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A threshold model for local volatility: evidence of leverage and mean-reversion effects on historical data

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In financial markets, low prices are generally associated with high volatilities and vice-versa, this well known stylized fact usually being referred to as leverage effect. We propose a local volatility model, given by a stochastic differential equation with piecewise constant coefficients, which accounts of leverage and mean-reversion effects in the dynamics of the prices. This model exhibits a regime switch in the dynamics accordingly to a certain threshold. It can be seen as a continuous-time version of the Self-Exciting Threshold Autoregressive (SETAR) model. We propose an estimation procedure for the volatility and drift coefficients as well as for the threshold level. Parameters estimated on the daily prices of 348 stocks of NYSE and S&P 500, on different time windows, show consistent empirical evidence for leverage effects. Mean-reversion effects are also detected, most markedly in crisis periods.

Keywords. Oscillating Brownian motion; leverage effect; realized volatility; mean-reversion; Self-Exciting Threshold Autoregressive model; Regime-Switch

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1 Introduction

Despite the predominance of the Black & Scholes model for the dynamics of asset prices, its deficiencies to reflect all the phenomena observed in the markets are well documented and subject to many studies. Some *stylized facts* not consistent with the Black & Scholes model are non-normality of log-returns, asymmetry, heavy tails, varying conditional volatilities, volatility clustering, ... (Cont, 2001). Regime switching is also consistently observed (Ang and Timmermann, 2012; Salhi et al., 2016). Besides, some assets and indices exhibit mean-reverting effects (see *e.g.* Meng et al. (2013), Monoyios and Sarno (2002), Lo and MacKinlay (1988), Poterba and Summers (1988), Su and Chan (2015a), Su and Chan (2017), and Spierdijk, Bikker, and Hoek (2012)).

By considering only the asset's price at discrete, fixed times $\{k\Delta t\}_{k=0,1,2,\dots}$, the log-returns $r_t = \log(S_{t+1}/S_t)$ of the Black & Scholes model $\{S_t\}_{t\geq 0}$ are nothing more than the simple time series

$$r_{t+1} = \left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}\epsilon_t \text{ with } \epsilon_t \sim \mathcal{N}(0,1). \tag{1}$$

Several models alternative to (1) have been proposed to take some of these stylized facts into account. Among the most popular ones, ARCH and GARCH models and their numerous variants reproduce volatility clustering effects (Engle, Focardi, and Fabozzi, 2012).

In this article, we focus on *leverage effects*, a term which refers to a negative correlation between the prices and the volatility. As observed for a long time, the lower the price, the higher the volatility. First explanations were given in Black (1976) and Christie (1982). Processes such as the *constant elasticity volatility* (CEV) were proposed to account of these phenomena (Christie, 1982). One common economic explanation of leverage effects is that when an asset price decreases, the ratio of the company's debt with respect to the equity value becomes larger, and as a consequence volatility increases; another explanation is that investors tend to become more nervous after a large negative return than after a large positive return; anyway, the origin of leverage effects is still subject to discussion (see *e.g.* Hens and Steude (2009)).

In the early '80s, H. Tong has proposed a broad class of time series, the *threshold autoregressive models* (TAR), with non-linear effects reproducing cyclical data (Tong, 1983; Tong, 2011; Tong, 2015). This class, which contains *Hidden Markov chains* (HMM) as well as *self-exciting threshold autoregressive* models (SETAR), produces a wide range of behaviors. HMM models rely on a temporal segmentation (they are good for crisis detection), while SETAR models rely on a spatial segmentation, with a regime change when the price goes below or above a threshold.

Time series of SETAR type capture leverage and mean-reverting effects by defining a threshold which separates two regimes (high/low volatility, positive/negative trend). Unlike models such as HMM, no external nor latent randomness is used.

In finance, various aspects of SETAR like models have been considered (Yadav, Pope, and Paudyal, 1994; Chen, So, and Liu, 2011; Meng et al., 2013; Siu, 2016; Rabemananjara and Zakoian, 1993). An alternative form to SETAR model is provided by *threshold stochastic volatility* models (Xu, 2012; So, Li, and Lam, 2002), where the volatility depends non-linearly on the price through a threshold model. Also other considerations such as *psychological barriers* (Jang et al., 2015; Kolb, 2016) lead to threshold models.

Continuous-time models could be seen as the limit of time series as the time step goes to 0. They have some advantages over time series, for allowing irregularly sampling, the use of stochastic calculus tools and possibly analytic or semi-analytic formulas for fast evaluation of option prices and risk estimation. Continuous-time threshold models (or *threshold diffusion*) have been studied in Siu (2016) and Su and Chan (2016) for option valuation, in Meng et al. (2013) for portfolio optimization, etc. Self-exciting variants of Vasiček and Cox-Ingersoll-Ross continuous-time models have also been proposed for interest rates (Decamps, Goovaerts, and Schoutens, 2006; Pai and Pedersen, 1999). In Brockwell and Williams (1997), a continuous-time equivalent of an integrated SETAR model is constructed and applied to financial data. Mota and Esquível (2014) and Esquível and Mota (2014) propose two continuous-time models which mimic the SETAR time series. In Mota and Esquível (2014), one of these models, referred to as the *Delay Threshold Regime Switching model* (DTRS), is tested on the daily prices of 21 companies over almost 5 years. For almost all the stocks, they found a regime-change for the volatility.

Contribution of the paper. We present the *Geometric Oscillating Brownian motion* (GOBM), a threshold local volatility model with piecewise constant volatility and drift, as in Gairat and Shcherbakov (2016). This model is an instance of the *tiled volatility model* of Lipton and Sepp (2011). We stress that the GOBM is the solution of a one-dimensional SDE. Therefore, it is simpler to manipulate than the DTRS of Mota and Esquível (2014), although having similar features. For the same reason, the market is complete under the GOBM. The GOBM can also be simulated by a standard Euler scheme (Yan, 2002; Chan and Stramer, 1998). Option valuation can be performed as well using semi-analytic approaches (Lipton and Sepp, 2011; Gairat and Shcherbakov, 2016; Pigato, 2017), and the related problem of estimating ex-ante volatilities from the call prices can be solved using Sturm-Liouville theory (Lipton and Sepp, 2011).

In the GOBM model, a fixed threshold separates two regimes for the prices. Both the volatility and the drift parameter can assume two possible values, according to the position of the stock price, above or below the threshold. Let us write σ_- for the volatility below the threshold, σ_+ for the volatility above the threshold, and similarly b_- and b_+ for the drift. Such model accounts of the leverage effect, when $\sigma_- > \sigma_+$. In this case, when prices are low, volatility increases, consistently with what is observed on empirical financial data. As in Mota and Esquível (2014), the dynamics has two regimes, one corresponding to the bull market, with prices above the threshold and low volatility, and one corresponding to the bear market, with prices below the threshold and high volatility. In this sense, the model displays an "endogenous" regime switch. A motivation for considering such price dynamics coming from a different viewpoint is given in Ankirchner, Blanchet-Scalliet, and Jeanblanc (2017): it is shown that the GOBM describes the price dynamics corresponding to the optimal strategy for a manager who can control, in a stylized setting, the volatility of the value of a firm, getting bonus payments when the value process performs better than a reference index.

After describing and motivating the usage of the model, we consider the estimation of volatilities, drifts and thresholds from discrete observations of historical stock prices. The estimation procedures used in Mota and Esquível (2014), Esquível and Mota (2014), and Brockwell and Williams (1997) are all derived from the ones designed for SETAR time series. Here, we approach the problem directly, proposing an estimation procedure based on stochastic calculus. The estimator of the volatility coefficients is inspired by the integrated volatility/realized variance estimator; for its theoretical analysis we refer the reader to Lejay and Pigato (2018b). Our estimator can be implemented straightforwardly, differently from the MLE, which is very hard to implement as there is no simple closed form for the density of the GOBM. On the other hand, the estimator of the drift coefficient is the maximum likelihood (MLE) one. Its implementation is also straightforward. Its asymptotic behavior is studied in Lejay and Pigato (2018a).

In the present paper, we discuss several issues regarding the quality of the estimation and propose a method for estimating the threshold, based on the Akaike Information Principle. In addition, we provide a hypothesis test to decide whether or not the volatility is constant. We test the performance of such methods via numerical experiments on simulated data. These tests are conclusive.

Finally, we look at empirical financial data. We first benchmark our model against the same dataset as Mota and Esquível (2014): 21 stock prices from the NYSE, on the time window 2005-2009. We find similar results: in particular, we consistently find leverage effects ($\sigma_- > \sigma_+$) and mean-reverting behavior ($b_- > 0, b_+ < 0$). Then, we apply our estimators to the empirical time series of the S&P 500, on the three separate five years windows $2003 - 2007$, $2008 - 2012$ and $2013 - 2017$, finding again consistent evidence of leverage effects. More specifically, we may say, based on the hypothesis test mentioned above and on the estimated ratios

$\frac{\sigma_-}{\sigma_+}$, that the leverage effect is particularly marked in the period $2008 - 2012$, most likely because it contains the 2008 financial crisis. The mean-reverting behavior is also quite clearly detectable in the $2008 - 2012$ period, less so in the periods $2003 - 2007$ and $2013 - 2017$, on which b_- is always positive but b_+ does not display a predominant sign. This seems to be in agreement with the finding that "…the speed at which stocks revert to their fundamental value is higher in periods of high economic uncertainty, caused by major economic and political events", as was shown in Spierdijk, Bikker, and Hoek (2012). We refer to the same paper and to Section 6 for the economic interpretation of this finding.

