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► **To cite this version:**

Jean-François Mercier, Colin Mietka, Florence Millot, Vincent Pagneux. Acoustic propagation in a vortical homentropic flow. 2017. hal-01663949

**HAL Id: hal-01663949**

**<https://inria.hal.science/hal-01663949v1>**

Preprint submitted on 14 Dec 2017

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# ACOUSTIC PROPAGATION IN A VORTICAL HOMETROPIC FLOW\*

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**Abstract.** This paper is devoted to the theoretical and the numerical studies of the radiation of an acoustic source in a general homentropic flow. As a linearized model, we consider Goldstein's Equations, which extend the usual potential model to vortical flows. The equivalence between Linearized Euler's Equations with general source terms and Goldstein's Equations is established, and the relations between unknowns, in each model, are analysed. A closed-form relation between the hydrodynamic phenomena and the acoustics is derived. Finally, numerical results are presented and the relevance of using Goldstein's Equations compared to the potential model is illustrated.

**Key words.** Acoustics, Hydrodynamics, Goldstein's Equations, Finite Element Method, Discontinuous Galerkin Element Method, Perfectly Matched Layers

**AMS subject classifications.** 65J10, 65N30, 65Z05, 35J50, 35Q35, 35Q31

**1. Introduction.** Aeroacoustics consists in determining the acoustic perturbations propagating in an imposed flow. It is mostly the need of noise reduction in aeronautics which creates an increasing interest in the field of aeroacoustics, the main application concerning the sound propagation inside the radiation and outside a plane engine. But let us cite also the need to reduce the sound emitted by exhaust pipes in the car industry or by ventilation ducts in the domestic industry. In the present paper, we focus on the propagation of the sound created by a known source in a given flow. In particular we do not address the mechanisms responsible of noise generation [1, 2], involving non-linear models. We consider linearized equations and we focus on the time harmonic regime.

The most studied case of acoustic propagation in a flow is the case of a curl free carrying flow. This case is simpler because for a uniform flow [3] or even for a flow  $\mathbf{v}_0$  potential and homentropic, the acoustic perturbations are also potential and the velocity potential  $\varphi$  satisfies the convected Helmholtz equation [4, 5]:

$$(1) \quad \frac{D}{Dt} \left( \frac{1}{c_0^2} \frac{D\varphi}{Dt} \right) = \frac{1}{\rho_0} \operatorname{div}(\rho_0 \nabla \varphi),$$

where  $\rho_0$  and  $c_0$  are respectively the density and the sound velocity of the flow and where

$$(2) \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \nabla,$$

is the convective derivative. This model is well adapted to a classical Finite Element discretization and is widely used in some industrial applications [6, 7] or in the analysis of the influence of liners on the acoustic propagation [8, 9, 10].

For a more complex flow, a scalar description of acoustic perturbations is no longer possible because of the coupling between acoustic and hydrodynamic phenomena. Such coupling requires more sophisticated vector models. For the time-domain

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\*This work was funded by the Fog Research Institute under contract no. FRI-454.

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38 problem, several methods have been developed to solve the Linearized Euler Equa-  
 39 tions, but the treatment of the artificial boundaries still raises open questions. On the  
 40 other hand, the time-harmonic problem in an undounded domain has not been con-  
 41 sidered for a general flow, excepting thanks to the Galbrun equation [11]. However,  
 42 the Galbrun equation [12, 13, 14] revealed not to be adapted to 3D configurations. To  
 43 go beyond this limitation, in this paper we consider the Goldstein equations [15, 16].  
 44 They have been widely used to model the development of perturbations in a swirling  
 45 flow [17, 18, 19, 20, 21, 22, 23] and we will show that they are well adapted for  
 46 aeroacoustics applications.

47 The main advantage of using Goldstein’s equations is that they are simpler than  
 48 alternative models, among which the Linearized Euler equations [24, 25, 26, 27], the  
 49 Galbrun equation or the Möhring equations [28, 29], because they are mainly scalar  
 50 and close to a usual wave equation. Indeed, they can be seen as a perturbation of  
 51 the simple convected Helmholtz equation (1) and they even exactly reduce to this  
 52 equation in the potential areas of the carrier flow, this last point being particularly  
 53 important for 3D applications.

54 In this paper, we study theoretically and numerically the radiation of general  
 55 sources in a general flow, using the Goldstein equations. A general flow means a flow  
 56 at least not potential, but also not only a parallel shear flow, for which Pridmore-  
 57 Brown equation [30] is more commonly used. In addition to the velocity potential,  
 58 Goldstein’s equations involve a supplementary unknown, namely the hydrodynamic  
 59 vector field  $\xi$ , which satisfies a transport equation. Our main contribution is to  
 60 determine the closed-form expression (14) of the Goldstein hydrodynamic unknown  
 61  $\xi$ . In particular we will show that for a potential flow and potential sources,  $\xi = \mathbf{0}$   
 62 and Eq. (1) is recovered. Note that Goldstein derived his equations only in the  
 63 case of a potential flow in presence of incident convected disturbances but without  
 64 explicit source term (see Eq. (9)). Since we are interested in a general configuration,  
 65 we derive the equations in the general case. Starting from Euler’s equations, we  
 66 first derive a generalization of the Goldstein equations that are valid in presence  
 67 of compactly supported source terms and of a vortical flow as well. Finally, since  
 68 the Goldstein equations are exact, we will quantify the errors when the convected  
 69 Helmholtz equation (1) is used in a non potential flow, such approximation being  
 70 often used for its simplicity.

71 The outline of the paper is the following. The generalized Goldstein equations are  
 72 derived in section 2 and we show they are equivalent to Euler’s equations. This section  
 73 ends up with the determination of the general expression of  $\xi$ . Section 3 is mainly  
 74 devoted to some 2D numerical experiments. First, we consider a parallel shear flow in  
 75 an unbounded domain. Such simple flow facilitates the presentation of the numerical  
 76 method, based on the introduction of Perfectly Matched Layers (PMLs) to bound  
 77 the calculation domain and on the coupling between continuous and discontinuous  
 78 Finite Elements. We also give some conditions under which the radiation problem  
 79 with PMLs is well-posed, ensuring the convergence of the Finite Element scheme.  
 80 Then we use Goldstein’s equations to study numerically the acoustic radiation of a  
 81 source in presence of such a flow. The solutions are validated by comparison with the  
 82 solutions obtained from the Galbrun equation and the influence of a vortical flow is  
 83 illustrated. Finally we consider a non-parallel flow and we illustrate the closed-form  
 84 relation giving  $\xi$ .

85 **2. Derivation of the Goldstein equations from the Euler equations.** Our  
 86 first purpose is to prove in a simple and comprehensive way the equivalence between

87 Euler's equations and Goldstein's equations in presence of source terms and of a  
 88 vortical flow. Although we are primary interested in the time harmonic regime, such  
 89 proof will be done in the time regime because the initial conditions enable us to have  
 90 uniqueness results when integrating transport equations. We start by recalling the  
 91 Linearized Euler Equations.

92 **2.1. Euler's equations.** Since the theory is not limited to a 2D configuration,  
 93 we work in 3D and we consider a propagation domain  $\Omega = \mathbb{R}^3$  or  $\Omega = \mathbb{R}^3 \setminus B$  where  $B$   
 94 represents obstacles or walls guide, on which a rigid boundary condition is imposed  
 95  $\tilde{\mathbf{v}} \cdot \mathbf{n} = 0$  with  $\tilde{\mathbf{v}}$  the total fluid velocity. We consider a subsonic inviscid flow of a  
 96 perfect compressible fluid with constant specific heats capacity  $c_p$  and  $c_v$  at constant  
 97 pressure and volume. The flow is the superposition of a steady flow and of small time-  
 98 dependent acoustic perturbations: the velocity  $\mathbf{v}$ , the density  $\rho$ , the pressure  $p$  and  
 99 the entropy  $s$ . The small perturbations satisfy in  $\Omega$  the Linearized Euler Equations.

