4D hyperbolic regular quantum codes

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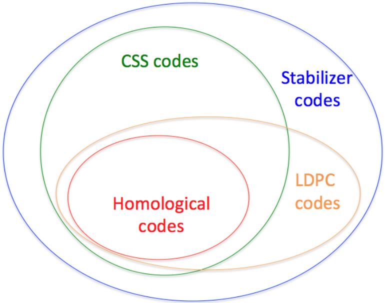
Table of contents

Quantum error correcting codes

- aim: protect k qubits of information.
- k logical qubits
  but n physical qubits ($n > k$)
  $\rightarrow$ codespace of dimension 2^k in a 2^n Hilbert space.
- distance d of the code proportional to the maximal number of physical qubits which can be corrupted without corrupting the information carried by the logical qubits.
  $\rightarrow$ $[\![n,k,d]\!]$ quantum error correcting code
- asymptotics of k and d when $n \rightarrow \infty$

Families of codes

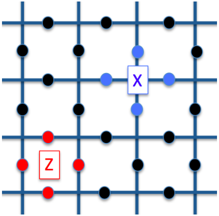
- stabilizer codes: the codespace is the 1-eigenspace of a set of commuting stabilizers.
- CSS codes: each stabilizer is a tensor product of X and I matrices or a tensor product of Z and I matrices.
- LDPC codes: each stabilizer has a bounded number of X or Z factors and each qubit is non trivially acted on by a bounded number of stabilizer.



- homological codes: The stabilizers are defined from a cellulation of a manifold.

Example of homological code: Toric code [Kitaev 2002]

- cellulation of the torus by squares.
- To each edge corresponds a physical qubit.
- To each face corresponds a Z-type stabilizer.
- To each vertex corresponds an X-type stabilizer.

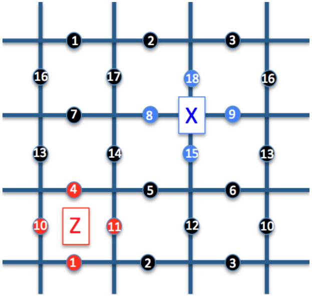


A torus is obtained by identifying left and right sides and identifying up and down sides of the square.

Stabilizers commute because each (face,vertex) pair shares an even number of edges.

Example of homological code: Toric code [Kitaev 2002]

- The Z-type stabilizer corresponding to the red face acts like Z on qubits 1, 10, 4 and 11.
- The X-type stabilizer corresponding to the blue vertex acts like X on qubits 8, 18, 9 and 15.



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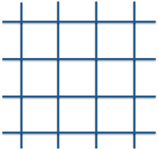
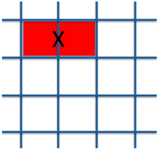
Detection of errors



represents a codestate $|\psi_0>$.
All stabilizers have eigenvalue 1.



represents the state $X_{17}|\psi_0>$.
red stabilizers have eigenvalue -1.

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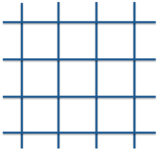

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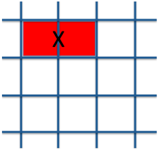
$$Z_{stab.}|\psi_0>= |\psi_0>$$



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$$Z_{red\ stab.}X_{17}|\psi_0>$$
$$= -X_{17}Z_{red\ stab.}|\psi_0>$$
$$= -X_{17}|\psi_0>$$

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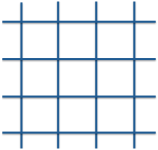

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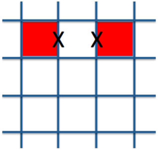
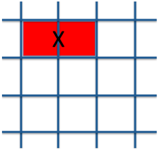


state $X_{17}X_{18}|\psi_0>$.
red stabilizers have eigenvalue -1.



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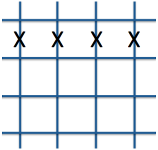
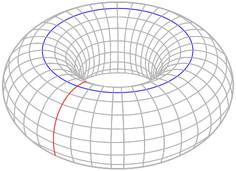


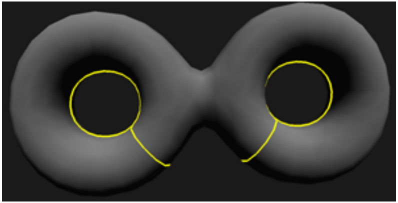
state $X_{16}X_{17}X_{18}X_{19}|\psi_0>$:
codestate different from $|\psi_0>$.

Geometric interpretation of k

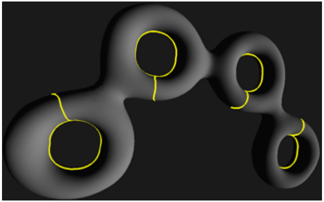
- The code dimension is the rank of the first homology group H_1 of the manifold.
- Informally, it is the number of different loops of the manifold.



$k = 2 = rank(H_1)$



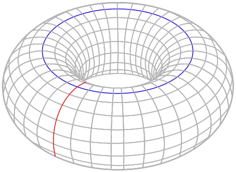
$k = 4 = rank(H_1)$



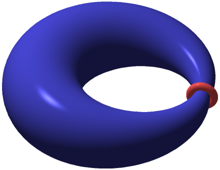
$k = 8 = rank(H_1)$

Geometric interpretation of n and d

- number of physical qubits n proportional to the area of the manifold.
- minimal distance d proportional to the systole of the manifold.
- the systole is the length of the shortest non contractible loop of the manifold.