As final consideration, we remark that the GOBM, despite its extreme simplicity and limited number of parameters, reproduces notable stylized facts of financial markets such as leverage effects and mean-reverting properties. Moreover, the application of the estimators described above to empirical data confirm the presence of such features in the dynamics of financial indices.

Outline. The GOBM is presented in Section 2. In Section 3 we consider the estimation procedures for the volatilities (Section 3.1), the drift (Section 3.2) and the threshold (Section 3.3). In Section 4, we benchmark the GOBM model against the DTRS model (in Section 4.1) introduced by Mota and Esquível (2014) by comparing the estimators on the same data sets (in Section 4.2). In Section 5, we present a hypothesis test to decide whether a leverage effect is present or not. Finally, in Section 6 we apply our estimators to the stock prices of the S&P 500, in three consecutive periods of 5 years, the second one containing the 2008 financial crisis. The article ends with a global conclusion in Section 7.

2 The (geometric) Oscillating Brownian motion

The model. The Geometric Oscillating Brownian motion (GOBM) is the solution of an SDE of type

$$S_t = x + \int_0^t \sigma(S_s) S_s \,\mathrm{d}B_s + \int_0^t \mu(S_s) S_s \,\mathrm{d}s, \tag{2}$$

where, for a threshold $m \in \mathbb{R}$,

$$\sigma(x) = \begin{cases} \sigma_+ & \text{if } x \geq m, \\ \sigma_- & \text{if } x < m \end{cases} \text{ and } \mu(x) = \begin{cases} \mu_+ & \text{if } x \geq m, \\ \mu_- & \text{if } x < m. \end{cases} \tag{3}$$

Remark that this is a local volatility model, meaning that the coefficients (and in particular the volatility coefficient σ) only depend on the stock price.

We use a solution S of (2) as a model for the evolution of the price of an asset. The *log-price* $X = \log(S)$ satisfies the SDE

$$X_t = x + \int_0^t \sigma(X_s)\,\mathrm{d}B_s + \int_0^t b(X_s)\,\mathrm{d}s, \tag{4}$$

with

$$\sigma(x) = \begin{cases} \sigma_+ & \text{if } x \geq r, \\ \sigma_- & \text{if } x < r \end{cases} \text{ and } b(x) = \begin{cases} b_+ = \mu_+ - \sigma_+^2/2 & \text{if } x \geq r, \\ b_- = \mu_- - \sigma_-^2/2 & \text{if } x < r \end{cases} \tag{5}$$

for a threshold $r = \log(m)$. Notice the slight abuse of notation in (3) and (5), due to the change of the value for the threshold when taking the logarithm.

When the drift $b = 0$ and $m = 0$, X is called an *Oscillating Brownian motion* (OBM, Keilson and Wellner (1978)), a name we keep even in presence of a two-valued drift and a threshold. When $\sigma_+ = \sigma_-$ and $b_+ = b_-$, then the price follows the Black & Scholes model. By extension, we still call the solution to (4) a *GOBM*.

The effect of the drift is discussed in Section 3.2. When $b_+ < 0$ and $b_- > 0$, the process is ergodic and mean-reverting. The convergence toward equilibrium differs from the ones in the Vašiček and Heston models in which the drift is linear.

Existence and uniqueness. The solution to (4) is an instance of a more general class of processes with discontinuous coefficients which was studied in Le Gall (1984). In particular, there exists a unique strong solution to (4), hence to (2).

The (geometric)-OBM can be easily manipulated with the standard tool of stochastic analysis, sometimes relying on the Itô-Tanaka formula instead of the sole Itô formula (See *e.g.* Étoré (2006)).

Properties of the market. Unlike in some regime switching models, there is no hidden randomness leading to incomplete markets, while offering some regime change properties.

Proposition 1. *Assuming the GOBM model for the returns process with a constant risk-free rate, the market is viable and complete.*

Proof. Using the results of Le Gall (1984), the Girsanov Theorem can be applied to the equation for the log-price. Hence, as for the Black & Scholes model, it is possible to reduce the discounted log-price to a martingale by removing the drift. Hence, there exists an equivalent martingale measure, meaning that the market is viable (Jeanblanc, Yor, and Chesney, 2009, Theorem 2.1.5.4, p. 89).

As any absolutely continuous measure could only be reached through a Girsanov transform (Le Gall, 1984), the risk neutral measure is unique, meaning that the market is complete. $\square$

Remark 1. The affine transform $\Phi(x) = x/\sqrt{\sigma_-}\mathbf{1}_{x<m} + x/\sqrt{\sigma_+}\mathbf{1}_{x>m}$ transforms the log-price X_t into the solution Y to the SDE with local time

$$Y_t = \Phi(X_0) + B_t + \int_0^t \frac{b_+}{\sigma_+}\mathbf{1}_{x>0}(Y_s)\,\mathrm{d}s + \int_0^t \frac{b_-}{\sigma_-}\mathbf{1}_{x<0}(Y_s)\,\mathrm{d}s + \kappa L_t^0(Y), \qquad (6)$$

where $(L_t^0(Y))_{t\geq 0}$ is the local time of Y as position 0 and $\kappa = (\sqrt{\sigma_-} - \sqrt{\sigma_+})/(\sqrt{\sigma_-} + \sqrt{\sigma_+})$. Eq. (6) is a drifted Skew Brownian motion (SBM). The local time part cannot be removed by a Girsanov transform (Le Gall, 1984). For the SBM, arbitrage could exists as shown in Rossello (2012). The GOBM may be generalized by considering a log-price solution to $\mathrm{d}X_t = x + \sigma(X_t)\,\mathrm{d}B_t + b(X_t)\,\mathrm{d}t + \eta\,\mathrm{d}L_t^r(X)$, where $L^r(X)$ is the local time of X at the threshold r. The effect of the coefficient $\eta \in (-1, 1)$ would be to "push upward" (if $\eta > 0$) or downward (if $\eta < 1$) the price, which corresponds to some directional predictability effect (Alvarez E. and Salminen, 2017). However, considering $\eta \neq 0$ radically changes the structure of the market with respect to classical SDEs.

Application to option pricing. The simple form of the coefficients in the GOBM allows one to perform explicit computations. For example, the resolvent has a rather simple closed-form expression. However, for a non-vanishing drift, the analytic form of the density is cumbersome in general (Lejay, Lenôtre, and Pichot, 2017). Notwithstanding, the explicit expressions of the density or the generator could be used to perform option pricing (Jang et al., 2015; Gairat and Shcherbakov, 2016; Decamps, De Schepper, and Goovaerts, 2004) or to estimate implied volatility (Lipton and Sepp, 2011).

Monte Carlo simulation. From Chan and Stramer (1998) or Yan (2002), the continuous-time Euler scheme

$$\overline{X}_t = \overline{X}_{\frac{kT}{n}} + \int_{kT/n}^t \sigma\left(\overline{X}_{\frac{kT}{n}}\right)\,\mathrm{d}B_s + b\left(\overline{X}_{\frac{kT}{n}}\right)\left(t - \frac{kT}{n}\right) \text{ for } \frac{kT}{n} \leq t < \frac{(k+1)T}{n} \qquad (7)$$

provides us with an approximation of X. Thus, the GOBM at times kT/n, $k = 0, 1, 2, \dots, n$ is very simple to simulate by $\overline{S}_{kT/n} = \exp(\overline{X}_{kT/n})$ through the recursive equation

$$\overline{X}_{\frac{(k+1)T}{n}} = \overline{X}_{\frac{kT}{n}} + \sigma\left(\overline{X}_{\frac{kT}{n}}\right)\xi_k + b\left(\overline{X}_{\frac{kT}{n}}\right)\frac{T}{n},\ k = 0, \dots, n-1 \qquad (8)$$

for a sequence $\xi_0, \dots, \xi_n$ of independent random variables with distribution $\mathcal{N}(0, T/n)$.

3 Estimation of the parameters from the observations of the stock prices

The GOBM X is defined by five parameters (volatility, drift and threshold, see Table 1) which we are willing to estimate. In Sections 3.1 and 3.2 we consider the estimation of $(\sigma_\pm, b_\pm)$ for fixed threshold r, by considering first the estimation of the ex-post volatility and then of the drift. Afterwards, in Section 3.3, the threshold is chosen through a model selection principle.

