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1 Delaunay triangulation of a random sample of a 2 good sample has linear size*

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8 — Abstract —

9 The *randomized incremental construction* (RIC) for building geometric data structures has been
10 analyzed extensively, from the point of view of worst-case distributions. In many practical situa-
11 tions however, we have to face nicer distributions. A natural question that arises is: do the usual
12 RIC algorithms automatically adapt when the point samples are nicely distributed. We answer
13 positively to this question for the case of the Delaunay triangulation of ε -nets.

14 ε -nets are a class of nice distributions in which the point set is such that any ball of radius ε
15 contains at least one point of the net and two points of the net are distance at least ε apart. The
16 Delaunay triangulations of ε -nets are proved to have linear size; unfortunately this is not enough
17 to ensure a good time complexity of the randomized incremental construction of the Delaunay
18 triangulation. In this paper, we prove that a uniform random sample of a given size that is taken
19 from an ε -net has a linear sized Delaunay triangulation in any dimension. This result allows
20 us to prove that the randomized incremental construction needs an expected linear size and an
21 expected $O(n \log n)$ time.

22 Further, we also prove similar results in the case of non-Euclidean metrics, when the point
23 distribution satisfies a certain *bounded expansion* property; such metrics can occur, for example,
24 when the points are distributed on a low-dimensional manifold in a high-dimensional ambient
25 space.

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30 **1** Introduction

31 An ε -net of some compact domain D is a point set of size n in D such that any ball centered in
32 D of radius ε contains at least a point and two points of the net are distance at least ε apart.
33 When we enforce such a hypothesis of “nice” distribution of the points in space, a volume
34 counting argument ensures that the local complexity of the Delaunay triangulation around
35 a vertex is bounded by a constant (dependent on the dimension d but not on the number
36 of points). Unfortunately, to be able to control the complexity of the usual randomized

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37 incremental algorithms [9, 2, 5, 1], it is not enough to control the final complexity of the
38 Delaunay triangulation, but also the complexity of the triangulation of a random subset.

39 One would expect that a random subsample of size k of an ε -net is also an ε' -net for
40 $\varepsilon' = \varepsilon \sqrt[2]{\frac{n}{k}}$ with high probability. Actually this is not quite true, it may happen with
41 reasonable probability that a ball of radius $O(\varepsilon')$ contains $\Omega(\log k / \log \log k)$ points or that
42 a ball of radius $\Omega(\varepsilon' \sqrt[2]{\log k})$ does not contain any point. Thus this approach can transfer the
43 complexity of an ε -net to the one of a random subsample of an ε -net but losing \log factors.

44 In this paper, we study the Delaunay triangulation of a random sample of an ε -net and
45 deduce results about the complexity of randomized incremental constructions. To avoid
46 technicalities due to boundary, we assume without real loss of generality that the ε -net is
47 taken from the flat torus of dimension d which is a compact manifold without boundary. If
48 we equip the flat torus with the Euclidean metric, we will show that the expected number of
49 d -simplices of the star of any point in the sample can be bounded by a constant that does
50 not depend on the number of points in the sample. It will follow that the complexity of the
51 randomized incremental construction of the Delaunay triangulation of an ε -net in general
52 position takes time $O(n \log n)$ in any dimension.

53 We will extend those results to non-Euclidean metrics that satisfy a certain *bounded*
54 *expansion* property; such metrics can occur, for example, when the points are well distributed
55 on a low-dimensional manifold in a high-dimensional ambient space.

56 The rest of the paper is as follows: In Section 2 we introduce the basic concepts of
57 Delaunay complex, net, growth-restricted measure, random sample, and state our results.
58 In Section 3, we bound the size of the d -skeleton of the Delaunay complex of a uniform
59 random sample of a given size extracted from an ε -net. In Section 4, we extend this result
60 to growth-restricted metrics. Finally, in Section 5, we use those size bounds to compute
61 the space and time complexity of the randomized incremental construction for constructing
62 Delaunay complexes of ε -nets.

63 2 Definitions, Notations, and Results

64 In this paper, we consider a finite set of points \mathcal{X} in the flat torus \mathbb{T}^d of dimension d , where
65 $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Since \mathbb{T}^d is a compact manifold without boundary, it will be possible to define
66 finite nets without having to care about boundary effects. We can associate to the flat torus
67 its infinite sheeted covering by \mathbb{R}^d obtained by periodically copying the points in $[0, 1]^d$ by
68 translations with integer coordinates.

69 We denote by $\Sigma(p, r)$ and $B(p, r)$ the sphere and the ball of center p and radius r
70 respectively and $\text{int}(B)$ the interior of a set B .

71 The volume of the unit Euclidean ball of dimension d is denoted V_d and the area of the
72 boundary of such a ball is denoted S_{d-1} . It is known that $V_d = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}$ and $S_d = 2\pi V_{d-1}$,
73 where $\Gamma(t) := \int_0^\infty e^{-x} x^{t-1} dx$, ($t > 0$) denotes the *Gamma function*. For $d \in \mathbb{Z}^+$, $\Gamma(d+1) = d!$.
74 We note that $2^d d^{-d/2} \leq V_d \leq d^{-d/2}$ (see e.g. [13]).

75 Given a discrete set A , $\#(A)$ denotes its cardinality and, for $k \in \mathbb{Z}^+$, $\binom{A}{k}$ denotes the set
76 of k -sized subsets of A .

77 2.1 Delaunay complexes

78 ► **Definition 1** (Delaunay complex). Given a set \mathcal{X} in some ambient metric space, the
79 Delaunay complex of \mathcal{X} is the (abstract) simplicial complex with vertex set \mathcal{X} which is the
80 nerve of the Voronoi diagram of \mathcal{X} : a simplex σ belongs to $\text{Del}(\mathcal{X})$ iff the Voronoi cells of

81 its vertices have a non empty common intersection. Equivalently, σ can be circumscribed
 82 by an empty ball, i.e. a ball whose bounding sphere contains the vertices of σ and whose
 83 interior contains no points of \mathcal{X} .

84 Our combinatorial results (Theorems 14, 15, 16) do not assume that the points of \mathcal{X} are in
 85 general position and we will consider Euclidean as well as more general metrics. Accordingly,
 86 we won't assume that the Delaunay complex embeds in the ambient space and we will provide
 87 bounds on the size of the d -skeleton of the Delaunay complex, i.e. the subcomplex consisting
 88 of the faces of the Delaunay complex of dimension at most d . Differently, our algorithmic
 89 result (Theorem 18) assumes that the Delaunay complex is a triangulation of \mathcal{X} , i.e. a
 90 geometric simplicial complex embedded in the ambient space containing \mathcal{X} .

