begin{aligned}
p_y^\beta(t, x, z) &= \frac{1}{\sqrt{2\pi t^{3/2}}} \int_0^\infty e^{\beta(|y-x| + \bar{y} - |z-y|) - \frac{\beta^2}{2} t} (\bar{y} + |z-y| + |y-x|) e^{-\frac{(\bar{y} + |z-y| + |y-x|)^2}{2t}} d\bar{y} \\
&\quad + \frac{1}{\sqrt{2\pi t}} e^{\beta(|y-x| - |z-y|) - \frac{\beta^2}{2} t} \left(e^{-\frac{(z-x)^2}{2t}} - e^{-\frac{(|z-y| + |y-x|)^2}{2t}} \right).
\end{aligned} \tag{30}$$

In particular,

$$p_y^\beta(t, x, y) = \frac{1}{\sqrt{2\pi t}} \int_{\frac{|x-y|}{\sqrt{t}}}^\infty z e^{-\frac{(z-\beta\sqrt{t})^2}{2}} dz. \tag{31}$$

Proof. Let f be a bounded continuous function. The Girsanov transform leads to

$$\mathbb{E}(f(X_t)) = \mathbb{E}(f(x + W_t) e^{\beta \int_0^t \text{sgn}(y - x - W_s) dW_s - \frac{\beta^2}{2} t}).$$

Let L_t^a be the Brownian local time. By Tanaka's formula ([9], p. 205):

$$|W_t - a| = |a| + \int_0^t \operatorname{sgn}(W_s - a) dW_s + L_t^a.$$

Therefore

$$\int_0^t \operatorname{sgn}(y - x - W_s) dW_s = |y - x| + L_t^a - |W_t - (y - x)|,$$

from which

$$\mathbb{E}(f(X_t)) = \mathbb{E}(f(x + W_t)) e^{\beta(|y-x| + L_t^{y-x} - |W_t - (y-x)|) - \frac{\beta^2}{2}t}.$$

recall that (W_t, L_t^a) has the following joint distribution (see [1, p.200, Eq.(1.3.8)]):

$$\begin{cases} \bar{y} > 0 : & \mathbb{P}(W_t \in dz, L_t^a \in d\bar{y}) = \frac{1}{\sqrt{2\pi t^{3/2}}} (\bar{y} + |z - a| + |a|) e^{-\frac{(\bar{y} + |z - a| + |a|)^2}{2t}} d\bar{y} dz. \\ \mathbb{P}(W_t \in dz, L_t^a = 0) = & \frac{1}{\sqrt{2\pi t}} e^{-\frac{z^2}{2t}} dz - \frac{1}{\sqrt{2\pi t}} e^{-\frac{(|z - a| + |a|)^2}{2t}} dz. \end{cases}$$

It comes:

$$\begin{aligned} \mathbb{E}(f(X_t)) &= \frac{1}{\sqrt{2\pi t^{3/2}}} \int_{\mathbb{R}} \int_0^\infty f(x + z) e^{\beta(|y-x| + \bar{y} - |z - (y-x)|) - \frac{\beta^2}{2}t} (\bar{y} + |z - (y-x)| + |y-x|) \\ &\quad e^{-\frac{(\bar{y} + |z - (y-x)| + |y-x|)^2}{2t}} d\bar{y} dz \\ &\quad + \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} f(x + z) e^{\beta(|y-x| - |z - (y-x)|) - \frac{\beta^2}{2}t} (e^{-\frac{z^2}{2t}} - e^{-\frac{(|z - (y-x)| + |y-x|)^2}{2t}}) dz. \end{aligned}$$

The change of variables $x + z = z'$ leads to

$$\begin{aligned} \mathbb{E}(f(X_t)) &= \frac{1}{\sqrt{2\pi t^{3/2}}} \int_{\mathbb{R}} f(z') \int_0^\infty e^{\beta(|y-x| + \bar{y} - |z' - y|) - \frac{\beta^2}{2}t} (\bar{y} + |z' - y| + |y-x|) e^{-\frac{(\bar{y} + |z' - y| + |y-x|)^2}{2t}} d\bar{y} dz' \\ &\quad + \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} f(z') e^{\beta(|y-x| - |z' - y|) - \frac{\beta^2}{2}t} (e^{-\frac{(z' - x)^2}{2t}} - e^{-\frac{(|z' - y| + |y-x|)^2}{2t}}) dz', \end{aligned}$$

from which the desired result follows. □

In the next Corollary we use the same notation as in the proof of Theorem 3.1.