The step by step procedure presented here is simple to implement and provides good results in practice. It is key that the theoretical results of Lejay and Pigato (2018b) on the estimation of $\sigma_\pm$ apply here, with no need for any previous knowledge of the drift, still unknown when estimating the volatility. One could also use a MLE for the volatility, based on discrete observations, but this would require explicit expressions for the transition density, which are very involved and depend on the drift parameters, still unknown at this point of the procedure. On the other hand, the integrated volatility estimators are simple to implement and widely studied in the framework of SDEs, so that this seems the most natural approach in our setting. Concerning the implementation of the estimator for the drift, our MLE (cf. (15)) does not directly involve the volatility parameter, so at first sight this may be implemented independently. At a closer look, we notice that such estimator involves the local time at the discontinuity, which we have to estimate from discrete observations of the underling process. To the best of our knowledge, all the estimators of the local time require the knowledge of the diffusivity/volatility parameter (cf. (17)), so in practice we need to estimate $\sigma_\pm$ first and then use it in the subsequent estimation of $b_\pm$.

To conclude this outline of the step-by-step estimation procedure we mention that, in order to implement the MLE of the drift, some kind of knowledge of the diffusion term is usually required. In Su and Chan (2017), for instance, a similar diffusion is considered, but with piecewise-affine drift. The authors make the assumption of a constant diffusion term. Another possible approach is to use a quasi-likelihood estimation. We refer, for instance, to Su and Chan (2015b), where no assumption is made on a functional form for the diffusion term. The authors propose in this case the quasi-likelihood approach, instead of using the true likelihood. Unlike with our setting, diffusions are assumed to be stationary, ergodic.

3.1 Estimation of the ex-post volatility

In this section we consider the estimation of the ex-post volatility for prices given by the model in (2), when the threshold $r = \log(m)$ is known. We recall the estimators

S	price of the stock
$X = \log(S)$	log-price
$\xi = X - r$	shifted log-price for a threshold r
r	threshold of X
$m = \exp(r)$	threshold of S
σ_-	volatility of X below r
σ_+	volatility of X above r
b_-	drift of X below r
b_+	drift of X above r
$\mu_- = b_- + \frac{\sigma_-^2}{2}$	appreciation rate of S below m
$\mu_+ = b_+ + \frac{\sigma_+^2}{2}$	appreciation rate of S above m
d	delay (DTRS only)

Table 1: Notations for the GOBM and DTRS models.

and the theoretical convergence results presented in Lejay and Pigato (2018b), and discuss their application in the framework of volatility modeling. Remark that the process $\xi := X - r = \log(S) - r$ is a drifted OBM.

The data. Our observations are $n+1$ daily data $\{\xi_k\}_{k=0,\dots,n}$ with $\xi_k = \log(S_k) - r$ for an *a priori* known threshold r.

Our aim is to estimate (σ_+, σ_-) from such observations.

Discrete brackets. For two processes Z, Z', we define the discrete brackets by

$$[Z, Z']_n := \sum_{i=1}^{n} (Z_i - Z_{i-1})(Z'_i - Z'_{i-1}) \text{ and } [Z]_n := [Z, Z]_n. \tag{9}$$

Occupation times. The occupation times below and above the threshold play a central role in our study.

Using the shifted log-price $\xi = \log(S) - r$, the positive and negative *occupation times* up to time T are $Q_T^{\pm} = \int_0^T \mathbf{1}_{\pm\xi_s \geq 0}\,\mathrm{d}s$.

We then define a Riemann type approximation of $Q_n^{\pm}$ by a Riemann approximation is then

$$Q_{\pm}(n) := \sum_{i=1}^{n} \mathbf{1}_{\pm\xi_i \geq 0}. \tag{10}$$

The estimators. For a process ξ, we write $\xi^+ = \max\{\xi, 0\}$ and $\xi^- = -\min\{\xi, 0\}$, its positive and negative part. Our estimators $\sigma_\pm(n)^2$ for $\sigma_\pm^2$ are

$$\sigma_\pm(n)^2 = \frac{[\xi^\pm, \xi]_n}{Q_\pm(n)}. \tag{11}$$

These estimators are natural generalizations of the *realized volatility estimators* (Barndorff-Nielsen and Shephard, 2002).

Proposition 2 (Lejay and Pigato (2018b))**.** *When $b_+ = b_- = 0$, then $(\sigma_-(n)^2, \sigma_+(n)^2)$ is a consistent estimator of (σ_-^2, σ_+^2). Besides, there exists a pair of unit Gaussian random variables (G_-, G_+) independent from the underlying Brownian motion B (hence of ξ) such that*

$$\begin{bmatrix} \sqrt{n}\sqrt{\dfrac{Q_-(n)}{n}}\left(\sigma_-(n)^2 - \sigma_-^2\right) \\ \sqrt{n}\sqrt{\dfrac{Q_+(n)}{n}}\left(\sigma_+(n)^2 - \sigma_+^2\right) \end{bmatrix} = \begin{bmatrix} \dfrac{[\xi^-, \xi]_n - Q_-(n)\sigma_-^2}{\sqrt{Q_-(n)}} \\ \dfrac{[\xi^+, \xi]_n - Q_+(n)\sigma_+^2}{\sqrt{Q_+(n)}} \end{bmatrix} \xrightarrow[n\to\infty]{law} \begin{bmatrix} \sqrt{2}\sigma_-^2 G_- \\ \sqrt{2}\sigma_+^2 G_+ \end{bmatrix}. \tag{12}$$

Dealing with a drift. Proposition 2 is actually proved on high-frequency data $\xi_{k,n} := \xi_{k/n}$, $k = 0, \dots, n$ on the time interval $[0, 1]$.

Using a scaling argument, for any constant $c > 0$, $\{c^{-1/2}\xi_{ct}\}_{t\geq 0}$ is equal in distribution to $\zeta^{(c)}$ solution to the SDE

$$\mathrm{d}\zeta_t^{(c)} = \sigma(\zeta_t^{(c)})\,\mathrm{d}W_t + \sqrt{c}b(\zeta_t^{(c)})\,\mathrm{d}t,\ \zeta_0^{(c)} = c^{-1/2}\xi_0 \tag{13}$$

for a Brownian motion W. With $c = n$, the problem of estimating the coefficients of $\{\xi_k\}_{k=0,\dots,n}$ is the same as the *high frequency* estimation of the coefficients of $\{\sqrt{n}\zeta_{k,n}^{(n)}\}_{k=0,\dots,n}$ on the time range $[0, 1]$.

Without drift, observing $\{\xi_{k,n}\}_{k=0,\dots,n}$ or $\{\xi_k\}_{k=0,\dots,n}$ leads to the same estimation. Using the Girsanov theorem, Proposition 2 stated for the high-frequency regime, that is on the observations $\{\zeta_{k,n}^{(1)}\}_{k=0,\dots,n}$ (since all the $\zeta^{(n)}$ are equal in distribution), is also valid in presence of a bounded drift.

With our data, the drift is very small compared to the ex-post volatility and the number n of observations is finite so that we still apply Proposition 2.

3.2 Estimation of the drift coefficients

To estimate the values $b_\pm$ of the drift, we consider that $\sigma_\pm$ has already been estimated and that the threshold $r = \log m$ is known (this issue is treated in Sect. 3.3). For the sake of simplicity, we still consider the shifted log-price process $\xi = X - r = \log S - r$.

Maximum likelihood estimation of the drift. A way to estimate the drift is to consider the drift among the possible ones which maximizes the Girsanov density $G(\xi)$ with respect to the solution to the driftless SDE $\mathrm{d}\xi_t = \sigma(\xi_t)\,\mathrm{d}W_t$ for a Brownian motion W. We then construct a maximum likelihood estimator (Iacus, 2008). The Girsanov density is

$$G(\xi) = \exp\left(\int_0^t \frac{b(\xi_s)}{\sigma(\xi_s)}\,\mathrm{d}W_s - \frac{1}{2}\int_0^t \frac{b^2(\xi_s)}{\sigma^2(\xi_s)}\,\mathrm{d}s\right). \tag{14}$$

As $b_\pm$ and $\sigma_\pm$ are piecewise constant, we can transform the stochastic integral $\int b(\xi_s)\sigma^{-1}(\xi_s)\,\mathrm{d}W_s$ using the Itô-Tanaka formula. It is then straightforward to establish that for our choice of model, the maximum of $G(\xi)$ with respect to a piecewise constant b is realized for

$$\beta_\pm(T) = \pm\frac{\xi_T^\pm - \xi_0^\pm - L_T/2}{Q_T^\pm(\xi)}, \tag{15}$$

where L_T is the symmetric local time of ξ at 0 and $Q_T^\pm$ are the occupation times of $\mathbb{R}^\pm$.

When the coefficients are constant, as for the log-price in the Black & Scholes model, where $\mathrm{d}\zeta_t = \sigma\,\mathrm{d}B_t + b\,\mathrm{d}t$, the constant drift coefficient b may be estimated through $b(T) = (\zeta_T - \zeta_0)/T$. Our estimator (15) generalizes this formula; the local time term appears because of the discontinuity in the coefficients.