100 Following Goldstein [15], it is assumed that the carrier flow is homentropic:  $s_0$   
 101 is constant. This is an hypothesis commonly used [17, 18, 19, 20], despite effects of  
 102 a mean entropy have been sometimes taken into account [21, 22]. For such a carrier  
 103 flow, Goldstein showed that the Linearized Euler Equations simplify in a system that  
 104 we extend by including general source terms  $f$ ,  $\mathbf{g}$  and  $h$ :

$$105 \quad (3) \quad \begin{cases} \frac{D\mathbf{v}^*}{Dt} + (\mathbf{v}^* \cdot \nabla)\mathbf{v}_0 + \nabla \left( \frac{p}{\rho_0} \right) = \mathbf{g}, \\ \frac{D}{Dt} \left( \frac{p}{\rho_0 c_0^2} \right) + \frac{1}{\rho_0} \operatorname{div}(\rho_0 \mathbf{v}) = -f, \\ \frac{Ds}{Dt} = h, \\ \mathbf{v}^* = \mathbf{v} - \frac{s\mathbf{v}_0}{2c_p}. \end{cases}$$

106  $D/Dt$  is defined in Eq. (2) and the boundary conditions are  $\mathbf{v} \cdot \mathbf{n} = 0$  on  $\partial\Omega$ . Source  
 107 terms are chosen compactly supported and such that  $f, h \in L^2(\Omega)$ ,  $\mathbf{g} \in (L^2(\Omega))^3$ . We  
 108 consider causal sources and null initial conditions:

$$109 \quad \begin{aligned} f(\mathbf{x}, t \leq 0) &= 0 = \mathbf{g}(\mathbf{x}, t \leq 0) = h(\mathbf{x}, t \leq 0), \\ \mathbf{v}^*(\mathbf{x}, 0) &= 0 = p(\mathbf{x}, 0) = s(\mathbf{x}, 0). \end{aligned}$$

110 In (3), the entropy perturbation  $s$  is decoupled from the other perturbations. Note  
 111 that this is no longer true if the carrier flow is not homentropic. Then the entropy  
 112 satisfies  $\frac{Ds}{Dt} + \mathbf{v} \cdot \nabla s_0 = h$  and as soon as  $s_0 \neq 0$ ,  $s$  depends on  $\mathbf{v}$ . For a homentropic  
 113 carrier flow,  $s$  can be eliminated by solving the following decoupled problem:

$$114 \quad (4) \quad \begin{cases} \frac{Ds}{Dt} = h, & \text{with } s(\mathbf{x}, 0) = 0, \end{cases}$$

115 whose unique solution is noted  $H(\mathbf{x}, t)$  (see remark 2). Then, noting  $\boldsymbol{\omega}^* = \mathbf{curl} \mathbf{v}^*$

116 the vorticity perturbation, we rewrite (3) in the more convenient form

$$117 \quad (5a) \quad \frac{\partial \mathbf{v}^*}{\partial t} + \nabla(\mathbf{v}_0 \cdot \mathbf{v}^*) + \boldsymbol{\omega}_0 \times \mathbf{v}^* + \boldsymbol{\omega}^* \times \mathbf{v}_0 + \nabla \left( \frac{p}{\rho_0} \right) = \mathbf{g},$$

$$118 \quad (5b) \quad \frac{D}{Dt} \left( \frac{p}{\rho_0 c_0^2} \right) + \frac{1}{\rho_0} \operatorname{div}(\rho_0 \mathbf{v}) = -f,$$

$$119 \quad (5c) \quad \mathbf{v}^* = \mathbf{v} - \frac{H \mathbf{v}_0}{2c_p},$$

$$120 \quad (5d) \quad \mathbf{v}^*(\mathbf{x}, 0) = 0 = p(\mathbf{x}, 0) = s(\mathbf{x}, 0).$$

122 with

$$123 \quad (6) \quad \boldsymbol{\omega}_0 = \operatorname{curl} \mathbf{v}_0,$$

124 the vorticity of the carrier flow and where we have used the vector identity Eq. (31).

125 **2.2. Goldstein's equation with general sources in a general flow.** In this  
 126 paragraph, we aim at obtaining the Goldstein equations from Eq. ((5a),(5b),(5c),(5d)),  
 127 with a general flow ( $\boldsymbol{\omega}_0 = \mathbf{0}$  or  $\neq \mathbf{0}$ ) and general sources  $f$  and  $\mathbf{g}$ . The Helmholtz  
 128 decomposition indicates that  $\mathbf{g}$  can be written  $\mathbf{g} = \nabla g_1 + \operatorname{curl} \mathbf{g}_2$ , with  $\mathbf{g}_2 = \mathbf{0}$  if  
 129  $\operatorname{curl} \mathbf{g} = \mathbf{0}$ . In 3D, the condition  $\operatorname{div} \mathbf{g}_2 = 0$  is added whereas in 2D,  $g_2$  becomes a  
 130 scalar. We will prove the theorem:

131 **THEOREM 1.** *Euler's Equations ((5a),(5b),(5c),(5d)) are equivalent to the gen-*  
 132 *eralized Goldstein equations*

$$133 \quad (7) \quad \begin{cases} \frac{D}{Dt} \left( \frac{1}{c_0^2} \frac{D\varphi}{Dt} \right) - \frac{1}{\rho_0} \operatorname{div}[\rho_0(\nabla\varphi + \boldsymbol{\xi})] = F + \frac{D}{Dt} \left( \frac{g_1}{c_0^2} \right), \\ \frac{D\boldsymbol{\xi}}{Dt} + (\boldsymbol{\xi} \cdot \nabla) \mathbf{v}_0 - \nabla\varphi \times \boldsymbol{\omega}_0 = \operatorname{curl} \mathbf{g}_2, \end{cases}$$

134 where

$$135 \quad F = f + \frac{\mathbf{v}_0 \cdot \nabla H}{2c_p},$$

136 associated to the initial conditions  $\varphi(\mathbf{x}, 0) = 0 = (\partial\varphi/\partial t)(\mathbf{x}, 0) = \boldsymbol{\xi}(\mathbf{x}, 0)$ , with

$$137 \quad (8) \quad \begin{cases} \mathbf{v} = \nabla\varphi + \boldsymbol{\xi} + \frac{H \mathbf{v}_0}{2c_p}, \\ p = \rho_0 \left( g_1 - \frac{D\varphi}{Dt} \right). \end{cases}$$

138 Before proving this theorem, let us make some remarks.

139 **REMARK 1.**

- 140 • If all the sources are vanished  $\mathbf{g} = 0 = f = h$  and if  $\boldsymbol{\omega}_0 = \mathbf{0}$ , Eq. (7) and Eq.  
 141 (8) degenerate to the result of Goldstein [15]:

$$142 \quad (9) \quad \begin{cases} \frac{D}{Dt} \left( \frac{1}{c_0^2} \frac{D\varphi}{Dt} \right) = \frac{1}{\rho_0} \operatorname{div}[\rho_0(\nabla\varphi + \boldsymbol{\xi})], \\ \frac{D\boldsymbol{\xi}}{Dt} = -(\boldsymbol{\xi} \cdot \nabla) \mathbf{v}_0. \end{cases}$$

- 143 • The definition usually used [15, 18, 20, 21],  $p = -\rho_0 D\varphi/Dt$ , is no longer  
144 valid if  $g_1 \neq 0$  (see Eq. (8)).
- 145 • the “acoustical” part  $g_1$  of the source  $\mathbf{g}$  appears as a source term only in  
146 the equation for the acoustic field  $\varphi$ . In a same way the “hydrodynamic”  
147 part  $g_2$  of the source  $\mathbf{g}$  appears as a source term only in the equation for the  
148 hydrodynamic field  $\xi$ .
- 149 • the decomposition of the velocity in Eq. (8) is not a Helmholtz decomposi-  
150 tion since we have not imposed  $\text{div}\left(\xi + \frac{H\mathbf{v}_0}{2c_p}\right) = 0$ . In fact in practice this  
151 quantity is not found to be equal to zero and it happens that the Helmholtz de-  
152 composition is not well-adapted to aeroacoustics. We have chosen to use the  
153 Goldstein’s decomposition  $\mathbf{v}^* = \nabla\varphi + \xi$ , but an alternative is to use Clebsch  
154 potentials [16].

155 *Proof.* It is straightforward to check that if  $\varphi$  and  $\xi$  are solutions of the coupled  
156 system of Goldstein’s equations (7), then  $p = \rho_0(g_1 - D\varphi/Dt)$  and  $\mathbf{v}^* = \nabla\varphi + \xi$  are  
157 solutions of Eq. ((5a),(5b),(5c),(5d)). It has been already done without any source  
158 term, for isentropic perturbations ( $s = 0$ ) [17, 18, 19, 20] or for general perturbations  
159 in a cylindrical geometry [21].

