



HAL
open science

Homological quantum error correcting codes and real projective space

Vivien Londe

► **To cite this version:**

Vivien Londe. Homological quantum error correcting codes and real projective space. Journées Codage et Cryptographie 2017, Apr 2017, La Bresse, France. hal-01671444

HAL Id: hal-01671444

<https://inria.hal.science/hal-01671444>

Submitted on 22 Dec 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Homological quantum error correcting codes and real projective space

Vivien Londe, Anthony Leverrier

Inria Paris, Equipe Secret

April 26, 2017

Outline

Quantum error correcting codes

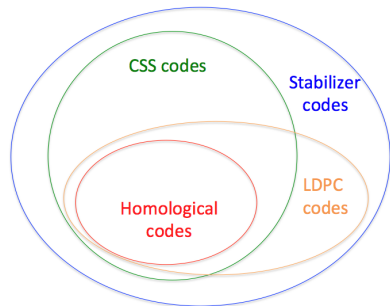
Quantum codes from real projective spaces

Quantum error correcting codes

- ▶ qubit: $\alpha |0\rangle + \beta |1\rangle$ where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$.
- ▶ two types of errors: $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
- ▶ $[[n, k, d]]$ error correcting code: **k logical qubits** but **n physical qubits** ($n > k$)
- ▶ **minimal distance d** of the code proportional to the maximal number of **errors** which can be **corrected**.

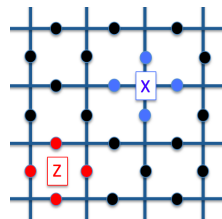
Families of codes

- ▶ stabilizer codes: the codespace is the 1-eigenspace of a set of commuting operators.
- ▶ CSS codes: can be specified by two orthogonal linear classical codes \mathcal{C}_X and \mathcal{C}_Z .
- ▶ LDPC codes: lines and columns of parity check matrices have bounded weights.
- ▶ homological codes: The two orthogonal classical codes are defined from a cellulation of a manifold.



Example of homological code: Toric code [Kitaev 2002]

- ▶ cellulation of the torus by squares.
- ▶ To each edge corresponds a physical bit.
- ▶ To each face corresponds a line of H_Z .
- ▶ To each vertex corresponds a line of H_X .

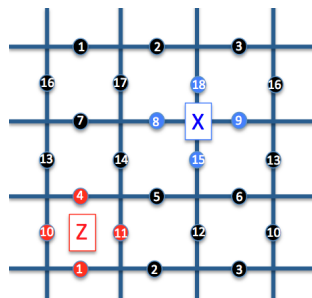


A torus is obtained by identifying left and right sides and identifying up and down sides of the square.

\mathcal{C}_X and \mathcal{C}_Z are orthogonal because each (face,vertex) pair shares an even number of edges.

Example of homological code: Toric code [Kitaev 2002]

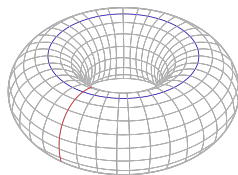
- ▶ The line of H_Z corresponding to the red face checks the parity of the bits 1, 10, 4 and 11.
- ▶ The line of H_X corresponding to the blue vertex checks the parity of the bits 8, 18, 9 and 15.



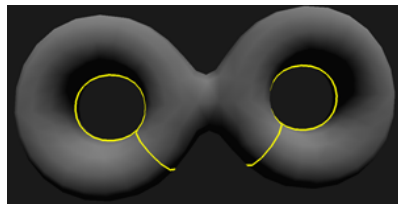
\mathcal{C}_X and \mathcal{C}_Z are orthogonal because each (face, vertex) pair shares an even number of edges.

