An algorithmic characterization of P-matricity II: corrections, refinements, and validation

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The paper "An algorithmic characterization of **P**-matricity" (*SIAM Journal on Matrix Analysis and Applications*, 34:3 (2013) 904–916, by the same authors as here) implicitly assumes that the iterates generated by the Newton-min algorithm for solving a linear complementarity problem of dimension n, which reads $0 \leqslant x \perp (Mx+q) \geqslant 0$, are uniquely determined by some index subsets of $[\![1,n]\!]$. Even if this is satisfied for a subset of vectors q that is dense in $\mathbb{R}^n$, this assumption is improper, in particular in the statements where the vector q is not subject to restrictions. The goal of the present contribution is to show that, despite this blunder, the main result of that paper is preserved. This one claims that a nondegenerate matrix M is a **P**-matrix if and only if the Newton-min algorithm does not cycle between two distinct points, whatever is q. The proof is hardly more complex, requiring only some additional refinements.

Keywords: linear complementarity problem, **NM**-matrix, Newton-min algorithm, **P**-matricity characterization, **P**-matrix, semismooth Newton method.

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1 Introduction

It was a bitter disappointment to notice, four years after its official publication, that an error has been made in a paper of ours [3; 2013]. Fortunately, its main result is preserved (it was actually "proved" twice in the faulty paper using different arguments, one of the "proofs" being less infected than the other by the careless confusion). The error in the reasoning is a systematic confusion between an implication and an equivalence, the latter being thought to be true because it is linked to a definition.

The present contribution is therefore of a special nature; it has an unusual contents. Its goal is twofold. On the one hand, it is important to provide a correct proof of the main result, which, we think, is still interesting. On the other hand, since the publication [3]

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cannot be removed, it is also instructive to point the finger at what is wrong in some of its claims. Both goals will be pursued simultaneously, since the path to the final result proposed in [3] is still appropriate. As far as possible, we will try to make the paper self-contained, except when we recall some results whose correct proof is in extenso in [3].

The *linear complementarity problem* we consider here and in [3] has a standard form [8], which can be described as follows. Being given a positive integer n, a real matrix $M \in \mathbb{R}^{n\times n}$, and a real vector $q \in \mathbb{R}^n$, the problem consists in determining a real vector $x \in \mathbb{R}^n$ such that one has in matrix notation

$$x \geqslant 0, \qquad Mx + q \geqslant 0, \qquad \text{and} \qquad x^{\mathsf{T}}(Mx+q) = 0,$$

where the inequalities have to be understood componentwise, the notation v^{T} is used to denote the transpose of the vector v, and $(u,v) \mapsto u^{\mathsf{T}}v := \sum_i u_i v_i$ is the Euclidean scalar product. We will usually refer to the problem by its abbreviated form, namely

$$\mathrm{LCP}(M,q): \qquad 0 \leqslant x \perp (Mx+q) \geqslant 0,$$

where $\perp$ denotes perpendicularity with respect to the Euclidean scalar product.

Notation and background. The set made of the first n positive integers is denoted by $[\![1,n]\!] := \{1,\dots,n\}$. If $I \subset [\![1,n]\!]$, $I^c := [\![1,n]\!] \smallsetminus I$ denotes its complement in $[\![1,n]\!]$. We denote by M_{IJ} the submatrix of the matrix $M \in \mathbb{R}^{n\times n}$ formed of its rows with indices in $I \subset [\![1,n]\!]$ and its columns with indices in $J \subset [\![1,n]\!]$. In [3] and here, it is always assumed that M is *nondegenerate*, meaning that the principal minors of M do not vanish (i.e., $\det M_{II} \neq 0$ for all $I \subset [\![1,n]\!]$; by convention $\det M_{\varnothing\varnothing} = 1$). The matrix M is a **P**-matrix if all its principal minors are positive (i.e., $\det M_{II} > 0$, for all $I \subset [\![1,n]\!]$). It is known that problem $\mathrm{LCP}(M,q)$ has a unique solution, whatever is q, if and only if M is a **P**-matrix [23, 8; 1958].

Definition 1.1 (node) For $I \subset [\![1,n]\!]$, we denote by $x^{(I)} \in \mathbb{R}^n$ the point defined by

$$x^{(I)}_{I^c} = 0 \qquad \text{and} \qquad (Mx^{(I)} + q)_I = 0, \tag{1.1a}$$

Such a point is called a *node*. When M is nondegenerate, the system (1.1a) defines $x^{(I)}$ unambiguously, since it has for unique solution

$$x^{(I)}_{I^c} = 0 \qquad \text{and} \qquad x^{(I)}_I = -M_{II}^{-1} q_I, \tag{1.1b}$$

where M_{II}^{-1} is a compact notation for $(M_{II})^{-1}$. □

Since there are 2^n distinct subsets of $[\![1,n]\!]$, there are at most 2^n nodes. Actually, this number of nodes depends on the vector q. For example, when $q = 0$, we see by (1.1b) that there is a single node: the zero vector.

The *Newton-min algorithm* is designed to find a solution to $\mathrm{LCP}(M,q)$. It computes the next iterate $x^+ \in \mathbb{R}^n$ from the current iterate $x \in \mathbb{R}^n$ by

$$x^+ := x^{(\mathcal{S}(x))}, \tag{1.2}$$

where the index selector $\mathcal{S} : \mathbb{R}^n \multimap [\![1,n]\!]$ is the multifunction defined at $x \in \mathbb{R}^n$ by

$$\mathcal{S}(x) := \{i \in [\![1,n]\!] : x_i > (Mx+q)_i\}. \tag{1.3}$$

Therefore, even if the first iterate is not a node, the next iterates are nodes. By (1.1b), each iteration requires computing the solution of a linear system of order $|\mathcal{S}(x)|$. We see that the Newton-min algorithm visits some of the potentially 2^n nodes of the problem, in the hope of finding a solution node, if any. We recall that, when M is a **P**-matrix, the Newton-min algorithm may cycle when $n \geqslant 3$ but not for $n \in \{1, 2\}$ [1, 2]. This algorithm is best viewed today as a semi-smooth Newton algorithm [21, 22] applied on the equation form of $\mathrm{LCP}(M, q)$ that reads $\min(x, Mx + q) = 0$ (the "min" operator also acts componentwise). We refer the reader to the paragraph 7 of the introduction of [2] for a discussion on the origin of the algorithm and to [7, 18, 12, 11, 5, 4, 19, 13, 17, 6, 9, 15, 16, 14] for other related contributions.

The paper is organized as follows. Section 2 presents the common source of the errors made in [3; 2013], as well as the strategy used here to adapt its results and to prove them adequately. Section 3 is dedicated to finding a valid version of the equivalence in the proposition 3.2 of [3]; it is obtained by strengthening one of its claim and weakening the other. Section 4 focuses on the proof of the main result, which remains correct and claims that a nondegenerate matrix M is a **P**-matrix if and only if the Newton-min algorithm does not cycle between two distinct points, whatever is q (but it may make cycles having 3 or more nodes).

References to the original paper. The references of the original paper [3] are specified here with the prefix [3]. Hence, "proposition [3].x.y" means proposition x.y of [3], "([3].a.b)" means formula (a.b) of [3], and section [3].α means section α of [3].