160 To prove the converse implication, that a solution of Euler’s equations is a solution  
161 of Goldstein’s equations, is more complicated and has never been done in the general  
162 case. A first difficulty is to define uniquely the Goldstein unknowns  $(\varphi, \xi)$  from the  
163 Euler unknowns  $(p, \mathbf{v}^*)$ . It has been done only for a potential flow and with incident  
164 fields as source terms [15]. To extend the results to a vortical flow and to the presence  
165 of sources, first we define  $\varphi$ , starting from (8) with the initial condition  $\varphi(\mathbf{x}, 0) = 0$   
166 (see remark 2). Then we deduce  $\xi$ . More explicitly we use the relations

$$167 \quad (10) \quad \begin{cases} \frac{D\varphi}{Dt} = g_1 - \frac{p}{\rho_0}, & \text{with } \varphi(\mathbf{x}, 0) = 0, \\ \xi = \mathbf{v}^* - \nabla\varphi, \end{cases}$$

168 to define the Goldstein’s unknowns  $(\varphi, \xi)$ .

169 Applying the change of unknowns Eq. (10), the mass conservation equation (5b)  
170 leads to

$$171 \quad (11) \quad \frac{D}{Dt} \left( \frac{1}{c_0^2} \frac{D\varphi}{Dt} \right) = \frac{1}{\rho_0} \text{div}[\rho_0(\nabla\varphi + \xi)] + f + \frac{D}{Dt} \left( \frac{g_1}{c_0^2} \right) + \frac{\mathbf{v}_0 \cdot \nabla H}{2c_p}.$$

172 Moreover, using  $\omega^* = \mathbf{curl} \xi$ , deduced from Eq. (10), (5a) leads to

$$173 \quad \nabla \left( \frac{\partial\varphi}{\partial t} + \mathbf{v}_0 \cdot \mathbf{v}^* + g_1 - \frac{D\varphi}{Dt} \right) + \frac{\partial\xi}{\partial t} + \omega_0 \times \mathbf{v}^* + \mathbf{curl} \xi \times \mathbf{v}_0 = \mathbf{g},$$

174 which simplifies in:

$$175 \quad \nabla(\mathbf{v}_0 \cdot \xi) + \frac{\partial\xi}{\partial t} + \omega_0 \times (\nabla\varphi + \xi) + \mathbf{curl} \xi \times \mathbf{v}_0 = \mathbf{curl} \mathbf{g}_2.$$

176 Finally using once again Eq. (31) is obtained the hydrodynamic equation

$$177 \quad (12) \quad \frac{D\xi}{Dt} = \nabla\varphi \times \omega_0 - (\xi \cdot \nabla)\mathbf{v}_0 + \mathbf{curl} \mathbf{g}_2. \quad \square$$

178 **REMARK 2.** *An efficient method to solve the first equation of (10) or to solve*  
 179 *Eq. (4) is to use a change of variable in order to get a family of ordinary differential*  
 180 *equations along the streamlines of the flow. For instance, for a parallel shear flow*  
 181  $\mathbf{v}_0(\mathbf{x}) = v_0(y)\mathbf{e}_x$  *with*  $\mathbf{x} = (x, y, z)$ , *the streamlines are the lines*  $y = \text{cste}$  *and*  $z = \text{cste}$   
 182 *and we get the explicit unique solutions:*

$$183 \quad \varphi(\mathbf{x}, t) = \int_0^t \zeta[x - v_0(y)(t - u), y, z, u] du \quad \text{where} \quad \zeta(\mathbf{x}, t) = g_1(\mathbf{x}, t) - \frac{p(\mathbf{x}, t)}{\rho_0(\mathbf{x})},$$

$$184 \quad H(\mathbf{x}, t) = \int_0^t h[x - v_0(y)(t - u), y, z, u] du.$$

185 **2.3. General expression of the hydrodynamic unknown.** The perturbation  
 186  $\varphi$  is governed by a classical convected wave-like equation (7), with source terms.  
 187 The main difficulty is to determine  $\boldsymbol{\xi}$ , which satisfies a less classical transport-like  
 188 equation. However there is a simplification: it is possible to find a simple expression  
 189 of  $\boldsymbol{\xi}$  which in particular predicts where  $\boldsymbol{\xi}$  vanishes. We present now the derivation of  
 190 such expression.

191 In the case of a potential flow  $\boldsymbol{\omega}_0 = \mathbf{0}$  and of potential sources  $\mathbf{g}_2 = \mathbf{0}$ , it is easy  
 192 to get from Eq. (12) that  $\boldsymbol{\xi} = \mathbf{0}$  and Goldstein's equations (7) reduce to the potential  
 193 model (Eq. (1) with the addition of source terms). In the general case of a non-  
 194 potential flow in presence of non-potential sources,  $\boldsymbol{\xi} \neq \mathbf{0}$  but a simple expression of  
 195  $\boldsymbol{\xi}$  can also be obtained. Indeed Eq. (12) may be written

$$196 \quad (13) \quad \frac{D\boldsymbol{\xi}}{Dt} + (\boldsymbol{\xi} \cdot \nabla)\mathbf{v}_0 = \nabla\varphi \times \boldsymbol{\omega}_0 + \mathbf{curl} \mathbf{g}_2.$$

197 This means that  $\boldsymbol{\xi}$  is induced by two different sources: the coupling between acoustic  
 198 perturbations  $\nabla\varphi$  and the flow vorticity  $\boldsymbol{\omega}_0$  and the vorticity source term  $\mathbf{curl} \mathbf{g}_2$ .  
 199 More precisely we will prove the following theorem:

200 **THEOREM 2.** *The general solution of Eq. (13) is*

$$201 \quad (14) \quad \boldsymbol{\xi} = \mathbf{u} \times \boldsymbol{\omega}_0 + \boldsymbol{\sigma},$$

202 *where*  $\mathbf{u}$  *is the displacement perturbation and where*  $\boldsymbol{\sigma}(\mathbf{g}_2)$  *is a term only due to the*  
 203 *hydrodynamic source*  $\mathbf{g}_2$  *(or equivalently to*  $\mathbf{curl} \mathbf{g}$ *).*

204  $\mathbf{u}$  is defined from the velocity through (see [11])

$$205 \quad (15) \quad \mathbf{v}^* = \frac{D\mathbf{u}}{Dt} - (\mathbf{u} \cdot \nabla)\mathbf{v}_0,$$

206 with the initial condition  $\mathbf{u}(\mathbf{x}, 0) = \mathbf{0}$ . To prove the Theorem 2 and the decomposition  
 207 (14), we introduce the temporary unknown

$$208 \quad (16) \quad \tilde{\boldsymbol{\xi}} = \mathbf{u} \times \boldsymbol{\omega}_0,$$

209 and we prove first the following lemma:

210 **LEMMA 3.** *The quantity*  $\tilde{\boldsymbol{\xi}}$  *defined in Eq. (16) satisfies the following equation*

$$211 \quad (17) \quad \frac{D\tilde{\boldsymbol{\xi}}}{Dt} = (\mathbf{v}^* - \tilde{\boldsymbol{\xi}}) \times \boldsymbol{\omega}_0 - (\tilde{\boldsymbol{\xi}} \cdot \nabla)\mathbf{v}_0.$$

212 Note that this is simply a vectorial relation, which has nothing to do with the problems  
 213 satisfied by  $\mathbf{u}$  and  $\mathbf{v}^*$ . We give the proof in the Appendix A.

214 Note that since  $\nabla\varphi = \mathbf{v}^* - \boldsymbol{\xi}$ ,  $\boldsymbol{\xi}$  in Eq. (12) with  $\mathbf{curl} \mathbf{g}_2 = \mathbf{0}$  satisfies the same  
 215 equation than  $\tilde{\boldsymbol{\xi}}$  in Eq. (17). Thanks to this remark, from Eq. (12) and Eq. (17) we  
 216 deduce that  $\boldsymbol{\zeta} = \boldsymbol{\xi} - \tilde{\boldsymbol{\xi}}$  satisfies

$$217 \quad (18) \quad \frac{D\boldsymbol{\zeta}}{Dt} = -\boldsymbol{\zeta} \times \boldsymbol{\omega}_0 - (\boldsymbol{\zeta} \cdot \nabla)\mathbf{v}_0 + \mathbf{curl} \mathbf{g}_2,$$

218 with  $\boldsymbol{\zeta}(\mathbf{x}, 0) = \mathbf{0}$ .