As (15) is applied to ξ, solution to $\mathrm{d}\xi_t = \sigma(\xi_t)\,\mathrm{d}B_t + b(\xi_t)\,\mathrm{d}t$, a direct application of the Itô-Tanaka formula to $x \mapsto x^\pm$ in (15) implies that for the martingales $M_t^\pm = \int_0^t \sigma_\pm(\xi_s)\mathbf{1}_{\pm\xi_s\geq 0}\,\mathrm{d}B_s$,

$$\beta_\pm(T) = b_\pm + \frac{M_T^\pm}{Q_T^\pm} \text{ with } \langle M^\pm\rangle_T = Q_T^\pm \text{ and } \langle M^+, M^-\rangle_T = 0. \tag{16}$$

Estimators. Although neither $Q_T^\pm$ nor L_T are observed, they can be approximated from the observations. The occupation time $Q_T^\pm$ is approximated by $Q_\pm(n)$ given by (10). The local time L_T could be approximated as in Lejay and Pigato (2018b) by

$$L(n)_T = \frac{-3\sqrt{n\pi}}{2\sqrt{2T}}\frac{\sigma_+ + \sigma-}{\sigma_+\sigma_-}[\xi^+, \xi^-]_T. \tag{17}$$

We then approximate $\beta_\pm(T)$ by

$$\beta_\pm(T) \approx b_\pm(n) := \pm\frac{\xi_T^\pm - \xi_0^\pm - L(n)_T/2}{Q_\pm(n)_T}. \tag{18}$$

The $b_\pm(n)$ are discrete times approximations of the continuous-time estimators, which are easily constructed from the observations.

Asymptotic properties. The drift estimator shall be studied for long time horizon. The asymptotic properties of $\beta_\pm(T)$ as $T \to \infty$, hence of $b_\pm(n)$, depend on the asymptotic behaviors of $Q_T^\pm$ from (16). We summarize in Table 2 the different cases that depend solely on the respective signs of b_+ and b_-.

	$b_+ < 0$	$b_+ = 0$	$b_+ > 0$
$b_- > 0$	ergodic (**E**)	null recurrent (**N1**)	transient (**T0**)
$b_- = 0$	null recurrent (**N1**)	null recurrent (**N0**)	transient (**T0**)
$b_- < 0$	transient (**T0**)	transient (**T0**)	transient (**T1**)

Table 2: Regime of ξ according to the respective signs of (b_-, b_+).

The ergodic case, which corresponds to a mean-reverting process, is of course the most favorable one. In the transient case, the estimators may not converge. We present quickly some of the results in Lejay and Pigato (2018a).

E The ergodic case is equivalent to the mean-reverting case. Thus $Q_T^\pm/T$ converges almost surely as $T \to \infty$. Therefore $(\beta_-(T), \beta_+(T))$ converges almost surely to (b_-, b_+). For two independent unit Gaussian random variables (N^-, N^+),

$$\sqrt{T}\begin{bmatrix}\beta_-(T) - b_- \\ \beta_+(T) - b_+\end{bmatrix} \xrightarrow[T\to\infty]{\text{law}} \sqrt{|b_-| + |b_+|}\begin{bmatrix}\frac{\sigma_-}{\sqrt{b_-}}N^- \\ \frac{\sigma_+}{\sqrt{b_+}}N^+\end{bmatrix}. \tag{19}$$

T0 If $b_+ > 0$, $b_- \geq 0$, $\lim_{T\to\infty} Q_T^- < +\infty$. Therefore, $\beta_+(T)$ converges to b_+ and $\sqrt{T}(\beta_+(T) - b_+)$ converges in distribution to $\sigma_+ N^+$ for $N^+ \sim \mathcal{N}(0,1)$. The estimator $\beta_-(T)$ of b_- does not converge to b_- and is then meaningless. The case $b_- < 0$, $b_+ \leq 0$ is treated by symmetry.

T1 If $b_+ > 0$ and $b_- < 0$, then with probability $p = \sigma_- b_+/(\sigma_+ b_- + \sigma_- b_+)$ it holds that $\lim_{T\to\infty} Q_T^+/T = 1$ a.s. and $\lim_{T\to\infty} Q_T^- < +\infty$, while with probability $1-p$, $\lim_{T\to\infty} Q_T^-/T = 1$ a.s. and $\lim_{T\to\infty} Q_T^+ < +\infty$. This asymptotic behavior is due to the fact that after a given random time, the process does not cross the threshold anymore. Given $\lim_{T\to\infty} Q_T^+/T = 1$, $\beta_+(T)$ converges almost surely to b_+ and $\sqrt{T}(\beta_+(T) - b_+)$ converges in distribution to $\sigma_+ N^+$ for $N^+ \sim \mathcal{N}(0,1)$. The alternative situation, happening with probability $1-p$, is treated by symmetry.

N0 Whatever $T > 0$, Q_T^+/T follows a variant of the ArcSine distribution (Keilson and Wellner, 1978; Lejay and Pigato, 2018b). Therefore, the distribution of $\sqrt{T}(\beta_-(T), \beta_+(T))$ does not depend on T. Then $\beta_\pm(T)$ are consistent estimators of $b_\pm = 0$.

N1 If $b_+ = 0$, $b_- > 0$, then $\lim_{T\to\infty} Q_T^+/T = 1$ almost surely. In addition $Q_T^-/\sqrt{T}$ converges in distribution to $\sigma_+|N|/b_-$ for $N \sim \mathcal{N}(0,1)$. Therefore, $(\beta_-(T), \beta_+(T))$ converges almost surely to (b_-, b_+). Besides, there exist independent unit Gaussian random variables N^- and N^+, also independent from N, such that

$$\begin{bmatrix} T^{1/4}(\beta_-(T) - b_-) \\ T^{1/2}\beta_+(T) \end{bmatrix} \xrightarrow[T\to\infty]{\text{law}} \begin{bmatrix} \sigma_- \sqrt{\frac{b_-}{\sigma_+}} \frac{N^-}{\sqrt{|N|}} \\ \sigma_+ N^+ \end{bmatrix}. \tag{20}$$

The case $b_+ < 0$, $b_- = 0$ is treated by symmetry.

3.3 Estimation of the threshold

The above estimators for σ and b assume that the value m of the threshold is known. Following Tong (1983) (see also Priestley, 1988, p. 79), we estimate m using a principle of *model selection* relying on the ideas of the Akaike Information Principle (AIC) (Akaike, 1973). Since the AIC involves the likelihood function, for which we do not necessarily have closed form expressions, we will need to work with approximations.

Approximation of the density. Given a threshold m as well as volatility and drift functions $x \mapsto \sigma(x)$ and $x \mapsto b(x)$, we first consider the density $y \mapsto p(\Delta t, x, y; m, \sigma, b)$ of $X_{t+\Delta t}$ given $X_t = x$ (the process is time-homogeneous so that p only depends on Δt, not on t). For a vanishing drift, a close form expression for p is know (Keilson and Wellner, 1978). In presence of drift, the expression may become cumbersome if not intractable (Lejay, Lenôtre, and Pichot, 2017). However, p can be approximated in a short time via the related Green function, easier to compute (See Lenôtre, 2015, Chapter 2). Alternatively, we assume that the drift is constant over the time interval $[t, t+\Delta t]$ and replace p by the density of $Y_{t+\Delta t} + b(x)\Delta t$ given $Y_t = x$, where Y has the same volatility of X yet with a vanishing drift. In the implementation, we use the latter approximation of p which we denote by $\widetilde{p}(t, x, \cdot; m, \sigma, b)$.

Selection of the threshold. The procedure to select the "best" threshold is then

1/ We fix $m^{(1)}, \dots, m^{(k)}$ possible thresholds in the range of the observed values $\{X_{t_i}\}_{i=0,\dots,T}$ of the log-price X.

2/ For each threshold $m^{(j)}$, we estimate the drift and volatilities $\widehat{\sigma}^{(j)}$ and $\widehat{b}^{(j)}$.

3/ We compute the approximate log-likelihood

$$\Lambda^{(j)} = \sum_{i=0}^{T-1} \log \widetilde{p}(t, X_{t_i}, X_{t_{i+1}}; m^{(j)}, \widehat{\sigma}^{(j)}, \widehat{b}^{(j)}). \tag{21}$$

4/ We select as threshold $\widehat{m}$ the value $m^{(\tilde{j})}$ where $\tilde{j}$ is the indice for which $\{\Lambda^{(j)}\}_{j=1,\dots,k}$ is minimal.

Comparison with other models. In the model selection based on the AIC, the best model is the one for which the log-likelihood corrected by a value depending on the number of parameters is minimized. Here, the number of parameters is fixed to 4 so that it is sufficient to use only approximations of the log-likelihoods. A similar procedure is used in Meng et al. (2013), yet with a density estimated through Monte Carlo, which is time-consuming. On the contrary, our procedure avoids any simulation step. With respect to the estimation for the SETAR model (Tong, 1983; Priestley, 1988), as well as the one of the DTRS model presented below, based on least squares (Mota and Esquível, 2014), there is no delay so that the dimension of the model is reduced by 1.