91 Delaunay [7] proved that, if the ambient space is \mathbb{R}^d equipped with the Euclidean metric
 92 and if \mathcal{X} is in general position¹, the Delaunay complex of \mathcal{X} is a triangulation called the
 93 Delaunay triangulation of \mathcal{X} . The case of the flat torus equipped with the Euclidean metric
 94 is slightly more complicated. The Delaunay complex then embeds in the infinite sheeted
 95 covering as a geometric triangulation when the points are in general position. However, the
 96 Delaunay complex does not embed in the flat torus in general since a point can have a copy
 97 of itself as one of its neighbours. Nevertheless, as shown by Caroli and Teillaud [4], the
 98 complex always embeds in the 3^d -sheeted covering of \mathbb{T}^d (consisting of 3^d copies of each point
 99 in $(\mathbb{R}/3\mathbb{Z})^d$) and embed in the one sheeted covering provided that the largest Delaunay sphere
 100 has diameter less than half the systole of the space, which is 1 for \mathbb{T}^d . More generally, if the
 101 faces of the Voronoi diagram of \mathcal{X} satisfy the so-called *closed ball property* (i.e. the faces are
 102 topological balls of the right dimension), then the Delaunay complex is a triangulation under
 103 the general position assumption [11]. A popular way to ensure the closed ball property is to
 104 assume that \mathcal{X} is a sufficiently dense net (see [10] for instance) which is consistent with the
 105 general approach taken in this paper.

106 2.2 ε -nets over the Euclidean metric

107 ► **Definition 2** (ε -net). A set \mathcal{X} of n points in \mathbb{T}^d is an ε -net if any ball of radius ε contains
 108 at least one point, and any two points are at least distance ε apart.

109 This definition applies for any metric. In the case of the Euclidean metric, we have the
 110 following properties.

111 ► **Lemma 3** (Maximum packing size). *Let $\rho \leq 1$, then any packing of the ball of radius $r \geq \rho$
 112 in dimension d by disjoint balls of radius $\rho/2$ has a number of balls smaller than $\left(\frac{3r}{\rho}\right)^d$.*

113 **Proof.** Consider a maximal set of disjoint balls of radius $\frac{\rho}{2}$ with center inside the ball $B(r)$
 114 of radius r . Then the balls with the same centers and radius ρ cover the ball $B(r)$ (otherwise
 115 it contradicts the maximality). By a volume argument we get that the number of balls is
 116 bounded from above by $\frac{V_d \times (r + \frac{\rho}{2})^d}{V_d \times (\frac{\rho}{2})^d} \leq \left(\frac{3r}{\rho}\right)^d$ for $\rho < 1$. ◀

117 ► **Lemma 4** (Minimum cover size). *Any covering of a ball of radius r in dimension d by balls
 118 of radius ρ has a number of balls greater than $\left(\frac{r}{\rho}\right)^d$.*

119 **Proof.** The volume argument gives a lower bound of $\frac{V_d \times r^d}{V_d \times \rho^d} = \left(\frac{r}{\rho}\right)^d$. ◀

¹ i.e. no subset of $d + 2$ points of \mathcal{X} lie on a same hypersphere.

120 A key property of ε -nets is that the d -skeletons of their Delaunay complexes have linear
121 size as stated in the next lemma.

122 ► **Lemma 5.** *Let $\varepsilon \in (0, 1]$ be given, and let \mathcal{X} be an ε -net over \mathbb{T}^d , where $d \in \mathbb{Z}^+$ is any
123 positive integer. Then the d -skeleton of the Delaunay complex of \mathcal{X} , $Del(\mathcal{X})$ has $4^{d^2} \varepsilon^{-d}$
124 simplices.*

125 **Proof.** Observe that, by the minimum distance property of the points in \mathcal{X} , the balls of radius
126 $\varepsilon/2$ centered around each point in \mathcal{X} , are disjoint, and by a volume argument there can be at
127 most $\frac{1}{V_d \times (\varepsilon/2)^d} \leq 2^{-d} d^{d/2} (\varepsilon/2)^{-d} = d^{d/2} \varepsilon^{-d}$ such balls in \mathbb{T}^d . The balls of radius ε centered
128 around each point in \mathcal{X} cover the space thus their number is at least $\frac{1}{V_d \times \varepsilon^d} \geq d^{d/2} 2^{-d} \cdot \varepsilon^{-d}$.
129 Thus

$$130 \quad \#(\mathcal{X}) \in [d^{d/2} 2^{-d} \cdot \varepsilon^{-d}, d^{d/2} \varepsilon^{-d}]. \quad (1)$$

131 Next, observe that the circumradius of any simplex in $Del(\mathcal{X})$ cannot be greater than ε ,
132 since this would imply the existence of a ball in \mathbb{T}^d of radius at least ε , containing no points
133 from \mathcal{X} . Therefore given a point $p \in \mathcal{X}$, any point which lies in a Delaunay simplex incident
134 to p , must be at most distance 2ε from p . Again by a volume argument, the number of such
135 points is at most $\frac{V_d \times (2\varepsilon + \varepsilon/2)^d}{V_d \times ((\varepsilon/2)^d)} = 5^d$. Thus, the number of Delaunay simplices of dimension at
136 most d that contain p , is at most the complexity of the d -skeleton of the Delaunay complex
137 in \mathbb{T}^d on 5^d vertices. This is at most $(5^d)^{\lceil d/2 \rceil}$. Thus we can conclude that the number of
138 simplices in $Del(\mathcal{X})$ is at most the cardinality of \mathcal{X} , times the maximum number of simplices
139 incident to any given point $p \in \mathcal{X}$, or $(5^d)^{\lceil d/2 \rceil} \cdot \#(\mathcal{X}) \leq (5^d)^{\frac{d+1}{2}} 2^{-d} d^{d/2} \varepsilon^{-d} \leq 4^{d^2} \varepsilon^{-d}$. ◀

140 2.3 Growth-restricted measures

141 We shall denote by $(\mathcal{M}, \mathcal{X})$ a metric measure space $\mathcal{M} = (U, d(\cdot, \cdot), \mu)$, where $(U, d(\cdot, \cdot))$ is
142 a metric space, and μ is a measure over the Borel algebra of U , together with an ε -net \mathcal{X}
143 over \mathcal{M} . We shall use for μ the counting measure with respect to \mathcal{X} , that is, for a Borel set
144 $S \subset \mathcal{M}$, $\mu(S) = \#(\mathcal{X} \cap S)$. Abusing notation slightly, we shall use $\mu(c, r)$ to mean $\mu(B(c, r))$.

145 ► **Definition 6** (see e.g. [6], [12]). A measure space $\mathcal{M} = (U, d(\cdot, \cdot), \mu)$ is said to be (ρ, dim) -
146 *growth-restricted*², for $dim, \rho > 0$, if and only if for all $x \in \mathcal{M}$ and all $r \geq \rho$,

$$147 \quad \mu(x, 2r) \leq 2^{dim} \cdot \mu(x, r).$$

148 2^{dim} will be referred to as the *expansion constant* of \mathcal{M} and dim as the expansion dimension.
149 A (ρ, dim) -growth-restricted measure space \mathcal{M} is *strongly growth-restricted* if there also exist
150 constants $\eta > 0$ and $r_0 > 0$, such that for all $x \in \mathcal{M}$ and $r < r_0$

$$151 \quad \mu(x, 2r) \geq (1 + \eta) \mu(x, r). \quad (2)$$

152 We shall refer to a (ρ, dim) -growth-restricted measure space simply as a growth-restricted
153 space, when (ρ, dim) are not explicitly required or are clear from the context.