219 We can now prove the Theorem 2 by solving Eq. (18), the solution depending of  
 220 the value of the source term:

- 221 • if  $\mathbf{curl} \mathbf{g}_2 = \mathbf{0}$ , then the solution of Eq. (18) is zero (and thus  $\boldsymbol{\xi} = \mathbf{u} \times \boldsymbol{\omega}_0$ ),  
 222 since, it is easy to get that

$$223 \quad \frac{d\mathcal{E}}{dt} \leq 2(\|\boldsymbol{\omega}_0\| + \|\nabla\mathbf{v}_0\|)\mathcal{E},$$

224 where  $\mathcal{E} = \int_{\Omega} \frac{\rho_0}{2} |\boldsymbol{\zeta}|^2 dx$ . Since  $\mathcal{E}(t = 0) = 0$ , we deduce thanks to Gronwall  
 225 lemma that  $\boldsymbol{\xi} = \tilde{\boldsymbol{\xi}}$  at any time.

- 226 • if  $\mathbf{curl} \mathbf{g}_2 \neq \mathbf{0}$ , if we note  $\boldsymbol{\sigma}(\mathbf{g}_2)$  the unique solution (see the previous item  
 227 for the homogeneous problem) of Eq. (18), then  $\boldsymbol{\xi} = \tilde{\boldsymbol{\xi}} + \boldsymbol{\sigma}$ .

228 **REMARK 3.** *In the case  $\mathbf{g} = \mathbf{0}$  and for homentropic perturbations  $s = 0$ , the first*  
 229 *part  $\boldsymbol{\xi} = \mathbf{u} \times \boldsymbol{\omega}_0$  has already been found [16] whereas Goldstein [15], considering the*  
 230 *case  $\boldsymbol{\omega}_0 = \mathbf{0}$  but with an incident field, acting like a source term  $\mathbf{curl} \mathbf{g} \neq \mathbf{0}$ , found*  
 231 *the second part  $\boldsymbol{\xi} = \boldsymbol{\sigma}$ .*

232 **3. Numerical experiments.** To illustrate the superiority of the Goldstein equa-  
 233 tions with respect to the potential model (1), we will now solve in the time harmonic  
 234 regime the radiation problem defined by equations (7). We use a Finite Element  
 235 method, which, contrary to a Finite Differences method, has the main advantage to  
 236 be well suitable for unstructured meshes, adapted to complex geometries. We will  
 237 restrict to the 2D case.

238 **3.1. Acoustic radiation in a parallel shear flow.** We start by considering  
 239 the simple case of a shear flow for two reasons. First it is a validation case because  
 240 for such flow, the link between different alternative models is simple. Second it is  
 241 a simple way to control the vorticity of the carrier flow (it is the derivative of the  
 242 velocity profile) and thus to quantify the domain of validity of the potential model  
 243 Eq. (1).

244 **3.1.1. Numerical scheme.** Let us consider a two dimensional shear flow in  $\mathbb{R}^2$

$$245 \quad (19) \quad \mathbf{v}_0(x, y) = v_0(y)\mathbf{e}_x,$$

246 with a velocity  $v_0$  continuously differentiable on  $\mathbb{R}$ . We impose an homentropic flow  
 247 ( $s_0 = cst$ ) and we get from the Linearized Euler Equations that the density  $\rho_0$ , the  
 248 pressure  $p_0$  and the sound velocity  $c_0$  are constant. We suppose the flow subsonic  
 249  $|\mathbf{v}_0| < c_0$  and we introduce the Mach number  $M_0(y) = v_0(y)/c_0$ . We consider har-  
 250 monic noise sources  $F(\mathbf{x})e^{-i\omega t}$  and  $\mathbf{g}(\mathbf{x})e^{-i\omega t}$  with the frequency  $\omega$  and we solve (7)  
 251 with (2) replaced by

$$252 \quad \frac{D}{Dt} = -i\omega + \mathbf{v}_0 \cdot \nabla.$$



253 Time harmonic regime seems easier to solve than transient regime since the time  
 254 variable has disappeared, but it introduces an extra major difficulty when solving nu-  
 255 merically a radiation problem: to consider a bounded domain, we have to define some  
 256 well-suited radiation conditions. In this aim, following the treatment of Galbrun's  
 257 equation [11], we surround the computational domain with perfectly matched layers  
 258 (PMLs) [31, 32]. The computational domain, schematized on Fig. 1, is  $\Omega = \Omega_L \cup \Omega_b$   
 259 where  $\Omega_b = ]x_m, x_p[ \times ]0, h[$  is the physical bounded domain containing at least the  
 260 sources,  $\Omega = ]x_m - L_1, x_p + L_1[ \times ]-L_2, h + L_2[$  is the computational domain and  
 261  $\Omega_L = \Omega \setminus \Omega_b$  is the PML domain. The velocity  $v_0$  is supposed uniform outside  $]0, h[$   
 262 in order to get no reflected waves from the domain outside  $\Omega_b$  (otherwise the PMLs  
 263 could not be introduced). The introduction of PMLs corresponds to define the scale  
 264 changes

$$265 \quad \forall (x, y) \in \Omega, \quad \begin{cases} \frac{\partial}{\partial x} \rightarrow \check{\alpha}_1(x) \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \rightarrow \check{\alpha}_2(y) \frac{\partial}{\partial y} \end{cases}$$

266 where the piecewise functions  $\check{\alpha}_i$  are defined by

$$267 \quad \check{\alpha}_1(x) = \begin{cases} 1 & \text{if } x \in ]x_m, x_p[, \\ \alpha \in \mathbb{C} & \text{otherwise.} \end{cases} \quad \check{\alpha}_2(y) = \begin{cases} 1 & \text{if } y \in ]0, h[, \\ \alpha \in \mathbb{C} & \text{otherwise.} \end{cases}$$

268 The complex number  $\alpha$  in the PMLs is chosen such that [11]

$$269 \quad (20) \quad \operatorname{Re}(\alpha) > 0 \quad \text{and} \quad \operatorname{Im}(\alpha) < 0.$$

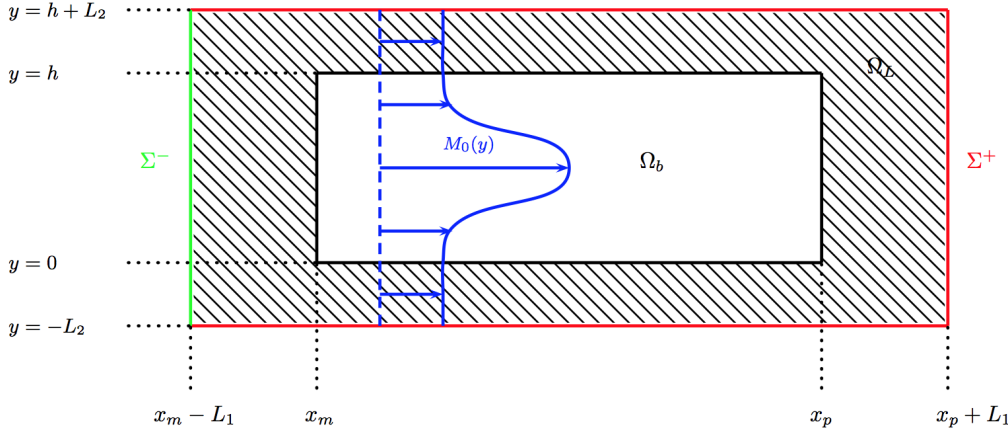


FIG. 1. Description of the two dimensional problem with PMLs

270

271 The purpose of these PMLs is to let the outgoing waves exit  $\Omega_b$  and to suppress  
 272 any reflected wave from the borders of  $\Omega_b$ , in order to simulate the propagation without  
 273 boundaries. The outgoing solution is selected by the PMLs by setting the boundary

274 conditions  $\varphi = 0$  on  $\Sigma^\pm$  and  $\boldsymbol{\xi} = \mathbf{0}$  on  $\Sigma^-$  (since  $\boldsymbol{\xi}$  satisfies a transport equation, it  
 275 just needs to be set upstream [11]).

