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# A new McKean-Vlasov stochastic interpretation of the parabolic-parabolic Keller-Segel model: The one-dimensional case.

Denis Talay\* and Milica Tomašević \*†

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**Abstract:** In this paper we propose a new type of stochastic interpretation of the parabolic-parabolic Keller-Segel systems. It involves an original type of McKean-Vlasov interaction kernel. At the particle level, each particle interacts with all the past of each other particle. At the mean-field level studied here, the McKean-Vlasov limit process interacts with all the past time marginals of its probability distribution. We here prove that the one-dimensional parabolic-parabolic Keller-Segel system in the whole Euclidean space and the corresponding McKean-Vlasov stochastic differential equation are well-posed for any values of the parameters of the model.

**Key words:** Chemotaxis model; Parabolic system; Keller-Segel system; Global solutions; Singular McKean-Vlasov dynamic.

## 1 Introduction

The standard  $d$ -dimensional parabolic-parabolic Keller-Segel model for chemotaxis describes the time evolution of the density  $\rho_t$  of a cell population and of the concentration  $c_t$  of a chemical attractant:

$$\begin{cases} \partial_t \rho(t, x) = \nabla \cdot (\frac{1}{2} \nabla \rho - \chi \rho \nabla c), & t > 0, x \in \mathbb{R}^d, \\ \alpha \partial_t c(t, x) = \frac{1}{2} \Delta c - \lambda c + \rho, & t > 0, x \in \mathbb{R}^d. \\ \rho(0, x) = \rho_0(x), c(0, x) = c_0(x), \end{cases} \quad (1)$$

See e.g. Corrias [3], Perthame [12] and references therein for theoretical results on this system of PDEs and applications to Biology.

Recently, stochastic interpretations have been proposed for a simplified version of the model, that is, the parabolic-elliptic model which corresponds to the value  $\alpha = 0$ . They all rely on the fact that, in the parabolic-elliptic case, the equations for  $\rho_t$  and  $c_t$  can be decoupled and  $c_t$  can be explicitated as the convolution of the initial condition  $c_0$  and the kernel  $k(x) = -\frac{x}{2\pi|x|^2}$ . Consequently, the stochastic process of McKean-Vlasov type whose  $\rho_t$  is the time marginal density involves the singular interaction kernel  $k$ . This explains why, so far, only partial results are obtained and heavy techniques are used to get them. Haškovec and Schmeiser [5] introduce a kernel with a cut-off parameter and obtain the tightness of the particle probability distributions w.r.t. the cut-off parameter and the number of particles. They also obtain partial results in the direction of the propagation of chaos. In the subcritical case, that is, when the parameter  $\chi$  of the parabolic-elliptic model is small enough, Fournier and Jourdain [4] obtain well-posedness of a particle system without cut-off. In addition, they obtain a consistency property which is weaker than the propagation of chaos. They also describe complex behaviours of the particle system in the sub and super critical cases. Cattiaux and Pédèches [2] revisit the well-posedness of the particle system by using Dirichlet forms.

We here propose a new type of stochastic interpretation which can be seen as the McKean-Vlasov representation of the mild formulation of the equation satisfied by  $\rho_t$ . This representation has the advantage to concern the parabolic-parabolic model ( $\alpha > 0$ ). It involves a quite different interaction kernel than the above mentioned approaches, and actually such an interaction does not seem to have been studied in the literature. At the particle level, each particle interacts with all the past of each other particle by means of

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a time integrated singular kernel (for the well-posedness and propagation of chaos property of the particle system, see our companion paper Jabir et al. [8]). At the mean-field level studied here, the McKean-Vlasov limit process interacts with all the past time marginals of its probability distribution. The system reads

$$\begin{cases} dX_t = b^\sharp(t, X_t)dt + \left\{ \int_0^t (K_{t-s}^\sharp \star p_s)(X_t)ds \right\} dt + dW_t, & t > 0, \\ p_s(y)dy := \mathcal{L}(X_s), & X_0 \sim \rho_0(x)dx, \end{cases} \quad (2)$$

where  $K_t^\sharp(x) := \chi e^{-\lambda t} \nabla \left( \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|x|^2}{2t}} \right)$  and  $b^\sharp(t, x) := \chi e^{-\lambda t} \nabla \mathbb{E} c_0(x + W_t)$ . Here,  $(W_t)_{t \geq 0}$  is a  $d$ -dimensional Brownian motion on a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))$  and  $X_0$  is an  $\mathcal{F}_0$ -measurable random variable. Notice that the formulation requires that the one dimensional time marginals of the law of the solution are absolutely continuous with respect to Lebesgue's measure.

The analysis of the well-posedness of this non-linear equation and the proof that  $p_s = \rho_s$  for any  $s$  are delicate, particularly in the multi-dimensional case when  $\chi$  is large enough to induce solutions with blow-ups in finite time. This theoretical work is still in progress [14]. However, as numerical simulations of the related particle system appear to be effective, it seems interesting to validate our approach in the one-dimensional case where we are able to obtain quite general existence and uniqueness results for both the deterministic system (1) and the stochastic dynamics (2).

The one-dimensional deterministic model was previously studied by [11, 7] in bounded intervals  $I$  with periodic boundary conditions while we here deal with the problem posed on the whole space  $\mathbb{R}$ . In [11] one assumes  $\rho_0 \in L^2(I) \cap L^1(I)$  and  $c_0 \in H^1(I)$ . In [7] one assumes  $\rho_0 \in L^\infty(I) \cap L^1(I)$  and  $c_0 \in W^{\sigma,p}(I)$ , where  $p$  and  $\sigma$  belong to a particular set of parameters. Our hypotheses below on  $\rho_0$  do not include, either  $L^2$ -norms, or Sobolev norms.

We also emphasize that we do not limit ourselves to the specific kernel  $K_t^\sharp(x)$  related to the Keller-Segel model. We below show that the mean-field PDE and stochastic differential equation of Keller-Segel type are well-posed for a whole class of suitable kernels.

The paper is organized as follows. In Section 2 we state our main results. In Section 3 we prove a preliminary estimate on the probability density of diffusions whose drift is only supposed Borel measurable and bounded. In Section 4 we study a non-linear McKean-Vlasov-Fokker-Planck equation. In Section 5 we prove the local existence and uniqueness of a solution to a non-linear stochastic differential equation more general than (2) (for  $d = 1$ ). In Section 6 we get the global well-posedness of this equation. In Section 7 we apply the preceding result to the specific case of the one-dimensional Keller-Segel model. The appendix section 8 concerns an explicit formula for the transition density of a particular diffusion.

**Notation.** In all the paper we denote by  $C_T$  any constant which may change from line to line but only depends on the time horizon  $T$ . In particular, it does not depend on the functions  $b, K, p_0$  introduced below.

## 2 Our main results

Our first main result concerns the well-posedness of a non-linear one-dimensional stochastic differential equation (SDE) with a non standard McKean-Vlasov interaction kernel which at each time  $t$  involves in a singular way all the time marginals up to time  $t$  of the probability distribution of the solution. As our technique of analysis is not limited to the above kernel  $K^\sharp$ , we consider the following McKean-Vlasov stochastic equation:

$$\begin{cases} dX_t = b(t, X_t)dt + \left\{ \int_0^t (K_{t-s} \star p_s)(X_t)ds \right\} dt + dW_t, & t \leq T, \\ p_s(y)dy := \mathcal{L}(X_s), & X_0 \sim p_0, \end{cases} \quad (3)$$

and in all the sequel we assume the following conditions on the interaction kernel.

**Hypothesis (H).** *The function  $K$  defined on  $\mathbb{R}^+ \times \mathbb{R}$  is such that for any  $T > 0$ :*

1.  $K \in L^1((0, T]; L^1(\mathbb{R})) \cap L^1((0, T]; L^2(\mathbb{R}))$ .
2. For any  $t > 0$  the function  $K_t(x)$  is a continuous function on  $\mathbb{R}$ .
3. The set of points  $x \in \mathbb{R}$  such that  $\lim_{t \rightarrow 0} K_t(x) < \infty$  has Lebesgue measure zero.

4. For any  $t > 0$ , the functions  $f_1(t) := \int_0^t \frac{\|K_{t-s}\|_{L^1(\mathbb{R})}}{\sqrt{s}} ds$  and  $f_2(t) := \int_0^t \frac{\|K_{t-s}\|_{L^2(\mathbb{R})}}{s^{1/4}} ds$  are well defined and bounded on  $[0, T]$ .
5. For any  $T > 0$  there exists  $C_T$  such that, for any probability density  $\phi$  on  $\mathbb{R}$ ,

$$\sup_{(t,x) \in (0,T] \times \mathbb{R}} \int \phi(y) \|K_{\cdot}(x-y)\|_{L^1(0,t)} dy \leq C_T.$$

6. Finally,

$$\sup_{0 \leq t \leq T} \int_0^T \|K_{T+t-s}\|_{L^1(\mathbb{R})} \frac{1}{\sqrt{s}} ds \leq C_T.$$

As emphasized in the introduction, the well-posedness of the system (3) cannot be obtained by applying known results in the literature.

Given  $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$  and a family of densities  $(p_t)_{t \leq T}$  we set

$$B(t, x; p) := \int_0^t (K_{t-s} \star p_s)(x) ds. \quad (4)$$

We now define the notion of a weak solution to (3).

**Definition 2.1.** The family  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), X, W)$  is said to be a weak solution to the equation (3) up to time  $T > 0$  if:

1.  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))$  is a filtered probability space.
2. The process  $X := (X_t)_{t \in [0, T]}$  is real-valued, continuous, and  $(\mathcal{F}_t)$ -adapted. In addition, the probability distribution of  $X_0$  has density  $p_0$ .
3. The process  $W := (W_t)_{t \in [0, T]}$  is a one-dimensional  $(\mathcal{F}_t)$ -Brownian motion.
4. The probability distribution  $\mathbb{P} \circ X^{-1}$  has time marginal densities  $(p_t, t \in [0, T])$  with respect to Lebesgue measure which satisfy

$$\forall 0 < t \leq T, \quad \|p_t\|_{L^2(\mathbb{R})} \leq \frac{C_T}{t^{1/4}}. \quad (5)$$

5. For all  $t \in [0, T]$  and  $x \in \mathbb{R}$ , one has that  $\int_0^t |b(s, x)| ds < \infty$ .
6.  $\mathbb{P}$ -a.s. the pair  $(X, W)$  satisfies (3).

**Remark 2.2.** Let  $T > 0$  and  $(s, x) \in (0, T] \times \mathbb{R}$ . Setting

$$I(\varepsilon) := \int_0^{s-\varepsilon} \int K_{s-\tau}(x-y) q_\tau(y) dy d\tau,$$

one has

$$|I(\varepsilon)| \leq \int_0^{s-\varepsilon} \|q_\tau\|_{L^2(\mathbb{R})} \|K_{s-\tau}\|_{L^2(\mathbb{R})} d\tau.$$

Inequality (5) allows one to let  $\varepsilon \rightarrow 0$  and get

$$\sup_{0 \leq t \leq T} \int_0^t |B(s, x, p)| ds < \infty.$$

The following theorem provides existence and uniqueness of the weak solution to (3).

**Theorem 2.3.** Let  $T > 0$ . Suppose that  $p_0 \in L^1(\mathbb{R})$  is a probability density function and  $b \in L^\infty([0, T] \times \mathbb{R})$  is continuous w.r.t. the space variable. Under the hypothesis (H), Eq. (3) admits a unique weak solution up to  $T$  in the class of probability measures on  $C([0, T], \mathbb{R})$  whose one dimensional marginal densities  $(p_t)_{t \leq T}$  satisfy the following estimate:

$$\forall 0 < t \leq T, \quad \|p_t\|_{L^\infty(\mathbb{R})} \leq \frac{C_T}{\sqrt{t}}. \quad (6)$$

Our next result concerns the well-posedness of the the one-dimensional parabolic-parabolic Keller-Segel model

$$\begin{cases} \frac{\partial \rho}{\partial t}(t, x) = \frac{\partial}{\partial x} \cdot \left( \frac{1}{2} \frac{\partial \rho}{\partial x} - \chi \rho \frac{\partial c}{\partial x} \right), & t > 0, x \in \mathbb{R}, \\ \frac{\partial c}{\partial t}(t, x) = \frac{1}{2} \frac{\partial^2 c}{\partial x^2} - \lambda c + \rho, & t > 0, x \in \mathbb{R}, \\ \rho(0, x) = \rho_0(x), & c(0, x) = c_0. \end{cases} \quad (7a)$$

$$\quad (7b)$$

The parameters  $\chi$  and  $\lambda$  are strictly positive. As this system preserves the total mass, that is,

$$\forall t > 0, \quad \int_{\Omega} \rho(t, x) dx = \int_{\Omega} \rho_0(x) dx =: M,$$

the new functions  $\tilde{\rho}(t, x) := \frac{\rho(t, x)}{M}$  and  $\tilde{c}(t, x) := \frac{c(t, x)}{M}$  satisfy the system (7) with the new parameter  $\tilde{\chi} := \chi M$ . Therefore, w.l.o.g. we may and do thereafter assume that  $M = 1$ .