2 Common source of the errors in [3]

Even though it is not expressed in that way, the following wrong equivalence is implicitly used several times in [3]:

$$x^+ = x^{(I)} \quad \overset{\text{(usually wrong)}}{\Longleftrightarrow} \quad I = \mathcal{S}(x), \tag{2.1}$$

where x^+ is supposed to be the node computed from $x \in \mathbb{R}^n$ by the Newton-min algorithm, $x^{(I)}$ is the node defined by (1.1), and $\mathcal{S}$ is the index selector defined around (1.3).

For example in the beginning of the proof of proposition [3].3.2, it is essentially written: "*by definition*, the Newton-min algorithm (1.2) generates the node $x^{(J)}$ from the node $x^{(I)}$, for some given distinct index sets I and $J \subset [\![1, n]\!]$, if *and only if* ([3].3.4) holds". Let us clarify this claim. Formula ([3].3.4) reads

$$\begin{gathered} -(M_{II}^{-1} q_I)_{J^c} \leqslant 0, \qquad (M_{II}^{-1} q_I)_J < 0, \\ q_{I^c \cap J} < M_{(I^c \cap J)I} M_{II}^{-1} q_I, \qquad \text{and} \qquad M_{(I \cup J)^c I} M_{II}^{-1} q_I \leqslant q_{(I \cup J)^c}. \end{gathered} \tag{2.2}$$

Since, from (1.1), $x^{(I)}$ satisfies

$$\left\{ \begin{array}{l} x_I^{(I)} = -M_{II}^{-1} q_I \\ x_{I^c}^{(I)} = 0 \end{array} \right. \qquad \text{and} \qquad \left\{ \begin{array}{l} (Mx^{(I)} + q)_I = 0 \\ (Mx^{(I)} + q)_{I^c} = q_{I^c} - M_{I^c I} M_{II}^{-1} q_I, \end{array} \right. \tag{2.3}$$

wee see that

$$(2.2) \qquad \Longleftrightarrow \qquad \begin{array}{ll} x_{I \cap J^c}^{(I)} \leqslant (Mx^{(I)} + q)_{I \cap J^c}, & x_{I \cap J}^{(I)} > (Mx^{(I)} + q)_{I \cap J}, \\ x_{I^c \cap J}^{(I)} > (Mx^{(I)} + q)_{I^c \cap J}, & x_{I^c \cap J^c}^{(I)} \leqslant (Mx^{(I)} + q)_{I^c \cap J^c} \end{array}$$

$$\begin{aligned}
&\iff \quad x^{(I)}_{J^c} \leqslant (Mx^{(I)}+q)_{J^c}, \quad x^{(I)}_J > (Mx^{(I)}+q)_J \\
&\iff \quad J = \mathcal{S}(x^{(I)}),
\end{aligned}$$

where we have used the definition (1.3) of the index selector $\mathcal{S}$. We quote this equivalence below for a future reference:

$$(2.2) \quad \iff \quad J = \mathcal{S}(x^{(I)}). \tag{2.4}$$

The sentence highlighted above, found in the beginning of the proof of proposition [3].3.2, therefore claims that the Newton-min computes $x^{(J)}$ from $x^{(I)}$ if and only if $J = \mathcal{S}(x^{(I)})$. After the change in notation $x^{(I)} \curvearrowright x$ and $x^{(J)} \curvearrowright x^+$, this corresponds to the alleged equivalence (2.1).

The right-to-left implication "$\Leftarrow$" in (2.1) certainly holds by the very definition (1.2) of the Newton-min algorithm, but the left-to-right implication "$\Rightarrow$" may fail, because the node x^+ may also be defined by an index set $I' \subset [\![1,n]\!]$ different from the given index set I: $x^+ = x^{(I)} = x^{(I')}$. We stress this observation with a counter-example that will help becoming acquainted with the problem and the Newton-min algorithm.

Counter-example 2.1 (left-to-right implication in (2.1) may fail) Consider problem LCP(M,q) with $n=2$, $M = I_2$, and $q = e^1 := (1 \;\; 0)^\mathsf{T}$. Then, the problem has only two distinct nodes, namely $x^{(\{1\})} = x^{(\{1,2\})} = -e^1$ and $x^{(\varnothing)} = x^{(\{2\})} = 0$, the latter being the solution to the problem. If one takes $I = \{1\}$ and $J = \{2\}$, the Newton-min algorithm goes indeed from the node $x^{(\{1\})}$ to the solution $x^{(\{2\})}$, but it is not true that $\{2\} = \mathcal{S}(x^{(\{1\})})$. To see this, write $x^{(\{1\})} = -e^1$ and $(Mx^{(\{1\})} + q) = 0$, from which and the definition (1.3) of $\mathcal{S}$, one concludes that $\mathcal{S}(x^{(\{1\})}) = \varnothing$. $\square$

In other words, in (1.2), $\mathcal{S}(x)$ is just one of the index sets I' that defines the new iterate x^+ as a node $x^{(I')}$, not necessarily the one that is fixed in the context where this wrong equivalence is used (in proposition [3].3.2 the index sets are fixed outside its claims (i) and (ii)). From this point of view, it is convenient to introduce the following definition.

Definition 2.2 (uniquely determined node) A node x of LCP(M,q) is said to be *uniquely determined* if there is a unique index set $I \subset [\![1,n]\!]$ such that $x = x^{(I)}$. $\square$

It is precisely because some reasonings in [3] neglect the fact that the considered nodes may not be uniquely determined that corrections and refinements are desirable. The fact that all the nodes are uniquely determined depends on q and one can show that the q's for which that property occurs is dense in $\mathbb{R}^n$ (see section 5). Nevertheless, this density property seems to us useless when Motzkin's theorem of the alternative plays a key role in the analysis, like in [3]. Therefore our strategy to amend or to validate the results of [3] does not consist in using that density property.

Despite the systematic misinterpretation (2.1) of the meaning of the definition (1.2) of the Newton-min iteration, the proofs of [3] are not meaningless. Our approach consists therefore to give a precise statement of what these proofs provide and next to give complements to enrich these results in order to make the outcomes as close as possible to the results claimed in [3] (these are sometimes erroneous). This approach is actually mainly used for proposition [3].3.2, whose role is prominent. Occasionally, these complements are even not necessary. In particular, the main result of [3] is valid: a nondegenerate matrix M is a **P**-matrix if and only if the Newton-min algorithm does not cycle between two different nodes, whatever is q.