276 The Goldstein equations, in time harmonic regime, with PMLs and in a parallel  
 277 shear flow become

$$278 \quad (21) \quad \begin{cases} D_\alpha^2 \varphi &= \operatorname{div}_\alpha (\nabla_\alpha \varphi + \boldsymbol{\xi}) + F + D_\alpha \left( \frac{g_1}{c_0} \right), \\ D_\alpha \xi_x &= -M'_0(y) \left( \check{\alpha}_2 \frac{\partial \varphi}{\partial y} + \xi_y \right) + \check{\alpha}_2 \frac{\partial g_2}{\partial y}, \\ D_\alpha \xi_y &= M'_0(y) \check{\alpha}_1 \frac{\partial \varphi}{\partial x} - \check{\alpha}_1 \frac{\partial g_2}{\partial x}, \end{cases}$$

279 where  $D_\alpha = -ik + M_0(y)\check{\alpha}_1\partial/\partial x$  with  $k = \omega/c_0$ . Thanks to the PMLs, we can prove  
 280 that problem (21) is of Fredholm type, which in particular ensures the convergence  
 281 of a Finite Element approximation of the solution (outside an eventual set of discrete  
 282 resonance frequencies). More precisely we can prove that (the demonstration is rather  
 283 technical and not given here):

284 THEOREM 4. *Problem (21) is of Fredholm type if*

$$285 \quad (22) \quad \min \left[ \Re(\alpha)(1 - s_0^2), \Re\left(\frac{1}{\alpha}\right) \right] - s_1 C_{\alpha, \Omega} > 0,$$

286 where  $s_0 = \max_{y \in [0, h]} |M_0(y)|$ ,  $s_1 = \max_{y \in [0, h]} \left| \frac{M'_0(y)}{M_0(y)} \right|$  and  $C_{\alpha, \Omega}$  is a constant depending of  
 287 the PML parameter and of the geometry of  $\Omega$ .

288 REMARK 4. *Note that thanks to Eq. (20), both  $\Re(\alpha)$  and  $\Re(1/\alpha)$  are positive.*  
 289 *Therefore the theorem requires that  $s_0 < 1$  (subsonic flow) and that  $s_1$  is small enough*  
 290 *(low shear flow). It is known that an incompressible shear flow with an inflection point*  
 291 *in the velocity profile may be unstable if the slope is large enough. The condition  $s_1$*   
 292 *small is certainly linked to this instability, although the condition  $M'_0 = 0$  does not*  
 293 *appear explicitly in the condition (22). The stability of compressible shear flows has*  
 294 *been less studied, results can be found in the low frequency limit [33], but no explicit*  
 295 *results on the influence of  $\max_{y \in [0, h]} |M'_0|$  are given in this reference.*

296 REMARK 5. *The condition (22) is impossible to fulfill if  $M_0$  vanishes, but we*  
 297 *have found an alternative condition that replaces the condition (22) for vanishing or*  
 298 *even low Mach number values. If  $M_0(y_0) = 0$ , then let us introduce the small layer*  
 299  *$y \in [y_0 - \varepsilon, y_0 + \varepsilon]$  where  $\varepsilon$  is chosen such that  $M_0$  is small in this layer. Then the*  
 300 *radiation problem (21) is found to be globally well posed if it is well-posed both*

- 301 • *in the "low Mach" layer  $y \in [y_0 - \varepsilon, y_0 + \varepsilon]$ , which is the case under the*  
 302 *condition:*

$$303 \quad (23) \quad \min \left[ \Re(\alpha)(1 - s_0^2), \Re\left(\frac{1}{\alpha}\right) \right] - \frac{\max_{y \in [y_0 - \varepsilon, y_0 + \varepsilon]} |M'_0(y)|}{k} > 0,$$

- 304 • *outside the "low Mach" layers, which is the case if the condition (22) is*  
 305 *satisfied by  $s_1 = \max_{y \notin [y_0 - \varepsilon, y_0 + \varepsilon]} \left| \frac{M'_0(y)}{M_0(y)} \right|$ .*

306 REMARK 6. *Conditions (22) and (23) are sufficient conditions to get a well-posed*  
 307 *radiation problem. In practice, the numerical method is stable for less restrictive*  
 308 *conditions.*

309 A numerical method for solving the Galbrun Equation has been developed in the  
 310 case of shear flows [34, 35], and then extended to slow [36] and general flows [11]. This  
 311 numerical approach relies on a Finite Element method coupling continuous and Dis-  
 312 continuous elements [37, 38, 39], these latter leading to a stable method to deal with  
 313 harmonic transport problems. The numerical scheme to solve Goldstein's equations  
 314 is not new, we have adapted the numerical scheme developed to solve Galbrun's  
 315 equation. Here we just sum up the numerical approach, for more details, see [11]. To  
 316 solve the Goldstein equations, we couple two Finite Element schemes: the potential  
 317 equation of (21) is discretized with Lagrange Finite Elements whereas the hydrody-  
 318 namic equations of (21) are discretized with Discontinuous Galerkin Elements. Both  
 319 quantities are discretized on the same mesh with around 70 000 nodes. They are  
 320 determined conjointly and time calculations are of the order of a few minutes.

321 **3.1.2. Parameters for the numerical simulations.** We work with variables  
 322 and unknowns without dimension which leads to  $\rho_0 = c_0 = 1$ . In most of the numerical  
 323 results, we consider a jet flow of Mach number given by

$$324 \quad (24) \quad M_0(y) = M_\infty + \mu \exp\left(-\frac{y^2}{R^2}\right),$$

325 where  $\mu$ ,  $M_\infty$  and  $R$  are parameters. We consider first  $F = 0$  in Eq. (21),  $g_2 = 0$   
 326 (thus the hydrodynamic field  $\boldsymbol{\xi}$  is equal to  $\mathbf{u} \times \boldsymbol{\omega}_0$ , see Eq. (14)) and two types of  
 327 acoustic sources  $g_1$ , such that  $\nabla g_1$  is a dipolar or a quadripolar source:

$$328 \quad (25) \quad \begin{aligned} g_1^*(x, y) &= \exp\left(-\frac{(x-x_c)^2 + (y-y_c)^2}{r_S^2}\right), \\ \tilde{g}_1(x, y) &= \frac{(x-x_c)(y-y_c)}{r^2} \exp\left(-\frac{(x-x_c)^2 + (y-y_c)^2}{r_S^2}\right). \end{aligned}$$

329  $(x_c, y_c)$  is the center of the source and  $r_S$  is its characteristic radius. For simplicity  
 330 we consider isentropic disturbances  $s = 0$ , which implies  $\mathbf{v}^* = \mathbf{v}$  in Eq. (3), and we  
 331 will compare three models, where  $D/Dt = M_0(y)\partial/\partial x - ik$  with  $k = \omega/c_0$ :

- 332 • the potential model

$$333 \quad \frac{D^2\varphi}{Dt^2} = \operatorname{div}(\nabla\varphi) + \frac{Dg_1}{Dt},$$

- 334 • the Goldstein model

$$335 \quad (26) \quad \begin{cases} \frac{D^2\varphi}{Dt^2} = \operatorname{div}(\nabla\varphi + \boldsymbol{\xi}) + \frac{Dg_1}{Dt}, \\ \frac{D\boldsymbol{\xi}}{Dt} = \nabla\varphi \times \boldsymbol{\omega}_0 - (\boldsymbol{\xi} \cdot \nabla)\mathbf{M}_0, \end{cases}$$

336 with  $\boldsymbol{\omega}_0 = \operatorname{curl} \mathbf{M}_0$ ,

- 337 • the Galbrun model (see [12, 11])

$$338 \quad (27) \quad \frac{D^2\mathbf{u}}{Dt^2} - \nabla(\nabla \cdot \mathbf{u}) = \nabla g_1.$$

339 Thanks to the introduction of the source term  $\nabla g_1$  in Eq. (27), it is possible to  
 340 prove that Eq. (26) and Eq. (27) are equivalent. Our aim is to illustrate the efficiency  
 341 of the Goldstein model. The Galbrun model is introduced as a reference method and  
 342 the potential model is used for its simplicity. More precisely:

343 • Since the Galbrun model has been implemented and tested on many cases  
 344 [11], it will be used to produce reference solutions to test the validity of the  
 345 numerical resolution of the Goldstein equations. To compare the numerical  
 346 results from the three models, we need to consider common quantities: the  
 347 pressure and the velocity. These quantities are given by

$$348 \quad p = g_1 - \frac{D\varphi}{Dt} = -\text{div } \mathbf{u} \quad \text{and} \quad \mathbf{v} = \nabla\varphi + \boldsymbol{\xi} = \frac{D\mathbf{u}}{Dt} - (\mathbf{u} \cdot \nabla)\mathbf{M}_0.$$

349 • The potential model is very attractive because its resolution is simple: only  
 350 continuous Finite Element are required. Although it is not valid for non-  
 351 potential flows, it can be solved for any flows and the comparison with Gold-  
 352 stein's model will indicate the importance of the induced error. The results  
 353 will be presented first for low shear flows, for which the error when using the  
 354 potential model is expected to be small, then for stronger shear flows and  
 355 finally for a potential flow with an hydrodynamic source.