Denote by  $g(t, x)$  the density of  $W_t$ . We define the notion of solution for the system (7):

**Definition 2.4.** *Given  $\chi > 0$  and  $T > 0$ , the pair  $(\rho, c)$  is said to be a solution to K-S model if  $\rho \in L^\infty([0, T]; L^1(\mathbb{R}))$ ,  $c \in L^\infty([0, T]; C_b^1(\mathbb{R}))$ , and for any  $t \in (0, T]$  we have*

$$\|\rho_t\|_{L^2(\mathbb{R})} \leq \frac{C}{t^{\frac{1}{4}}}, \quad \|\rho_t^1\|_{L^\infty(\mathbb{R})} \leq \frac{C_T}{\sqrt{t}},$$

and  $(\rho_t)$  satisfies the following equation in the sense of the distributions:

$$\rho_t = g(t, \cdot) * \rho_0 + \chi \int_0^t \frac{\partial g}{\partial y}(t - s, \cdot) * \left( \frac{\partial c}{\partial x}(s, \cdot) \rho_s \right) ds,$$

with

$$c_t = e^{-\lambda t} g(t, \cdot) * c_0 + \int_0^t e^{-\lambda s} \rho_{t-s} * g(s, \cdot) ds.$$

These solutions are known as integral solutions and they have already been studied in PDE literature for the two-dimensional Keller-Segel model for which sub-critical and critical regimes exist depending on the parameters of the model (see [3] and references therein).

In the one-dimensional case there is no critical regime.

**Theorem 2.5.** *Assume that  $\rho_0 \in L^1(\mathbb{R})$  and  $c_0 \in C_b^1(\mathbb{R})$ . Given any  $\chi > 0$  and  $\lambda \geq 0$ , there exists a unique global solution  $(\rho, c)$  to (7) in the sense of Definition 2.4.*

The condition on  $c_0$  can be replaced by any other condition that ensures that  $\frac{\partial}{\partial x} e^{-\lambda t} \mathbb{E}(c_0(x + W_t))$  is bounded by a deterministic constant.

Below we will also get some regularity results: If  $\rho_0 \in L^\infty(\mathbb{R})$ , then  $\rho \in L^\infty([0, T]; L^1 \cap L^\infty(\mathbb{R}))$ . If  $\rho_0 \in L^2(\mathbb{R})$ , then  $\rho \in L^\infty([0, T]; L^1(\mathbb{R}))$  and  $t^{1/4} \|\rho_t\|_{L^\infty(\mathbb{R})} \leq C$ .

Our last main result shows the relationship between the non-linear SDE (3) and the Keller-Segel system (7).

**Corollary 2.6.** *Let the functions  $c_0$  and  $\rho_0$  be in  $C_b^1(\mathbb{R})$  and  $L^1(\mathbb{R})$ , respectively. The kernel  $K_t^{\frac{1}{2}}(x) := \chi e^{-\lambda t} \frac{-x}{2\pi t^{3/2}} e^{-\frac{x^2}{2t}}$  and the function  $b(t, x) := \chi e^{-\lambda t} \mathbb{E} c'_0(x + W_t)$  satisfy the hypothesis of Theorem 2.3. Denote by  $\rho_t \equiv p_t$  the time marginals of the probability distribution of the corresponding solution to Eq. (2) with  $d = 1$ . Set*

$$c_t := e^{-\lambda t} g(t, \cdot) * c_0 + \int_0^t e^{-\lambda s} \rho_{t-s} * g(s, \cdot) ds.$$

*Then the pair  $(\rho_t, c_t)$  is the unique global solution to the Keller-Segel system (7) in the sense of Definition 2.4.*

### 3 Preliminary: A density estimate

Let  $X^{(b)}$  be a process defined by

$$X_t^{(b)} = X_0 + \int_0^t b(s, X_s^{(b)}) ds + W_t, \quad t \in [0, T]. \quad (8)$$

To obtain  $L^\infty(\mathbb{R})$  estimates for the transition probability density  $p^{(b)}(t, x, y)$  of  $X^{(b)}$  under the only assumption that the drift  $b(t, x)$  is measurable and uniformly bounded we slightly extend the estimate proven in [13] for time homogeneous drift coefficients  $b(x)$ . We here propose a proof different from the original one in [13]. It avoids the use of densities of pinned diffusions and the claim that  $p^{(b)}(t, x, y)$  is continuous w.r.t. all the variables which does not seem obvious to us. We here adapt Makhlouf's method [10], the main difference between [10] and our calculation being that instead of the Wiener measure our reference measure is the probability distribution of the particular diffusion process  $X^\beta$  introduced in [13] and defined by

$$X_t^\beta = x + \beta \int_0^t \operatorname{sgn}(y - X_s^\beta) ds + W_t.$$

In the sequel we denote by  $p_y^\beta(t, x, z)$  the transition density of  $X^\beta$  and use Qian and Zheng's explicit representation for it (see [13] and our appendix section 8).

**Theorem 3.1.** *Let  $X^{(b)}$  be the process defined in (8). Assume  $\beta := \sup_{t \in [0, T]} \|b(t, \cdot)\|_\infty < \infty$ . Then for all  $x, y \in \mathbb{R}$  and  $t \in (0, T]$  it holds that*

$$p^{(b)}(t, x, y) \leq \frac{1}{\sqrt{2\pi t}} \int_{\frac{|x-y|}{\sqrt{t}}}^\infty z e^{-\frac{(z-\beta\sqrt{t})^2}{2}} dz. \quad (9)$$

*Proof.* Let  $f \in \mathcal{C}_K^\infty(\mathbb{R})$  and fix  $t \in (0, T]$ . Consider the parabolic PDE driven by the infinitesimal generator of  $X^\beta$ :

$$\begin{cases} \frac{\partial u}{\partial t}(s, x) + \frac{1}{2} \frac{\partial^2 u}{\partial x^2}(s, x) + \beta \operatorname{sgn}(y - x) \frac{\partial u}{\partial x}(s, x) = 0, & 0 \leq s < t, \quad x \in \mathbb{R}, \\ u(t, x) = f(x), & x \in \mathbb{R}. \end{cases} \quad (10)$$

In view of Veretennikov [15, Thm.1] there exists a solution  $u(s, x) \in W_p^{1,2}([0, t] \times \mathbb{R})$ . Applying the Itô-Krylov formula to  $u(s, X_s^\beta)$  we obtain that

$$u(s, x) = \int f(z) p_y^\beta(t - s, x, z) dz.$$

In view of (34) from our appendix for some constant  $C$  depending on  $\beta$ ,  $T$  and  $y$  only we thus have,  $|\frac{\partial p_y^\beta}{\partial x}(t - s, x, z)| \leq \frac{C}{t-s}$ . This allows us to differentiate under the integral sign:

$$\frac{\partial u}{\partial x}(s, x) = \int f(z) \frac{\partial p_y^\beta}{\partial x}(t - s, x, z) dz, \quad \forall 0 \leq s < t \leq T.$$

Fix  $0 < \varepsilon < t$ . Now apply the Itô-Krylov formula to  $u(s, X_s^{(b)})$  for  $0 \leq s \leq t - \varepsilon$  and use the PDE (10). It comes:

$$\mathbb{E}(u(t - \varepsilon, X_{t-\varepsilon}^{(b)})) = u(0, x) + \mathbb{E} \int_0^{t-\varepsilon} (b(s, X_s^{(b)}) - \beta \operatorname{sgn}(y - X_s^{(b)})) \frac{\partial u}{\partial x}(s, X_s^{(b)}) ds.$$

In view of Corollary 8.2 in the appendix there exists a function  $h \in L^1([0, t] \times \mathbb{R})$  such that

$$\forall 0 < s < t \leq T, \quad \forall y, z \in \mathbb{R}, \quad \mathbb{E} \left| \frac{\partial}{\partial x} p_y^\beta(t - s, X_s^{(b)}, z) \right| \leq C_{T, \beta, x_0, y} h(s, z). \quad (11)$$

Consequently,

$$\begin{aligned} \mathbb{E}(u(t - \varepsilon, X_{t-\varepsilon}^{(b)})) &= \int f(z) p_y^\beta(t, x, z) dz \\ &+ \int f(z) \int_0^{t-\varepsilon} \mathbb{E} \left\{ (b(s, X_s^{(b)}) - \beta \operatorname{sgn}(y - X_s^{(b)})) \frac{\partial}{\partial x} p_y^\beta(t - s, X_s^{(b)}, z) \right\} ds dz. \end{aligned}$$

Let now  $\epsilon$  tend to 0. By Lebesgue's dominated convergence theorem we obtain

$$\begin{aligned} \int f(z)p^{(b)}(t, x, z) &= \int f(z)p_y^\beta(t, x, z)dz \\ &+ \int f(z) \int_0^t \mathbb{E} \left\{ (b(s, X_s^{(b)}) - \beta \operatorname{sgn}(y - X_s^{(b)})) \frac{\partial}{\partial x} p_y^\beta(t-s, X_s^{(b)}, z) \right\} ds dz. \end{aligned}$$

Therefore the density  $p^{(b)}$  satisfies:

$$p^{(b)}(t, x, z) = p_y^\beta(t, x, z) + \int_0^t \mathbb{E} \left\{ (b(s, X_s^{(b)}) - \beta \operatorname{sgn}(y - X_s^{(b)})) \frac{\partial}{\partial x} p_y^\beta(t-s, X_s^{(b)}, z) \right\} ds.$$

As noticed in [13], in view of Formula (35) from our appendix we have

$$(b(s, x) - \beta \operatorname{sgn}(y - x)) \frac{\partial}{\partial x} p_y^\beta(t-s, x, y) \leq 0.$$

This leads us to choose  $z = y$  in the preceding equality, which leads us to

$$p^{(b)}(t, x, y) = p_y^\beta(t, x, y) + \int_0^t \mathbb{E} \left\{ (b(s, X_s^{(b)}) - \beta \operatorname{sgn}(y - X_s^{(b)})) \frac{\partial}{\partial x} p_y^\beta(t-s, X_s^{(b)}, y) \right\} ds,$$

from which

$$\forall t \leq T, \quad p^{(b)}(t, x, y) \leq p_y^\beta(t, x, y).$$

□

**Corollary 3.2.** Assume  $X_0^{(b)}$  is distributed according to the probability density function  $p_0$  on  $\mathbb{R}$ . Denote by  $p(t, y)dy$  the probability density of  $X_t^{(b)}$

1. If  $p_0 \in L^\infty(\mathbb{R})$ , then for any  $t \in (0, T]$  we have

$$\|p(t, \cdot)\|_\infty \leq 2\|p_0\|_\infty + \beta. \quad (12)$$

2. If  $p_0 \in L^p(\mathbb{R})$  for some  $p \geq 1$ , then for any  $t \in (0, T]$  we have

$$\|p(t, \cdot)\|_\infty \leq \frac{C\|p_0\|_{L^p(\mathbb{R})}}{t^{\frac{1}{2p}}} + \beta. \quad (13)$$

*Proof.* In view of (9) we have

$$\begin{aligned} p(t, y) &\leq \frac{1}{\sqrt{2\pi t}} \int p_0(x) \int_{\frac{|x-y|}{\sqrt{t}}}^\infty z e^{-\frac{(z-\beta\sqrt{t})^2}{2}} dz dx \\ &\leq \frac{1}{\sqrt{2\pi t}} \int p_0(x) \int_{\frac{|x-y|}{\sqrt{t}} - \beta\sqrt{t}}^\infty (z + \beta\sqrt{t}) e^{-\frac{z^2}{2}} dz dx \\ &= \frac{1}{\sqrt{2\pi t}} \left( \int p_0(x) e^{-\frac{(|x-y|-\beta t)^2}{2t}} dx + \beta\sqrt{t} \int p_0(x) \int_{\frac{|x-y|}{\sqrt{t}} - \beta\sqrt{t}}^\infty e^{-\frac{z^2}{2}} dz dx \right) \\ &\leq \frac{1}{\sqrt{2\pi t}} \int p_0(x) e^{-\frac{(|y-x|-\beta t)^2}{2t}} dx + \beta. \end{aligned}$$

If  $p_0 \in L^\infty(\mathbb{R})$ , we then use that

$$\frac{1}{\sqrt{2\pi t}} \int p_0(x) e^{-\frac{(|y-x|-\beta t)^2}{2t}} dx \leq 2\|p_0\|_\infty.$$

If  $p_0 \in L^p(\mathbb{R})$ ,  $p \geq 1$ , we apply Hölder's inequality and get

$$\frac{1}{\sqrt{2\pi t}} \int p_0(x) e^{-\frac{(|y-x|-\beta t)^2}{2t}} dx \leq \frac{\|p_0\|_{L^p(\mathbb{R})}}{\sqrt{2\pi t}} \left( \int e^{-q\frac{(|y-x|-\beta t)^2}{2t}} dx \right)^{1/q} \leq \frac{C_q t^{\frac{1}{2q}}}{\sqrt{t}} = \frac{C_q}{t^{\frac{1}{2p}}}.$$

□

## 4 A non-linear McKean–Vlasov–Fokker–Planck equation

**Proposition 4.1.** *Let  $T > 0$ . Assume  $p_0 \in L^1(\mathbb{R})$ ,  $b \in L^\infty((0, T) \times \mathbb{R})$  and Hypothesis (H). Let  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), X, W)$  be a weak solution to (3) until  $T$ . Then,*

1. *The marginals  $(p_t)_{t \in (0, T]}$  satisfy in the sense of the distributions the mild equation*

$$\forall t \in (0, T], \quad p_t = g(t, \cdot) * p_0 + \int_0^t \frac{\partial}{\partial y} g(t-s, \cdot) * (p_s(b(s, \cdot) + B(s, \cdot; p))) ds. \quad (14)$$

2. *Equation (14) admits at most one solution  $p = (p_t)_{t \in [0, T]}$  which for any  $t \in (0, T]$  belongs to  $L^1(\mathbb{R})$  and satisfies (5) and (6).*

*Proof.* We successively prove (14) and the uniqueness of its solution in  $L^1(\mathbb{R})$ .