154 ► **Lemma 7.** *If $(\mathcal{M}, \mathcal{X})$ has expansion dimension at most dim , then for all $c \in U$, $r > 0$,*

$$155 \quad \mu(c, r) \leq \left(\frac{4r}{\varepsilon} \right)^{dim}.$$

² Also known as *doubling measure*, *Federer measure*, or *diametrically regular measure* in the literature.

156 **Proof.** Let $c \in U$ be any point, and $r > 0$. Consider the ball $B = B(c, \varepsilon/2)$. Since \mathcal{X} is an
 157 ε -net, by the triangle inequality, there can be at most a single point of \mathcal{X} inside B , that is,
 158 $\mu(c, \varepsilon/2) \leq 1$. Therefore, by the bound on \dim , we have that $\mu(c, \varepsilon) \leq 2^{\dim}$. Applying this
 159 argument repeatedly $k = \lceil \log(r/\varepsilon) \rceil$ times, we have that
 160 $\mu(c, r) \leq \mu(c, 2^k \varepsilon) \leq 2^{k \cdot \dim} \leq 2^{(\log(r/\varepsilon)+1)\dim} \leq (2r/\varepsilon)^{\dim}$. ◀

161 ▶ **Lemma 8.** *Given $(\mathcal{M}, \mathcal{X})$, $p, c \in U$, and $r > r_1 > 0$, then if $B(c, r_1) \subseteq B(p, r)$, then*
 162 $\mu(c, r_1) \geq 2^{-\lceil \log(2r/r_1) \rceil \cdot \dim} \cdot \mu(p, r)$.

163 **Proof.** Any point $q \in B(p, r)$, satisfies $d(c, q) \leq 2r$. Therefore, $B(p, r) \subseteq B(c, 2r) =$
 164 $B(c, 2^k r_1)$, where $k = \lceil \log(2r/r_1) \rceil$. Applying the restricted growth condition k times, we
 165 get $\mu(c, r_1) \geq 2^{-k \cdot \dim} \mu(c, 2r) \geq 2^{-k \dim} \mu(p, r)$. ◀

166 We now give a lemma that shows the connection between growth-restricted and strongly
 167 growth-restricted measure spaces, when the underlying domain is compact.

▶ **Lemma 9.** *Given a (ρ, \dim) growth-restricted metric measure space $(\mathcal{M}, \mathcal{X})$, $\mathcal{M} =$
 $(U, d(\cdot, \cdot), \mu)$ where U is a compact domain, then $(\mathcal{M}, \mathcal{X})$ is also strongly growth-restricted,
 i.e. for $r \leq r_{max}/2$, where r_{max} is at most half the systole of U , any $p \in U$ and $r > \rho$ satisfy*

$$\mu(p, 2r) \geq (1 + \eta) \cdot \mu(p, r), \quad \text{where } \eta \geq 2^{-4\dim}.$$

168 **Proof.** Consider the ball $B_1 = B(p, 2r)$, where $r \geq 4\rho$. Since $r \leq r_{max}/2$, where r_{max} is at
 169 most half the systole of U , $\Sigma(p, 2r)$ is not self-intersecting. Therefore, no point in $B(p, 2r)$
 170 overlaps itself. Now, since U is a continuous domain, there exists a point $c \in B(p, 2r)$ such
 171 that $d(p, c) = 3r/2$. Then the ball $B(c, r/4) \subseteq B(p, 2r)$, since any point $s \in B(c, r/4)$ satisfies
 172 $d(p, s) \leq d(p, c) + d(c, s) \leq 7r/4$. Further, $B(p, r) \cap B(c, r/4) = \emptyset$, i.e. the balls $B(c, r/2)$ and
 173 $B(p, r)$ must be disjoint, since if there existed any point $q \in B(c, r/2) \cap B(p, r)$, then $d(p, q) \leq r$
 174 and $d(q, c) \leq r/4$, and therefore we would have that $d(p, c) \leq 5r/4 < 3r/2$, which would
 175 contradict our assumption $d(p, c) = 3r/2$. Applying Lemma 8(ii) to $B(c, r/4) \subset B(p, 2r)$, we
 176 have that

$$177 \quad \mu(c, r/4) \geq 2^{-\lceil \log(\frac{2(2r)}{r/4}) \rceil \dim} \cdot \mu(p, 2r) = 2^{-4\dim} \mu(p, 2r) \geq 2^{-4\dim} \mu(p, 2r) \geq 2^{-4\dim} \mu(p, r).$$

Since $B(p, r) \cap B(c, r/4) = \emptyset$ and $B(p, r) \cup B(c, r/4) \subset B(p, 2r)$, we have that

$$\mu(p, 2r) \geq \mu(c, r/4) + \mu(p, r) \geq (1 + 2^{-4\dim}) \mu(p, r),$$

178 which completes the proof of the lemma. ◀

179 2.4 Random samples

180 In this paper, we consider two types of random subsets of ε -nets.

181 ▶ **Definition 10** (Bernoulli random sample). A subset \mathcal{Y} of a set \mathcal{X} is a Bernoulli sample of
 182 \mathcal{X} of parameter α if each point of \mathcal{X} belongs to \mathcal{Y} with probability α independently.

183 ▶ **Definition 11** (Uniform random sample). A subset \mathcal{Y} of set \mathcal{X} is a uniform random sample
 184 of \mathcal{X} of size s if \mathcal{Y} is any possible subset of \mathcal{X} of size s with equal probability.

185 In order to work with uniform random samples, we shall require some lemmas on the
 186 uniformly random sampling distribution stated below.

187 ► **Lemma 12** (*j*-moment). Given $a, b, c \in \mathbb{Z}^+$, and a set C of size c , a fixed subset $B \subseteq C$ of
 188 size $b \leq c$ and a uniformly random sample A of size a , then for $j \leq \min\{c/2, \sqrt{c}\}$, the j -th
 189 moment of the random variable $X := \sharp(A \cap B)$ is given by $\mathbb{E}[X^j] \leq 3\mathbb{E}[X]^j$.

190 The proof is provided in the Appendix.

191 ► **Lemma 13.** Given $a, b, c \in \mathbb{Z}^+$, with $b \leq \min(\frac{a}{2}, \sqrt{c})$, the probability that the random
 192 sample A having cardinality a of a set C of cardinality c , contains B having cardinality b ,
 193 and is disjoint from another fixed set T , which is disjoint from B and has cardinality t , is at
 194 most $(1 + \frac{2b^2}{c}) (\frac{a}{c})^b e^{-t\frac{a}{2c}}$.