356 **3.1.3. Comparisons for a low vorticity flow.** First we consider a jet flow  
 357 associated to a low vorticity  $|\boldsymbol{\omega}_0|$ , defined by  $R = 0.35$ ,  $M_\infty = 0.1$  and  $\mu = 0.2$   
 358 in the domain  $\Omega_b = ]0, 3[\times ]-1, 1[$ . The source is  $g_1^*$  (see Eq. (25)) with  $(x_c, y_c) = (1, 0)$ ,  
 359  $r_S = 0.05$  and  $k = 12$ . For small  $|\boldsymbol{\omega}_0|$  values,  $\boldsymbol{\xi} = \mathbf{u} \times \boldsymbol{\omega}_0$  is small and the potential  
 360 model is expected to be a rather good approximation. More quantitatively, the term  
 361  $\nabla\varphi$  is dominant over the hydrodynamic term  $\boldsymbol{\xi}$  in the acoustic velocity decomposition  
 362 Eq. (10). Indeed the vorticity  $\boldsymbol{\omega}_0 = -dM_0/dy\mathbf{e}_z$  is bounded by

$$363 \quad \boldsymbol{\omega}_0 \equiv |\boldsymbol{\omega}_0| = \frac{2\mu}{R^2}y \exp\left(-\frac{y^2}{R^2}\right) \leq \sqrt{\frac{2}{e}}\frac{\mu}{R}.$$

364 From the second equation of (26), we get the rough estimate  $\omega|\boldsymbol{\xi}| \sim \omega_0|\nabla\varphi|$  and  
 365 therefore  $|\boldsymbol{\xi}|/|\nabla\varphi| \sim \omega_0/\omega \leq 0.04$ .

366 On Fig. 2 and Fig. 3 are represented respectively the horizontal acoustic velocity  
 367 and the pressure radiated by the source for the three models. The horizontal white  
 lines delimit the area with a strong shear of the flow. We obtain very similar numerical

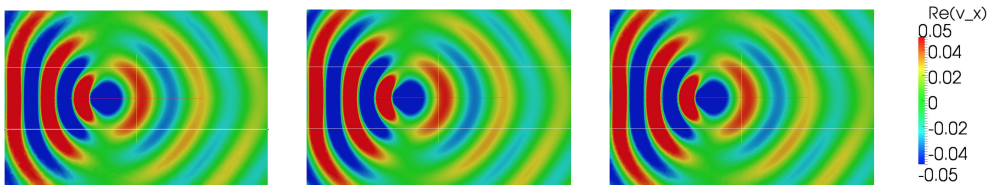


FIG. 2. Real part of the velocity perturbation  $Re(v_x)$  in a low shear jet flow Eq. (24) for Goldstein (left), Galbrun (center) and the potential model (right). The shear area is between the white lines. The source corresponds to the central disc. For such a low shear flow, the solutions of the three different equations are found very similar

368 solutions, which confirms that the potential model may be used as a good approxima-  
 369 tion in this low shear case. Actually, there are small differences in the vorticity areas  
 370 (between the white lines) on the acoustic velocity but no difference on the pressure.  
 371 The impact of these differences on the far-fields is found very limited.

373 **3.1.4. Comparisons for a larger vorticity flow.** We consider now for the  
 374 same geometry and the same source a jet flow with a stronger vorticity,  $R = 0.15$  and

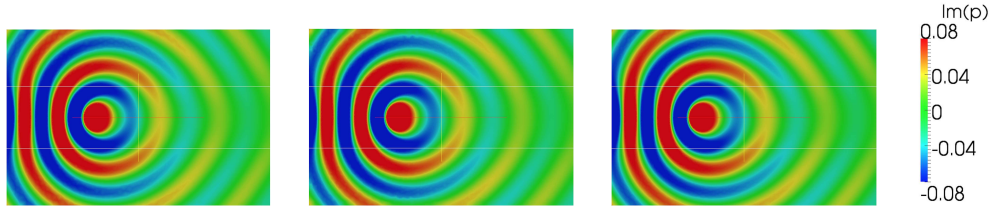


FIG. 3. *Imaginary part of the pressure perturbation  $Im(p)$  for the same configuration than in Fig. 2 for: Goldstein (left), Galbrun (center) and the potential model (right). Since the pressure is not an hydrodynamic quantity, the three solutions are nearly the same.*

375 thus  $|\xi|/|\nabla\varphi| \sim \omega_0/\omega \leq 0.1$ , for which only the Goldstein and the Galbrun models  
 376 are rigorously valid. We have checked that Goldstein's and Galbrun's models give the  
 377 same results and we focus here on the comparison between Goldstein's and potential  
 378 models: on Fig. 4 are represented  $Re(v_x)$  radiated by a source from the Goldstein  
 379 model on the left and from the potential model on the right at  $k = 8$ . Differences  
 380 between the two solutions, especially in the vorticity area (between the white lines, in  
 the vicinity of  $y = 0$ ), are observed. In particular the Goldstein equations are able to

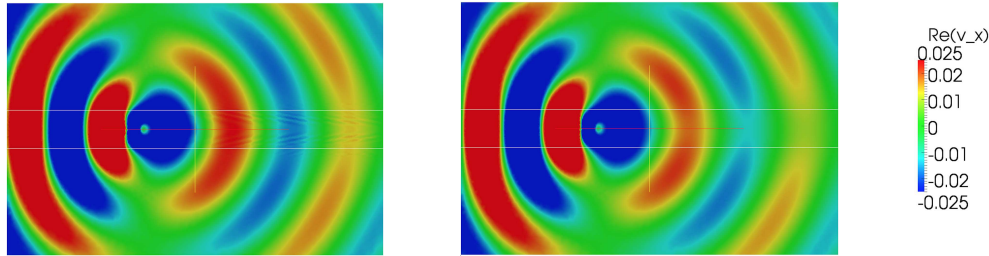


FIG. 4. *Real part of the velocity perturbation  $Re(v_x)$  in a strong shear jet flow for Goldstein (left) and the potential model (right). Hydrodynamics patterns, appearing as inclined lines with a small wavelength, are forgotten by the potential model.*

381  
 382 capture the hydrodynamic phenomena, neglected by the potential model: oscillations  
 383 of  $v_x$  (inclined lines of small wavelength downstream the source, due to oscillations  
 384 of  $\xi$ ) are seen on Fig. 4 left part, as a result of the transport phenomenon (see [11]).  
 385 Indeed the solution of  $D\xi/Dt = 0$  is  $\xi = \mathcal{A}(y) \exp(ikx/M_0(y))$ : for each  $y$  value, it  
 386 corresponds to patterns moving at  $M_0(y)$  with a wavelength  $2\pi M_0(y)/k$ .