1. Notice that, in view of Hypothesis (H), for all  $t \in (0, T]$  and  $x \in \mathbb{R}$  one has  $|B(t, x; p)| \leq C_T$ . Now, for  $f \in C_b^2(\mathbb{R})$  consider the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial s} G + \frac{1}{2} \frac{\partial^2 G}{\partial y^2} = 0, & 0 \leq s < t, \quad y \in \mathbb{R}, \\ \lim_{s \rightarrow t^-} G(s, y) = f(y). \end{cases} \quad (15)$$

The function

$$G_{t,f}(s, y) = \int f(x) g(t-s, y-x) dx$$

is a smooth solution to (15). Applying Itô's formula we get

$$\begin{aligned} G_{t,f}(t, X_t) - G_{t,f}(0, X_0) &= \int_0^t \frac{\partial G_{t,f}}{\partial s}(s, X_s) ds + \int_0^t \frac{\partial G_{t,f}}{\partial y}(s, X_s) (b(s, X_s) + B(s, X_s; p)) ds \\ &\quad + \int_0^t \frac{\partial}{\partial y} G_{t,f}(s, X_s) dW_s + \frac{1}{2} \int_0^t \frac{\partial^2}{(\partial y)^2} G_{t,f}(s, X_s) ds. \end{aligned}$$

Using (15) we obtain

$$\mathbb{E} f(X_t) = \mathbb{E} G_{t,f}(0, X_0) + \int_0^t \mathbb{E} \left[ \frac{\partial}{\partial y} G_{t,f}(s, X_s) (b(s, X_s) + B(s, X_s; p)) \right] ds =: I + II. \quad (16)$$

On the one hand one has

$$I = \int \int f(x) g(t, x-y) dx p_0(y) dy = \int f(x) (g(t, \cdot) * p_0)(x) dx.$$

On the second hand one has

$$\begin{aligned} II &= \int_0^t \int \frac{\partial}{\partial y} \left[ \int f(x) g(t-s, x-y) dx \right] (b(s, y) + B(s, y; p)) p_s(y) dy ds \\ &= \int_0^t \int \int f(x) \frac{\partial}{\partial y} g(t-s, x-y) dx (b(s, y) + B(s, y; p)) p_s(y) dy ds \\ &= \int f(x) \int_0^t \left[ \frac{\partial}{\partial y} g(t-s, \cdot) * ((b(s, \cdot) + B(s, \cdot; p)) p_s) \right](x) ds dx. \end{aligned}$$

Thus (16) can be written as

$$\int f(x) p_t(x) dx = \int f(x) (g(t, \cdot) * p_0)(x) dx + \int f(x) \int_0^t \left[ \frac{\partial}{\partial y} g(t-s, \cdot) * ((b(s, \cdot) + B(s, \cdot; p)) p_s) \right](x) ds dx,$$

which is the mild equation (14).



2. Assume  $p_t^1$  and  $p_t^2$  are two mild solutions in the sense of the distributions to (14) which satisfy

$$\exists C > 0, \forall t \in (0, T], \quad \|p_t^1\|_{L^2(\mathbb{R})} + \|p_t^2\|_{L^2(\mathbb{R})} \leq \frac{C_T}{t^{1/4}}$$

In view of Hypothesis (H.4) we have

$$\sup_{t \in [0, T]} \|B(t, x; p^1)\|_{L^\infty(\mathbb{R})} + \sup_{t \in [0, T_0]} \|B(t, x; p^2)\|_{L^\infty(\mathbb{R})} \leq C_T.$$

Now,

$$\begin{aligned} \|p_t^1 - p_t^2\|_{L^1(\mathbb{R})} &\leq \int_0^t \left\| \frac{\partial g}{\partial y}(t-s, \cdot) * [B(s, \cdot; p^1)p_s^1 - B(s, \cdot; p^2)p_s^2] \right\|_{L^1(\mathbb{R})} ds \\ &\quad + \int_0^t \left\| \frac{\partial g}{\partial y}(t-s, \cdot) * [b(s, \cdot)(p_s^1 - p_s^2)] \right\|_{L^1(\mathbb{R})} ds \\ &\leq \int_0^t \left\| \frac{\partial g}{\partial y}(t-s, \cdot) * [(B(s, \cdot; p^1) - B(s, \cdot; p^2))p_s^1] \right\|_{L^1(\mathbb{R})} ds \\ &\quad + \int_0^t \left\| \frac{\partial g}{\partial y}(t-s, \cdot) * [(p_s^1 - p_s^2)B(s, \cdot; p^2)] \right\|_{L^1(\mathbb{R})} ds \\ &\quad + \int_0^t \left\| \frac{\partial}{\partial y} g(t-s, \cdot) * [b(s, \cdot)(p_s^1 - p_s^2)] \right\|_{L^1(\mathbb{R})} ds \\ &=: I + II + III. \end{aligned}$$

As

$$\left\| \frac{\partial}{\partial y} g(t-s, \cdot) \right\|_{L^1(\mathbb{R})} \leq \frac{C_T}{\sqrt{t-s}},$$

the convolution inequality  $\|f * h\|_{L^1(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})} \|h\|_{L^1(\mathbb{R})}$  leads to

$$II \leq \int_0^t \left\| \frac{\partial}{\partial y} g(t-s, \cdot) \right\|_{L^1(\mathbb{R})} \|(p_s^1 - p_s^2)B(s, \cdot; p^2)\|_{L^1(\mathbb{R})} ds \leq C_T \int_0^t \frac{\|p_s^1 - p_s^2\|_{L^1(\mathbb{R})}}{\sqrt{t-s}} ds.$$

As  $b$  is bounded, we also have

$$|III| \leq C_T \int_0^t \frac{\|p_s^1 - p_s^2\|_{L^1(\mathbb{R})}}{\sqrt{t-s}} ds.$$

We now turn to  $I$ . Notice that

$$\|B(s, \cdot; p^1) - B(s, \cdot; p^2)\|_{L^1(\mathbb{R})} \leq \int_0^s \|K_{s-\tau}\|_{L^1(\mathbb{R})} \|p_\tau^1 - p_\tau^2\|_{L^1(\mathbb{R})} d\tau,$$

from which

$$\begin{aligned} I &\leq \int_0^t \frac{C_T}{\sqrt{t-s}\sqrt{s}} \int_0^s \|K_{s-\tau}\|_{L^1(\mathbb{R})} \|p_\tau^1 - p_\tau^2\|_{L^1(\mathbb{R})} d\tau ds \\ &\leq \int_0^t \|p_\tau^1 - p_\tau^2\|_{L^1(\mathbb{R})} \int_\tau^t \frac{C_T}{\sqrt{t-s}\sqrt{s}} \|K_{s-\tau}\|_{L^1(\mathbb{R})} ds d\tau. \end{aligned}$$

In addition, using Hypothesis (H.4),

$$\int_\tau^t \frac{1}{\sqrt{t-s}\sqrt{s}} \|K_{s-\tau}\|_{L^1(\mathbb{R})} ds \leq \frac{1}{\sqrt{\tau}} \int_\tau^t \frac{1}{\sqrt{t-s}} \|K_{s-\tau}\|_{L^1(\mathbb{R})} ds = \frac{1}{\sqrt{\tau}} \int_0^{t-\tau} \frac{\|K_s\|_{L^1(\mathbb{R})}}{\sqrt{t-\tau-s}} ds \leq \frac{C_T}{\sqrt{\tau}}.$$

It comes:

$$I \leq C_T \int_0^t \frac{\|p_\tau^1 - p_\tau^2\|_{L^1(\mathbb{R})}}{\sqrt{\tau}} d\tau.$$

Gathering the preceding estimates we obtain

$$\|p_t^1 - p_t^2\|_{L^1(\mathbb{R})} \leq C_T \int_0^t \frac{\|p_s^1 - p_s^2\|_{L^1(\mathbb{R})}}{\sqrt{t-s}} ds + C_T \int_0^t \frac{\|p_s^1 - p_s^2\|_{L^1(\mathbb{R})}}{\sqrt{s}} ds.$$

Applying a Singular Gronwall's Lemma (see Lemma 4.2 below), we conclude

$$\forall t \in (0, T], \quad \|p_t^1 - p_t^2\|_{L^1(\mathbb{R})} = 0.$$

This implies that  $p_t^1$  and  $p_t^2$  are equal almost everywhere for every  $t \in (0, T]$ .  $\square$

In the above proof we have used the following result:

**Lemma 4.2.** *Let  $(u(t))_{t \geq 0}$  be a non-negative bounded function such that for a given  $T > 0$ , there exists a positive constant  $C_T$  such that for any  $t \in (0, T]$ :*

$$u(t) \leq C_T \int_0^t \frac{u(s)}{\sqrt{s}} ds + C_T \int_0^t \frac{u(s)}{\sqrt{t-s}} ds. \quad (17)$$

Then,  $u(t) = 0$  for any  $t \in (0, T]$ .

*Proof.* Set  $u_t^* := \sup_{s \leq t} u(s)$ . The inequality (17) implies that

$$u_t^* \leq 4C_T \sqrt{t} u_t^*.$$

Set  $T^* := \frac{1}{32C_T^2}$ . If  $T \leq T^*$ , then for  $t \leq T$ , we have  $u_t^* \leq \frac{u_t^*}{\sqrt{2}}$ . Thus,  $u_t^* = 0$  for every  $t \leq T$  and the lemma is proved. If  $T > T^*$ , for  $T^* < t < T$ ,

$$u(t) \leq C_T \int_0^{T^*} \left( \frac{1}{\sqrt{t-s}} + \frac{1}{\sqrt{s}} \right) u(s) ds + \int_{T^*}^t \left( \frac{1}{\sqrt{t-s}} + \frac{1}{\sqrt{s}} \right) u(s) ds.$$

The first integral is null, since  $u(T^*) = 0$ . Thus,

$$u(t) \leq \int_0^{t-T^*} \left( \frac{1}{\sqrt{t-T^*-\theta}} + \frac{1}{\sqrt{T^*+\theta}} \right) u(T^*+\theta) d\theta \leq \int_0^{t-T^*} \left( \frac{1}{\sqrt{t-T^*-\theta}} + \frac{1}{\sqrt{T^*}} \right) u(T^*+\theta) d\theta.$$

For  $0 < s \leq T - T^*$ , define  $v(s) := u(s + T^*)$ . The previous inequality becomes:

$$v(t - T^*) \leq C_T \int_0^{t-T^*} \left( \frac{1}{\sqrt{t-T^*-\theta}} + \frac{1}{\sqrt{T^*}} \right) v(\theta) d\theta \leq \frac{\sqrt{T^*} + \sqrt{T-T^*}}{\sqrt{T^*}} \int_0^{t-T^*} \frac{v(\theta)}{\sqrt{t-T^*-\theta}} d\theta.$$

Setting  $t - T^* =: \tau$ , we thus have

$$v(\tau) \leq \frac{\sqrt{T^*} + \sqrt{T-T^*}}{\sqrt{T^*}} \int_0^\tau \frac{v(\theta)}{\sqrt{\tau-\theta}} d\theta, \quad \forall 0 \leq \tau < T - T^*.$$

Now we are in a position to apply the singular Gronwall's lemma (see [6, Lemma 7.1.1]) and conclude that  $v(\tau) = 0$  for  $0 \leq \tau \leq T - T^*$ . Thus,  $u(t) = 0$  for  $T^* \leq t \leq T$ .  $\square$

## 5 A local existence and uniqueness result for Equation (3)

Set

$$D(T) := \int_0^T \int_{\mathbb{R}} |K_t(x)| dx dt < \infty. \quad (18)$$

The main result in this section is the following theorem.