195 **Proof.** The total number of ways of choosing the random sample A is $\binom{c}{a}$. The number
 196 of ways of choosing A such that $B \subset A$ and $T \cap A = \emptyset$, is $\binom{c-b-t}{a-b}$. Therefore the required
 197 probability is

$$\begin{aligned}
 198 \quad \mathbb{P}[B \subset A, T \cap A = \emptyset] &= \frac{\binom{c-b-t}{a-b}}{\binom{c}{a}} \\
 199 &= \frac{\prod_{i=0}^{b-1} (a-i) \prod_{i=b}^{a-1} (a-i)}{\prod_{i=0}^{b-1} (c-i) \prod_{i=b}^{a-1} (c-i)} \cdot \frac{\prod_{i=0}^{a-b-1} (c-b-t-i)}{\prod_{i=0}^{a-b-1} (a-b-i)} \\
 200 &= \frac{\prod_{i=0}^{b-1} (a-i) \prod_{i=0}^{a-b-1} (c-b-t-i)}{\prod_{i=0}^{b-1} (c-i) \prod_{i=0}^{a-b-1} (c-b-i)} \\
 201 &= (a/c)^b \frac{\prod_{i=0}^{b-1} (1-i/a)}{\prod_{i=0}^{b-1} (1-i/c)} \prod_{i=0}^{a-b-1} \left(1 - \frac{t}{c-b-i}\right) \\
 202 &\leq (a/c)^b \frac{\prod_{i=0}^{b-1} (1-i/a)}{\prod_{i=0}^{b-1} (1-i/c)} \left(1 - \frac{t}{c-b}\right)^{a-b} \\
 203 &= (a/c)^b \frac{\prod_{i=0}^{b-1} (1-i/a)}{\exp(\sum_{i=0}^{b-1} \ln(1-i/c))} \left(1 - \frac{t}{c-b}\right)^{a-b} \\
 204 &\leq (a/c)^b \frac{1}{\exp(\sum_{i=0}^{b-1} \ln(1-i/c))} \left(\exp\left(-\frac{t}{c-b}\right)\right)^{a-b} \tag{3}
 \end{aligned}$$

$$\begin{aligned}
 205 &= (a/c)^b \exp\left(-\sum_{i=0}^{b-1} \ln(1-i/c)\right) \left(\exp\left(-\frac{t}{c-b}\right)\right)^{a-b} \\
 206 &\leq (a/c)^b \exp\left(\sum_{i=0}^{b-1} \frac{2i}{c}\right) \left(\exp\left(-\frac{t}{c-b}\right)\right)^{a-b} \tag{4}
 \end{aligned}$$

$$\begin{aligned}
 207 &= (a/c)^b \exp\left(\frac{b(b-1)}{c}\right) \exp\left(-\frac{t(a-b)}{c-b}\right) \\
 208 &\leq (a/c)^b \left(1 + \frac{2b^2}{c}\right) e^{-\frac{ta}{2c}}, \tag{5}
 \end{aligned}$$

209 where (3) uses $1 - x < e^{-x}$ for $x > 0$, (4) comes from the fact that $-\ln(1-x) \leq 2x$ for
 210 $x \in (0, 1/2]$, and that $b < a/2 < c/2$, and (5) comes from $e^x < 1 + 2x$ for $x \in [0, 1]$ and
 211 $b^2 < c$ for the second factor and for the third factor the fact that when $0 \leq b \leq a/2$ we have
 212 $\frac{a-b}{c-b} \geq \frac{a-a/2}{c-0} = \frac{a}{2c}$. ◀

213 2.5 Our Results

214 We first consider the Euclidean case and provide a bound on the number of Delaunay simplices
 215 containing a given point in a random subsample of an ε -net (Theorem 14). Although the

216 bound for the Euclidean case given will be generalized in Theorem 15 and Theorem 16, we
 217 state it and prove it separately since its proof is simpler and the constants are better.

218 ► **Theorem 14** (Euclidean metric). *Given an ε -net \mathcal{X} in $(\mathbb{T}^d, \|\cdot\|)$, where $\varepsilon \in (0, \frac{1}{4}]$, the
 219 expected number of simplices incident to a point $p \in \mathcal{X}$, in the d -skeleton of the Delaunay
 220 complex of a uniform random sample $S \subset \mathcal{X}$ of size $s \geq (2\sqrt{d})^d d^3 + 1$ containing p , is less
 221 than $2^{d(4d+1)+2}$.*

222 Theorem 14 will be proved in Section 3. Next, we generalize Theorem 14 to hold for
 223 growth-restricted metrics.

224 ► **Theorem 15.** *Given $\varepsilon \in (0, 1]$, a metric distance function $d(\cdot, \cdot)$ over \mathbb{T}^d , and an ε -net \mathcal{X} ,
 225 such that $(\mathbb{T}^d, d(\cdot, \cdot), \mu)$ is a growth-restricted measure space, having expansion constant at
 226 most 2^{dim} , it holds that the expected number of d -simplices incident to a point $p \in \mathcal{X}$, in the
 227 Delaunay complex of a uniform random sample $S \subset \mathcal{X}$ of size $s \geq 4d$, is at most*

228
$$\mathbb{E} [\sharp(\text{star}(p))] \leq 2^{2d-dim+3d} + \left(\frac{2^{d+2+3(d-dim)}}{d!} \right) \cdot \left[\sum_{k=1}^{\infty} g(k)^d \cdot e^{-g(k)} \right],$$

229 where $g(k) = q \cdot \frac{\mu(p, 2^k \delta)}{2^{2dim+1}}$, with $q := \frac{s-1}{n-1}$, and δ as the least y such that $q \cdot \mu(p, 2y) \geq 2^{2dim+1}d$.

230 The following theorem is a corollary of Theorem 15.

231 ► **Theorem 16.** *For a strongly growth-restricted metric space $(\mathcal{M}, \mathcal{X})$, if \mathcal{M} has a compact
 232 domain U then the expected number of simplices in the star of $p \in \mathcal{X}$ is at most*

233
$$\mathbb{E} [\sharp(\text{star}(p))] \leq 2^{2ddim+3d} + 2^{d+3+3(d-dim)}.$$

234 Our most general result, on growth-restricted metrics over \mathbb{T}^d , follows as a simple
 235 consequence of Theorem 16 and Lemma 9.