387 In the previous example, the impact of neglecting  $\xi$  is found very limited: indeed,  
 388 the radiated patterns for the Goldstein and potential models in Fig. 4 are very  
 389 similar. This is because  $|\xi|/|\nabla\varphi| \leq 0.1$ . But in general it is important not to neglect  
 390 hydrodynamic phenomena since strong differences in the far field can occur. Such  
 391 differences are easier to see, considering a new geometry: we consider a larger domain  
 392  $\Omega_b = ]-3, 3[^2$ ,  $k = 12$  and we place the quadripolar source  $\tilde{g}_1$  of Eq. (25) at  $(x_c, y_c) =$   
 393  $(-1, 1.5)$  with  $r_S = 0.1$ . The jet flow corresponds to  $R = 0.1$ ,  $M_\infty = 0$  and  $\mu = 0.3$ .  
 394 The results obtained from the three models are presented on Fig. 5. The far fields  
 395 obtained for the Goldstein and the Galbrun models are equivalent and both different  
 396 from the result given by the potential flow model. The far field of the potential  
 397 solution is clearly different downstream the source, in particular for the rays reflected  
 398 by the jet core, highlighted by dashed lines.

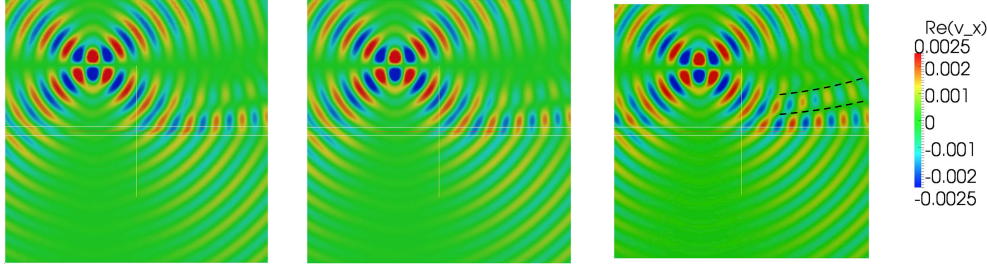


FIG. 5. Real part of the velocity perturbation  $Re(v_x)$  for the radiation of a quadrupolar source for Goldstein (left), Galbrun (center) and the potential model (right). A large domain enables us to see the far field. The dashed black lines highlight the main differences between the potential model and the other models. In particular an extra reflected "ray", non physical, is obtained by the potential model.

399 Finally, to illustrate the validity of the relation  $\boldsymbol{\xi} = \mathbf{u} \times \boldsymbol{\omega}_0$  where  $\mathbf{u}$  is the  
 400 perturbation displacement, we consider on Fig. 6 the case  $\Omega_b = ]0, 3[ \times ] - 1, 1[$ ,  $k = 6$ ,  
 401 the jet flow characterized by  $R = 0.25$ ,  $M_\infty = 0.2$ ,  $\mu = 0.1$  and the source  $g_1^*$ ,  
 represented by a black circle.  $Re(\xi_x)$  from the Goldstein equations and  $Re((\mathbf{u} \times \boldsymbol{\omega}_0)_x)$

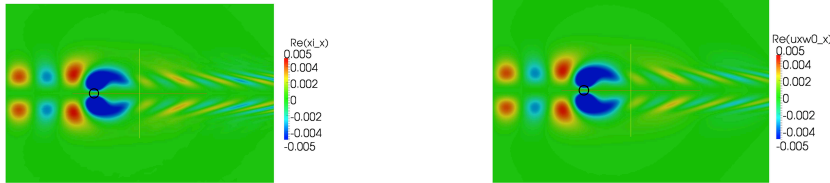


FIG. 6. Validation of the relation  $\boldsymbol{\xi} = \mathbf{u} \times \boldsymbol{\omega}_0$  for the radiation of a dipolar source, represented as a black circle, in a jet flow. We represent  $Re(\xi_x)$  for Goldstein (left) and  $Re((\mathbf{u} \times \boldsymbol{\omega}_0)_x)$  for Galbrun (right).  $\xi_x$  develops only in the shear flow areas. Note that  $\xi_x$  does not vanish upstream the source.

402 from the Galbrun Equation are found exactly equal. Note that  $\boldsymbol{\xi}$  is not only convected  
 403 by the flow, it takes non zero values upstream the source. This is due to the fact that  
 404 the acoustic field  $\mathbf{u}$  is radiated in all directions by the source.  
 405

406 **3.2. Case of a 2D flow.** In this section, we extend our illustrations to non-  
 407 shear flow and non-potential sources. We focus on Goldstein's configuration [15] and  
 408 we consider a potential flow with a vortical source ( $g_2 \neq 0$ ). We consider the flow  
 409 around a circular obstacle of radius  $R$ ,  $\mathbf{M}_0(x, y) = \nabla \varphi_0$  with

$$410 \quad \varphi_0 = M_\infty \left( \frac{r}{R} + \frac{R}{r} \right) \cos \theta,$$

411 in polar coordinates ( $x = r \cos \theta, y = r \sin \theta$ ). Such flow and source lead to the  
 412 following equations:

$$413 \quad \begin{cases} \frac{D^2 \varphi}{Dt^2} = \operatorname{div}(\nabla \varphi + \boldsymbol{\xi}) + \frac{Dg_1}{Dt}, \\ \frac{D\boldsymbol{\xi}}{Dt} = -(\boldsymbol{\xi} \cdot \nabla) \mathbf{M}_0 + \operatorname{curl} g_2, \end{cases}$$

with  $D/Dt = \mathbf{M}_0 \cdot \nabla - ik$  and where we take  $g_1 = g_1^* = g_2$  (see Eq. (25)). From

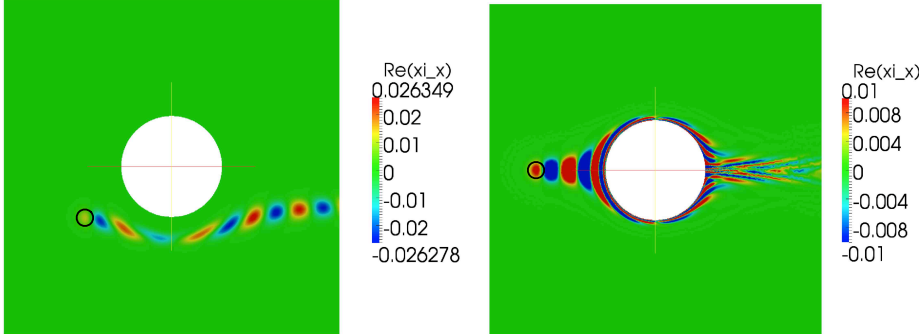


FIG. 7. Radiation of a vortical source in the potential flow around a disc. Are represented  $Re(\xi_x)$  obtained from the resolution of Goldstein's equations for two different positions of an hydrodynamic source, represented as a black circle. As expected,  $Re(\xi_x)$  is convected along the flow streamlines. Note that the potential model neglects  $\xi_x$  (it assumes it is zero).

414 Eq. (14), since for a potential flow  $\mathbf{u} \times \boldsymbol{\omega}_0 = \mathbf{0}$ , we have  $\boldsymbol{\xi} = \boldsymbol{\sigma}$ . On Fig. 7, for a  
 415 flow coming horizontally from the left, is represented  $\sigma_x = Re(\xi_x)$  in  $\Omega_b = ]-1, 1[^2$ ,  
 416 for  $R = 0.3$ ,  $M_\infty = 0.3$ ,  $k = 6$  and for two different positions of the source. We see  
 417  $\boldsymbol{\xi}$ , created by the source whose location is indicated by a black circle, and following  
 418 the flow streamlines. On Fig. 7 (left) the source is off-centered from the flow whereas  
 419 on Fig. 7 (right) the source is centered on the stream line associated to a stop point  
 420 on the disc. In the second case the pattern of  $Re(\xi_x)$  is rather complex and behind  
 421 the disc, the wavelength of  $Re(\xi_x)$  is very small because the velocity is weak. Note  
 422 that the potential model would ignore these patterns since this model consists in  
 423 taking  $\boldsymbol{\xi} = \mathbf{0}$  in the Goldstein equation and therefore the potential model is a bad  
 424 approximation when  $g_2$  is not small.  
 425

426 **4. Concluding remarks.** In order to study the acoustic radiation of a general  
 427 source in a complex flow, we have derived the generalized Goldstein equations in the  
 428 presence of a vortical flow and source terms. These equations extend the potential  
 429 model, valid for potential flows, to general flows, by including a vector hydrodynamic  
 430 field.