**Theorem 5.1.** *Let  $T_0 > 0$  be such that  $D(T_0) < 1$ . Assume  $p_0 \in L^1(\mathbb{R})$  and  $b \in L^\infty((0, T_0) \times \mathbb{R})$  continuous w.r.t. space variable. Under Hypothesis (H), Equation (3) admits a unique weak solution up to  $T_0$  in the class of probability measures on  $C([0, T_0], \mathbb{R})$  whose one dimensional marginal densities satisfy (6).*

**Iterative procedure** We will work with the following sequence of SDE's. For  $k = 1$

$$\begin{cases} dX_t^1 = b(t, X_t^1) dt + \left\{ \int_0^t (K_{t-s} \star p_0)(X_t^1) ds \right\} dt + dW_t, \\ X_0^1 \sim p_0. \end{cases} \quad (19)$$

Denote the drift of this equation by  $b^1(t, x)$ . Supposing that, in the step  $k - 1$ , the one dimensional time marginals of the law of the solution have densities  $p^{k-1}(s, y)$ , we define the drift in the step  $k$  as

$$b^k(t, x, p^{k-1}) = b(t, x) + B(t, x; p^{k-1}).$$

The corresponding SDE is

$$\begin{cases} dX_t^k = b^k(t, X_t^k, p^{k-1}) dt + dW_t, \\ X_0^k \sim p_0. \end{cases} \quad (20)$$

In order to prove the desired local existence and uniqueness result we set up the non-linear martingale problem related to (3).

**Definition 5.2.** A probability measure  $\mathbb{Q}$  on the canonical space  $C([0, T_0]; \mathbb{R})$  equipped with its canonical filtration is a solution to the non-linear martingale problem  $(MP(p_0, T_0, b))$  if:

(i)  $\mathbb{Q}_0 = p_0$ .

(ii) For any  $t \in (0, T_0]$ , the one dimensional time marginals of  $\mathbb{Q}$ , denoted by  $\mathbb{Q}_t$ , have densities  $q_t$  w.r.t. Lebesgue measure on  $\mathbb{R}$ . In addition, they satisfy  $q_t \in L^2(\mathbb{R})$  and

$$\sup_{t \in (0, T_0]} \|q_t\|_{L^2(\mathbb{R})} \leq \frac{C_{T_0}}{t^{1/4}}. \quad (21)$$

(iii) For any  $f \in C_K^2(\mathbb{R})$  the process  $(M_t)_{t \leq T_0}$ , defined as

$$M_t := f(x(t)) - f(x(0)) - \int_0^t \left[ \frac{1}{2} \frac{\partial^2 f}{(\partial x)^2}(x(u)) + \frac{\partial f}{\partial x}(x(u))(b(u, x(u)) + \int_0^u \int K_{u-\tau}(x(u)-y) q_\tau(y) dy d\tau) \right] du$$

is a  $\mathbb{Q}$ -martingale.

Notice that the arguments in Remark 2.2 justify that all the integrals in the definition of  $M_t$  are well defined.

We start with the analysis of Equations (19)-(20).

**Proposition 5.3.** Same assumptions as in Theorem (5.1). Then, for any  $k \geq 1$ , Equations (19)-(20) have a unique weak solution  $(X, \mathbb{P}^k)$  up to  $T_0$ . Moreover, for  $t \in (0, T_0]$ , the time marginals  $\mathbb{P}_t^k$  of  $\mathbb{P}^k$  have densities  $p_t^k$  w.r.t. Lebesgue measure on  $\mathbb{R}$ . Setting  $\beta^k = \sup_{t \leq T_0} \|b^k(t, \cdot)\|_{L^\infty(\mathbb{R})}$ , one has

$$\sup_{t \leq T_0} \|p^k\|_{L^\infty(\mathbb{R})} \leq \frac{C_{T_0}}{\sqrt{t}} \quad \text{and} \quad \beta^k \leq C(b_0, T_0).$$

*Proof.* In all the sequel we set  $b_0 := \|b\|_{L^\infty(\mathbb{R})}$ . We proceed by induction.

**Case  $k = 1$**  In view of (H.5), one has  $\beta^1 \leq b_0 + C_{T_0}$ . This implies that the equation (19) has a unique weak solution in  $[0, T_0]$  with time marginal densities  $(p^1(t, y) dy)_{t \leq T_0}$  which in view of (13) satisfy

$$\forall t \in (0, T_0], \quad \|p^1(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{C_{T_0}}{\sqrt{t}} + \beta^1.$$

**Case  $k > 1$ .** Assume now that the equation for  $X^k$  has a unique weak solution and assume  $\beta^k$  is finite. In addition, suppose that the one dimensional time marginals satisfy

$$\forall t \in (0, T_0], \quad \|p^k(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{C_{T_0}}{\sqrt{t}} + \beta^k.$$

In view of (H.4), the new drift satisfies

$$|b^{k+1}(t, x; p^k)| \leq b_0 + \int_0^t \|p^k(s, \cdot)\|_{L^\infty(\mathbb{R})} \|K_{t-s}\|_{L^1(\mathbb{R})} ds \leq b_0 + \int_0^t \left(\frac{1}{\sqrt{2\pi s}} + \beta^k\right) \|K_{t-s}\|_{L^1(\mathbb{R})} ds \leq b_0 + C_{T_0} + \beta^k D(T_0).$$

Thus, we conclude that  $\beta^{k+1} \leq b_0 + C_{T_0} + \beta^k D(T_0)$ . Therefore, there exists a unique weak solution to the equation for  $X^{k+1}$ . Furthermore, by (13):

$$\forall t \in (0, T_0], \quad \|p^{k+1}(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{C_{T_0}}{\sqrt{t}} + \beta^{k+1}.$$

Notice that

$$\forall k > 1, \quad \beta^{k+1} \leq b_0 + C_{T_0} + \beta^k D(T_0) \quad \text{and} \quad \beta^1 \leq b_0 + C_{T_0}.$$

Thus, by hypothesis  $D(T_0) < 1$ , we have

$$\forall k \geq 1, \quad \beta^k \leq \frac{b_0 + C_{T_0}}{1 - D(T_0)} + b_0 + C_{T_0} \quad (22)$$

and

$$\|p^k(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{C_{T_0}}{\sqrt{t}} + \beta^k \leq \frac{C_{T_0}}{\sqrt{t}} + \frac{b_0 + C_{T_0}}{1 - D(T_0)} + b_0 + C_{T_0}. \quad (23)$$

□

**Proposition 5.4.** *Same assumptions as in Theorem (5.1). Then,*

- 1) *The family of probabilities  $(\mathbb{P}^k)_{k \geq 1}$  is tight.*
- 2) *Any weak limit  $\mathbb{P}^\infty$  of a convergent subsequence of  $(\mathbb{P}^k)_{k \geq 1}$  solves  $(MP(p_0, T_0, b))$ .*

*Proof.* In view of (22), we obviously have

$$\exists C_{T_0} > 0, \quad \sup_k |X_t^k - X_s^k|^4 \leq C_{T_0} |t - s|^2, \quad \forall 0 \leq s \leq t \leq T_0.$$

This is a sufficient condition for tightness (see e.g. [9, Chap.2, Pb.4.11]). Therefore, there exists a weakly convergent subsequence of  $(\mathbb{P}^k)_{k \geq 1}$  that we will still denote by  $(\mathbb{P}^k)_{k \geq 1}$ . Denote its limit by  $\mathbb{P}^\infty$ . Let us check that  $\mathbb{P}^\infty$  solves the martingale problem  $(MP(p_0, T_0, b))$ .

- i) Each  $\mathbb{P}_0^k$  has density  $p_0$ , and therefore  $\mathbb{P}_0^\infty$  also has density  $p_0$ .
- ii) From (23) for any  $k \geq 1$  and  $0 < t \leq T_0$ , we have that

$$p_t^k \in L^2(\mathbb{R}) \quad \text{and} \quad \|p_t^k\|_{L^2(\mathbb{R})} \leq \frac{C(b_0, T_0)}{t^{1/4}}. \quad (24)$$

Define the functional  $T_t(g)$  by

$$T_t(g) := \int g(y) \mathbb{P}_t^\infty(dy), \quad g \in C_K(\mathbb{R}).$$

By weak convergence we have

$$T_t(g) = \lim_{k \rightarrow \infty} \int g(y) p_t^k(y) dy,$$

and thus

$$|T_t(g)| \leq \frac{C(b_0, T_0)}{t^{1/4}} \|g\|_{L^2(\mathbb{R})}.$$

Therefore, for each  $0 < t \leq T_0$ ,  $T_t$  is a bounded linear functional on a dense subset of  $L^2(\mathbb{R})$ . Thus,  $T_t$  can be extended to a linear functional on  $L^2(\mathbb{R})$ . By Riesz-representation theorem, there exists a unique  $p_t^\infty \in L^2(\mathbb{R})$  such that  $\|p_t^\infty\|_{L^2(\mathbb{R})} \leq \frac{C(b_0, T_0)}{t^{1/4}}$  and  $p_t^\infty$  is the probability density of  $\mathbb{P}_t^\infty(dy)$ .

iii) Set

$$M_t := f(x(t)) - f(x(0)) - \int_0^t \left[ \frac{1}{2} \frac{\partial^2 f}{(\partial x)^2}(x(u)) + \frac{\partial f}{\partial x}(x(u))(b(u, x(u)) + \int_0^u (K_{u-\tau} \star p_\tau^\infty)(x(u)) d\tau \right] du,$$

We have to prove

$$\mathbb{E}_{\mathbb{P}^\infty}[(M_t - M_s)\phi(x(t_1), \dots, x(t_N))] = 0, \quad \forall \phi \in C_b(\mathbb{R}^N) \text{ and } 0 \leq t_1 < \dots < t_N < s \leq t \leq T_0, N \geq 1.$$

The process

$$M_t^k := f(x(t)) - f(x(0)) - \int_0^t \left[ \frac{1}{2} \frac{\partial^2 f}{(\partial x)^2}(x(u)) + \frac{\partial f}{\partial x}(x(u))(b(u, x(u)) + \int_0^u (K_{u-\tau} \star p_\tau^{k-1})(x(u)) d\tau \right] du$$

is a martingale under  $\mathbb{P}^k$ . Therefore, it follows that

$$\begin{aligned} 0 &= \mathbb{E}_{\mathbb{P}^k}[(M_t^k - M_s^k)\phi(x(t_1), \dots, x(t_N))] = \\ &= \mathbb{E}_{\mathbb{P}^k}[\phi(\dots)(f(x(t)) - f(x(s)))] + \mathbb{E}_{\mathbb{P}^k}[\phi(\dots) \int_s^t \frac{1}{2} \frac{\partial^2 f}{(\partial x)^2}(x(u)) du] \\ &+ \mathbb{E}_{\mathbb{P}^k}[\phi(\dots) \int_s^t \frac{\partial f}{\partial x}(x(u)) b(u, x(u)) du] + \mathbb{E}_{\mathbb{P}^k}[\phi(\dots) \int_s^t \frac{\partial f}{\partial x}(x(u)) \int_0^u (K_{u-\tau} \star p_\tau^{k-1})(x(u)) d\tau du]. \end{aligned}$$

Since  $(\mathbb{P}^k)$  weakly converges to  $\mathbb{P}^\infty$ , the first two terms on the r.h.s. converge. Now, observe that

$$\begin{aligned} &\mathbb{E}_{\mathbb{P}^k}[\phi(\dots) \int_s^t \frac{\partial f}{\partial x}(x(u)) \int_0^u (K_{u-\tau} \star p_\tau^{k-1})(x(u)) d\tau du] \\ &- \mathbb{E}_{\mathbb{P}^\infty}[\phi(\dots) \int_s^t \frac{\partial f}{\partial x}(x(u)) \int_0^u (K_{u-\tau} \star p_\tau^\infty)(x(u)) d\tau du] \\ &= (\mathbb{E}_{\mathbb{P}^k}[\phi(\dots) \int_s^t \frac{\partial f}{\partial x}(x(u)) \int_0^u (K_{u-\tau} \star p_\tau^{k-1})(x(u)) d\tau du] \\ &- \mathbb{E}_{\mathbb{P}^k}[\phi(\dots) \int_s^t \frac{\partial f}{\partial x}(x(u)) \int_0^u (K_{u-\tau} \star p_\tau^\infty)(x(u)) d\tau du]) \\ &+ (\mathbb{E}_{\mathbb{P}^k}[\phi(\dots) \int_s^t \frac{\partial f}{\partial x}(x(u)) \int_0^u (K_{u-\tau} \star p_\tau^\infty)(x(u)) d\tau du] \\ &- \mathbb{E}_{\mathbb{P}^\infty}[\phi(\dots) \int_s^t \frac{\partial f}{\partial x}(x(u)) \int_0^u (K_{u-\tau} \star p_\tau^\infty)(x(u)) d\tau du]) \\ &=: I + II. \end{aligned}$$

Now,

$$\begin{aligned} |I| &\leq \|\phi\|_{L^\infty(\mathbb{R})} \int_s^t \int_0^u \int \left| \frac{\partial f}{\partial x}(x) \right| (K_{u-\tau} \star (p_\tau^{k-1} - p_\tau^\infty))(x) |p_u^k(x)| dx d\tau du \\ &\leq \|\phi\|_{L^\infty(\mathbb{R})} \int_s^t \frac{C_T}{u^{1/4}} \int_0^u \sqrt{\int \left( \frac{\partial f}{\partial x}(x) \right)^2 \left( \int K_{u-\tau}(x-y) (p_\tau^{k-1}(y) - p_\tau^\infty(y)) dy \right)^2 dx} d\tau du. \end{aligned}$$

Then, fix  $u \in [s, t]$  and  $\tau < u$ . By (H), the function  $f_1(y) = \mathbb{1}\{\tau < u\} K_{u-\tau}(x-y)$  is a bounded and continuous function of  $y$ . By weak convergence

$$\forall \tau < u, \quad \lim_{k \rightarrow \infty} \int K_{u-\tau}(x-y) p_\tau^{k-1}(y) dy = \int K_{u-\tau}(x-y) p_\tau^\infty(y) dy.$$

In view of (24), ii) and (H), one has

$$\left( \frac{\partial f}{\partial x}(x) \right)^2 \left( \int K_{u-\tau}(x-y) (p_\tau^{k-1}(y) - p_\tau^\infty(y)) dy \right)^2 \leq \left( \frac{\partial f}{\partial x}(x) \right)^2 \|K_{u-\tau}\|_{L^2(\mathbb{R})}^2 \frac{C_T}{\sqrt{\tau}}. \quad (25)$$

As  $f \in C_K^2(\mathbb{R})$ , we apply Lebesgue's dominated convergence to obtain

$$\sqrt{\int \left(\frac{\partial f}{\partial x}(x)\right)^2 \left(\int K_{u-\tau}(x-y)(p_\tau^{k-1}(y) - p_\tau^\infty(y))dy\right)^2 dx} \xrightarrow{k \rightarrow \infty} 0.$$

Estimate (25) and (H) enable us to apply again Lebesgue's dominated convergence to deduce that  $I \rightarrow 0$  as  $k \rightarrow \infty$ .