236 ► **Corollary 17.** *Given $\varepsilon \in (0, 1]$, a metric distance function $d(\cdot, \cdot)$ over \mathbb{T}^d , and an ε -net \mathcal{X} ,
 237 such that $(\mathbb{T}^d, d(\cdot, \cdot), \mu)$ is a growth-restricted measure space, having expansion constant at
 238 most 2^{dim} , then*

239
$$\mathbb{E} [\sharp(\text{star}(p))] \leq 2^{2ddim+3d} + 2^{d+3+3(d-dim)}.$$

240 We next use the above bounds to get the space and time complexity of the randomized
 241 incremental construction of the d -skeleton of the Delaunay complex of an ε -net:

242 ► **Theorem 18** (Randomized incremental construction). *Let \mathcal{X} be an ε -net over a strongly
 243 growth-restricted metric space $(\mathbb{T}^d, d(\cdot, \cdot), \mu)$, where $\varepsilon \in (0, 1]$. If the faces of the Voronoi
 244 diagram of \mathcal{X} satisfy the closed-ball property, then the randomized incremental construction
 245 of the d -skeleton of the Delaunay complex needs $O(n \log n)$ expected time and $O(n)$ expected
 246 space, where $n = \sharp(\mathcal{X})$ and d is considered as a constant in the big O .*

247 Theorem 18 will be proved in Section 5.

248 ► **Remark.** 1. Theorem 14 and Theorem 16 also work for the case when the random sample
 249 is a Bernoulli sample of parameter q . Observe that in this case $\mathbb{P} [E(\tau)]$ is just the
 250 probability that the d points of τ are chosen in \mathcal{Y} , and the points inside $B(c_\sigma, r_\sigma)$ are not
 251 chosen in \mathcal{Y} . Again, the number of points in $B(c_\sigma, r_\sigma) \cap \mathcal{X}$ is at least $(r_\sigma/\varepsilon)^d$. Therefore,
 252 $\mathbb{P} [E(\tau)]$ is simply $q^d(1-q)^{(r_\sigma/\varepsilon)^d} < q^d e^{-q(r_\sigma/\varepsilon)^d}$, i.e. less than the bound in inequality (6).
 253 The rest of the proof follows as before.

254 2. Our results can be extended to other types of good samples, e.g. the weaker notion of
 255 (ε, κ) -samples for which any ball of radius ε contains at least one point and at most κ
 256 points. If we fix $\kappa = \kappa_0 = 2^{O(d)}$, we get exactly the same result. The bounds can be
 257 straightforwardly adapted to accommodate other values of κ .

258 **3** The Euclidean case (Proof of Theorem 14)

259 In this section, we prove that a subsample of a given size s , drawn randomly from an ε -net
 260 $\mathcal{X} \subset \mathbb{T}^d$, has a Delaunay complex with a d -skeleton of linear complexity, with a constant of
 261 proportionality bounded by 2^{cd^2} , where c is a constant independent of ε and d .

262 We shall focus on computing the expected number of d -dimensional simplices. (The
 263 expected number of i -dimensional simplices can be computed analogously for each i).

264 Let n denote $\sharp(\mathcal{X})$; from the volume argument in the proof of Lemma 5 we have upper and
 265 lower bounds on n at Equation (1). Let the uniform random sample of size s denoted by
 266 \mathcal{Y} . Also, let us fix a point $p \in \mathcal{Y}$; we shall upper bound the number of d -simplices incident
 267 in the Delaunay complex $Del(\mathcal{Y})$ incident to p , that is $star_{Del(\mathcal{Y})}^{(d)}(p)$, or $star(p)$ in short.

268 Consider a d -tuple of points in \mathcal{X} : $\tau \in \mathcal{X}^d$, such that the d -simplex formed by the points in
 269 $\sigma := \tau \cup \{p\}$, whose circumcenter and circumradius are denoted c_σ and r_σ . Then, given that
 270 $p \in \mathcal{Y}$, the event $E^{(\tau)} := \sigma \in Del(\mathcal{Y})$ could occur only if the following events occur

271 (i) $E_1^{(\sigma)} := \forall p' \in \tau, p' \in \mathcal{Y}$, and

272 (ii) $E_2^{(\sigma)} := \text{int}(B(c_\sigma, r_\sigma)) \cap \mathcal{Y} = \emptyset$.

273 Observe that once the points of σ are fixed, the points in $\text{int}(B(c_\sigma, r_\sigma)) \cap \mathcal{X}$ are uniquely
 274 determined. Given that $p \in \mathcal{Y}$, the distribution of $\mathcal{Y} \setminus \{p\}$ is now that of a uniformly random
 275 sample of size $s - 1$ from $\mathcal{X} \setminus \{p\}$. The event $E^{(\tau)}$ now fits the setting of Lemma 13, with
 276 the universe $C = \mathcal{X} \setminus \{p\}$, the random sample $A = \mathcal{Y} \setminus \{p\}$, the set of points required to
 277 be contained in the sample $B = \tau$, and the disjoint set of points required to be not
 278 contained in the sample, $T = B(c_\sigma, r_\sigma) \cap \mathcal{X}$. Since \mathcal{X} is an ε -net, we have an ε -covering of
 279 $B(c_\sigma, r_\sigma)$. Therefore from Lemma 4, we have that $t = \sharp(T) = \sharp(B(c_\sigma, r_\sigma) \cap \mathcal{X}) \geq \left(\frac{r_\sigma}{\varepsilon}\right)^d$. We
 280 can also assume that $c = n - 1$, $a = s - 1$, and $b = d$ satisfy the conditions of Lemma 13, that
 281 (i) $b \leq \min\left(\frac{a}{2}, \sqrt{c}\right)$ since otherwise $s \leq \max(2d + 1, d^2 + 1)$, and so the worst-case complexity
 282 is $s^{d+1} \leq 2d^{2(d+1)}$, which is a constant; and (ii) $b \leq a$, since $s \geq (2\sqrt{d})d^3 + 1$. Let $q := \frac{s-1}{n-1}$
 283 and $\delta := \varepsilon \cdot \left(\frac{2d}{q}\right)^{1/d}$. Therefore, applying Lemma 13 with $t \geq (r_\sigma/\varepsilon)^d$, the probability that
 284 $\sigma \in Del(\mathcal{Y})$ given $p \in \mathcal{Y}$ can be upper-bounded by:

$$\begin{aligned}
 285 \quad \mathbb{P}\left[E^{(\tau)} | p \in \mathcal{Y}\right] &= \mathbb{P}\left[E_1^{(\sigma)} \wedge E_2^{(\sigma)}\right] \leq \left(1 + \frac{2d^2}{n-1}\right) \times q^d \times \exp\left(-\frac{q}{2}(r_\sigma/\varepsilon)^d\right) \\
 286 &\leq 3 \times q^d \times \exp\left(-\frac{q}{2}(r_\sigma/\varepsilon)^d\right).
 \end{aligned}$$

287 The first inequality follows by applying Lemma 13 with $c = n - 1$, $a = s - 1$, $b = d$,
 288 and $t \geq (r_\sigma/\varepsilon)^d$. Then we use the fact that we are working in the range $b \leq \sqrt{c}$, i.e.
 289 $\frac{2b^2}{c} = \frac{2d^2}{n-1} \leq 2$.