Corollary 8.2. *Let $0 < s < t \leq T$. Then for any $z, y \in \mathbb{R}$, there exists $C_{T, \beta, x, y}$ such that*

$$\mathbb{E} \left| \left(\frac{\partial}{\partial x} p_y^\beta \right) (t - s, X_s^{(b)}, z) \right| \leq C_{T, \beta, x, y} h(s, z),$$

where h belongs to $L^1([0, t] \times \mathbb{R})$.

Proof. By Girsanov's theorem, for some constant $C_{T, \beta}$ we have

$$\mathbb{E} \left| \left(\frac{\partial}{\partial x} p_y^\beta \right) (t - s, X_s^{(b)}, z) \right| \leq C_{T, \beta} \sqrt{\mathbb{E} \left| \left(\frac{\partial}{\partial x} p_y^\beta \right) (t - s, W_s^x, z) \right|^2}.$$

Observe that

$$\begin{aligned} \frac{\partial}{\partial \bar{x}} p_y^\beta (t - s, \bar{x}, z) &= \frac{\beta}{\sqrt{2\pi(t-s)}} e^{-2\beta|z-y|} e^{-\frac{(|z-y| + |y-\bar{x}| - \beta(t-s))^2}{2(t-s)}} \operatorname{sgn}(\bar{x} - y) \\ &\quad + \frac{\beta}{\sqrt{2\pi(t-s)}} e^{-\beta|z-y| - \frac{\beta^2}{2}(t-s)} e^{\beta|y-\bar{x}| - \frac{(z-\bar{x})^2}{2(t-s)}} \operatorname{sgn}(\bar{x} - y) \\ &\quad + \frac{z - \bar{x}}{2\pi(t-s)^{3/2}} e^{-\beta|z-y| - \frac{\beta^2}{2}(t-s)} e^{\beta|y-\bar{x}| - \frac{(z-\bar{x})^2}{2(t-s)}}. \end{aligned}$$

The sum of the absolute values of the first two terms in the right-hand side is bounded from above by

$$\frac{\beta}{\sqrt{2\pi(t-s)}} e^{-2\beta|z-y|+\beta|y-\bar{x}|}.$$

Thus,

$$\mathbb{E} \left| \left(\frac{\partial}{\partial x} p_y^\beta \right) (t-s, X_s^{(b)}, z) \right| \leq \frac{C_{T,\beta}}{\sqrt{2\pi(t-s)}} \sqrt{\mathbb{E} e^{2\beta|y-W_s^x|}} + \frac{C_{T,\beta}}{(t-s)^{3/2}} \sqrt{\mathbb{E} (|z-W_s^x|^2 e^{2\beta|y-W_s^x| - \frac{(z-W_s^x)^2}{t-s}})} =: B+A.$$

Notice that

$$A \leq \frac{C_{T,\beta}}{(t-s)^{3/2}} (\mathbb{E}[|z-W_s^x|^4 e^{-2\frac{(z-W_s^x)^2}{t-s}}] \mathbb{E}[e^{4\beta|y-W_s^x|}])^{1/4} =: \frac{C_{T,\beta}}{(t-s)^{3/2}} (A_1 A_2)^{1/4}.$$

Firstly, as there exists an $\alpha > 0$ such that $|a|^4 e^{-a^2} \leq C e^{-\alpha a^2}$, one has

$$A_1 \leq C(t-s)^2 \int e^{-\alpha \frac{(z-u)^2}{t-s}} g_s(u-x) du \leq \frac{(t-s)^{2+1/2}}{\sqrt{s+(t-s)/(2\alpha)}} e^{-\frac{(z-x)^2}{2(s+(t-s)/(2\alpha))}}.$$