Now, to prove that  $II \rightarrow 0$ , it suffices to prove that the functional  $F : C([0, T_0]; \mathbb{R}) \rightarrow \mathbb{R}$  defined by

$$x(\cdot) \mapsto \phi(x(t_1), \dots, x(t_N)) \int_s^t \frac{\partial f}{\partial x}(x(u)) \int_0^u \int K_{u-\tau}(x(u) - y) p_\tau^\infty(y) dy d\tau du$$

is continuous. Let  $(x_n)$  a sequence converging in  $C([0, T_0]; \mathbb{R})$  to  $x$ . Since  $\phi$  is a continuous function, it suffices to show that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_s^t \frac{\partial f}{\partial x}(x_n(u)) \int_0^u \int K_{u-\tau}(x_n(u) - y) p_\tau^\infty(y) dy d\tau du \\ &= \int_s^t \frac{\partial f}{\partial x}(x(u)) \int_0^u \int K_{u-\tau}(x(u) - y) p_\tau^\infty(y) dy d\tau du. \end{aligned} \quad (26)$$

For  $(u, \tau) \in [s, t] \times [0, t]$ , set

$$f_3^n(u, \tau) := \mathbb{1}\{\tau < u\} \frac{\partial f}{\partial x}(x_n(u)) \int K_{u-\tau}(x_n(u) - y) p_\tau^\infty(y) dy.$$

Since  $x_n$  converges pointwise to  $x$  and  $K$  satisfies (H), by dominated convergence,

$$\lim_{n \rightarrow \infty} f_3^n(u, \tau) = \mathbb{1}\{\tau < u\} \frac{\partial f}{\partial x}(x(u)) \int K_{u-\tau}(x(u) - y) p_\tau^\infty(y) dy.$$

Furthermore,

$$|f_3^n(u, \tau)| \leq C \mathbb{1}\{\tau < u\} \|p_\tau^\infty\|_{L^2(\mathbb{R})} \|K_{u-\tau}\|_{L^2(\mathbb{R})} \leq \frac{C}{\tau^{1/4}} \mathbb{1}\{\tau < u\} \|K_{u-\tau}\|_{L^2(\mathbb{R})}.$$

In view of (H),

$$\frac{1}{\tau^{1/4}} \mathbb{1}\{\tau < u\} \|K_{u-\tau}\|_{L^2(\mathbb{R})} \in L^1([s, t] \times [0, t]),$$

so, by Lebesgue's dominated convergence theorem, we get

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_s^t \int_0^t \mathbb{1}\{\tau < u\} \frac{\partial f}{\partial x}(x_n(u)) \int K_{u-\tau}(x_n(u) - y) p_\tau^\infty(y) dy d\tau du \\ &= \int_s^t \int_0^t \mathbb{1}\{\tau < u\} \frac{\partial f}{\partial x}(x(u)) \int K_{u-\tau}(x(u) - y) p_\tau^\infty(y) dy d\tau du, \end{aligned}$$

which ends the proof. □

### Proof of Theorem 5.1

Proposition 5.4 implies the existence of a weak solution  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), X, W)$  to (3) up to time  $T_0$ . As the time marginals  $p_t^\infty$  satisfy  $\|p_t^\infty\|_{L^2(\mathbb{R})} \leq \frac{C}{t^{1/4}}$ ,  $t \in (0, T_0]$  and (H) holds,

$$|B(t, x; p^\infty)| \leq C(T_0).$$

Therefore the drift coefficient of  $X$  is bounded, so that Corollary 3.2 implies that for any  $t \in (0, T_0]$

$$\|p_t^\infty\|_{L^\infty(\mathbb{R})} \leq \frac{C(T_0, b_0)}{\sqrt{t}}. \quad (27)$$

Now, Proposition 4.1 characterizes these marginals as unique solutions to the Equation (14). Let  $\tilde{b}(t, x) := b(t, x) + \int_0^t (K_{t-s} * p_s^\infty)(x) ds$  and consider the SDE

$$d\tilde{X}_t = \tilde{b}(t, \tilde{X}_t) dt + dW_t, \quad t \leq T_0. \quad (28)$$

Any weak solution has probability distribution  $\tilde{\mathbb{P}}$ . Now, assume that there exists another weak solution  $\mathbb{P}$  to (3) up to  $T_0$ , such that  $\mathbb{P}_t(dy) = p_t(y)dy$  and  $p_t$  satisfies (5) for any  $t \leq T_0$ . Then, by Proposition 4.1,  $p_t(y)dy = p_t^\infty(y)dy$ . Therefore,  $\mathbb{P}$  is another weak solution to (28). We conclude that  $\mathbb{P} = \mathbb{P}^\infty$ . The Theorem 5.1 is proved.

## 6 Proof of Theorem 2.3: A global existence and uniqueness result for Equation (3)

We now construct a solution for an arbitrary time horizon  $T > 0$ . We will do it by restarting the equation after the already fixed  $T_0$ . We start with  $T = 2T_0$ . Then, we will see how to generalize this procedure for an arbitrary  $T > 0$ . Throughout this section, we denote by  $\mathbb{P}^1$  the measure on  $C([0, T_0]; \mathbb{R})$  constructed to be the law of the local unique weak solution to (3) in the previous section.

### 6.1 Solution on $[0, 2T_0]$

**Proposition 6.1.** *Let  $T_0 > 0$  be such that  $D(T_0) < 1$ . Assume  $p_0 \in L^1(\mathbb{R})$  and let  $b \in L^\infty([0, 2T_0] \times \mathbb{R})$  be continuous w.r.t. the space variable. Under the hypothesis (H), Equation (3) admits a unique weak solution up to  $2T_0$  in the class of probability measures on  $C([0, 2T_0]; \mathbb{R})$  whose one dimensional marginals are densities that satisfy (6).*

We start with analyzing the dynamics of (3) after  $T_0$  and informally explaining the construction of a solution between  $T_0$  and  $2T_0$ . Assume, for a while, that Proposition 6.1 holds true. Denote the density of  $X_t$  by  $p_t^1$ , for  $t \leq T_0$  and by  $p_t^2$ , for  $t \in (T_0, 2T_0]$ . Let  $t \geq 0$ . In view of Equation (3), we would have

$$X_{T_0+t} = X_{T_0} + \int_{T_0}^{T_0+t} b(s, X_s) ds + \int_{T_0}^{T_0+t} \int_0^s (K_{s-\theta} * p_\theta)(X_s) d\theta ds + W_{T_0+t} - W_{T_0}.$$

Observe that

$$\begin{aligned} \int_{T_0}^{T_0+t} \int_0^s (K_{s-\theta} * p_\theta)(X_s) d\theta ds &= \int_{T_0}^{T_0+t} \int_0^{T_0} (K_{s-\theta} * p_\theta^1)(X_s) d\theta ds + \int_{T_0}^{T_0+t} \int_{T_0}^s (K_{s-\theta} * p_\theta^2)(X_s) ds d\theta \\ &=: B_1 + B_2. \end{aligned}$$

In addition,

$$B_1 = \int_0^t \int_0^{T_0} (K_{T_0+s'-\theta} * p_\theta^1)(X_{T_0+s'}) d\theta ds',$$

and

$$B_2 = \int_0^t \int_{T_0}^{T_0+s'} (K_{T_0+s'-\theta} * p_\theta^2)(X_{T_0+s'}) d\theta ds' = \int_0^t \int_0^{s'} (K_{s'-\theta'} * p_{T_0+\theta'}^2)(X_{T_0+s'}) d\theta' ds'.$$

Now set  $Y_t := X_{T_0+t}$  and  $\tilde{p}_t(y) := p_{T_0+t}^2(y)$ . Consider the new Brownian motion  $B_t := W_{T_0+t} - W_{T_0}$ . It comes:

$$Y_t = Y_0 + \int_0^t b(s + T_0, Y_s) ds + \int_0^t \int_0^{T_0} (K_{T_0+s'-\theta} * p_\theta^1)(Y_s) d\theta ds + \int_0^t \int_0^{s'} (K_{s'-\theta'} * \tilde{p}_{\theta'})(Y_s) d\theta' ds + B_t,$$

for  $t \in [0, T_0]$ . Setting

$$b_1(t, x, T_0) := \int_0^{T_0} (K_{T_0+t-s} * p_s^1)(x) ds \quad \text{and} \quad \tilde{b}(t, x) := b(T_0 + t, x),$$

we have

$$\begin{cases} dY_t = \tilde{b}(t, Y_t) dt + b_1(t, Y_t, T_0) dt + \left\{ \int_0^t (K_{t-s} * \tilde{p}_s)(Y_t) ds \right\} dt + dB_t, & t \leq T_0 \\ Y_0 \sim p_{T_0}^1(y) dy, & Y_s \sim \tilde{p}_s(y) dy. \end{cases} \quad (29)$$

To prove Proposition 6.1 we therefore construct a weak solution to (29) on  $[0, T_0]$  and paste its probability distribution with the measure  $\mathbb{P}^1$ . We then prove that the so defined measure is a solution to (3) on  $[0, 2T_0]$ . Notice that the process in (29) is of the same type as (3). Therefore,

**Lemma 6.2.** *Same assumptions as in Proposition 6.1. Let  $\mathbb{P}^1$  the measure on  $C([0, T_0]; \mathbb{R})$  constructed to be the law of the unique weak solution to (3) up to  $T_0$ . Denote by  $p_t^1$  its time marginals and define  $b_1(t, x, T_0) := \int_0^{T_0} (K_{T_0+t-s} \star p_s^1)(x) ds$  and  $\tilde{b}(t, x) := b(T_0 + t, x)$ . Then, Equation (29) admits a unique weak solution up to  $T_0$  in the class of probability measures on  $C([0, T_0]; \mathbb{R})$  whose one dimensional marginals are densities that satisfy (6).*

*Proof.* Let us check that we may apply Theorem 5.1 to (29). Firstly, the initial law  $p_{T_0}^1$  of  $Y$  satisfies by construction the assumption of Theorem 5.1. Then, the role of the additional drift  $b$  is here played by the sum of the two linear drifts,  $\tilde{b}$  and  $b_1$ . Let us show it is bounded and continuous w.r.t. space variable. By hypothesis,  $\tilde{b}$  is bounded in  $[0, T_0] \times \mathbb{R}$  and continuous in the space variable. Using the bounds on  $p^1$  and (H), we conclude that  $b_1$  is bounded uniformly in  $t$  and  $x$

$$|b_1(t, x, T_0)| \leq C(T_0, b_0) \int_0^{T_0} \frac{\|K_{T_0+t-s}\|_{L^1(\mathbb{R})}}{\sqrt{s}} ds \leq C(2T_0, b_0).$$

The continuity in the space variable comes from the fact that, by (H),  $K_t$  is bounded and continuous function on  $\mathbb{R}$  if  $t > 0$ . Then we apply the dominated convergence to show that for  $x_n \rightarrow x$ , we have that  $b_1(t, x_n, T_0) \rightarrow b_1(t, x, T_0)$  for all  $t \in [0, T_0]$ .