3 On proposition [3].3.2

Lemma 3.1 below gives the correct expression of the outcome of the proof of proposition [3].3.2. The conditions (i) of proposition [3].3.2 and lemma 3.1 are identical, but their conditions (ii) are very different. In particular, condition (ii) below is compatible with a cycle $x^{(I)} \to x^{(J)} \to x^{(I)}$ that would be made by the Newton-min algorithm for some q, while condition (ii) in proposition [3].3.2 claims that such a cycle does not occur. The latter claim is wrong! To see this, take $q = 0$ and arbitrary distinct index sets I and $J \subset [\![1, n]\!]$; then $x^{(I)} = x^{(J)} = 0$ and $Mx^{(I)} + q = Mx^{(J)} + q = 0$, so that the Newton-min algorithm makes the cycle $0 \to 0 \to 0$. In contrast, the conclusion in (ii) below is correct when $q = 0$, without having to use (i), since one cannot have $J = \mathcal{S}(x^{(I)})$ and $I = \mathcal{S}(x^{(J)})$ because $\mathcal{S}(x^{(I)})$ and $\mathcal{S}(x^{(J)})$ are both empty and $I \neq J$ by assumption. It is important to require $I \neq J$ in the assumption, otherwise (i) does not provide any information.

Recall that the *symmetric difference* of the two index sets I and $J \subset [\![1, n]\!]$ is defined and denoted by

$$I \mathbin{\triangle} J := (I \cap J^c) \cup (I^c \cap J) = (I \cup J) \smallsetminus (I \cap J).$$

Lemma 3.1 (NSC for $\boldsymbol{J \neq \mathcal{S}(x^{(I)})}$ or $\boldsymbol{I \neq \mathcal{S}(x^{(J)})}$) *Suppose that $M \in \mathbb{R}^{n\times n}$ is nondegenerate and let I and $J \subset [\![1, n]\!]$ be two different index sets. Then, the following conditions are equivalent:*

(i) *there is an $\alpha \in \mathbb{R}_+^{|I \triangle J|} \smallsetminus \{0\}$ such that*

$$\begin{aligned} &\begin{pmatrix} M_{(I\cap J^c)(I\cap J^c)} & -M_{(I\cap J^c)(I^c\cap J)} \\ -M_{(I^c\cap J)(I\cap J^c)} & M_{(I^c\cap J)(I^c\cap J)} \end{pmatrix}^{\mathsf{T}} \alpha \\ &\qquad \geqslant \begin{pmatrix} -M_{(I\cap J)(I\cap J^c)} & M_{(I\cap J)(I^c\cap J)} \end{pmatrix}^{\mathsf{T}} M_{(I\cap J)(I\cap J)}^{-\mathsf{T}} \begin{pmatrix} -M_{(I\cap J^c)(I\cap J)} \\ M_{(I^c\cap J)(I\cap J)} \end{pmatrix}^{\mathsf{T}} \alpha, \end{aligned} \tag{3.1}$$

where the right hand side is zero when $I \cap J = \varnothing$,

(ii) *whatever is q, one cannot have $J = \mathcal{S}(x^{(I)})$ and $I = \mathcal{S}(x^{(J)})$.*

Proof. We only sketch the proof, since it is very similar to the one of proposition [3].3.2. Only the equivalence (3.2) below differs, since it takes into account the fact that the equivalence (2.1) does not hold. Actually, instead of expressing ([3].3.4) and ([3].3.5) as the presence of a cycle $x^{(I)} \to x^{(J)} \to x^{(I)}$, which is not correct, we express it by its meaning derived from (2.4).

From (2.4), ([3].3.4) reads $J = \mathcal{S}(x^{(I)})$. Since ([3].3.5) can be obtained from ([3].3.4) by switching I and J, it reads $I = \mathcal{S}(x^{(J)})$. Therefore, for a fixed $q \in \mathbb{R}^n$, there holds:

$$([3].3.4) \text{ and } ([3].3.5) \quad \Longleftrightarrow \quad J = \mathcal{S}(x^{(I)}) \text{ and } I = \mathcal{S}(x^{(J)}). \tag{3.2}$$

Next, it is shown in the proof of proposition [3].3.2, using Motzkin's theorem of the alternative allowed, that

$$\begin{array}{rl} & \exists\, q \in \mathbb{R}^n \text{ satisfying ([3].3.4) and ([3].3.5)} \\ \iff & \nexists\ (\alpha,\alpha',\alpha'',\beta) \in \mathbb{R}_+^{|I\,\triangle\,J|} \times \mathbb{R}_+^{|I\cap J|} \times \mathbb{R}_+^{|I\cap J|} \times \mathbb{R}_+^{|I\,\triangle\,J|} \\ & \text{that satisfies } (\alpha,\alpha',\alpha'') \neq 0 \text{ and ([3].3.6).} \end{array} \tag{3.3}$$

Finally, it is shown in points 2 and 3 of the proof of proposition [3].3.2 that (3.3) simplifies in

$$\begin{array}{rl} & \exists\, q \in \mathbb{R}^n \text{ satisfying ([3].3.4) and ([3].3.5)} \\ \iff & \nexists\ \alpha \in \mathbb{R}_+^{|I\,\triangle\,J|} \setminus \{0\} \text{ that satisfies (3.1).} \end{array} \tag{3.4}$$

One can now show the equivalence between (i) and (ii). Indeed, by the contrapositive of (3.4), (i) holds if and only if there is no $q \in \mathbb{R}^n$ satisfying ([3].3.4) and ([3].3.5). Finally, the contrapositive of (3.2) shows that this is equivalent to saying that, whatever is $q \in \mathbb{R}^n$, one cannot satisfy both $J = \mathcal{S}(x^{(I)})$ and $I = \mathcal{S}(x^{(J)})$. This is precisely (ii). □

The next proposition is the correct version of proposition [3].3.2. The equivalence between its claims (i) and (ii) is obtained by making point (i) of proposition [3].3.2 stronger and point (ii) of proposition [3].3.2 weaker. Point (i) below is stronger since there must exists a nonnegative α verifying (3.5) for various index sets I and J, not only for the index sets I_0 and J_0 given in the introduction of the proposition. Point (ii) below is weaker since the q's are restricted to those ensuring that $x^{(I_0)}$ differs from $x^{(J_0)}$ (in particular, $q = 0$ is not among the q's accepted in (ii), since then $x^{(I_0)} = x^{(J_0)} = 0$).

Proposition 3.2 (no cycle $x^{(I)} \to x^{(J)} \to x^{(I)}$) *Suppose that $M \in \mathbb{R}^{n\times n}$ is nondegenerate and let I_0 and $J_0 \subset [\![1,n]\!]$ be two different index sets. Then, the following conditions are equivalent:*

(i) *for any two different index sets I and $J \subset [\![1,n]\!]$ such that $x^{(I)} = x^{(I_0)}$ and $x^{(J)} = x^{(J_0)}$, there is an $\alpha \in \mathbb{R}_+^{|I\,\triangle\,J|} \setminus \{0\}$ such that*

$$\begin{pmatrix} M_{(I\cap J^c)(I\cap J^c)} & -M_{(I\cap J^c)(I^c\cap J)} \\ -M_{(I^c\cap J)(I\cap J^c)} & M_{(I^c\cap J)(I^c\cap J)} \end{pmatrix}^{\mathsf{T}} \alpha$$
$$\geqslant \begin{pmatrix} -M_{(I\cap J)(I\cap J^c)} & M_{(I\cap J)(I^c\cap J)} \end{pmatrix}^{\mathsf{T}} M_{(I\cap J)(I\cap J)}^{-\mathsf{T}} \begin{pmatrix} -M_{(I\cap J^c)(I\cap J)} \\ M_{(I^c\cap J)(I\cap J)} \end{pmatrix}^{\mathsf{T}} \alpha, \tag{3.5}$$

where the right hand side is zero when $I \cap J = \varnothing$,

(ii) *whatever is q such that $x^{(I_0)} \neq x^{(J_0)}$, the Newton-min algorithm does not make the cycle $x^{(I_0)} \to x^{(J_0)} \to x^{(I_0)}$ when it is used to solve* $\mathrm{LCP}(M,q)$.