290 Let $I_0 := [0, \delta)$, $I_k := [2^{k-1}\delta, 2^k\delta)$ for $k \in \mathbb{N}$. By the triangle inequality, if $\sigma \in Del(\mathcal{Y})$
 291 has a circumradius r_σ , then all the points in σ must lie in the ball $B(p, 2r_\sigma)$. This ball is
 292 not self intersecting in \mathbb{T}^d if $r_\sigma \leq \frac{1}{4}$, which allows to relate the number of points inside to its
 293 volume. Therefore by Lemma 3, the number of potential d -tuples which can contribute to
 294 $star(p)$ is at most $\binom{(3 \cdot 2r_\sigma/\varepsilon)^d}{d} \leq \frac{(6r_\sigma/\varepsilon)^{d^2}}{d!}$.

295 Let

$$296 \quad Z_p(k) := \sharp(\{\sigma \in Del(\mathcal{Y}) : p \in \sigma, r_\sigma \in I_k\}), p \in \mathcal{Y}.$$

297 denote the number of Delaunay simplices incident to $p \in \mathcal{Y}$ and which have circumradius
 298 $r_\sigma \in I_k$.

299 **Bound on $Z_p(0)$**

300 Firstly, consider the range $r_\sigma \in I_0$. If $q \in \sigma \in \text{star}(p)$ such that $r_\sigma \in I_0$, then by the triangle
 301 inequality, q lies in the ball $B(p, 2\delta) \cap \mathcal{X}$. By Lemma 3, the expected number of points in
 302 this ball is at most

$$\begin{aligned} 303 \quad \mathbb{E} [\#(\text{int}(B(p, 2\delta)) \cap \mathcal{Y})] &\leq (6\delta/\varepsilon)^d q \\ 304 &= 6^d \left(\frac{(2d/q)^{1/d} \cdot \varepsilon}{\varepsilon} \right)^d q \\ 305 &= 2d \cdot 6^d \leq 2^{3d+1} \cdot d. \end{aligned}$$

306 By Lemma 12, we have that

$$307 \quad \mathbb{E} [\#(\text{int}(B(p, 2\delta)) \cap \mathcal{Y})^d] \leq 3 \mathbb{E} [\#(\text{int}(B(p, 2\delta)) \cap \mathcal{Y})]^d \leq 2^{3d^2+d} \cdot d^d.$$

308 For $k = 0$, we have $\mathbb{E} [Z_p(0)] \leq \frac{\mathbb{E}[\#(\text{int}(B(p, 2\delta)) \cap \mathcal{Y})^d]}{d!} \leq 3 \cdot 2^{3d^2+d} \cdot e^{d-1} < 2^{3d^2+3d}$, where in the
 309 penultimate inequality we used Stirling's approximation, $d! \geq e(d/e)^d$.

310 **Bound on $Z_p(k), k \geq 1$**

311 For $k \geq 1$, to apply the above bound on the number of potential simplices yielding a sphere
 312 of radius in I_k , we need to limit $r_\sigma \leq \frac{1}{4}$. In this case we have

$$\begin{aligned} 313 \quad \mathbb{E} [Z_p(k)] &\leq \sum_{\tau \in (\mathcal{X} \cap \text{int}(B_d(p, 2^k \delta)))} \mathbb{P} [E(\tau)] \leq \frac{(6 \cdot 2^{k+1} \delta / \varepsilon)^{d^2}}{d!} \cdot 3 \cdot q^d e^{-\frac{q}{2}(2^{k-1} \delta / \varepsilon)^d} \\ 314 &\leq 3 \cdot 2^{3d^2} \frac{(2^k \delta / \varepsilon)^{d^2}}{d!} \cdot q^d \cdot e^{-\frac{q}{2}(2^{k-1} \delta / \varepsilon)^d}. \end{aligned}$$

315 **Bound on k_{\max}**

316 Let $I_{k_{\max}} = [2^{k_{\max}} \delta, \infty) \supset [1/4, \infty)$, so that $k_{\max} := \lfloor \log(1/4\delta) \rfloor$. From Equation (1), we
 317 have that $n = \#(\mathcal{X}) \leq 2^{-d} d^{d/2} \varepsilon^{-d}$. Therefore, by Lemma 4, any ball B of radius at least
 318 $2^{k_{\max}} \delta \geq 1/8$, has at least $(2^{k_{\max}} \delta / \varepsilon)^d = (1/4\varepsilon)^d \geq \left(\frac{n}{2^d d^{d/2}}\right)$ points in its interior, i.e.

$$319 \quad \#(\text{int}B \cap \mathcal{X}) \geq \left(\frac{n}{(2\sqrt{d})^d} \right).$$

320 The maximum number of d -tuples which can possibly form a Delaunay d -simplex with p , is
 321 at most $\binom{n-1}{d} \leq (n-1)^d / d!$. Each of these simplices yields less than 3^d possible Delaunay
 322 sphere in the \mathbb{T}^d . Therefore, the expected number of simplices having radius at least $2^{k_{\max}} \delta$,
 323 is at most

$$\begin{aligned} 324 \quad \mathbb{E} [Z_p(k_{\max})] &= 3^d \frac{(n-1)^d}{d!} \cdot \mathbb{P} [E(\tau) | p \in \mathcal{Y}] \\ 325 &= 3^d \frac{(n-1)^d}{d!} \cdot q^d \cdot \exp \left(-\frac{q}{2} \cdot \frac{n}{(2\sqrt{d})^d} \right) \\ 326 &\leq 3^d \frac{(s-1)^d}{d!} \cdot \exp \left(-\frac{s-1}{2(2\sqrt{d})^d} \right). \end{aligned} \tag{6}$$

327 For $s > s_0 = 2(2\sqrt{d})^d \cdot d^3 + 1$ this function is decreasing in term of s and it is easy to check
 328 that the value in s_0 is smaller than 4. Thus we have $\mathbb{E} [Z_p(k_{\max})] \leq 4$.

329 Therefore for $s \geq s_0$, we only need to sum k upto k_{\max} .

330 **Summing** $\mathbb{E}[Z_p(k)]$

$$\begin{aligned}
331 \quad \sum_{k=1}^{k_{\max}-1} \mathbb{E}[Z_p(k)] &\leq \sum_{k=1}^{\infty} \frac{2^{3d^2}}{d!} e^{-\frac{q}{2}(2^{k-1}\delta/\varepsilon)^d} \cdot 3 \cdot ((2^k\delta/\varepsilon)^d q)^d \\
332 &\leq \sum_{k=1}^{\infty} \frac{2^{3d^2}}{d!} e^{-g(k)} \cdot 3 \cdot (g(k))^d \cdot 2^{d^2+d} \\
333 &\leq 3 \cdot \frac{2^{4d^2+d}}{d!} \sum_{k=1}^{\infty} e^{-g(k)} \cdot (g(k))^d,
\end{aligned}$$