Secondly,

$$\begin{aligned} A_2 &= \int e^{4\beta|y-u|} g_s(u-x) du = e^{-4\beta y} \int_y^\infty e^{4\beta u} \frac{1}{\sqrt{s}} e^{-\frac{(u-x)^2}{2s}} du + e^{4\beta y} \int_{-\infty}^y e^{-4\beta u} \frac{1}{\sqrt{s}} e^{-\frac{(u-x)^2}{2s}} du \\ &= e^{4\beta(x-y)} e^{8\beta^2 s} \int_y^\infty \frac{1}{\sqrt{s}} e^{-\frac{(u-x-4\beta s)^2}{2s}} du + e^{4\beta(y-x)} e^{8\beta^2 s} \int_{-\infty}^y \frac{1}{\sqrt{s}} e^{-\frac{(u-x+4\beta s)^2}{2s}} du \leq e^{8\beta^2 s} C_{\beta,x,y}. \end{aligned}$$

Therefore,

$$A \leq C_{T,\beta,x,y} \frac{1}{(t-s)^{7/8}} g_{s+(t-s)/(2\alpha)}(z-x).$$

The term B is treated in the similar way as A_2 . □

References

- [1] BORODIN, A. N., AND SALMINEN, P. *Handbook of Brownian motion—facts and formulae*. Probability and its Applications. Birkhäuser Verlag, Basel, 1996.
- [2] CATTIAUX, P., AND PÉDÈCHES, L. The 2-D stochastic Keller-Segel particle model: existence and uniqueness. *ALEA Lat. Am. J. Probab. Math. Stat.* 13, 1 (2016), 447–463.
- [3] CORRIAS, L., ESCOBEDO, M., AND MATOS, J. Existence, uniqueness and asymptotic behavior of the solutions to the fully parabolic Keller-Segel system in the plane. *J. Differential Equations* 257, 6 (2014), 1840–1878.
- [4] FOURNIER, N., AND JOURDAIN, B. Stochastic particle approximation of the Keller–Segel equation and two-dimensional generalization of Bessel processes. *Ann. Appl. Probab.* 27, 5 (2017), 2807–2861.
- [5] HAŠKOVEC, J., AND SCHMEISER, C. Convergence of a stochastic particle approximation for measure solutions of the 2D Keller-Segel system. *Comm. Partial Differential Equations* 36, 6 (2011), 940–960.
- [6] HENRY, D. *Geometric theory of semilinear parabolic equations*, vol. 840 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1981.
- [7] HILLEN, T., AND POTAPOV, A. The one-dimensional chemotaxis model: global existence and asymptotic profile. *Math. Methods Appl. Sci.* 27, 15 (2004), 1783–1801.
- [8] JABIR, J.-F., TALAY, D., AND TOMAŠEVIĆ, M. Mean–field limit of a particle approximation of the one-dimensional parabolic–parabolic Keller-Segel model without smoothing. *submitted* (2017).
- [9] KARATZAS, I., AND SHREVE, S. E. *Brownian motion and stochastic calculus*, second ed., vol. 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991.
- [10] MAKHLOUF, A. Representation and Gaussian bounds for the density of Brownian motion with random drift. *Commun. Stoch. Anal.* 10, 2 (2016), 151–162.
- [11] OSAKI, K., AND YAGI, A. Finite dimensional attractor for one-dimensional Keller-Segel equations. *Funkcial. Ekvac.* 44, 3 (2001), 441–469.
- [12] PERTHAME, B. PDE models for chemotactic movements: parabolic, hyperbolic and kinetic. *Appl. Math.* 49, 6 (2004), 539–564.
- [13] QIAN, Z., AND ZHENG, W. Sharp bounds for transition probability densities of a class of diffusions. *C. R. Math. Acad. Sci. Paris* 335, 11 (2002), 953–957.
- [14] TOMAŠEVIĆ, M. Ph.d. thesis. *In progress*.
- [15] VERETENNIKOV, A. Y. Parabolic equations and itô’s stochastic equations with coefficients discontinuous in the time variable. *Mathematical notes of the Academy of Sciences of the USSR* 31, 4 (1982), 278–283.