Proof. $[(i) \Rightarrow (ii)]$ We prove the contrapositive. Let $q \in \mathbb{R}^n$ be such that $x^{(I_0)} \neq x^{(J_0)}$ and such that the Newton-min algorithm makes the cycle $x^{(I_0)} \to x^{(J_0)} \to x^{(I_0)}$ when it is used to solve $\mathrm{LCP}(M,q)$. Define $J := \mathcal{S}(x^{(I_0)})$ and $I := \mathcal{S}(x^{(J_0)})$. Then, by the definition (1.2) of the Newton-min algorithm, this means that the algorithm computes $x^{(J)}$ from $x^{(I_0)}$ and $x^{(I)}$ from $x^{(J_0)}$. By the existence of the cycle $x^{(I_0)} \to x^{(J_0)} \to x^{(I_0)}$, it follows that $x^{(I)} = x^{(I_0)}$ and $x^{(J)} = x^{(J_0)}$, implying in turn that $J := \mathcal{S}(x^{(I)})$ and $I := \mathcal{S}(x^{(J)})$. Of course $I \neq J$ since $x^{(I)} \neq x^{(J)}$. Hence, we have shown that the condition (ii) of lemma 3.1 does not hold for the determined and distinct index sets I and J. By the contrapositive of the implication $(i) \Rightarrow (ii)$ of that lemma, there exists *no* $\alpha \in \mathbb{R}_+^{|I \,\triangle\, J|} \setminus \{0\}$ such that (3.1) holds. This last claim contradicts point (i) of the present proposition.

[$(ii) \Rightarrow (i)$] Suppose now that (ii) holds. Let I and $J \subset [\![1,n]\!]$ be any two *different* subsets such that $x^{(I)} = x^{(I_0)}$ and $x^{(J)} = x^{(J_0)}$. Let us show that the condition (ii) of lemma 3.1 holds for those I and J. Choose any $q \in \mathbb{R}^n$. We consider two cases.

- If $x^{(I)} \neq x^{(J)}$, one cannot have $J = \mathcal{S}(x^{(I)})$ and $I = \mathcal{S}(x^{(J)})$, since otherwise we would have the cycle $x^{(I)} \to x^{(J)} \to x^{(I)}$, identical to the cycle $x^{(I_0)} \to x^{(J_0)} \to x^{(I_0)}$ between the two distinct points $x^{(I_0)}$ and $x^{(J_0)}$, which would contradict (ii).
- If $x^{(I)} = x^{(J)}$, one cannot have $J = \mathcal{S}(x^{(I)})$ and $I = \mathcal{S}(x^{(J)})$, since otherwise one would have $I = J$, which would contradict the assumption $I \neq J$ made above.

Since condition (ii) of lemma 3.1 holds for the index sets I and J, its implication $(ii) \Rightarrow (i)$ shows that point (i) of the present proposition holds with the chosen I and J. □

The next counter-example shows that, in point (i) of proposition 3.2, one must consider all the different index sets I and $J \subset [\![1,n]\!]$ such that $x^{(I)} = x^{(I_0)}$ and $x^{(J)} = x^{(J_0)}$, not only I_0 and J_0 like in proposition [3].3.2, in order to get point (ii) of proposition 3.2.

Counter-example 3.3 ($(i) \not\Rightarrow (ii)$ without all adequate I and J in (i)) Consider the instance of problem LCP(M,q) with $n = 2$,

$$M = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \text{and} \qquad q = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

The matrix M is nondegenerate but not a $\mathbf{P}$-matrix (since $M_{11} \leqslant 0$). Therefore the Newton-min algorithm is well defined but may cycle between two distinct nodes (as this will be confirmed in theorem 4.2 below), depending on the value of q, and it does indeed for the given q (the problem LCP(M,q) has no solution actually).

The nodes corresponding to the index sets $I_0 := \varnothing$ and $J_0 := \{1,2\}$ read

$$x^{(I_0)} = 0 \qquad \text{and} \qquad x^{(J_0)} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Since

$$Mx^{(I_0)} + q = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad Mx^{(J_0)} + q = 0,$$

it follows that $\mathcal{S}(x^{(I_0)}) = \{1\}$ and $\mathcal{S}(x^{(J_0)}) = \varnothing$. Now, $x^{(\{1\})} = x^{(J_0)}$, so that the Newton-min algorithm makes the cycle $x^{(I_0)} \to x^{(J_0)} \to x^{(I_0)}$ between the two distinct nodes $x^{(I_0)}$ and $x^{(J_0)}$. Hence condition (ii) of proposition 3.2 does not hold.

Consider now the condition (i) of proposition 3.2 for this problem and the chosen index sets I_0 and J_0. To be true, the condition (i) requires in particular that, for the two different index sets $I = I_0$ and $J = J_0$, there exists a nonzero $\alpha \in \mathbb{R}_+^2$ such that (3.5) holds, that is $M^{\mathsf{T}}\alpha \geqslant 0$. This is clearly possible, for example by taking $\alpha = (1 \;\; 1)^{\mathsf{T}}$. Therefore the implication $(i) \Rightarrow (ii)$ would be false if (i) was expressed with only $I = I_0$ and $J = J_0$.

Now the requirement on α in condition (i) is not satisfied if one takes $I = I_0 := \varnothing$ and $J := \{1\}$. This is an acceptable choice of I and J, since $I \neq J$, $x^{(I)} = x^{(I_0)}$, and $x^{(J)} = x^{(J_0)}$.

For that choice, the requirement in (i) reduces to "$M_{11}\alpha \geqslant 0$ for some $\alpha > 0$", a property that is clearly false. Therefore, for the considered problem, (i) does not hold, like (ii).

In conclusion, one cannot fixed $I = I_0$ and $J = J_0$ in condition (i) of proposition 3.2 but must verify the existence of a nonzero nonnegative α verifying (3.5) for all the different index sets I and $J \subset [\![1, n]\!]$ such that $x^{(I)} = x^{(I_0)}$ and $x^{(J)} = x^{(J_0)}$. □

4 Revision of section [3].4

The next lemma reformulates the contrapositive of Lemma [3].4.1 in terms of the index selector $\mathcal{S}$ defined in (1.3). It is now viewed as a condition such that the Newton-min algorithm computes a nonzero displacement from $x^{(I)}$. This lemma is no longer used in the proof of theorem 4.2, like this was the case in theorem [3].4.2, but in proposition 4.4.