4 Benchmarking the model

We apply now our estimators to empirical financial data. We benchmark our model against the *Delay and Threshold Regime Switching model* (DTRS) of Mota and Esquível (2014) by using the same data. We start this section shortly presenting the DTRS model.

4.1 The Delay and Threshold Regime Switching model

Mota and Esquível (2014) introduce a *regime switching model with delay and threshold* (DTRS). First, they consider two sets of (functional) parameters (σ_1, μ_1) and (σ_2, μ_2), as well as a diffusion solution to the stochastic differential equation

$$\mathrm{d}S_t = \mu_{J_t}(t, S_t)\,\mathrm{d}t + \sigma_{J_t}(t, S_t)\,\mathrm{d}B_t \tag{22}$$

for a Brownian motion B, where J is a non-anticipative process with values in the set of indices $\{1, 2\}$.

The rule for J to switch is based on a threshold m, a delay d as well as a small parameter $\epsilon > 0$. Assume $S_0 \leq m$ and $J_0 = 1$. The process evolves according to the parameters (σ_1, μ_1) until it reaches the level $m + \epsilon$ at a (random) time τ_1. Then it evolves according to the parameters (σ_1, μ_1) up to time $\tau_1 + d$ where it switches

AAPL	Apple	ADBE	Adobe
AMZN	Amazon	C	CitiGroup
CA	CA	CSCO	Cisco
GOOG	Google	HP	Hewlett-Packard
IBM	IBM	JPM	JP Morgan
KO	Coca-cola	MCD	McDonalds
MON	Monsanto	MSFT	Microsoft
MSI	Motorola	NYT	New-York Times
PCG	PG&E	PG	P & G
PM	Philip Morris	PFE	Pfizer
SBUX	Starbucks		

Table 3: Abbreviations of the names of the stocks (in Yahoo Finance).

to parameters (σ_2, μ_2) (that is $J_{\tau_1+d} = 2$) until it reaches the level m at time τ_2. Then it switches again to the state 1 after a delay d ($J_{\tau_2+d} = 1$) and so on.

The parameter ϵ prevents an accumulation of "immediate" switches so that S can be constructed on a rigorous basis (Esquível and Mota, 2014). With respect to simulation or estimation, ϵ is of no practical importance as S is only observed or simulated at discrete times.

More specifically, the DTRS model considered in Mota and Esquível (2014) assumes that the μ_i and σ_i ($i = 1, 2$) are

$$\begin{cases} \sigma_1(t,x) = \sigma_- \cdot x & \text{if } x < m, \\ \sigma_2(t,x) = \sigma_+ \cdot x & \text{if } x \geq m \end{cases} \text{and} \begin{cases} \mu_1(t,x) = \mu_- \cdot x & \text{if } x < m, \\ \mu_2(t,x) = \mu_+ \cdot x & \text{if } x \geq m \end{cases} \tag{23}$$

for some constants $\sigma_\pm > 0$ and $\mu_\pm$. Hence, on each regime, the price S follows a dynamic of Black & Scholes type. We also define $b_\pm = \mu_\pm - \sigma_\pm^2/2$ so that $b_\pm$ are the possible values of the drift for the log-price.

Adapting the estimation approach for the SETAR (Tong, 1983), Mota and Esquível (2014) propose a consistent estimation procedure of the parameters, based on least squares.

Results for the DTRS. This estimator is applied to the daily log-prices of 21 stock prices of the NYSE, from January 2005 to November 2009 (presented in Table 3). In Table 4, we report the estimated values of $\sigma_\pm$, m, $\mu_\pm$ (or $b_\pm$) and d found in Mota and Esquível (2014). These values have to be compared with the ones in Table 5.

For most of the data, a leverage effect is observed: the ex-post volatility below the threshold is higher than above it. In Mota and Esquível (2014), option prices

Delay Threshold Regime Switching (DTRS) (Mota and Esquível, 2014)									
Index	d	m [$]	σ_- [%]	σ_+ [%]	μ_- [%]	μ_+ [%]	b_- [%]	b_+ [%]	signs
AAPL	8	173.5	54.1	45.6	57.5	−112.4	42.8	−122.7	+−
ADBE	1	41.5	44.0	25.1	31.8	−74.3	21.9	−77.4	+−
AMZN	1	77.7	51.4	45.4	54.2	−462.2	41.1	−472.5	+−
C	2	43.1	120.6	16.8	39.8	−3.5	−32.8	−5.0	−−
CA	1	22.1	53.0	24.6	44.4	−23.2	30.2	−26.2	+−
CSCO	1	16.3	56.0	31.3	313.5	0.0	297.6	−5.0	+−
GOOG	13	642.0	37.0	40.2	40.6	−148.7	33.8	−156.7	+−
HP	1	46.9	39.2	27.1	42.3	−78.9	34.8	−82.4	+−
KO	1	10.0	78.6	29.1	398.2	−5.3	367.4	−9.6	+−
IBM	1	124.3	25.1	20.3	12.9	−93.5	9.6	−95.5	+−
JPM	2	25.0	131.3	47.5	715.4	−0.3	629.2	−11.6	+−
MCD	1	54.6	22.4	25.7	38.8	−29.5	36.3	−32.8	+−
MSFT	1	22.9	54.0	25.1	81.1	6.0	66.5	3.0	++
MSI	14	21.9	49.5	28.4	7.8	−47.9	−4.3	−51.9	−−
MON	1	112.0	47.0	49.4	57.5	−145.4	46.6	−157.5	+−
NYT	4	32.5	49.5	17.3	−9.8	−89.5	−22.2	−91.0	−−
PCG	6	35.3	49.4	21.6	170.1	−4.0	158.0	−6.6	+−
PFE	2	16.7	40.2	24.0	67.0	−14.6	59.0	−17.4	+−
PG	1	61.9	20.3	20.2	19.2	−28.0	17.1	−30.0	+−
PM	1	42.0	44.4	31.9	121.2	−40.3	111.4	−45.4	+−
SBUX	15	33.6	45.4	26.0	11.6	−39.8	1.3	−43.1	+−

Table 4: Estimated daily parameters in % per year found by Mota and Esquível (2014) for the DTRS model on daily data from January 2005 to November 2009 with the notations given in Table 1 (In the original table, volatilities and drift are expressed in % per day). The last column *signs* contains the respective signs of b_-, b_+ (a $+-$ indicates a mean-reversion effect).

of European calls are also computed using a Monte Carlo procedure. The resulting prices are in good agreement with the ones of the market.

Comparison between the DTRS model and the GOBM. In spirit, the GOBM is similar to DTRS of Mota and Esquível (2014) or to the model in Esquível and Mota (2014). Yet it avoids all the difficulties related to the "gluing" and regime change that involves a very thin layer which serves to avoid infinitely many immediate switches. Hottovy and Stechmann (2015) discuss the asymptotic behavior of the process as the width of the layer decreases to 0.

The GOBM has 5 parameters while the DTRS has 6 parameters because it also involves a delay. For most of the data, the estimated delay in the DTRS is $d = 1$, which means that the switching occurs without delay. Otherwise, the delay means a slow decreasing auto-correlation, or a long memory effect. Yet, for long delay, how to discriminate a leverage effect from sudden changes due to external parameters such as crisis? The presence of a delay increases the possibility of miss-specifications in the estimation procedure.

4.2 Estimation of the parameters of the GOBM

In Table 5, we estimate the parameters for the GOBM on the same time series as for the DTRS. The complete numerical results may be found in the side report Lejay and Pigato (2017).

Although we use the same source (Yahoo Finance) as Mota and Esquível (2014), it seems that KO is a different time series than in this article.

The volatilities (σ_-, σ_+) are in good agreement for both models. The respective signs of b_- and b_+ are consistent with the ones of Mota and Esquível (2014) and suggest a mean-reversion effect ($b_- > 0$, $b_+ < 0$) for most of the stock prices. The magnitudes of b_- and b_+ are also consistent with the ones of Mota and Esquível (2014). As the number of data is rather small ($n = 1217$) and the considered period is only 5 years, it is not reasonable to aim for a more accurate description of the drift.

The threshold estimations are also in good agreement for 11 stocks out of 21.

For both the DTRS and GOBM models, $\mu_- > 0$ excepted for C, NYT and MSI for the GOBM model, and NYT for the DTRS model. Moreover, $\sigma_- > \sigma_+$ for all the stocks, the only exceptions being MCD for both models and GOOG for the DTRS model. In the latter situation, σ_- is close to σ_+. This indicates that below the threshold, the ex-post volatility is higher and the drift is upward oriented.