334 where $g(k) := (2^{k-1}\delta/\varepsilon)^d q/2$. Observe that by the definition of δ , we have that $g(1) =$
335 $(\delta/\varepsilon)^d q/2 = d$. Further, for all $k \in \mathbb{Z}^+$, $g(k+1) = 2^d \times g(k)$, i.e. $g(k)$ is a strictly increasing
336 function of k . Therefore, for all $k > 1$, $g(k) > d$. Since the function $f(x) = x^d e^{-x}$ is
337 maximised at $x = d$, and is monotone decreasing for $x > d$, the summation $\sum_{k=1}^{\infty} (g(k))^d e^{-g(k)}$
338 can be upper bounded by the integral $\int_{x=0}^{\infty} x^d e^{-x} dx$, by substituting $x = g(k)$. Define
339 $Z_p := \sum_{k=1}^{\infty} Z_p(k)$ to be the number of Delaunay simplices in $star(p)$. Thus we get

$$\begin{aligned}
340 \quad \mathbb{E}[Z_p] &\leq Z_p(0) + Z_p(k_{\max}) + 3 \cdot \frac{2^{4d^2+d}}{d!} \int_{x=0}^{\infty} e^{-x} \cdot x^d dx \\
341 &\leq 2^{3d^2+3d+2} + 4 + 3 \cdot \frac{2^{4d^2+d}}{d!} \Gamma(d+1) \\
342 &= 2^{3d^2+3d+2} + 4 + 3 \cdot 2^{4d^2+d} \leq 2^{4d^2+d+2},
\end{aligned} \tag{7}$$

343 where in inequality (7) we used the identity $\Gamma(d+1) = d!$. This completes the proof of
344 Theorem 14. Therefore, the expected size of $Del(\mathcal{Y})$ is given by

$$\begin{aligned}
345 \quad \mathbb{E}[\#(Del(\mathcal{Y}))] &\leq \sum_{p \in \mathcal{X}} \mathbb{P}[p \in \mathcal{Y}] \mathbb{E}[Z_p | p \in \mathcal{Y}] \\
346 &\leq \sum_{p \in \mathcal{X}} \binom{s}{n} \cdot 2^{(4d+1)d+2} \leq s \cdot 2^{4d^2+d+2}.
\end{aligned} \tag{8}$$

347 **4 Growth-Restricted Measures**

348 In this section we generalise Theorem 14 to growth-restricted metrics, and prove Theorem 15.

349 **Proof of Theorem 16.** By the definition of δ in the statement of Theorem 15, $g(1) = d$.
350 Therefore, $g(k) \geq d$ for all $k \in [1, k_{max}]$. Also, by the condition (2), for $k \in [1, k_{max}]$, we
351 have

$$352 \quad g(k+1) = \frac{q\mu(p, 2^{k+1}\delta)}{2^{2dim+1}} \geq (1+\eta) \cdot \left(\frac{q\mu(p, 2^k\delta)}{2^{2dim+1}} \right) = (1+\eta) \cdot g(k).$$

353 Since $g(k) \geq d$ inductively, we get that $g(k)$ is a strictly increasing function of k . Substituting
354 $x = g(k)$, we get $x \in [d, \infty)$ for $k \geq 1$. Since $x^d e^{-x}$ is decreasing for all $x \in [d, \infty)$, therefore
355 the sum $\sum_{k=1}^{\infty} (g(k))^d e^{-g(k)}$ can be upper bounded by the integral $\int_{x=0}^{\infty} x^d e^{-x} dx = \Gamma(d+1) =$
356 $d!$. Using Stirling's approximation now gives the theorem. \blacktriangleleft

357 **Proof of Theorem 15.** The proof proceeds in similar fashion to that of Theorem 14. Consider
358 a simplex $\sigma \in \binom{\mathcal{X}}{d+1}$, such that $p \in \sigma$, $\sigma \setminus \{p\} = \tau$, having circumcentre c_σ and circumradius
359 r_σ .

360 As in the proof of Theorem 14, we can again assume that $c = n - 1$, $a = s - 1$, and $b = d$
 361 satisfy the conditions of Lemma 13, that (i) $b \leq \min(\frac{a}{2}, \sqrt{c})$ and (ii) $b \leq a$, since $s \geq 4d + 1$.
 362 Again applying Lemma 13, we have

$$363 \quad \mathbb{P}[\sigma \in Del_d(S)] \leq \left(1 + \frac{2d^2}{n-1}\right) \cdot q^d e^{-q \cdot \mu(c_\sigma, r_\sigma)/2} \leq 3 \cdot q^d e^{-q \cdot \mu(c_\sigma, r_\sigma)/2}. \quad (9)$$

364 Let $Z_p(0)$ denote the number of Delaunay simplices incident to p in the uniformly random
 365 sample $S \setminus \{p\}$, with circumradius $r_\sigma \in [0, \delta)$. For $k \in \mathbb{Z}^+$, let $Z_p(k)$ denote the number of
 366 Delaunay simplices incident to p in $S \setminus \{p\}$ with circumradius $r_\sigma \in [2^{k-1}\delta, 2^k\delta)$, i.e.

$$367 \quad Z_p(k) = \#\{\sigma \in star(p) : r_\sigma \in [2^{k-1}\delta, 2^k\delta)\}.$$

368 Let $p' \in \sigma \in star(p)$, then $p' \in B(c_\sigma, r_\sigma)$. Applying Lemma 8, we get that $B(c_\sigma, r_\sigma) \subset$
 369 $B(p, 2r_\sigma) \subset B(c_\sigma, 4r_\sigma)$. Therefore we get

$$370 \quad \mathbb{E}[Z_p(0)] \leq \frac{\mathbb{E}[\#\{(B(p, 2\delta) \cap \mathcal{Y})^d\}]}{d!}$$

$$371 \quad \leq 3 \frac{\mathbb{E}[\#\{(B(p, 2\delta) \cap \mathcal{Y})^d\}]}{d!} \quad (10)$$

$$372 \quad \leq 3 \frac{(q\mu(p, 2\delta))^d}{d!} \leq 3 \frac{(2^{2dim+1}d)^d}{d!} \quad (11)$$

$$373 \quad \leq 3 \cdot (2^{2d \cdot dim + d}) \frac{d^d}{e(d/e)^d} \leq 2^{2d \cdot dim + d + 2d} \leq 2^{2d \cdot dim + 3d}.$$

374 where the inequality (10) was by applying Lemma 12, inequality (11) followed from the
 375 definition of δ , and the last line followed from the definition of δ and Stirling's approximation.
 376 Next, to bound $\mathbb{E}[Z_p(k)]$ for non-zero values of k , by the definition of expansion dimension,
 377 we get

$$378 \quad \mu(c_\sigma, r_\sigma) \geq 2^{-2dim} \cdot \mu(c_\sigma, 4r_\sigma) \geq 2^{-2dim} \cdot \mu(p, 2r_\sigma), \quad (12)$$