Then, by Theorem 5.1, there exists a unique weak solution to (29) up to  $T_0$  in the class of probability measures on  $C[0, T_0]$  whose one dimensional marginals are densities that satisfy (6).  $\square$

Notice that the law of  $Y$  is the solution to the martingale problem  $(MP(p_{T_0}^1, T_0, \tilde{b} + b_1))$ .

**A new measure on  $C([0, 2T_0]; \mathbb{R})$**  Denote by  $\mathbb{P}^1$  the probability measure on  $\Omega_0 := C([0, T_0]; \mathbb{R})$  constructed in Section 5 and by  $\mathbb{P}^2$  the probability measure on  $\Omega_0$  constructed above. Then, denote their one dimensional densities by  $p_t^1$  and  $p_t^2$ , respectively. The Borel  $\sigma$ -field on  $\Omega_0$  is denoted by  $\mathcal{B}^0$ .  $\mathbb{P}^1$  and  $\mathbb{P}^2$ , respectively, solve the non linear martingale problems  $(MP(p_0, T_0, b))$  and  $(MP(p_{T_0}^1, T_0, \tilde{b} + b_1))$ .

Remember that  $\mathbb{P}_0^2 = \mathbb{P}_{T_0}^1$ , i.e.  $p_0^2(z)dz = p_{T_0}^1(z)dz$ . Define the mapping  $X^0$  from  $\Omega_0$  to  $\mathbb{R}$  as  $X^0(w) := w_0$ . By the means of Theorem 3.19 from Chapter 5 in [9] about regular conditional probabilities, we conclude that for each  $y \in \mathbb{R}$ , there exists a probability measure  $\mathbb{P}_y^2$  on  $(\Omega_0, \mathcal{B}^0)$  such that, for any  $A \in \mathcal{B}^0$ ,

$$\mathbb{P}_y^2(A) = P^2(A|X^0 = y).$$

In particular,

$$\mathbb{P}_y^2(w \in \Omega_0, w_0 = y) = 1.$$

For  $w^1, w^2 \in \Omega_0$  we define the concatenation  $w = w^1 \otimes_{T_0} w^2 \in \Omega$  of these two paths as

$$\begin{cases} w_\theta = w_\theta^1, & 0 \leq \theta \leq T_0, \\ w_{\theta+T_0} = w_{T_0}^1 + w_\theta^2 - w_0^2, & 0 \leq \theta \leq t - T_0. \end{cases}$$

On the other hand, for a given path  $w \in \Omega$ , the two paths  $w^1, w^2 \in \Omega_0$  such that  $w = w^1 \otimes_{T_0} w^2$  are

$$\begin{cases} w_\theta^1 = w_\theta, & 0 \leq \theta \leq T_0, \\ w_\theta^2 = w_{T_0+\theta}, & 0 \leq \theta \leq T_0. \end{cases}$$

We define the probability distribution  $\mathbb{Q}$  on  $C([0, 2T_0]; \mathbb{R})$  equipped with its Borel  $\sigma$ -field as follows. For any continuous and bounded functional  $\varphi$  on  $\Omega$ ,

$$\mathbb{E}_{\mathbb{Q}}[\varphi] = \int_{\Omega} \varphi(w) \mathbb{Q}(dw) := \int_{\Omega_0} \int_{\mathbb{R}} \int_{\Omega_0} \varphi(w^1 \otimes_{T_0} w^2) \mathbb{P}_y^2(dw^2) p_{T_0}^1(y) dy \mathbb{P}^1(dw^1). \quad (30)$$

Notice that if  $\varphi$  acts only on the part of the path up to  $t \leq T_0$  of any  $x \in \Omega$ , then

$$\mathbb{E}_{\mathbb{Q}}[\varphi((w_\theta)_{\theta \leq t})] = \int_{\Omega_0} \varphi((w_\theta)_{\theta \leq t}) \mathbb{P}^1(dx) = \mathbb{E}_{\mathbb{P}^1}[\varphi((w_\theta)_{\theta \leq t})]. \quad (31)$$



In addition, in view of (31) for any  $T_0 \leq t \leq 2T_0$  it holds that

$$\mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}[\varphi((w_{\theta})_{\theta \leq t})|\mathcal{B}_{T_0}]] = \mathbb{E}_{\mathbb{Q}}[\Phi((w_{\theta}^1)_{\theta \leq T_0})] = \int_{\Omega_0} \Phi(w^1) d\mathbb{P}^1(dw^1).$$

On the other hand, by (30),

$$\mathbb{E}_{\mathbb{Q}}[\varphi((w_{\theta})_{\theta \leq t})] = \int_{\Omega_0} \int_{\mathbb{R}} \int_{\Omega_0} \varphi(w^1 \otimes_{T_0} w^2) \mathbb{P}_y^2(dw^2) p_{T_0}^2(y) dy \mathbb{P}^1(dw^1).$$

By [9, Section 5, Thm 5.3.18] there exists a unique regular conditional probability for  $\mathbb{Q}$  given  $\mathcal{B}_{T_0}$ . Therefore, for any  $w^1 \in \Omega_0$ :

$$\mathbb{E}_{\mathbb{Q}}[\varphi((w_{\theta})_{\theta \leq t})|\mathcal{B}_{T_0}] = \int_{\mathbb{R}} \int_{\Omega_0} \varphi(w^1 \otimes_{T_0} w^2) \mathbb{P}_y^2(dw^2) p_{T_0}^2(y) dy. \quad (32)$$

**Proof of Proposition 6.1** Let us prove that the probability measure  $\mathbb{Q}$  solves the non linear martingale problem  $(MP(p_0, 2T_0, b))$  on the canonical space  $C([0, 2T_0]; \mathbb{R})$ .

- i) By (31), it is obvious that  $\mathbb{Q}_0 = \mathbb{P}_0^1$ . By construction,  $P_0^1 = p_0$ .
- ii) Next, let us characterize the one dimensional time marginals of  $\mathbb{Q}$ . For  $f \in C_b(\mathbb{R})$  and  $t \in [0, 2T_0]$ , consider the functional  $\varphi(x) = f(x_t)$ , for any  $x \in C([0, 2T_0]; \mathbb{R})$ . For  $t \leq T_0$ , by (31),

$$\mathbb{E}_{\mathbb{Q}}[\varphi(x)] = \int_{\Omega_0} f(x_t) \mathbb{P}^1(dx) = \int_{\mathbb{R}} f(z) p_t^1(z) dz.$$

Therefore,  $Q_t(dz) = p_t^1(z) dz$ .

For  $T_0 \leq t \leq 2T_0$ , by (30),

$$\mathbb{E}_{\mathbb{Q}}[\varphi(x)] = \int_{\Omega_0} \int_{\mathbb{R}} \int_{\Omega_0} \varphi(w_{t-T_0}^2) \mathbb{P}_y^2(dw^2) p_{T_0}^2(y) dy \mathbb{P}^1(dw^1) = \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(z) \mathbb{P}_{y, t-T_0}^2(dz) p_{T_0}^1(y) dy.$$

By Fubini's theorem:

$$\mathbb{E}_{\mathbb{Q}}[\varphi(x)] = \int_{\mathbb{R}} f(z) \int_{\mathbb{R}} \mathbb{P}_{y, t-T_0}^2(dz) p_{T_0}^1(y) dy.$$

Since  $\mathbb{P}_0^2 = p_{T_0}^1$  we deduce

$$\mathbb{E}_{\mathbb{Q}}[\varphi(x)] = \int_{\mathbb{R}} f(z) p_{t-T_0}^2(z) dz,$$

which shows that  $Q_t(dz) = p_{t-T_0}^2(z) dz$ .

Therefore, the one dimensional marginals of  $\mathbb{Q}$  have densities  $q_t$  which, by construction, belong to  $\mathbf{L}^2(\mathbb{R})$  and satisfy  $\|q_t\|_{L^2(\mathbb{R})} \leq \frac{C}{t^{1/4}}$ .

- iii) It remains to show that  $M_t$  defined as

$$M_t := f(x(t)) - f(x(0)) - \int_0^t \left[ \frac{1}{2} \frac{\partial^2 f}{(\partial x)^2}(x(u)) + \frac{\partial f}{\partial x}(x(u))(b(u, x(u))) + \int_0^u \int_{\mathbb{R}} K_{u-\tau}(x(u)-y) q_{\tau}(y) dy d\tau \right] du$$

is a  $\mathbb{Q}$ -martingale, i.e.  $\mathbb{E}_{\mathbb{Q}}(M_t|\mathcal{B}_s) = M_s$ .

- (a) Let  $s \leq t \leq T_0$ :

For any  $n \in \mathbb{N}$ , any continuous bounded functional  $\phi$  on  $\mathbb{R}^n$ , and any  $t_1 \leq \dots \leq t_n < s \leq t \leq T_0$ , by (31):

$$\mathbb{E}_{\mathbb{Q}}(\phi(x_{t_1}, \dots, x_{t_n})(M_t - M_s)) = \mathbb{E}_{\mathbb{P}^1}(\phi(x_{t_1}, \dots, x_{t_n})(M_t - M_s)).$$

As  $\mathbb{P}^1$  solves the  $(MP(p_0, T_0, b))$  up to  $T_0$ ,

$$\mathbb{E}_{\mathbb{Q}}(\phi(x_{t_1}, \dots, x_{t_n})(M_t - M_s)) = 0.$$

(b) For  $s \leq T_0 \leq t \leq 2T_0$ ,

$$\mathbb{E}_{\mathbb{Q}}(M_t|\mathcal{B}_s) = \mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}(M_t|\mathcal{B}_{T_0})|\mathcal{B}_s].$$

Let us prove that  $\mathbb{E}_{\mathbb{Q}}(M_t|\mathcal{B}_{T_0}) = M_{T_0}$ . Notice that

$$\begin{aligned} M_t - M_{T_0} &= f(x_t) - f(x_{T_0}) - \int_{T_0}^t \frac{1}{2} \frac{\partial^2 f}{(\partial x)^2}(x(u)) du - \int_{T_0}^t \frac{\partial f}{\partial x}(x(u)) b(u, x(u)) du \\ &\quad - \int_{T_0}^t \frac{\partial f}{\partial x}(x(u)) \int_0^u \int K_{u-\tau}(x(u) - y) q_{\tau}(y) dy d\tau du. \end{aligned}$$

Write the last integral as

$$\begin{aligned} &\int_{T_0}^t \frac{\partial f}{\partial x}(x(u)) \int_0^u \int K_{u-\tau}(x(u) - y) q_{\tau}(y) dy d\tau du = \int_{T_0}^t \frac{\partial f}{\partial x}(x(u)) \int_0^{T_0} \int K_{u-\tau}(x(u) - y) p_{\tau}^1(y) dy d\tau du \\ &+ \int_{T_0}^t \frac{\partial f}{\partial x}(x(u)) \int_{T_0}^u \int K_{u-\tau}(x(u) - y) p_{\tau-T_0}^2(y) dy d\tau du \\ &=: I_1 + I_2. \end{aligned}$$

Now,

$$I_1 = \int_0^{t-T_0} \frac{\partial f}{\partial x}(x(u+T_0)) \int_0^{T_0} \int K_{u+T_0-\tau}(x(u+T_0) - y) p_{\tau}^1(y) dy d\tau du.$$

For  $x \in \Omega$  identify  $x^1, x^2 \in \Omega_0$  such that  $x = x^1 \otimes_{T_0} x^2$ . Then,

$$I_1 = \int_0^{t-T_0} \frac{\partial f}{\partial x}(x^2(u)) \int_0^{T_0} (K_{u+T_0-\tau} \star p_{\tau}^1)(x^2(u)) d\tau du = \int_0^{t-T_0} \frac{\partial f}{\partial x}(x^2(u)) b_1(u, x^2(u), T_0) du.$$

Proceeding as above,

$$\begin{aligned} I_2 &= \int_0^{t-T_0} \frac{\partial f}{\partial x}(x(u+T_0)) \int_0^u \int K_{u-\tau}(x(u+T_0) - y) p_{\tau}^2(y) dy d\tau du \\ &= \int_0^{t-T_0} \frac{\partial f}{\partial x}(x^2(u)) \int_0^u \int K_{u-\tau}(x^2(u) - y) p_{\tau}^2(y) dy d\tau du. \end{aligned}$$

Similarly

$$\begin{aligned} &\int_{T_0}^t \frac{\partial f}{\partial x}(x(u)) b(u, x(u)) du = \int_0^{t-T_0} \frac{\partial f}{\partial x}(x(u+T_0)) b(u+T_0, x(u+T_0)) du \\ &= \int_0^{t-T_0} \frac{\partial f}{\partial x}(x^2(u)) b(u+T_0, x^2(u)) du = \int_0^{t-T_0} \frac{\partial f}{\partial x}(x^2(u)) \tilde{b}(u, x^2(u)) du. \end{aligned}$$