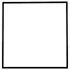
$d = systole \approx 20$
$n = 2 \times area \approx 800$



d = systole = length of red circle

Schläfli symbols

Schläfli symbols are defined recursively:



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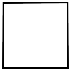
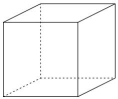
- $\{4\}$ is a square.
- $\{4{,}3\}$ is the regular polyhedron such that each vertex is incident to 3 squares ($\{4\}$): the cube.





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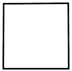
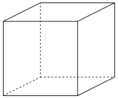
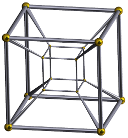
- $\{4{,}3\}$ is the regular polyhedron such that each vertex is incident to 3 squares ($\{4\}$): the cube.
- $\{4{,}3{,}3\}$ is the regular polytope such that each edge is incident to 3 cubes ($\{4{,}3\}$): the hypercube.

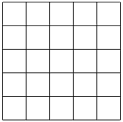




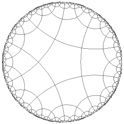


Regular tessellation

- $\{4,4\}$ is the regular tessellation of euclidean plane such that each vertex is incident to 4 squares. It is the grid of the toric code.



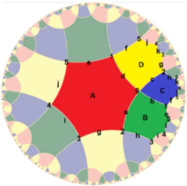
- $\{5,4\}$ is the regular tessellation such that each vertex is incident to 5 squares. It is a tessellation of the hyperbolic plane.



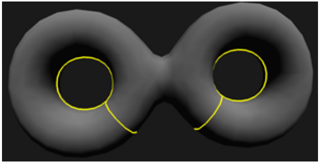
  - The automorphism group Γ of the $\{5,4\}$ tessellation admits the following presentation:
    $\Gamma =< \rho_0, \rho_1, \rho_2 \,|\, \rho_0^2 = \rho_1^2 = \rho_1^2 = (\rho_0\rho_1)^5 = (\rho_1\rho_2)^4 = id >$

Regular map

- Quotienting by a normal subgroup of the automorphism group of the regular tessellation Γ yields a closed surface with the local structure of the regular tessellation.



$\{6, 4\}_3$ regular map



topology of the $\{6, 4\}_3$ regular map

- To this tessellated closed surface corresponds a quantum error correcting code.

Limits of surface codes

- Tradeoffs for reliable quantum information storage in surface codes and color codes [Delfosse 2013]:
  $kd^2 \leq C(\ln k)^2 n$
- toric codes family: $k = 2$ and $d = \theta(\sqrt{n})$
- hyperbolic codes family: $k = \theta(n)$ and $d = \theta(\ln n)$

Codes based on cellulations of 4-manifolds can overcome this bound.

4-dimensional codes

- Cellulation of a 4-dimensional manifold by polytopes.
  - To each 2-cell corresponds a qubit.
  - To each 3-cell corresponds a Z-type stabilizer.
  - To each 1-cell (edge) corresponds an X-type stabilizer.

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- Logical qubits correspond to 2-dimensional holes.
  $\rightarrow$ the number k of logical qubits is the rank of the second homology group of the manifold.
- Minimal distance d is proportional to the 2-systole of the manifold.
  $\rightarrow$ it is the smallest area of a surface surrounding a 2-dimensional hole.

4-dimensional tessellations

- $\{4,3,3,4\}$ is the canonical grid of euclidean 4-space.
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- Congruence subgroups of this representation yield normal subgroups of Γ.

Codes associated to these 4-manifolds

- We thus obtain closed 4-manifold with the $\{4,3,3,5\}$ local structure.
- $k = \theta(n)$ and $d \geq n^{0.2}$ asymptotically.
- $kd^2 \geq (lnk)^2 n$ asymptotically.
- regularity useful for decoding algorithms.

Advantages of 4-dimensional hyperbolic regular codes

- Syndromes are cycles of edges.
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- Hyperbolicity makes syndromes more redundant than in euclidean space.
  $\rightarrow$ since the boundary of a hyperbolic disk is linear in its area.
- Regularity of the tessellation allows one to design explicit efficient decoding algorithms.

Normal subgroups of $\Gamma_{\{4,3,3,5\}}$

- $\Gamma_{\{4,3,3,5\}} =< R_i \mid i \in \{0, ..., 4\} >$

  with $PR_iP^{-1} \in \mathcal{M}_{5,5}(\mathcal{O}_{\mathbb{Q}[\sqrt{5}]})$ for $i \in \{0, ..., 4\}$
  $\mathcal{O}_{\mathbb{Q}[\sqrt{5}]}$ is the ring of integers of the number field $\mathbb{Q}[\sqrt{5}]$.

- If $\mathcal{I}$ is an ideal of $\mathcal{O}_{\mathbb{Q}[\sqrt{5}]}$, there is a group homomorphism:
  $\phi_{\mathcal{I}} : \mathcal{M}_{5,5}(\mathcal{O}_{\mathbb{Q}[\sqrt{5}]}) \to \mathcal{M}_{5,5}(\mathcal{O}_{\mathbb{Q}[\sqrt{5}]}/\mathcal{I})$
  Ker$(\phi_{\mathcal{I}})$ is a normal subgroup of $\Gamma_{\{4,3,3,5\}}$ (called a congruence subgroup).

Summary and Outlooks

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- To a tessellated closed manifold and an integer i corresponds a quantum code where qubits are identified with i-faces of the manifold.

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Thank you for your attention!