Index	m [$]	σ_- [%]	σ_+ [%]	μ_- [%]	μ_+ [%]	b_- [%]	b_+ [%]	signs
AAPL	119.4	59.86	40.48	55.37	1.51	37.46	−6.69	+−
ADBE	26.2	69.21	47.53	132.52	−13.43	108.57	−24.72	+−
AMZN	39.7	38.71	54.57	84.00	14.71	76.51	−0.18	+−
C	40.1	118.58	17.35	−30.86	−12.47	−101.17	−13.98	−−
CA	21.5	51.27	25.54	49.30	−13.12	36.15	−16.38	+−
CSCO	16.9	60.48	30.70	285.88	−9.70	267.59	−14.41	+−
GOOG	373.8	44.69	33.03	100.82	0.20	90.83	−5.26	+−
HP	57.6	66.18	41.09	50.75	−113.55	28.86	−121.99	+−
IBM	115.2	26.04	20.21	17.64	−20.18	14.25	−22.22	+−
JPM	33.2	124.90	41.11	230.79	−4.64	152.79	−13.09	+−
KO	47.8	23.70	17.88	13.97	3.20	11.16	1.60	++
MCD	51.4	20.23	28.02	30.57	2.28	28.52	−1.64	+−
MSFT	23.2	51.20	25.90	142.74	−20.36	129.64	−23.71	+−
MSI	14.2	66.44	25.99	−7.82	−1.03	−29.90	−4.41	−−
MON	114.1	54.39	44.85	39.96	−135.09	25.17	−145.15	+−
NYT	16.0	78.46	25.85	−7.19	−25.35	−37.97	−28.69	−−
PFE	19.1	39.68	20.55	8.89	−6.28	1.02	−8.39	+−
PCG	34.9	62.28	22.72	343.06	−4.89	323.67	−7.47	+−
PG	51.9	29.62	20.12	54.17	2.05	49.78	0.02	++
PM	45.6	42.21	27.89	18.87	−7.79	9.97	−11.68	+−
SBUX	13.0	71.88	46.51	45.75	−13.04	19.91	−23.86	+−

Table 5: Estimated parameters in % per year for the GOBM model on the daily data from January 2005 to November 2009 with the notations given in Table 1. The last column *signs* contains the respective signs of b_-, b_+ (a $+-$ indicates a mean-reversion effect).

5 Is there some leverage effect?

Our aim is to test whether or not $\sigma_+ = \sigma_-$ when $b_- = b_- = 0$ (on daily data, b_- and b_+ have small values with respect to σ_- and σ_+). Our Hypothesis test is then

(H_0) (null hypothesis) $\sigma_- = \sigma_+$;

(H_1) (alternative hypothesis) $\sigma_- \neq \sigma_+$.

5.1 Construction of a confidence region

For the sake of simplicity, let us set $S_\pm := \sigma_\pm^2$ and $S(n)_\pm := (\sigma_\pm(n))^2$.

For two elements $f_\pm$, such as $S_\pm$, we also define the two dimensional vector $\mathbf{f} := (f_-, f_+)'$.

Given the occupation time $Q_T^{\pm}$ below and above the threshold, we define $O_{\pm} = Q_T^{\pm}/T$ as the renormalized occupation time. The asymptotic result (12) of Proposition 2 is rewritten as

$$\mathbf{S}^n \approx \mathbf{S} + \frac{\sqrt{2}}{\sqrt{n}} \begin{bmatrix} S_- \frac{G_-}{\sqrt{O_-}} \\ S_+ \frac{G_+}{\sqrt{O_+}} \end{bmatrix} = \mathbf{S} + \frac{1}{\sqrt{n}} M\mathbf{G} \text{ with } M = \sqrt{2} \begin{bmatrix} S_-/\sqrt{O_-} & 0 \\ 0 & S_+/\sqrt{O_+} \end{bmatrix}, \tag{24}$$

where $\mathbf{G} = \mathcal{N}(0, \mathrm{Id})$ is a Gaussian vector independent from the process X. The stable convergence means that the limit term in (24) involves a double randomness, and M is a measurable function of X. Replacing $\mathbf{S}$ by its approximation $\mathbf{S}(n)$ as well as $\mathbf{O}$ by its Riemann sum approximation $\mathbf{O}(n)$ constructed from the observations, we set

$$M^n = \sqrt{2} \begin{bmatrix} S_-(n)/\sqrt{O_-(n)} & 0 \\ 0 & S_+(n)/\sqrt{O_+(n)} \end{bmatrix}. \tag{25}$$

As the Gaussian vector $\mathbf{G}$ is isotropic, we define for a level of confidence α the quantity q_α by $\mathbb{P}[|\mathbf{G}| \leq q_\alpha] = 1 - \alpha$. This quantity is easily computed since $|\mathbf{G}|^2$ follows a χ^2 distribution with two degrees of freedom. Our *confidence region* of level α is the ellipsis

$$\mathcal{R}_\alpha = \left\{ \mathbf{S}(n) + \frac{q_\alpha}{\sqrt{n}} M^n \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \,\middle|\, \theta \in [0, 2\pi) \right\}. \tag{26}$$

Our **rule of decision** is then: reject the Null Hypothesis (H_0) if the diagonal line $s : [0, +\infty) \mapsto (s, s)$ does not cross $\mathcal{R}_\alpha$.

5.2 Numerical simulations

We perform numerical simulations to check the reliability of the estimation of (σ_-, σ_+, r) as well the hypothesis test "$\sigma_+ = \sigma_-$". For this, we simulate $N = 1000$ paths with daily data over 5 years for the 3 sets of parameters given in Table 6. The density of the estimated values of σ_-, σ_+ and r are shown in Figure 1. The proportion of rejection of the hypothesis test "$\sigma_+ = \sigma_-$" are given in Table 7. A good agreement is then observed for the parameters σ_-, σ_+ and the threshold.

The estimation of μ_+ and μ_-, not shown here, presents a large variance as expected since when $\mu_+ = \mu_- = 0$, the process is only null recurrent. This is discussed in full details in Lejay and Pigato (2018a).

5.3 Empirical result on the 2005-2009 data

As the drift is small, it should not affect this test. Therefore we assume through all this section that $b_- = b_+ = 0$.

		σ_-	σ_+	μ_-	μ_+	m
set 1	$\sigma_+ \ll \sigma_-$	80 %/yr	30 %/yr	0	0	1
set 2	$\sigma_+ \approx \sigma_-$	50 %/yr	30 %/yr	0	0	1
set 3	$\sigma_+ = \sigma_-$	30 %/yr	30 %/yr	0	0	1
			$S_0 = 1\,\$$			

Table 6: Set of yearly parameters used for simulations.

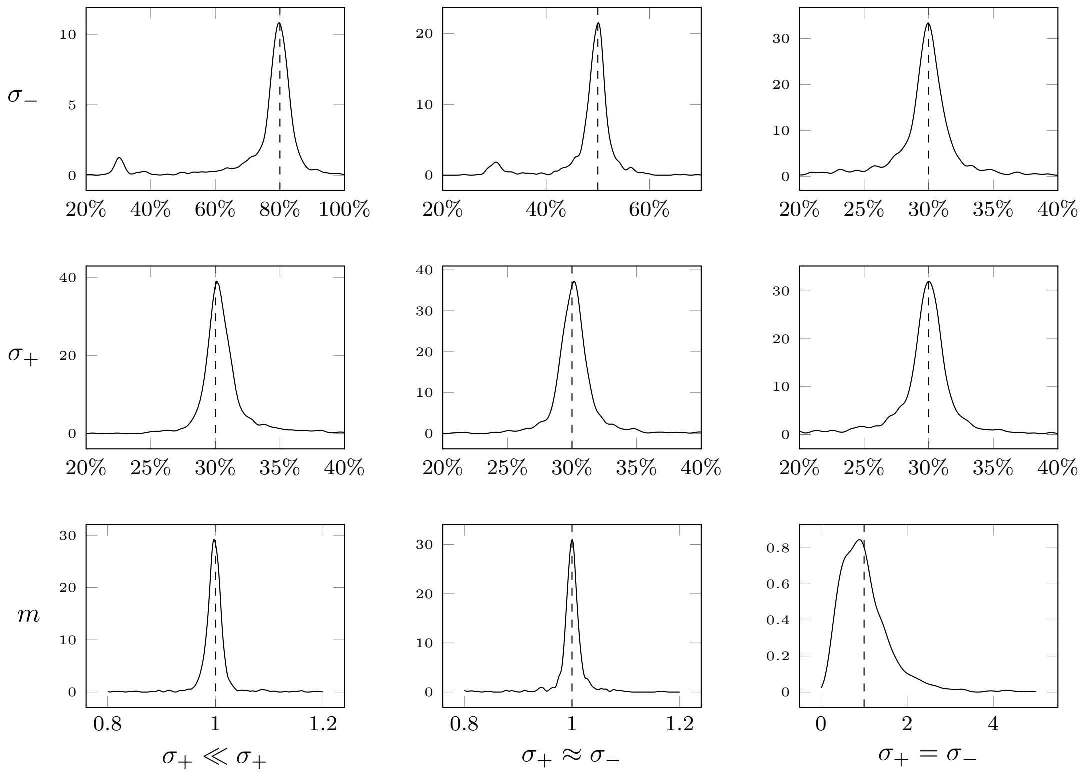

Figure 1: Density of the estimated values of σ_-, σ_+ (yearly) and the threshold m for the 3 sets of parameters of Table 6, with 1000 simulations per set.