431 Firstly, by taking some acoustic and hydrodynamic source terms into account,  
 432 we have proved that the Goldstein equations are equivalent to the Linearized Euler  
 433 model. For source terms  $f$  and  $\mathbf{g} = \nabla g_1 + \operatorname{curl} g_2$  introduced in Eq. (5a), we have  
 434 established that the generalized Goldstein equations can be written (see Eq. (7)) in  
 435 the following way:

$$436 \quad \begin{cases} A\varphi + B\boldsymbol{\xi} = S, \\ C\boldsymbol{\xi} + D\varphi = \operatorname{curl} g_2. \end{cases}$$

437 The operators  $A$ ,  $B$ ,  $C$  and  $D$  are defined as:

$$438 \quad \left\{ \begin{array}{l} A\varphi = \frac{D}{Dt} \left( \frac{1}{c_0^2} \frac{D\varphi}{Dt} \right) - \frac{1}{\rho_0} \operatorname{div} [\rho_0 \nabla \varphi], \\ B\xi = -\frac{1}{\rho_0} \operatorname{div} (\rho_0 \xi), \\ C\xi = \frac{D\xi}{Dt} + (\xi \cdot \nabla) \mathbf{v}_0, \\ D\varphi = -\nabla \varphi \times \boldsymbol{\omega}_0. \end{array} \right.$$

439 We have introduced the general source term

$$440 \quad S = f + \frac{D}{Dt} \left( \frac{g_1}{c_0^2} \right) + \frac{\mathbf{v}_0 \cdot \nabla H}{2c_p},$$

441 where  $H$  denotes the entropic perturbation. Moreover the solution  $\xi$  of the hydrodynamic equation of Eq. (7) may be written in the form (Eq. (14)):

$$443 \quad \xi = \mathbf{u} \times \boldsymbol{\omega}_0 + \boldsymbol{\sigma}(\mathbf{g}_2),$$

444 where  $\boldsymbol{\sigma}$  is the solution to (Eq. (18)):

$$445 \quad \frac{D\boldsymbol{\sigma}}{Dt} + (\boldsymbol{\sigma} \cdot \nabla) \mathbf{v}_0 + \boldsymbol{\sigma} \times \boldsymbol{\omega}_0 = \mathbf{curl} \mathbf{g}_2.$$

446 Our conclusions depend on the value of the vorticity of the flow  $\boldsymbol{\omega}_0$  and of the value  
447 of  $\mathbf{curl} \mathbf{g}$  in the momentum conservation law of Euler's equations. An overview of  
448 these conclusions is summarized in the following table:

	$\mathbf{curl} \mathbf{g} = 0$	$\mathbf{curl} \mathbf{g} \neq 0$
$\boldsymbol{\omega}_0 = 0$	$A\varphi = S$ $\xi = \mathbf{0}$ Blokhintzev's model[4]	$A\varphi = S - B\boldsymbol{\sigma}$ $\xi = \boldsymbol{\sigma}$ Goldstein's model[15]
$\boldsymbol{\omega}_0 \neq 0$	$A\varphi + B\xi = S$ $C\xi + D\varphi = \mathbf{0}$ $\xi = \mathbf{u} \times \boldsymbol{\omega}_0$ Visser's model[16]	$A\varphi + B\xi = S$ $C\xi + D\varphi = \mathbf{curl} \mathbf{g}_2$ $\xi = \mathbf{u} \times \boldsymbol{\omega}_0 + \boldsymbol{\sigma}$ New model

450 After the first theoretical section, we have presented a numerical method to solve  
451 the Goldstein equations in the time harmonic regime. PMLs have been introduced in  
452 order to bound the computational domain in the case of a two dimensional flow. The  
453 numerical method has been validated by a comparison with the Galbrun Equation and  
454 the importance of the hydrodynamic unknown has been emphasized by a comparison  
455 with the potential model.

456 A natural extension of this work is to consider more complicated boundary condi-  
457 tions than rigid boundaries, like impedance boundary conditions. Another perspective  
458 is to extend the numerical method to the 3D case. A possible strategy to decrease  
459 the computational cost, due to the introduction of many degrees of freedom by the  
460 Discontinuous Galerkin Element method, would be to compute  $\varphi$  and  $\xi$  on different  
461 meshes:  $\xi$  would be determined on a coarser mesh, since it is associated to small  
462 wavelengths, but on a mesh of small extension since only restricted to the vortical  
463 areas of the carrier flow.



464 **Acknowledgments.** The authors gratefully acknowledge the Agence Nationale  
465 de la Recherche (AEROSON project, ANR-09-BLAN-0068-02 program) for financial  
466 support.

467 **Appendix A. Hydrodynamic field.**

468 Here we prove the lemma 3:  $\tilde{\xi} = \mathbf{u} \times \boldsymbol{\omega}_0$  satisfies

$$469 \quad \frac{D\tilde{\xi}}{Dt} = (\mathbf{v}^* - \tilde{\xi}) \times \boldsymbol{\omega}_0 - (\tilde{\xi} \cdot \nabla) \mathbf{v}_0.$$

470 *Proof.* The carrier flow satisfies the Euler Equations:

$$471 \quad (28) \quad \begin{cases} \operatorname{div}(\rho_0 \mathbf{v}_0) = 0, \\ \rho_0(\mathbf{v}_0 \cdot \nabla) \mathbf{v}_0 + \nabla p_0 = 0, \end{cases}$$

472 associated to the state law of the fluid  $p_0 = \nu(\rho_0)$ , given by

$$473 \quad (29) \quad \nu(\rho_0) = \kappa \rho_0^\gamma,$$

474 where  $\kappa$  is a constant.

475 First the usual vorticity equation for  $\boldsymbol{\omega}_0$  is recovered, by writing the second equa-  
476 tion of (28), using Eq. (29), in the form

$$477 \quad (30) \quad \nabla \left( \frac{|\mathbf{v}_0|^2}{2} \right) + \boldsymbol{\omega}_0 \times \mathbf{v}_0 + \nabla \left( \frac{\gamma}{\gamma - 1} \frac{p_0}{\rho_0} \right) = \mathbf{0},$$

478 where we have used the vector identity

$$479 \quad (31) \quad (\mathbf{a} \cdot \nabla) \mathbf{b} + (\mathbf{b} \cdot \nabla) \mathbf{a} = \nabla(\mathbf{a} \cdot \mathbf{b}) + (\nabla \times \mathbf{a}) \times \mathbf{b} + (\nabla \times \mathbf{b}) \times \mathbf{a},$$

480 with  $\mathbf{a} = \mathbf{b} = \mathbf{v}_0$ . Taking the curl of (30) leads to:

$$481 \quad (32) \quad \frac{D\boldsymbol{\omega}_0}{Dt} = (\boldsymbol{\omega}_0 \cdot \nabla) \mathbf{v}_0 - (\operatorname{div} \mathbf{v}_0) \boldsymbol{\omega}_0,$$

482 (the flow is stationary and thus  $D/Dt = \mathbf{v}_0 \cdot \nabla$ ). Using the relations (15) and (32),  
483 we get (it is also Eq. (52) of [16])

$$484 \quad (33) \quad \frac{D(\mathbf{u} \times \boldsymbol{\omega}_0)}{Dt} = \mathbf{v}^* \times \boldsymbol{\omega}_0 + \underbrace{[(\mathbf{u} \cdot \nabla) \mathbf{v}_0] \times \boldsymbol{\omega}_0 + \mathbf{u} \times [(\boldsymbol{\omega}_0 \cdot \nabla) \mathbf{v}_0] - (\operatorname{div} \mathbf{v}_0)(\mathbf{u} \times \boldsymbol{\omega}_0)}_{= \tilde{\nabla}(\mathbf{u} \cdot (\mathbf{v}_0 \times \boldsymbol{\omega}_0))}$$

485 in which we have used  $\tilde{\nabla}$ , the nabla operator applying only on  $\mathbf{v}_0$  (while  $\mathbf{u}$  and  $\boldsymbol{\omega}_0$   
486 are considered constant)(see [16]). Therefore we get

$$487 \quad \frac{D\tilde{\xi}}{Dt} = \mathbf{v}^* \times \boldsymbol{\omega}_0 - \tilde{\nabla}(\mathbf{v}_0 \cdot \tilde{\xi}).$$

488 The vector identity (31), valid also for the operator  $\tilde{\nabla}$ , applied to  $\tilde{\xi}$  and  $\mathbf{v}_0$  leads us  
489 to the result of the lemma.  $\square$

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