Lemma 4.1 (nonzero displacement) *Suppose that M is nondegenerate and let be given $q \in \mathbb{R}^n$ and $I \subset [\![1, n]\!]$. Then*

$$\mathcal{S}(x^{(I)}) \smallsetminus I \neq \varnothing \qquad \Longrightarrow \qquad x^{(\mathcal{S}(x^{(I)}))} \neq x^{(I)}.$$

Proof. Let $J \coloneqq \mathcal{S}(x^{(I)})$. On the one hand, since by assumption $I^c \cap J = J \smallsetminus I$ is nonempty, there holds

$$\begin{aligned} (Mx^{(I)} + q)_{I^c \cap J} &< x^{(I)}_{I^c \cap J} \qquad [\text{definition of } J = \mathcal{S}(x^{(I)}) \text{ and } (1.3)] \\ &= 0 \qquad [\text{the components in } I^c \text{ of } x^{(I)} \text{ vanish by the definition } 1.1]. \end{aligned}$$

On the other hand,

$$(Mx^{(J)} + q)_J = 0,$$

by the definition 1.1 of the node $x^{(J)}$. Therefore $(Mx^{(I)} + q)_{I^c \cap J} \neq (Mx^{(J)} + q)_{I^c \cap J}$, since the first vector is negative and the second vanishes. Since $I^c \cap J \neq \varnothing$, this certainly implies that $x^{(J)} \neq x^{(I)}$. □

Let us now consider the revision of theorem [3].4.2, which is given in theorem 4.2 below. It is the main result of the paper. The statement of the latter theorem is almost identical to the former, except that in (iii) the two considered nodes are said to be distinct. The changes in the proof are the following.

- The proof of the implication $(i) \Rightarrow (ii)$ has been changed to take into account the fact that the equivalence (2.1) does not necessarily hold. Nevertheless, the argument is essentially the same after a redefinition of the index sets associated with the nodes of the considered cycle.
- With the small change in the statement of (iii), the implication $(ii) \Rightarrow (iii)$ becomes straightforward and no longer uses lemma 4.1.
- The implication $(iii) \Rightarrow (iv)$ is proved similarly, but with the updated version of the implication $(ii) \Rightarrow (i)$ of proposition 3.2, whose condition (ii) selects the q's for which $x^{(I_0)} \neq x^{(J_0)}$.

- We have taken the opportunity of this new proof to be a little more explicit in the proof of the implication $(iv) \Rightarrow (i)$.

For the reader's convenience, we have reproduced in full the parts of proof of theorem [3].4.2 that need no modification.

Let us recall some notation and associated properties. We denote by $\operatorname{cof}(M)$ the *cofactor matrix* of a matrix $M \in \mathbb{R}^{n\times n}$, whose element $[\operatorname{cof}(M)]_{ij}$ is the *cofactor* $\operatorname{cof}(M_{ij})$ of the element M_{ij} of M, that is

$$\operatorname{cof}(M_{ij}) \coloneqq (-1)^{i+j} \det M_{([\![1,n]\!]\smallsetminus\{i\})([\![1,n]\!]\smallsetminus\{j\})}. \tag{4.1}$$

We use the notation $\operatorname{cof}_{II}(M_{ij})$ for the cofactor of the element M_{ij} in M_{II}, assuming that both i and $j \in I$. Recall [20; 1987, chapter VI] that for any index i and j:

$$\det M = \sum_{i'} M_{i'j} \operatorname{cof}(M_{i'j}) = \sum_{j'} M_{ij'} \operatorname{cof}(M_{ij'}) \tag{4.2}$$

and that

$$M^{-1} = (\det M)^{-1} \operatorname{cof}(M^{\mathsf{T}}). \tag{4.3}$$

We also recall the following characterization of $\mathbf{P}$-matricity [10, 8; 1962]:

$$M \in \mathbf{P} \qquad \Longleftrightarrow \qquad \text{any } x \text{ verifying } x \cdot (Mx) \leqslant 0 \text{ vanishes}, \tag{4.4}$$

where we have denoted by $u \cdot v$ the *Hadamard product* of the vectors u and v, which is the vector whose ith component is $u_i v_i$.

Theorem 4.2 (a characterization of P-matricity) *Suppose that $M \in \mathbb{R}^{n\times n}$ is nondegenerate. Then the following conditions are equivalent:*

(i) *$M \in \mathbf{P}$,*

(ii) *for any q, the Newton-min algorithm does not cycle between two distinct nodes when it is used to solve* $\operatorname{LCP}(M, q)$,

(iii) *for any q, for any subset $J \subset [\![1,n]\!]$, and for any index $i \in [\![1,n]\!] \smallsetminus J$, such that $x^{(J)} \neq x^{(J\cup\{i\})}$, the Newton-min algorithm does not cycle between the nodes $x^{(J)}$ and $x^{(J\cup\{i\})}$ when it is used to solve* $\operatorname{LCP}(M, q)$,

(iv) *for any subset $J \subset [\![1,n]\!]$ and any index $i \in [\![1,n]\!] \smallsetminus J$, there holds*

$$M_{ii} \geqslant M_{\{i\}J} M_{JJ}^{-1} M_{J\{i\}}, \tag{4.5}$$

where the right hand side is zero when $J = \varnothing$.

Proof. $[(i) \Rightarrow (ii)]$ We prove the contrapositive, assuming that the algorithm visits in order the following nodes $x^{(I_0)} \to x^{(J_0)} \to x^{(I_0)}$, for some I_0 and $J_0 \subset [\![1,n]\!]$ and some $q \in \mathbb{R}^n$ such that $x^{(I_0)} \neq x^{(J_0)}$. We simplify the notation by setting $x^1 \coloneqq x^{(I_0)}$ and $x^2 \coloneqq x^{(J_0)}$.

The proof below uses the index set $I \coloneqq \mathcal{S}(x^2)$ instead of I_0 and the index set $J \coloneqq \mathcal{S}(x^1)$ instead of J_0 (the inequalities in (4.6) may not hold if I is replaced by I_0 and J by J_0, unlike what was claimed in [1]). By the definition (1.3) of these index sets, there hold

$$x^1_{J^c} \leqslant (Mx^1 + q)_{J^c} \qquad \text{and} \qquad x^1_J > (Mx^1 + q)_J, \tag{4.6a}$$

$$x^2_{I^c} \leqslant (Mx^2 + q)_{I^c} \qquad \text{and} \qquad x^2_I > (Mx^2 + q)_I. \tag{4.6b}$$

We now express the fact that the cycle $x^1 \to x^2 \to x^1$ occurs: by the definition (1.2) of the Newton-min algorithm, there hold $x^1 = x^{(\mathcal{S}(x^2))} = x^{(I)}$ and $x^2 = x^{(\mathcal{S}(x^1))} = x^{(J)}$, so that by the definition 1.1 of a node:

$$x^1_{I^c} = 0 \qquad \text{and} \qquad (Mx^1+q)_I = 0, \tag{4.7a}$$
$$x^2_{J^c} = 0 \qquad \text{and} \qquad (Mx^2+q)_J = 0. \tag{4.7b}$$