379 where the second inequality is from containment. Thus the expected value of $Z_p(k)$, $k \geq 1$ is
 380 bounded by

$$381 \quad \mathbb{E}[Z_p(k)] \leq \sum_{\sigma \in \binom{int B(p, 2^{k+1}\delta) \cap \mathcal{X}}{d+1}; p \in \sigma} \mathbb{P}[\sigma \in Del_d(S)]$$

$$382 \quad \leq \sum_{\sigma \in \binom{int B(p, 2^{k+1}\delta) \cap \mathcal{X}}{d+1}; p \in \sigma} 3 \cdot q^d e^{-q \cdot \mu(c_\sigma, r_\sigma)/2} \quad (13)$$

$$383 \quad \leq \sum_{\sigma \in \binom{int B(p, 2^{k+1}\delta) \cap \mathcal{X}}{d+1}; p \in \sigma} 3 \cdot q^d e^{-q \cdot \mu(p, 2^k\delta)/(2 \cdot 2^{2dim})} \quad (14)$$

$$384 \quad \leq 3 \cdot \left(\frac{\mu(p, 2^{k+1}\delta)^d}{d!}\right) \cdot q^d e^{-q \cdot \mu(p, 2^k\delta)/(2 \cdot 2^{2dim})},$$

385 where line (13) follows from (9), and line (14) follows from (12).

386 Bounding $\mu(p, 2^{k+1}\delta)$ from above by $2^{dim} \cdot \mu(p, 2^k\delta)$, and using the definition of $g(k) =$
 387 $\frac{q\mu(p, 2^k\delta)}{2^{2dim+1}}$, we get for $k \geq 1$

$$388 \quad \mathbb{E}[Z_p(k)] \leq 3 \cdot \frac{2^{d+3d \cdot dim}}{d!} (g(k))^d e^{-g(k)} \leq \left(\frac{2^{2+d+3d \cdot dim}}{d!}\right) (g(k))^d e^{-g(k)} \quad (15)$$

389 Therefore, the number of simplices in $star(p)$ is given by

$$390 \quad Z(p) := \#(star(p)) = \sum_{k=0}^{\infty} Z_p(k).$$

391 Taking expectations, we get

$$392 \quad \mathbb{E}[Z(p)] \leq \sum_{k=0}^{\infty} \mathbb{E}[Z_p(k)],$$

$$393 \quad \leq 2^{2d \cdot dim + 3d} + \left(\frac{2^{3d \cdot dim + d + 2}}{d!} \right) \sum_{k=1}^{\infty} (g(k))^d \cdot e^{-g(k)},$$

394 which completes the proof of Theorem 15. ◀

395

396 **5 Randomized Incremental Construction (Proof of Theorem 18)**

397 In this section we show how the results in Section 3 imply bounds on the algorithmic
 398 complexity of d -skeleton of the Delaunay complex of ε -nets. We state a general version of
 399 a theorem for the complexity, in terms of time and space requirements, of the randomized
 400 incremental construction of the d -skeleton of the Delaunay complex of a given point set in
 401 \mathbb{R}^d .

402 ► **Theorem 19** (Boissonnat-Yvinec [3], Devillers [8]). *Let $F(s)$ denote the expected number of*
 403 *simplices that appear in the d -skeleton of the Delaunay complex of a uniform random sample*
 404 *of size s , from a given point set P . Then*

405 (i) [3], Theorem 5.2.3(1):

406 *The expected number of simplices that appear in the d -skeleton of the Delaunay*
 407 *complex of a point set $P \in \mathbb{R}^d$ during the randomized incremental construction is*

$$408 \quad O\left(\sum_{s=1}^n \frac{F(s)}{s}\right),$$

409 (ii) [8], Theorem 5(1):

410 *If $F(s) = O(s)$, the expected space complexity is $O(n)$.*

411 (iii) [8], Theorem 5(2):

412 *If $F(s) = O(s)$, then the expected time complexity is $\sum_{s=1}^n \frac{n-s}{s} = O(n \log n)$.*

413 We can now prove Theorem 18.

414 **Proof of Theorem 18.** We first observe that if the faces of the Voronoi diagram of \mathcal{X} satisfy
 415 the closed-ball property, then the usual randomized incremental algorithm is correct. We
 416 analyze now its complexity. If $\epsilon > \frac{1}{3\sqrt{d}}$ then, by Equation (1), $s = O(1)$ and the number of
 417 Delaunay simplices is also bounded by a constant (for a fixed d).

418 Otherwise, the expected space complexity follows from Theorem 14 and Theorem 16,
 419 by applying Theorem 19(i): the expected number of simplices that appear at any time in
 420 the duration of the algorithm is $\sum_{s \geq 1} \frac{F(s)}{s} \leq \sum_{s \geq 1} \frac{(2\sqrt{d})^d d^3}{s} \frac{1}{s} (d^3 (2\sqrt{d})^d)^d + \sum_{s \geq 4d}^n 2^{d(4d+1)+2} \leq$
 421 $n 2^{d(4d+1)+2} = O(n)$. From Theorem 19 (ii), the expected space complexity is also bounded
 422 by $O(n)$. The time complexity also follows directly by application of Theorem 19 (iii). We
 423 thus get Theorem 18. ◀

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427 **Appendix**

428 **Proof of Lemma 12.** The j -th moment $\mathbb{E}[X^j]$ is the sum over all ordered j -tuples $l = (l_1, l_2, \dots, l_j) \in B^j$
 429 with $l_i \in B$, $i = 1, \dots, j$, of the probability that a j -tuple gets chosen in the random sample A . This
 430 probability is

$$\begin{aligned}
 \mathbb{P}[l \in A^j] &= \frac{\binom{c-j}{a-j}}{\binom{c}{a}} = \frac{\prod_{i=0}^{j-1} (a-i)}{\prod_{i=0}^{j-1} (c-i)} = \frac{a^j}{c^j} \times \frac{\prod_{i=0}^{j-1} (1-i/a)}{\prod_{i=0}^{j-1} (1-i/c)} \\
 &\leq \frac{a^j}{c^j} \times \frac{1}{\prod_{i=0}^{j-1} \exp(\ln(1-i/c))} = \frac{a^j}{c^j} \times \exp\left(-\sum_{i=0}^{j-1} \ln(1-i/c)\right) \\
 &\leq \frac{a^j}{c^j} \times \exp\left(\sum_{i=0}^{j-1} \frac{2i}{c}\right) \leq \frac{a^j}{c^j} \times e^{\frac{j^2}{c}} \tag{16}
 \end{aligned}$$

$$\leq \frac{a^j}{c^j} \times \left(1 + \frac{2j^2}{c}\right) \leq \frac{3a^j}{c^j}. \tag{17}$$

435 where in step (16) we used that $-\ln(1-x) \leq 2x$, for $x \in [0, 1/2]$, and in step (17) we used $e^x \leq 1 + 2x$,
 436 for $x \in [0, 1]$. Now since the number of such tuples A^j is no more than b^j , the expected number of chosen
 437 tuples is given by $\mathbb{E}[X^j] \leq 3 \cdot \left(\frac{ab}{c}\right)^j = 3(\mathbb{E}[X])^j$. ◀

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