It comes:

$$\begin{aligned} M_t - M_{T_0} &= f(x^2(t-T_0)) - f(x^2(0)) - \int_0^{t-T_0} \frac{1}{2} \frac{\partial^2 f}{(\partial x)^2}(x^2(u)) du - \int_0^{t-T_0} \frac{\partial f}{\partial x}(x^2(u)) \tilde{b}(u, x^2(u)) du \\ &\quad - \int_0^{t-T_0} \frac{\partial f}{\partial x}(x^2(u)) [b_1(u, x^2(u), T_0) + \int_0^u \int K_{u-\tau}(x^2(u) - y) p_{\tau}^2(y) dy d\tau] du. \end{aligned}$$

By definition of the measure  $\mathbb{Q}$ ,

$$\mathbb{E}^{\mathbb{Q}}(\phi(x_{t_1}, \dots, x_{t_n})(M_t - M_{T_0})) = \int_{\Omega_0} \phi(x_{t_1}^1, \dots, x_{t_n}^1) \int_{\mathbb{R}} \int_{\Omega_0} \varphi(x^2) \mathbb{P}_y^2(dx^2) p_{T_0}^1(y) dy \mathbb{P}^1(dx^1). \quad (33)$$

On the other hand, since  $\mathbb{P}^2$  solves  $(MP(p_{T_0}^1, T_0, \tilde{b} + b_1))$  one has

$$\mathbb{E}_{\mathbb{P}^2}(\varphi) = \int_{\Omega_0} \varphi(x_2) \mathbb{P}^2(dx^2) = 0.$$

By the definition of  $\mathbb{P}_y^2$ :

$$0 = \mathbb{E}_{\mathbb{P}^2}(\varphi) = \int_{\mathbb{R}} \mathbb{E}_{\mathbb{P}_y^2}(\varphi) p_0^2(y) dy = \int_{\mathbb{R}} \int_{\Omega_0} \varphi(x_2) \mathbb{P}_y^2(dx^2) p_{T_0}^1(y) dy.$$

Coming back to (33):

$$\mathbb{E}^{\mathbb{Q}}(\phi(x_{t_1}, \dots, x_{t_n})(M_t - M_{T_0})) = 0.$$

Finally, we conclude that  $\mathbb{E}_{\mathbb{Q}}(M_t | \mathcal{B}_{T_0}) = M_{T_0}$  and therefore,  $\mathbb{E}_{\mathbb{Q}}(M_t | \mathcal{B}_s) = M_s$ .

(c) For  $T_0 \leq s \leq t \leq 2T_0$  : we may rewrite the difference  $M_t - M_s$  in the same manner:

$$\begin{aligned} M_t - M_s &= f(x^2(t - T_0)) - f(x^2(s - T_0)) - \int_{s-T_0}^{t-T_0} \frac{1}{2} \frac{\partial^2 f}{(\partial x)^2}(x^2(u)) du \\ &\quad - \int_{s-T_0}^{t-T_0} \frac{\partial f}{\partial x}(x^2(u)) [b(u, x^2(u)) + b_1(u, x^2(u), T_0) + \int_0^u \int K_{u-\tau}(x(u+T_0) - y) p_{\tau}^2(y) dy d\tau] du \\ &=: \varphi(x^2). \end{aligned}$$

Now, take  $t_1 \leq \dots \leq t_n < s$ . Let us suppose that the first  $m$  are before  $T_0$  and others after. We have that

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}(\phi(x_{t_1}, \dots, x_{t_n})(M_t - M_s)) &= \\ &= \int_{\Omega_0} \int_{\mathbb{R}} \int_{\Omega_0} \phi(x_{t_1}^1, \dots, x_{t_m}^1, x_{t_{m+1}-T_0}^2, \dots, x_{t_n-T_0}^2) \varphi(x^2) d\mathbb{P}_y^2(dx^2) p_{T_0}^1(y) dy \mathbb{P}^1(dx^1). \end{aligned}$$

Since  $\mathbb{P}^2$  solves the  $(MP(p_{T_0}^1, T_0, \tilde{b} + b_1))$ , one has that  $\mathbb{E}_{\mathbb{P}^2}(\phi'(x_{t'_1}^2, \dots, x_{t'_n}^2) \varphi) = 0$  for any continuous bounded functional  $\phi'$  on  $\mathbb{R}^n$ , any  $n \in \mathbb{N}$  and any  $t'_1 \leq \dots \leq t'_n < s - T_0$ . Taking  $\phi'(x_{t'_1}^2, \dots, x_{t'_n}^2) = \phi(x_{t_1}^1, \dots, x_{t_m}^1, x_{t_{m+1}-T_0}^2, \dots, x_{t_n-T_0}^2)$  for a fixed  $x^1$ , we conclude that

$$\int_{\mathbb{R}} \int_{\Omega_0} \phi(x_{t_1}^1, \dots, x_{t_m}^1, x_{t_{m+1}-T_0}^2, \dots, x_{t_n-T_0}^2) \varphi(x_2) \mathbb{P}_y^2(dx^2) p_{T_0}^1(y) dy = 0.$$

Therefore,

$$\mathbb{E}^{\mathbb{Q}}(\phi(x_{t_1}, \dots, x_{t_n})(M_t - M_s)) = 0.$$

Thus,  $\mathbb{E}^{\mathbb{Q}}(M_t | \mathcal{B}_s) = M_s$ .

Now, we conclude the existence of a weak solution to (3) up to  $2T_0$ . As the one dimensional marginals of this solution are densities that satisfy (5), in view of Corollary 13, Estimate 6 is satisfied as soon as one concludes that the drift of  $X$  is bounded.

As for weak uniqueness, Section 4 holds true for any  $T > 0$ , so the discussion in Theorem 5.1 is to be repeated here. This concludes the proof of Proposition 6.1.

## 6.2 End of the proof of Theorem 2.3: construction of the global solution

Given any finite time horizon  $T > 0$ , split the interval  $[0, T]$  into  $n = \lceil \frac{T}{T_0} \rceil + 1$  intervals of length  $T_0$  and repeat  $n$  times the procedure applied for the case  $T = 2T_0$ . By construction, the time marginals of the thus constructed solution has probability densities which satisfy (5).

**Remark 6.3.** Repeating all the previous arguments we can, as well, construct a solution to (3) with the initial condition  $p_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ . In that case, we use Inequality (12) in the iterative procedure. The solution is unique in the space of measures whose one dimensional marginals are densities that belong to  $L^\infty((0, T); L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}))$ .

If  $p_0 \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$  we use Inequality (13) in the iterative procedure. The solution is unique in the space of probability measures whose time marginals have densities  $(p_t)_{t \geq 0}$  which satisfy Inequality (5).

## 7 Application to the Keller–Segel model

In this section we prove Corollary 2.6. We start with checking that  $K^\sharp$  satisfies Hypothesis (H).

1. Notice that

$$\|K_t^\sharp\|_{L^1(\mathbb{R})} \leq \frac{C_T \chi}{\sqrt{t}} \int |z| e^{-\frac{z^2}{2}} dz$$

and

$$\|K_t^\sharp\|_{L^2(\mathbb{R})} \leq \frac{C_T}{t^{3/4}}.$$

The condition (H.1) is thus satisfied.

2. From the definition of  $K^\sharp$  it is clear that for  $t > 0$ ,  $K_t^\sharp$  is a bounded and continuous function on  $\mathbb{R}$ .

3. The condition (H.3) is obviously satisfied.

4. As already noticed,

$$\|K_{t-s}\|_{L^1(\mathbb{R})} = \frac{C}{\sqrt{t-s}},$$

from which, by definition of the  $\beta$ -function,

$$f_1(t) = \int_0^t \frac{\|K_{t-s}\|_{L^1(\mathbb{R})}}{\sqrt{s}} ds = C \int_0^t \frac{1}{\sqrt{s}\sqrt{t-s}} ds = C\beta(1/2, 1/2),$$

where  $C$  is a universal constant. With similar arguments one easily checks that the function  $f_2$  is bounded on any compact time interval.

5. Let  $\phi$  be a probability density on  $\mathbb{R}$ . For  $(t, x) \in (0, T] \times \mathbb{R}$ , one has

$$\begin{aligned} \int \phi(y) \|K^\sharp(x-y)\|_{L^1(0,t)} dy &\leq C \int \phi(y) |x-y| \int_0^t \frac{1}{s^{3/2}} e^{-\frac{|x-y|^2}{2s}} ds dy \\ &= \int \phi(y) |x-y| \int_{\frac{|x-y|}{\sqrt{t}}}^\infty \frac{z^3}{|x-y|^3} e^{-\frac{z^2}{2}} \frac{|x-y|^2}{z^3} dz dy \\ &= \int \phi(y) \int_{\frac{|x-y|}{\sqrt{t}}}^\infty e^{-\frac{z^2}{2}} dz dy. \end{aligned}$$

This shows that (H.5) is satisfied.

Now, since  $c_0 \in C_b^1(\mathbb{R})$ , we have

$$|\chi e^{-\lambda t} \mathbb{E} c'_0(x + W_t)| \leq \chi \|c'_0\|_{L^\infty(\mathbb{R})}.$$

In addition, it is continuous w.r.t. the space variable. Thus, in view of Theorem 2.3, Equation (2) with  $d = 1$  is well-posed. Denote by  $\rho_t \equiv p_t$  the time marginals of the constructed probability distribution. Set

$$c_t := e^{-\lambda t} g(t, \cdot) * c_0 + \int_0^t e^{-\lambda s} \rho_{t-s} * g(s, \cdot) ds.$$

For any  $t \in (0, T]$ , by Inequality (5),  $c_t$  is well defined, continuous, and

$$\|c_t\|_{L^\infty(\mathbb{R})} \leq \|c_0\|_{L^\infty(\mathbb{R})} + C_T.$$

In addition,

$$\frac{\partial}{\partial x} c(t, x) = \frac{\partial}{\partial x} (e^{-\lambda t} \mathbb{E}(c_0(x + W_t))) + \frac{\partial}{\partial x} \left( \mathbb{E} \int_0^t e^{-\lambda s} p_{t-s}(x + W_s) ds \right).$$

Then observe that

$$\begin{aligned} \frac{\partial}{\partial x} \mathbb{E} \int_0^t e^{-\lambda s} \rho_{t-s}(x + W_s) ds &= \frac{\partial}{\partial x} \int_0^t e^{-\lambda s} \int \rho_{t-s}(x+y) \frac{1}{\sqrt{2\pi s}} e^{-\frac{y^2}{4s}} dy ds \\ &= \frac{\partial}{\partial x} \int_0^t e^{-\lambda(t-s)} \int \rho_s(y) \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(y-x)^2}{4(t-s)}} dy ds =: \frac{\partial}{\partial x} \int_0^t f(s, x) ds. \end{aligned}$$

As

$$|\frac{\partial}{\partial x}(\rho_s(y) \frac{1}{\sqrt{2\pi(t-s)}} e^{\frac{-(y-x)^2}{2(t-s)}})| \leq \rho_s(y) \frac{|y-x|}{2\sqrt{2\pi}(t-s)^{3/2}} e^{\frac{-(y-x)^2}{2(t-s)}} \leq \frac{C_T \rho_s(y)}{t-s},$$

we have

$$\frac{\partial}{\partial x} f(s, x) = e^{-\lambda(t-s)} \int \rho_s(y) \frac{y-x}{2\sqrt{2\pi}(t-s)^{3/2}} e^{\frac{-(y-x)^2}{2(t-s)}} dy.$$

Now, we repeat the same argument for  $\frac{\partial}{\partial x} \int_0^t f(s, x) ds$ . In order to justify the differentiation under the integral sign we notice that

$$|\frac{\partial}{\partial x} f(s, x)| \leq \frac{C_T}{\sqrt{(t-s)s}}.$$

Gathering the preceding calculations we have obtained

$$\frac{\partial}{\partial x} c(t, x) = e^{-\lambda t} \mathbb{E}(c'_0(x + W_t) + \int_0^t e^{-\lambda(t-s)} \int \rho_s(y) \frac{y-x}{\sqrt{2\pi}(t-s)^{3/2}} e^{\frac{-(y-x)^2}{2(t-s)}} dy ds).$$

Using the assumption on  $c_0$  and Inequality (5), for any  $t \in (0, T]$  one has

$$\|\frac{\partial}{\partial x} c(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \|c'_0\|_{L^\infty(\mathbb{R})} + C_T.$$

In addition, by Lebesgue's Dominated Convergence Theorem,  $\frac{\partial}{\partial x} c(t, x)$  is continuous on  $\mathbb{R}$ . Thus  $c_t \in C_b^1(\mathbb{R})$  for all  $t \in [0, T]$ .

As  $\chi \frac{\partial}{\partial x} c(t, x)$  is the drift coefficient in the SDE (3), in view of Proposition 4.1 the function  $\rho_t$  satisfies

$$\rho_t = g_t * \rho_0 + \chi \int_0^t \frac{\partial}{\partial y} g_{t-s} * (\frac{\partial}{\partial x} c(s, \cdot) \rho_s) ds$$

in the sense of the distributions. Therefore, we the pair  $(p, c)$  is a solution to the Keller Segel system (7) in the sense of Definition 2.4. We now check the uniqueness of this solution.