Geometric interpretation of k

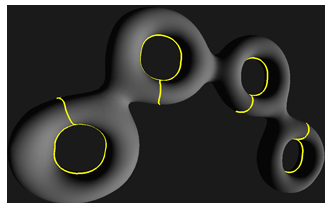
- ▶ The code dimension is the rank of the first homology group H_1 of the manifold.
- ▶ Informally, it is the number of different loops of the manifold.



$$k = 2 = \text{rank}(H_1)$$



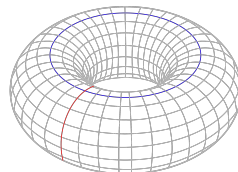
$$k = 4 = \text{rank}(H_1)$$



$$k = 8 = \text{rank}(H_1)$$

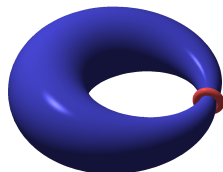
Geometric interpretation of n and d

- ▶ the code length n is proportional to the area of the manifold.
- ▶ the minimal distance d is proportional to the systole of the manifold.
- ▶ the systole is the length of the shortest non contractible loop of the manifold.



$$d = \text{systole} \approx 20$$

$$n = 2 \times \text{area} \approx 800$$

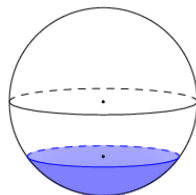


$$d = \text{systole} = \text{length of red circle}$$

positive curvature and the sphere S^2 goal: family of codes satisfying $n = o(d^2)$

$$\text{area}(\text{disk}_{\text{euclidean}}(r)) = \pi r^2$$

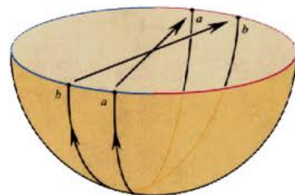
$$\begin{aligned}\text{area}(\text{disk}_{\text{spherical}}(r)) &= 2\pi(1 - \cos(r)) \\ &= \pi r^2 - \frac{\pi}{12}r^4 + O(r^6)\end{aligned}$$

 $\text{area}(\text{disk}_{\text{spherical}}) < \text{area}(\text{disk}_{\text{euclidean}})$ 

But every loop on the sphere is contractible to a point.
Hence the dimension k of a code defined on the sphere is zero.

The real projective plane

Identify every pair of antipodal points.
Some loops cannot be contracted to a point.

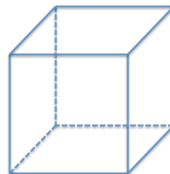


- ▶ $\text{systole}_{\text{projective plane}} = \pi$
- ▶ $\text{area}_{\text{projective plane}} = 2\pi$
- ▶ $(\text{systole}_{\text{projective plane}})^2 > \text{area}_{\text{projective plane}}$

But the area of a real projective plane of constant curvature $+1$ is bounded above by 2π .
A solution is to increase dimension.

A discrete model of the real projective plane

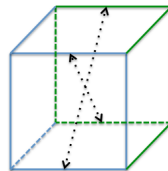
Identify pair of antipodal points of the cube.



A discrete model of the real projective plane

Identify pair of antipodal points of the cube.

► $n = 6$



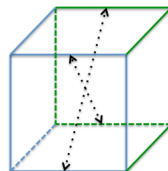
opposite edges are identified

A discrete model of the real projective plane

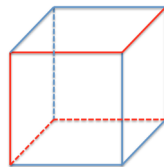
$$H_X = \text{mat}_{\text{adjacency}}(\text{vertices}, \text{edges})$$

Identify pair of antipodal points of the cube.

- ▶ $n = 6$
- ▶ $d_X = 3$



opposite edges are identified



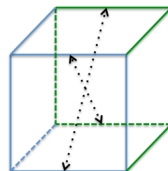
least weight codeword for the code \mathcal{C}_X

A discrete model of the real projective plane

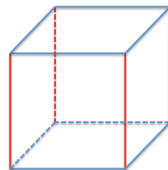
$$H_Z = \text{mat}_{\text{adjacency}}(\text{faces}, \text{edges})$$

Identify pair of antipodal points of the cube.

- ▶ $n = 6$
- ▶ $d_X = 3$
- ▶ $d_Z = 2$
- ▶ $d = \min(d_X, d_Z) = 2$



opposite edges are identified



least weight codeword for the code \mathcal{C}_Z

A discrete model of the real projective $2m$ -space

Identify pair of antipodal points of the hypercube of dimension $2m+1$.

Bits are identified with m -faces of the hypercube.