Using $(4.7a)_1$ and $(4.7b)_1$, we get, after a possible rearrangement of the component order

$$x^2 - x^1 = \begin{pmatrix} 0_{I\cap J^c} \\ x^2_{I\cap J} \\ x^2_{I^c\cap J} \\ 0_{I^c\cap J^c} \end{pmatrix} - \begin{pmatrix} x^1_{I\cap J^c} \\ x^1_{I\cap J} \\ 0_{I^c\cap J} \\ 0_{I^c\cap J^c} \end{pmatrix} = \begin{pmatrix} -x^1_{I\cap J^c} \\ (x^2-x^1)_{I\cap J} \\ x^2_{I^c\cap J} \\ 0_{I^c\cap J^c} \end{pmatrix}. \quad \begin{matrix} [+] \\ [?] \\ [-] \\ [0] \end{matrix}$$

The extra column on the right gives the sign of each component, when this is possible: the components of x^2-x^1 with indices in $I\cap J^c$ are nonnegative since $-x^1_{I\cap J^c} \geqslant -(Mx^1+q)_{I\cap J^c}$ [by $(4.6a)_1$] $= 0$ [by $(4.7a)_2$] and the components of x^2-x^1 with indices in $I^c\cap J$ are nonpositive since $x^2_{I^c\cap J} \leqslant (Mx^2+q)_{I^c\cap J}$ [by $(4.6b)_1$] $= 0$ [by $(4.7b)_2$]. Furthermore, by $(4.7a)_2$ and $(4.7b)_2$, there holds

$$M(x^2-x^1) = \begin{pmatrix} (Mx^2)_{I\cap J^c} \\ -q_{I\cap J} \\ -q_{I^c\cap J} \\ (Mx^2)_{I^c\cap J^c} \end{pmatrix} - \begin{pmatrix} -q_{I\cap J^c} \\ -q_{I\cap J} \\ (Mx^1)_{I^c\cap J} \\ (Mx^1)_{I^c\cap J^c} \end{pmatrix} = \begin{pmatrix} (Mx^2+q)_{I\cap J^c} \\ 0_{I\cap J} \\ -(Mx^1+q)_{I^c\cap J} \\ (M(x^2-x^1))_{I^c\cap J^c} \end{pmatrix}. \quad \begin{matrix} [-] \\ [0] \\ [+] \\ [?] \end{matrix}$$

The extra column on the right gives the sign of each component, when this is possible: the components of $M(x^2-x^1)$ with indices in $I\cap J^c$ are nonpositive since $(Mx^2+q)_{I\cap J^c} \leqslant x^2_{I\cap J^c}$ [by $(4.6b)_2$] $= 0$ [by $(4.7b)_1$] and the components of $M(x^2-x^1)$ with indices in $I^c\cap J$ are nonnegative since $-(Mx^1+q)_{I^c\cap J} \geqslant -x^1_{I^c\cap J}$ [by $(4.6a)_2$] $= 0$ [by $(4.7a)_1$]. Therefore

$$(x^2-x^1)\cdot M(x^2-x^1) \leqslant 0.$$

Since $x^1 \neq x^2$, M cannot be a $\mathbf{P}$-matrix (see (4.4)).

[$(ii) \Rightarrow (iii)$] Straightforward, since (iii) is just (ii) for the particular distinct nodes $x^{(J)}$ and $x^{(J\cup\{i\})}$.

[$(iii) \Rightarrow (iv)$] Let J and i be like in (iv), and set $I = J\cup\{i\}$. By (iii), whatever is q such that $x^{(I)} \neq x^{(J)}$, the Newton-min algorithm does not cycle between the nodes $x^{(I)}$ and $x^{(J)}$ when it is used to solve LCP(M,q). Then, the implication $(ii) \Rightarrow (i)$ of proposition 3.2 shows that there is a scalar $\alpha > 0$ such that (3.5) holds. Since $I\cap J^c = \{i\}$, $I^c \cap J = \varnothing$, $I\cap J = J$, $I \mathbin{\triangle} J = \{i\}$, this inequality (3.5) simplifies in

$$M_{ii}\alpha \geqslant \left(-M_{J\{i\}}\right)^{\mathsf{T}} M_{JJ}^{-\mathsf{T}} \left(-M_{\{i\}J}\right)^{\mathsf{T}} \alpha.$$

Now α is a positive scalar that can be eliminated and the right-hand side is a scalar (hence equal to its transpose), so that the above inequality becomes (4.5). In case $J=\varnothing$, inequality (3.5) simply yields $M_{ii} \geqslant 0$.

[$(iv) \Rightarrow (i)$] We prove by induction that $\det M_{II} > 0$ for any $I \subset [\![1,n]\!]$, which is equivalent to $M \in \mathbf{P}$. By applying (iv) with $J=\varnothing$, we obtain $M_{ii} > 0$ for a nondegenerate matrix, so that $\det M_{II} > 0$ when $|I| = 1$. Now, assume that J and i are chosen like in (iv), that $I = J \cup \{i\}$, that $\det M_{JJ} > 0$ (induction assumption), and let us show that $\det M_{II} > 0$, which will conclude the proof of $(iv) \Rightarrow (i)$.

Let us denote the indices in J by j_k, $k \in [\![1, |J|]\!]$, and let us label the elements of M_{JJ} by their indices in J. Using the cofactor matrix of M_{JJ} in (4.5) and the induction assumption $\det M_{JJ} > 0$, one gets (see the explanation of (4.8) and (4.9) below)

$$
\begin{aligned}
0 &\leqslant M_{ii} \det M_{JJ} - M_{\{i\}J} \operatorname{cof}(M_{JJ}^{\mathsf{T}}) M_{J\{i\}} \qquad [(4.5),\ (4.3),\ \det M_{JJ} > 0] \\
&= M_{ii} \det M_{JJ} - \sum_{k=1}^{|J|} \sum_{l=1}^{|J|} M_{ij_k} \operatorname{cof}_{JJ}([M_{JJ}]_{j_l j_k}) M_{j_l i} \\
&= M_{ii} \det M_{JJ} - \sum_{k=1}^{|J|} \sum_{l=1}^{|J|} M_{ij_k} (-1)^{l+k} \det M_{(J\smallsetminus\{j_l\})(J\smallsetminus\{j_k\})} M_{j_l i} \qquad [(4.1)] \\
&= M_{ii} \det M_{JJ} + \sum_{k=1}^{|J|} (-1)^{k+|J|+1} M_{ij_k} \sum_{l=1}^{|J|} M_{j_l i} (-1)^{l+|J|} \det M_{(J\smallsetminus\{j_l\})(J\smallsetminus\{j_k\})} \\
&= M_{ii} \det M_{JJ} + \sum_{k=1}^{|J|} (-1)^{k+|J|+1} M_{ij_k} \det \begin{pmatrix} M_{J(J\smallsetminus\{j_k\})} & M_{J\{i\}} \end{pmatrix} \qquad [(4.2)_1] \qquad (4.8) \\
&= \det M_{II} \qquad [(4.2)_2]. \qquad (4.9)
\end{aligned}
$$

Formula (4.8) comes from the computation of the determinant of the $|J| \times |J|$ matrix

$$\begin{pmatrix} M_{J(J\smallsetminus\{j_k\})} & M_{J\{i\}} \end{pmatrix}$$

using the first identity in (4.2) on its last column. Formula (4.9) computes the determinant of the $|I| \times |I|$ matrix

$$\begin{pmatrix} M_{JJ} & M_{J\{i\}} \\ M_{\{i\}J} & M_{ii} \end{pmatrix}$$

using the second identity in (4.2) on its last row. This one is indeed the determinant of M_{II} after permutations of two rows and two columns to put the row i and column i at the right place in I (this does not affect the sign of the determinant). Finally, using the nondegeneracy of M, we get $\det M_{II} > 0$. □

Let **NM** be the class of nondegenerate matrices $M \in \mathbb{R}^{n\times n}$ such that the Newton-min algorithm converges, when it is used to solve $\mathrm{LCP}(M, q)$, whatever is q and the initial point. The corollary [3].4.3 is still valid and reads as follows. We omit its proof, which needs no change.