Assume there exists another pair  $(\rho^1, c^1)$  satisfying Definition (2.4) with the same initial condition as  $(\rho, c)$ . Let us prove  $(\rho^1, c^1) = (\rho, c)$  a.e.

Using Definition 2.4,

$$\begin{aligned} \|\rho_t^1 - \rho_t\|_{L^1(\mathbb{R})} &\leq \int_0^t \|\frac{\partial}{\partial y} g_{t-s} * (\frac{\partial}{\partial x} c^1(s, \cdot) \rho_s^1 - \frac{\partial}{\partial x} c(s, \cdot) \rho_s)\|_{L^1(\mathbb{R})} ds \\ &\leq \int_0^t \|\frac{\partial}{\partial y} g_{t-s} * (\frac{\partial}{\partial x} c^1(s, \cdot) (\rho_s^1 - \rho_s))\|_{L^1(\mathbb{R})} ds + \int_0^t \|\frac{\partial}{\partial y} g_{t-s} * (\rho_s (\frac{\partial}{\partial x} c^1(s, \cdot) - \frac{\partial}{\partial x} c(s, \cdot)))\|_{L^1(\mathbb{R})} ds \\ &=: I + II. \end{aligned}$$

Using standard convolution inequalities and  $\|\frac{\partial}{\partial y} g_{t-s}\|_{L^1(\mathbb{R})} \leq \frac{C}{\sqrt{t-s}}$  we deduce:

$$I \leq C \int_0^t \frac{\|\rho_s^1 - \rho_s\|_{L^1(\mathbb{R})}}{\sqrt{t-s}} ds \quad \text{and} \quad II \leq C \int_0^t \frac{\|\frac{\partial}{\partial x} c^1(s, \cdot) - \frac{\partial}{\partial x} c(s, \cdot)\|_{L^1(\mathbb{R})}}{\sqrt{t-s}\sqrt{s}} ds.$$

Therefore

$$\|\frac{\partial}{\partial x} c^1(s, \cdot) - \frac{\partial}{\partial x} c(s, \cdot)\|_{L^1(\mathbb{R})} \leq \int_0^s \|(\rho_u^1 - \rho_u) * \frac{\partial}{\partial x} g_{s-u}\|_{L^1(\mathbb{R})} du \leq C \int_0^s \frac{\|\rho_u^1 - \rho_u\|_{L^1(\mathbb{R})}}{\sqrt{s-u}} du,$$

from which

$$\begin{aligned} II &\leq C \int_0^t \frac{1}{\sqrt{s}\sqrt{t-s}} \int_0^s \frac{\|\rho_u^1 - \rho_u\|_{L^1(\mathbb{R})}}{\sqrt{s-u}} du ds \\ &\leq C \int_0^t \|\rho_u^1 - \rho_u\|_{L^1(\mathbb{R})} \int_u^t \frac{1}{\sqrt{s}\sqrt{s-u}\sqrt{t-s}} ds du \leq C_T \int_0^t \frac{\|\rho_u^1 - \rho_u\|_{L^1(\mathbb{R})}}{\sqrt{u}} du. \end{aligned}$$

Gathering the preceding bounds for  $I$  and  $II$  we get

$$\|p_t^1 - p_t\|_{L^1(\mathbb{R})} \leq C_T \int_0^t \frac{\|p_s^1 - p_s\|_{L^1(\mathbb{R})}}{\sqrt{t-s}} ds + C_T \int_0^t \frac{\|p_s^1 - p_s\|_{L^1(\mathbb{R})}}{\sqrt{s}} ds.$$

In view of Lemma 4.2, for all  $t \leq T$

$$\|\rho_t^1 - \rho_t\|_{L^1(\mathbb{R})} = 0.$$

We deduce that  $c_t^1$  and  $c_t$  are equal almost everywhere for all  $t \leq T$ . This completes the proof of Corollary 2.6.

## 8 Appendix

We here propose a light simplification of the calculations in [13].

**Proposition 8.1.** *Let  $y \in \mathbb{R}$  and let  $\beta$  be a constant. Denote by  $p_y^\beta(t, x, z)$  the transition probability density (with respect to the Lebesgue measure) of the unique weak solution to*

$$X_t = x + \beta \int_0^t \operatorname{sgn}(y - X_s) ds + W_t.$$

Then

$$\begin{aligned} p_y^\beta(t, x, z) &= \frac{1}{\sqrt{2\pi t^{3/2}}} \int_0^\infty e^{\beta(|y-x|+\bar{y}-|z-y|)-\frac{\beta^2}{2}t} (\bar{y} + |z-y| + |y-x|) e^{-\frac{(\bar{y}+|z-y|+|y-x|)^2}{2t}} d\bar{y} \\ &\quad + \frac{1}{\sqrt{2\pi t}} e^{\beta(|y-x|-|z-y|)-\frac{\beta^2}{2}t} \left( e^{-\frac{(z-x)^2}{2t}} - e^{-\frac{(|z-y|+|y-x|)^2}{2t}} \right). \end{aligned} \quad (34)$$

In particular,

$$p_y^\beta(t, x, y) = \frac{1}{\sqrt{2\pi t}} \int_{\frac{|x-y|}{\sqrt{t}}}^\infty z e^{-\frac{(z-\frac{\beta\sqrt{t}}{2})^2}{2}} dz. \quad (35)$$

*Proof.* Let  $f$  be a bounded continuous function. The Girsanov transform leads to

$$\mathbb{E}(f(X_t)) = \mathbb{E}(f(x + W_t) e^{\beta \int_0^t \operatorname{sgn}(y-x-W_s) dW_s - \frac{\beta^2}{2}t}).$$

Let  $L_t^a$  be the Brownian local time. By Tanaka's formula ([9], p. 205):

$$|W_t - a| = |a| + \int_0^t \operatorname{sgn}(W_s - a) dW_s + L_t^a.$$

Therefore

$$\int_0^t \operatorname{sgn}(y-x-W_s) dW_s = |y-x| + L_t^a - |W_t - (y-x)|,$$

from which

$$\mathbb{E}(f(X_t)) = \mathbb{E}(f(x + W_t) e^{\beta(|y-x|+L_t^{y-x}-|W_t-(y-x)|)-\frac{\beta^2}{2}t}).$$

recall that  $(W_t, L_t^a)$  has the following joint distribution (see [1, p.200, Eq.(1.3.8)]):

$$\begin{cases} \bar{y} > 0 : & \mathbb{P}(W_t \in dz, L_t^a \in d\bar{y}) = \frac{1}{\sqrt{2\pi t^{3/2}}} (\bar{y} + |z-a| + |a|) e^{-\frac{(\bar{y}+|z-a|+|a|)^2}{2t}} d\bar{y} dz. \\ & \mathbb{P}(W_t \in dz, L_t^a = 0) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{z^2}{2t}} dz - \frac{1}{\sqrt{2\pi t}} e^{-\frac{(|z-a|+|a|)^2}{2t}} dz. \end{cases}$$

It comes:

$$\begin{aligned} \mathbb{E}(f(X_t)) &= \frac{1}{\sqrt{2\pi t^{3/2}}} \int_{\mathbb{R}} \int_0^\infty f(x+z) e^{\beta(|y-x|+\bar{y}-|z-(y-x)|)-\frac{\beta^2}{2}t} (\bar{y} + |z-(y-x)| + |y-x|) \\ &\quad e^{-\frac{(\bar{y}+|z-(y-x)|+|y-x|)^2}{2t}} d\bar{y} dz \\ &\quad + \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} f(x+z) e^{\beta(|y-x|-|z-(y-x)|)-\frac{\beta^2}{2}t} \left( e^{-\frac{z^2}{2t}} - e^{-\frac{(|z-(y-x)|+|y-x|)^2}{2t}} \right) dz. \end{aligned}$$

The change of variables  $x+z = z'$  leads to

$$\begin{aligned} \mathbb{E}(f(X_t)) &= \frac{1}{\sqrt{2\pi t^{3/2}}} \int_{\mathbb{R}} f(z') \int_0^\infty e^{\beta(|y-x|+\bar{y}-|z'-y|)-\frac{\beta^2}{2}t} (\bar{y} + |z'-y| + |y-x|) e^{-\frac{(\bar{y}+|z'-y|+|y-x|)^2}{2t}} d\bar{y} dz' \\ &\quad + \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} f(z') e^{\beta(|y-x|-|z'-y|)-\frac{\beta^2}{2}t} \left( e^{-\frac{(z'-x)^2}{2t}} - e^{-\frac{(|z'-y|+|y-x|)^2}{2t}} \right) dz', \end{aligned}$$

from which the desired result follows. □

**Corollary 8.2.** *Let  $0 < s < t \leq T$ . Then for any  $z, y \in \mathbb{R}$ , there exists  $C_{T,\beta,x,y}$  such that*

$$\mathbb{E}(|\frac{\partial}{\partial x} p_y^\beta(t-s, X_s^{(b)}, z)|) \leq C_{T,\beta,x,y} h(s, z),$$

where  $h$  belongs to  $L^1([0, t] \times \mathbb{R})$ .

*Proof.* Observe that

$$\begin{aligned} \frac{\partial}{\partial x} p_y^\beta(t-s, \bar{x}, z) &= \frac{\beta}{\sqrt{2\pi(t-s)}} e^{-2\beta|z-y|} e^{-\frac{(|z-y|+|y-\bar{x}|-\beta(t-s))^2}{2(t-s)}} \operatorname{sgn}(\bar{x}-y) \\ &\quad + \frac{\beta}{\sqrt{2\pi(t-s)}} e^{-\beta|z-y|-\frac{\beta^2}{2}(t-s)} e^{\beta|y-\bar{x}|-\frac{(z-\bar{x})^2}{2(t-s)}} \operatorname{sgn}(\bar{x}-y) \\ &\quad + \frac{z-\bar{x}}{2\pi(t-s)^{3/2}} e^{-\beta|z-y|-\frac{\beta^2}{2}(t-s)} e^{\beta|y-\bar{x}|-\frac{(z-\bar{x})^2}{2(t-s)}}. \end{aligned}$$

The sum of the absolute values of the first two terms in the right-hand side is bounded from above by

$$\frac{\beta}{\sqrt{2\pi(t-s)}} e^{-2\beta|z-y|+\beta|y-\bar{x}|}.$$

Now, by Girsanov's theorem, for some constant  $C$  depending on  $T$  and  $\beta$  only we have

$$\mathbb{E} \left| \frac{\partial}{\partial x} p_y^\beta(t-s, X_s^{(b)}, z) \right| \leq C \sqrt{\mathbb{E} \left| \frac{\partial}{\partial x} p_y^\beta(t-s, W_s^x, z) \right|^2}.$$

Let us treat the following term:

$$A := \frac{C_\beta}{(t-s)^{3/2}} \sqrt{\mathbb{E}(|z - W_s^x|^2 e^{2\beta|y-W_s^x|-\frac{(z-W_s^x)^2}{t-s}})}.$$

Notice that,

$$A \leq \frac{C_\beta}{(t-s)^{3/2}} (\mathbb{E}[|z - W_s^x|^4 e^{-2\frac{(z-W_s^x)^2}{t-s}}] \mathbb{E}[e^{4\beta|y-W_s^x|}])^{1/4} =: \frac{C_\beta}{(t-s)^{3/2}} (A_1 A_2)^{1/4}.$$

As there exists an  $\alpha > 0$ , such that  $|a|^4 e^{-a^2} \leq C e^{-\alpha a^2}$ , one has

$$A_1 \leq C(t-s)^2 \int e^{-\alpha \frac{(z-u)^2}{t-s}} g_s(u-x) du \leq \frac{(t-s)^{2+1/2}}{\sqrt{s+(t-s)/(2\alpha)}} e^{-\frac{(z-x)^2}{2(s+(t-s)/(2\alpha))}}.$$

On the other hand,

$$A_2 = \int e^{4\beta|y-u|} g_s(u-x) du = e^{-4\beta y} \int_y^\infty e^{4\beta u} \frac{1}{\sqrt{s}} e^{-\frac{(u-x)^2}{2s}} du + e^{4\beta y} \int_{-\infty}^y e^{-4\beta u} \frac{1}{\sqrt{s}} e^{-\frac{(u-x)^2}{2s}} du.$$

Completing the squares in the exponents,

$$A_2 = e^{4\beta(x-y)} e^{8\beta^2 s} \int_y^\infty \frac{1}{\sqrt{s}} e^{-\frac{(u-x-4\beta s)^2}{2s}} du + e^{4\beta(y-x)} e^{8\beta^2 s} \int_{-\infty}^y \frac{1}{\sqrt{s}} e^{-\frac{(u-x+4\beta s)^2}{2s}} du,$$

we conclude that

$$A_2 \leq e^{8\beta^2 s} C_{\beta,x,y}.$$

Therefore,

$$A \leq C_{\beta,x,y} \frac{1}{(t-s)^{7/8}} g_{s+(t-s)/(2\alpha)}(z-x).$$

The other term that stays is treated as  $A_2$ . Thus, we can bound the expression in the lemma with a function integrable in time and space.  $\square$

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