In Figure 2, we apply this rule to our data. The null hypothesis (H_0) "$\sigma_- = \sigma_+$" is rejected for all the stocks except for PCG, meaning that $\sigma_- \neq \sigma_+$ should be considered for 20 out of 21 stocks. The normalized occupation time O_+ for PCG is close to 99 %. This may explain the elongated shape of the associated confidence region.

In Figure 3, we plot the approximated log-likelihood $\Lambda^{(i)}$ in function of $r^{(i)} =$

$\sigma_+ \ll \sigma_-$	$\sigma_+ \approx \sigma_-$	$\sigma_+ = \sigma_-$
81%	81%	14%

Table 7: Proportion of rejection of the null hypothesis (H_0) "$\sigma_+ = \sigma_-$" with a 95% confidence level for the 3 sets of parameters of Table 6, with 1000 simulations per set.

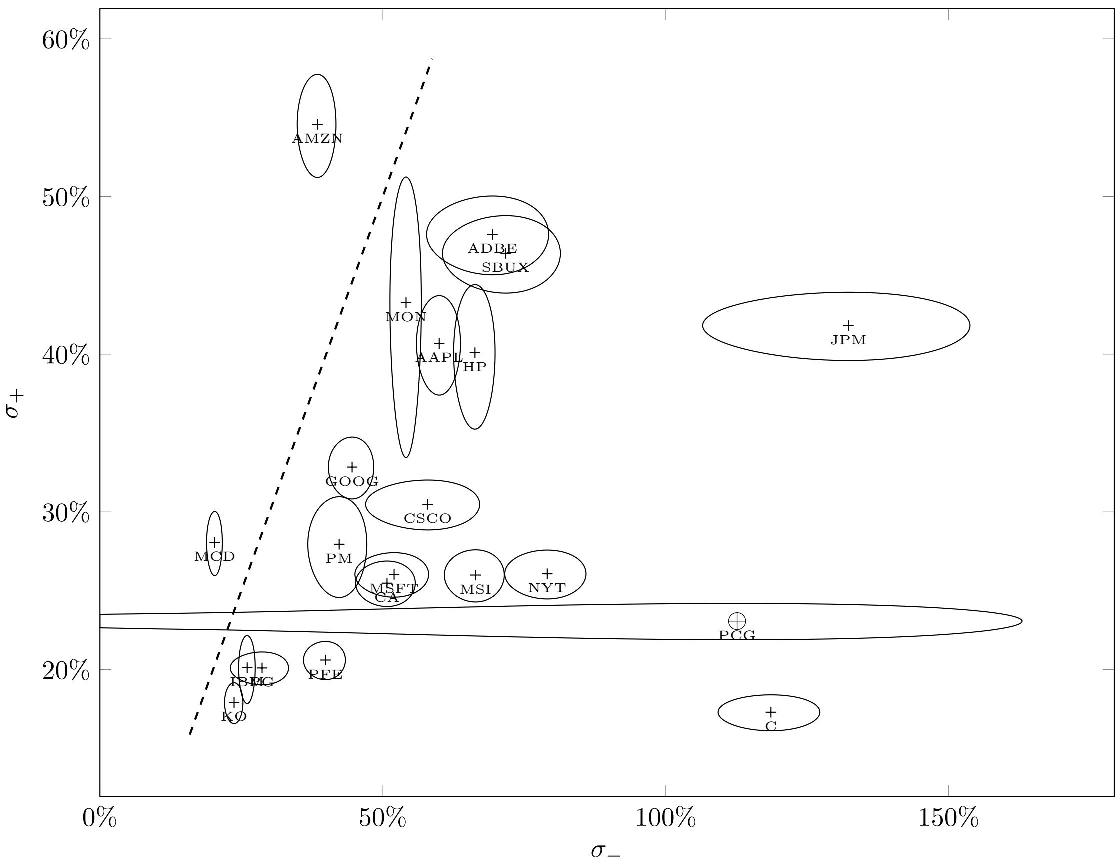

Figure 2: Confidence regions for (σ_-, σ_+): Each point is the value of the stock in the (σ_-, σ_+)-plane. Confidence regions at 95 % are the ellipsis in the (σ_-, σ_+)-plane around the points. Points marked by $\oplus$ are the ones for which the Hypothesis $\sigma_- = \sigma_+$ is not rejected. Points marked by $+$ are the ones for which this Hypothesis is rejected.

$\log m^{(i)}$ for 3 stocks. We see that $\Lambda^{(i)}$ may have one main peak (for CSCO), two main peaks (for GOOG) or be "flat" as for PCG. A steep peak means that it is clear where the threshold level should be taken, and the procedure is more stable. In these cases, σ_- is likely to differ from σ_+ and leverage effect occurs.

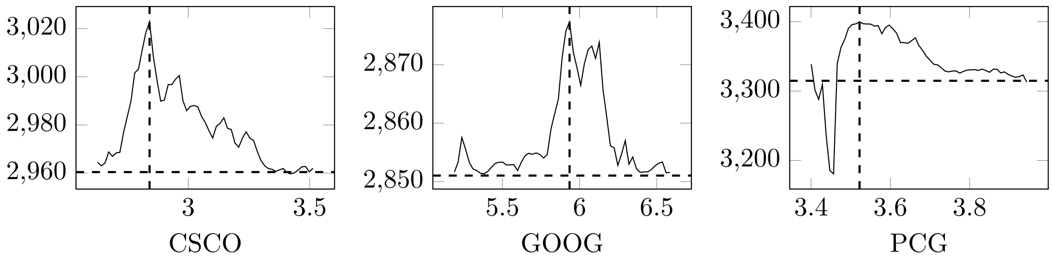

Figure 3: The (approximated) log-likelihood $\Lambda^{(i)}$ given by (21) in function of possible threshold $r^{(i)} = \log m^{(i)}$ for the stocks CSCO, GOOG and PCG. The vertical dashed line represents the threshold $r^{(i)}$ which maximizes $\Lambda^{(i)}$, hence the estimated r. The horizontal dashed line represents the value of the log-likelihood of the drifted Brownian motion (that is $\sigma_- = \sigma_+$ and $b_- = b_+$).

5.4 Comparison with a non-parametric estimator

Non-parametric estimation assumes nothing on the underlying volatility and drift coefficients (Kutoyants, 2004; Iacus, 2008). The Nadaraya-Watson estimator provides us with such an estimator (Iacus, 2008). We then compare graphically our estimations with the non-parametric estimation of the coefficients of the log-price. For this, we use the `R` package `sde` (Iacus, 2008). In Figure 4, we present the results for the 3 stocks already used in Figure 3 (more figures may be found in Lejay and Pigato (2017)). Most of the stocks seem to exhibit a behavior similar to the one presented here, with a sharp variation of both the volatility and the drift. Again, this reinforces the idea that regime switching holds for most of the stocks.

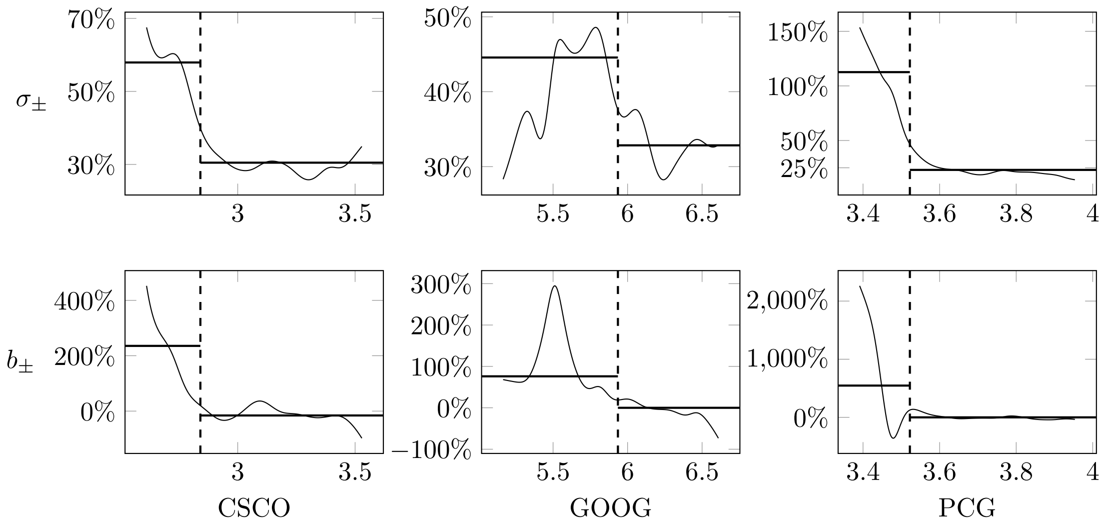

Figure 4: Non-parametric estimation of σ and b for the log-price of the stocks CSCO, GOOG and PCG with a Nadaraya-Watson estimator. The vertical dashed line represents the choice of the threshold. The horizontal lines represent the estimated values of $(\sigma_-(n), \sigma_+(n))$ (top) and $(b_-(n), b_+(n))$ (bottom).