Corollary 4.3 (NM is included in P) *The set of nondegenerate matrices M ensuring the convergence of the Newton-min algorithm when it is used to solve* $\mathrm{LCP}(M, q)$*, whatever are the vector q and the initial point, is included in* **P***. More compactly*

$$\mathbf{NM} \subset \mathbf{P}. \qquad (4.10)$$

To be complete, we reproduce proposition [3].4.4 with one additional property, which is that the points $x^{(I)}$ and $x^{(J)}$ introduced in the proposition are different. The interest of that property is that the proposition can then be used to prove the contrapositive of the implication $(iii) \Rightarrow (i)$ of theorem 4.2; see the comment after the proposition.

Proposition 4.4 (2-cycle for $M \notin \mathbf{P}$) *Suppose that the nondegenerate matrix M is not a $\mathbf{P}$-matrix. Then*

1) *there are two index sets I and $J \subset [\![1,n]\!]$ and an index $i \in [\![1,n]\!]$ such that $I = J \cup \{i\}$, $\det M_{II} < 0$, and $\det M_{JJ} > 0$,*
2) *for any two index sets I and $J \subset [\![1,n]\!]$ and an index $i \in [\![1,n]\!]$ having the properties given in point 1, the Newton-min algorithm cycles between the two distinct nodes $x^{(I)}$ and $x^{(J)}$ when the components of q are determined in order as follows*

$$q_J = -M_{JJ} e^J, \tag{4.11}$$

$$q_i = -M_{iJ} e^J - \varepsilon, \quad \text{with } 0 < \varepsilon < \frac{|\det M_{II}|}{\max_{j \in J} [\operatorname{cof}_{II}(M_{ij})]^+}, \tag{4.12}$$

$$q_{I^c} \geqslant \max\left(M_{I^cJ} M_{JJ}^{-1} q_J, M_{I^cI} M_{II}^{-1} q_I\right), \tag{4.13}$$

where e^J is the vector of all ones in $\mathbb{R}^{|J|}$.

Proof. The proof given in [3] is still valid, so that we only have to show that $x^{(I)} \neq x^{(J)}$ in point 2, which is straightforward. Indeed, the proof in [3] shows that the Newton-min algorithm makes the cycle $x^{(I)} \to x^{(J)} \to x^{(I)}$, with $J = \mathcal{S}(x^{(I)})$ and $I = \mathcal{S}(x^{(J)})$. Since $\mathcal{S}(x^{(J)}) \setminus J = I \setminus J = \{i\} \neq \varnothing$, lemma 4.1 implies that $x^{(I)} \neq x^{(J)}$. □

To conclude this section, let us show how proposition 4.4 can be used to prove the contrapositive of the implication $(iii) \Rightarrow (i)$ of theorem 4.2. Indeed, if (i) does not hold, M is not a $\mathbf{P}$-matrix (but is nondegenerate by assumption). By proposition 4.4, one can find two index sets I and $J \subset [\![1,n]\!]$ and an index $i \in [\![1,n]\!]$ such that $I = J \cup \{i\}$, as well a vector q, such that $x^{(I)} \neq x^{(J)}$ and the Newton-min algorithm makes the cycle $x^{(I)} \to x^{(J)} \to x^{(I)}$. Hence (iii) does not hold.

5 Uniquely determined nodes

This section highlights conditions under which the nodes of a particular instance of problem $\mathrm{LCP}(M,q)$ are *uniquely determined*, a concept introduced by definition 2.2. In particular, it is shown that, given a nondegenerate matrix M, the set of q's in $\mathbb{R}^n$ such that all the nodes are uniquely determined is dense in $\mathbb{R}^n$ (proposition 5.5).

Lemma 5.1 (2^n nodes) *Consider problem $\mathrm{LCP}(M,q)$ with a nondegenerate $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$. Then the following properties are equivalent*

(i) *problem $\mathrm{LCP}(M,q)$ has 2^n distinct nodes,* *(ii) the map* $\mathcal{X} : I \subset [\![1,n]\!] \mapsto x^{(I)} \in \mathbb{R}^n$ *is bijective,*

and these properties imply that

(iii) whatever is $x \in \mathbb{R}^n$*, the equivalence* (2.1) *holds, with* x^+ *being the point generated by the Newton-min algorithm from* x.

PROOF. $[(i) \Rightarrow (ii)]$ By construction, the map $\mathcal{X}$ is surjective. Since there are 2^n distinct intervals $I \subset [\![1,n]\!]$, the map $\mathcal{X}$ must also be injective if the are 2^n distinct nodes $x^{(I)}$.

$[(ii) \Rightarrow (i)]$ If the map $\mathcal{X}$ is bijective, the number of nodes $x^{(I)}$ is equal to the number of elements in $\mathcal{P}([\![1,n]\!])$, which is 2^n.

$[(ii) \Rightarrow (iii)]$ The implication "$\Leftarrow$" in (2.1) holds by definition of the Newton-min algorithm. For the reverse implication "$\Rightarrow$" in (2.1), suppose that $x^+ = x^{(I)}$ for some I. By definition of the algorithm, $x^+ = x^{(\mathcal{S}(x))}$, so that $x^{(I)} = x^{(\mathcal{S}(x))}$. By (ii), there holds $I = \mathcal{S}(x)$. $\square$

For the purpose of clarification, let us introduce the following notion. We say that a node $x^{(I)}$ is *reachable* if there is an $x \in \mathbb{R}^n$ such that $x^{(I)} = x^{(\mathcal{S}(x))}$ or, equivalently, if there is an $x \in \mathbb{R}^n$ such that the Newton-min algorithm starting at x computes $x^{(I)}$ as its next iterate. Let us denote by $\mathcal{N}_r$ the set of reachable nodes. This one is the range space of $\mathcal{X} \circ \mathcal{S}$:

$$\mathcal{N}_r = (\mathcal{X} \circ \mathcal{S})(\mathbb{R}^n).$$

Remark 5.2 (reachable nodes) *Not all the nodes of problem* LCP(M,q) *are reachable.* For example, if $M = I_n$ and $q > 0$, whatever is $x \in \mathbb{R}^n$, there holds $\mathcal{S}(x) = \varnothing$, so that $x^+ = x^{(\varnothing)} = 0$. Now there are 2^n distinct nodes for that problem, so that $2^n - 1$ nodes are not reachable.