With the 5-hypercube:

- ▶ $n = 40$
- ▶ $d_X = 10$
- ▶ $d_Z = 4$
- ▶ $d = \min(d_X, d_Z) = 4$

A discrete model of the real projective $2m$ -space

Identify pair of antipodal points of the hypercube of dimension $2m+1$.

Bits are identified with m -faces of the hypercube.

With the 5-hypercube:

- ▶ $n = 40$
- ▶ $d_X = 10$
- ▶ $d_Z = 4$
- ▶ $d = \min(d_X, d_Z) = 4$

Conjecture for the $(2m+1)$ -hypercube:

- ▶ $n = \binom{2m+1}{m} 2^m$
- ▶ $d_X = \binom{2m+1}{m}$
- ▶ $d_Z = 2^m$

Remark: $n = d_X d_Z$

A discrete model of the real projective $2m$ -space

Identify pair of antipodal points of the hypercube of dimension $2m+1$.

Bits are identified with m -faces of the hypercube.

With the 5-hypercube:

- ▶ $n = 40$
- ▶ $d_X = 10$
- ▶ $d_Z = 4$
- ▶ $d = \min(d_X, d_Z) = 4$

Conjecture for the $(2m+1)$ -hypercube:

- ▶ $n = \binom{2m+1}{m} 2^m$
- ▶ $d_X = \binom{2m+1}{m}$
- ▶ $d_Z = 2^m$

Remark: $n = d_X d_Z$

Comparison with toric codes

- ▶ High dimensional projective code:
 - ▶ $d \leq \sqrt{n}$
 - ▶ By identifying bits with ℓ -faces of the hypercube, $\ell > m$, one could reach $d = \sqrt{n}$.
 - ▶ logarithmically LDPC.
- ▶ High dimensional toric code:
 - ▶ $d = n^{\frac{1}{2+\ln(4)/\ln(\ell)}}$, ℓ is the length of the hypercube.
 - ▶ logarithmically LDPC.

Comparison with toric codes

- ▶ High dimensional projective code:

- ▶ $d \leq \sqrt{n}$
- ▶ By identifying bits with ℓ -faces of the hypercube, $\ell > m$, one could reach $d = \sqrt{n}$.
- ▶ logarithmically LDPC.

- ▶ High dimensional toric code:

- ▶ $d = n^{\frac{1}{2+\ln(4)/\ln(\ell)}}$, ℓ is the length of the hypercube.
- ▶ logarithmically LDPC.

- ▶ 2 dimensional toric code:

- ▶ $d = \sqrt{n}$
- ▶ strictly LDPC.

The high dimensional projective code is better than the high dimensional toric code but not as good as the two dimensional toric code.

Comparison with toric codes

- ▶ High dimensional projective code:

- ▶ $d \leq \sqrt{n}$
- ▶ By identifying bits with ℓ -faces of the hypercube, $\ell > m$, one could reach $d = \sqrt{n}$.
- ▶ logarithmically LDPC.

- ▶ High dimensional toric code:

- ▶ $d = n^{\frac{1}{2+\ln(4)/\ln(\ell)}}$, ℓ is the length of the hypercube.
- ▶ logarithmically LDPC.

- ▶ 2 dimensional toric code:

- ▶ $d = \sqrt{n}$
- ▶ strictly LDPC.

The high dimensional projective code is better than the high dimensional toric code but not as good as the two dimensional toric code.

Conclusions

- ▶ Positive curvature leads to greater minimal distances.
- ▶ High dimensional manifolds lead to smaller minimal distances.
- ▶ Homological quantum error correcting codes can be understood geometrically.

The only known family of codes satisfying $d = \omega(\sqrt{n})$ is a homological code built on a four dimensional manifold with non zero curvature [Freedman 2002].

Conclusions

- ▶ Positive curvature leads to greater minimal distances.
- ▶ High dimensional manifolds lead to smaller minimal distances.
- ▶ Homological quantum error correcting codes can be understood geometrically.

The only known family of codes satisfying $d = \omega(\sqrt{n})$ is a homological code built on a four dimensional manifold with non zero curvature [Freedman 2002].

Thank you for your attention! Questions?