6 Leverage and mean-reversion effects for the S&P 500 stocks

We apply our estimators to the S&P 500 stock prices over the 3 periods of 5 years each: 1/1/2003 – 31/12/2007, 1/1/2008 – 31/12/2012 and 1/1/2013 – 31/12/2017. The second period involves the 2008 financial crisis.

Estimators are thus computed on the three periods for 327 of the stocks out of the 500, failures being due to the lack of complete data or to stock prices for which the number of days below or above the threshold is smaller than 62 over 1259 samples (less than 5 %). The latter exclusion aims at avoiding outliers.

In Figure 5, we plot the estimated values of (σ_-, σ_+) and (b_-, b_+).

In the 2008-2012 period, containing the 2008 financial crisis, the stock prices exhibit higher ratio of σ_-/σ_+, hence stronger leverage effects. For the 2008-2012 period, the hypothesis test "$\sigma_- = \sigma_+$" is rejected for 318 over 399 of the stocks. For the 2008-2012 period, the test is rejected for all the stocks. For the 2013-2017 period, this hypothesis test is rejected for 320 over 327 of the stocks.

These findings seem to be consistent with Aït-Sahalia, Fan, and Li (2013), which shows leverage effect in an aggregated form through the correlation between the

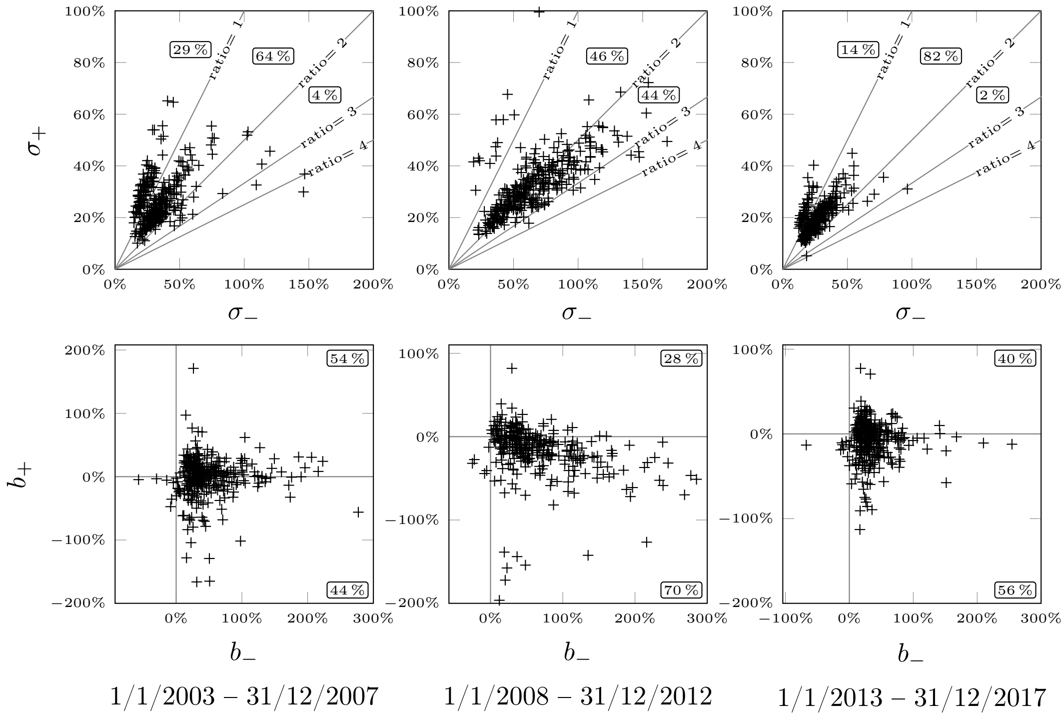

Figure 5: Yearly estimated values of (σ_-, σ_+) and (b_-, b_+) on the daily close prices for the stocks in the S&P 500. The percentages in boxes indicates the proportions of stocks in each region delimited by the solid gray lines.

VIX (involving the ex-ante volatility) and the log-return of the indices of the S&P 500 for the period 2004/1/1 – 2007/12/12.

Let us now look at the drift coefficient. First, we notice that b_- is almost always positive, but b_+ can be positive or negative. For the 2008-2012 period, which contains the 2008 financial crisis, 243 stocks show a mean-reverting behavior ($b_- > 0, b_+ < 0$) while for the 2003-2007 period (resp. the 2013-2017), only 144 (resp. 191) show a mean-reverting behavior. A possible interpretation of such results is that, on periods not involving financial crisis, prices tend to increase in time and therefore also the estimated value of b_+ is positive relatively often. When a crisis occurs, prices oscillate more, going down as well as up, and thus giving in most cases estimated values of $b_+ < 0$, $b_- > 0$. This seems to be consistent with the results of Section 4 and Mota and Esquível (2014).

Let us also report here a similar finding of Spierdijk, Bikker, and Hoek (2012), together with one of its economic interpretations given in the same paper: “Our findings suggest that expected returns diverge away from their long-term value and converge back to this level relatively quickly during periods of high economic uncertainty; much faster than in more tranquil periods. When the economic uncertainty dissolves, expected returns are likely to show a substantial increase in value during a relatively short time period, which could account for such high mean-reversion speed. Measures and interventions by financial and government institutions to restore financial stability may also speed up the adjustment process."

In Figure 6, we plot the log-threshold m over the mean log-price (in order to get a normalized value) for each of the period. We observe that this ratio ranges between 0.8 and 1.2 for each stock and each period. Besides, this ratio is higher for 312 of the 399 stocks for the 2013-2017 period than for the 2008-2012 period.

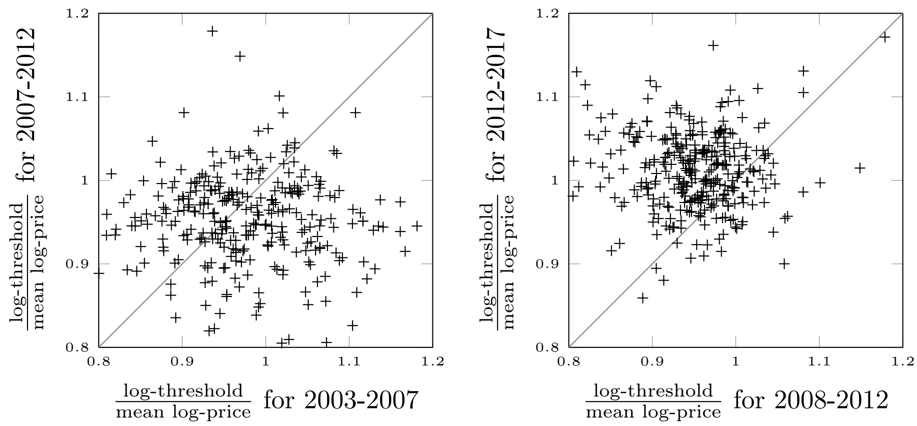

Figure 6: Ratio of the log-threshold m over the mean log-price for the 2007-2012 against the similar value for the 2013-2017 period. The solid line is $y = x$.

These facts seem to indicate than in the crisis period, a stronger leverage effect is more likely to occur at a lower threshold.

7 Conclusion

Leverage effects in finance have been the subject of a large literature with many empirical evidences. The *Geometric Oscillating Brownian motion* (GOBM) studied in this article mimics such leverage effect. This model can be interpreted as a continuous-time version of the self-exciting threshold autoregressive model (SETAR).

We have shown its validity on real data and exhibited evidence in favor of leverage effects. We detect a mean-reverting behavior for most of the stocks in periods of financial crisis, less so in periods not containing major events, in agreement with

Spierdijk, Bikker, and Hoek (2012). Our estimations are consistent with the ones of M. Esquível and P. Mota based on least squares.

Our model is simple and does not aim at capturing other stylized facts. It could serve as a basic building brick for more complex models. Our rationale is that the GOBM is really tractable while offering more flexibility than the Black & Scholes model:

- The estimation procedure is simple to set up.
- Simulations are easily performed.
- The market is complete.
- Option pricing could be performed through analytic or semi-analytic approach without relying on Monte Carlo simulations.

In addition, our model and estimation procedure could serve other purposes. In this model the leverage effect is a consequence of a spatial segmentation in which the dynamics of the price changes according to a threshold. The same estimation procedure could also be applied in short time windows in order to detect sharp changes, hence reflecting temporal changes, as for regime switching models involving Hidden Markov models.

Another possible application of the GOBM, and more generally of local volatilities with discontinuities, would be to introduce such features in more complex models. The properties we showed in the present paper and their capability of reproducing extreme skews in the implied volatility (Pigato, 2017) suggest that such discontinuities could be a tractable way to introduce asymmetries and regime changes in other models (Decamps, Goovaerts, and Schoutens, 2006).

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