Remark 5.3 *It is not true that the implication* $(iii) \Rightarrow (i)$ *or* (ii) *holds in the previous lemma.* Consider for example the case when

$$n = 2, \qquad M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \text{and} \qquad q = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{5.1}$$

The nodes of the problem are

$$x^{(\varnothing)} = 0, \qquad x^{(\{1\})} = x^{(\{1,2\})} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad x^{(\{2\})} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Only 2 nodes can be reached by the Newton-min algorithm, the nodes $x^{(\varnothing)}$ and $x^{(\{2\})}$. Indeed, the first node is reached when $x_1 \geqslant -1$ (in this case $\mathcal{S}(x)$ is indeed $\varnothing$), while the second is reached when $x_1 < -1$ (in this case $\mathcal{S}(x)$ is indeed $\{2\}$). These reachable nodes are uniquely identified by an index set:

$$\begin{aligned} x^+ = 0 = x^{(\varnothing)} \quad &\Longrightarrow \quad \varnothing = \mathcal{S}(x), \\ x^+ = (0,-1) = x^{(\{2\})} \quad &\Longrightarrow \quad \{2\} = \mathcal{S}(x). \end{aligned}$$

Therefore the equivalence (2.1) holds, although there are less than 2^n nodes. $\square$

Lemma 5.4 (uniquely determined reachable nodes) *Consider problem* LCP (M,q) *with a nondegenerate* $M \in \mathbb{R}^{n\times n}$ *and* $q \in \mathbb{R}^n$*. Then, the following properties are equivalent*

(i) *for any* $x^+ \in \mathcal{N}_r$*, there is a unique* $I \subset [\![1,n]\!]$ *such that* $x^+ = x^{(I)}$*,*

(ii) *whatever is* $x \in \mathbb{R}^n$ *and* x^+ *generated by the Newton-min algorithm from* x*, the equivalence* (2.1) *holds.*

Proof. $[(i) \Rightarrow (ii)]$ The implication "$\Leftarrow$" in (2.1) holds by definition of the Newton-min algorithm. For the reverse implication "$\Rightarrow$" in (2.1), suppose that $x^+ = x^{(I)}$ for some I. By definition of the algorithm, $x^+ = x^{(\mathcal{S}(x))}$, so that $x^{(I)} = x^{(\mathcal{S}(x))}$. By (i) and the fact that $x^{(I)}$ is reachable node, there holds $I = \mathcal{S}(x)$.

$[(ii) \Rightarrow (i)]$ Let $x^+ \in \mathcal{N}_r$. Then, there is some $x \in \mathbb{R}^n$ such that x^+ is the iterate computed from x by the Newton-min algorithm, which reads $x^+ = x^{(\mathcal{S}(x))}$. Now, if $x = x^{(I)}$, there holds $I = \mathcal{S}(x)$ by (ii), implying the uniqueness of the set I such that $x = x^{(I)}$. □

Let us introduce the set

$$\mathcal{Q}(M) \coloneqq \{q \in \mathbb{R}^n : \mathrm{LCP}(M,q) \text{ has } 2^n \text{ distinct nodes}\}.$$

Proposition 5.5 (properties of $\mathcal{Q}(M)$) *Suppose that* $M \in \mathbb{R}^{n\times n}$ *is nondegenerate. Then*

1) $\mathcal{Q}(M) \neq \mathbb{R}^n$*,*
2) $q \in \mathcal{Q}(M)$ *if and only if the nodes* $x^{(I)}$*,* $I \subset [\![1,n]\!]$*, of* LCP(M,q) *have the property that the components of* $x_I^{(I)}$ *are nonzero,*
3) $\mathcal{Q}(M)$ *is open and dense in* $\mathbb{R}^n$*.*

Proof. Denote $\mathcal{Q} \coloneqq \mathcal{Q}(M)$.

1) When $q = 0$, problem LCP(M,q) has 0 as single node. Therefore $0 \notin \mathcal{Q}$ and $\mathcal{Q}$ always differs from $\mathbb{R}^n$.

2) $[\Rightarrow]$ If, for some $J \subsetneq I$, the components of $x^{(I)}$ with index in $I \smallsetminus J$ vanish then $x^{(I)}$ satisfies

$$x^{(I)}_{J^c} = 0 \qquad \text{and} \qquad (Mx^{(I)} + q)_J = 0.$$

This system has for unique solution $x^{(J)}$, so that $x^{(I)} = x^{(J)}$. Since $I \neq J$, $\mathcal{X}$ is not injective and the implication $(ii) \Rightarrow (i)$ of lemma 5.1 shows that problem LCP(M,q) has not 2^n distinct nodes, meaning that $q \notin \mathcal{Q}$.

$[\Leftarrow]$ If, for all $I \subset [\![1,n]\!]$, the components of $x_I^{(I)}$ are nonzero, all the nodes differ by their zero components, so that problem LCP(M,q) has 2^n distinct nodes, meaning that $q \in \mathcal{Q}$.

3) [$\mathcal{Q}$ *is open*] We prefer showing that the complementary set $\mathcal{Q}^c$ of $\mathcal{Q}$ is closed, since its description intervenes in the proof of the density of $\mathcal{Q}$. Fix $I \subset [\![1,n]\!]$ and consider the map

$$\xi_I : q \in \mathbb{R}^n \mapsto (0_{I_c}, -M_{II}^{-1} q_I) \in \mathbb{R}^n,$$

which provides the node $x^{(I)}$ as a function of q. We have seen in point 2 that $q \notin \mathcal{Q}$ if and only if, for some $I \subset [\![1,n]\!]$ and some nonempty subset I_0 of I, $(\xi_I(q))_{I_0} = 0$. Since ξ_I is linear, $(\xi_I(q))_{I_0} = 0$ when q belongs to a subspace S_{I,I_0} of $\mathbb{R}^n$. Therefore, one can write

$$\mathcal{Q}^c = \bigcup \{S_{I,I_0} : I \subset [\![1,n]\!],\ I_0 \text{ non empty subset of } I\}.$$

Since S_{I,I_0} is a closed set, $\mathcal{Q}^c$ is closed as a finite union of closed sets.

[$\mathcal{Q}$ *is dense*] Let $q_0 \in \mathcal{Q}^c$ and $\varepsilon > 0$. We have to show that the ball $B(q_0, \varepsilon)$ centered at q_0 with radius ε intersects $\mathcal{Q}$. This is clear, since otherwise $\mathcal{Q}^c$ would contained the ball $B(q_0, \varepsilon)$, which is not possible for the finite union of strict subspaces of $\mathbb{R}^n$ that is $\mathcal{Q}^c$. □

Conclusion

This contribution brings some corrections and complements to the paper [3] by the same authors. Corrections are necessary because it was implicitly assumed in [3] that the nodes of the given problem (i.e., the points that the Newton-min algorithm can visit) were all different. This implicit assumption was not compatible with some claims, without bringing appropriate nuances, which is what is done in the present paper. The main result, according to which a nondegenerate matrix M is a **P**-matrix if and only if the Newton-min algorithm does not cycle between two distinct points, whatever is q and the starting point, is preserved. It is also shown that the set of vectors q ensuring the existence of 2^n distinct nodes is dense in $\mathbb{R}^n$.

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