A Gaussian de Finetti theorem and application to truncations of random Haar matrices

*arXiv:*1612.05080

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Inria Paris

Workshop on "Probabilistic techniques and Quantum Information Theory"

Outline of the talk

- de Finetti theorems: classical and quantum
- Representation of the unitary group on Fock spaces
- Which density matrices are unitarily-invariant?
- Truncation of Haar random unitaries

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Finite de Finetti theorems (classical version)

infinite version due to Bruno de Finetti (Annales IHP 1937).

Classical version: Diaconis, Freedman (Annals of Prob 1980)

Let $P(x_1, \cdots, x_n)$ be a permutation invariant distribution on S^n:

$$P(x_1, \cdots, x_n) = P(x_{\pi(1)}, \cdots, x_{\pi(n)}) \quad \forall \pi \in \mathfrak{S}_n.$$

Let P_k be the law of $(x_1, \cdots, x_k)$ on S^k. Then, for $1 \leq k \leq n$, there exists a measure dm on distributions p on S such that

$$\|P_k - \int p^{\times k} dm(p)\|_{TV} \leq 2|S|\frac{k}{n}.$$

proof follows from the total variation distance between hypergeometric and multinomial distributions:

- bag with n balls of $|S|$ colors, pick k balls
- *without replacement:* hypergeometric distribution (P is a convex mixture of such distributions corresponding to different bags)
- *with replacement:* multinomial distribution

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Finite de Finetti theorems (quantum version)

infinite version due to Caves, Fuchs, Schack (JMP 2002)

Quantum version: Christandl, König, Mitchison, Renner (CMP 2007)

Let ρ^{n} be a permutation invariant density operator on $(\mathbb{C}^{d})^{\otimes n}$:

$$\pi\rho^{n}\pi^{\dagger} = \rho^{n} \quad \forall \pi \in \mathfrak{S}_{n}.$$

Then there exists a measure dm on the set of mixed states on $\mathbb{C}^{d}$ such that

$$\|\mathrm{tr}_{n-k}\,\rho^{n} - \int \sigma^{\otimes k}\mathrm{dm}(\sigma)\|_{tr} \leq 2d^{2}\frac{k}{n}.$$

- d^2 improved to d if ρ^n is assumed to be pure
- implies the classical version
- *applications:* security of quantum cryptography (qkd), tomography of quantum states

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beyond discrete distributions and permutation invariance

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Let $P(x_1, \cdots, x_n)$ be an *orthogonally invariant* distribution on $\mathbb{R}^n$:

$$P(\vec{x}) = P(R\vec{x}) \quad \forall R \in O(n).$$

Let P_k be the law of $(x_1, \cdots, x_k)$ on $\mathbb{R}^k$. Then, for $1 \leq k \leq n-3$, there exists a measure dm on $\mathbb{R}_+$ such that

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with p_σ the law of a Gaussian variable $\mathcal{N}(0, \sigma^2)$

- ex: random vector on the sphere
- Similar results hold for other symmetries on $\mathbb{R}_+^n$ or $\mathbb{N}^n$, with normal distribution replaced by exponential or geometric.

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quantum version for unitary invariance: baby version

(Easy) quantum version: Leverrier, Cerf (PRA 2009)

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with σ_{x} is a thermal state with energy x.

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A further generalization

Let $X \in \mathbb{C}^{n\times p}$, $Y \in \mathbb{C}^{n\times q}$ be *unitarily invariant random matrices*:

$$P(UX, \overline{U}Y) = P(X, Y), \quad \forall U \in \mathcal{U}(n).$$

Let P_k be the distribution of the first k rows. What can we say about $P_k(X, Y)$?

Example: truncation of random Haar matrices

Let $\mathcal{H}_{k,k}$ be the distribution over $k \times k$ complex matrices obtained by drawing a random unitary $U \sim \mathrm{Haar}(\mathcal{U}(n))$ and outputting $\sqrt{n}U_{k,k}$ where $U_{k,k}$ is the $k \times k$ upper-left submatrix of U.

Let $\mathcal{G}^{k\times k}$ be the distribution over $k \times k$ complex matrices where each entry $\sim \mathcal{N}_{\mathbb{C}}(0, 1)$.

- Aaronson, Arkhipov (STOC 2011): $\|\mathcal{H}_{k,k} - \mathcal{G}^{k\times k}\|_{TV} = O\left(\frac{k^6}{n}\right)$
- expected right scaling (Jiang, Ma 2017): convergence for $k = o(n^{1/2})$
- this work (with a quantum approach):

$$\|\mathcal{H}_{k,k} - \mathcal{G}^{k\times k}\|_{TV} \leq \frac{2k^3}{n-k}$$

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Fock spaces

Let H be a finite-dimensional Hilbert space.

$$\mathcal{F}(\mathrm{H}) := \bigoplus_{\mathrm{k}=0}^{\infty} \mathbf{Sym}^{\mathrm{k}}(\mathrm{H}),$$

with $\mathbf{Sym}^{\mathrm{k}}(\mathrm{H})$: the symmetric part of $\mathrm{H}^{\otimes \mathrm{k}}$ (system with k excitations).

Ex: d*-mode space:* $\mathrm{H} \cong \mathbb{C}^{\mathrm{d}}$

- orthonormal basis of $\mathcal{F}(\mathbb{C}^{\mathrm{d}})$: $\{|\mathrm{k}_1, \mathrm{k}_2, \ldots, \mathrm{k}_{\mathrm{d}}\rangle \,:\, \mathrm{k}_{\mathrm{i}} \in \mathbb{N}\}$
- a pair of annihilation/creation operators is associated with each mode: $[\mathrm{a}_{\mathrm{i}}, \mathrm{a}_{\mathrm{i}}^{\dagger}] = \mathbb{1}$.
- states can be expressed as functions of creation operators applied to the vacuum:
$$|\mathrm{k}_1, \mathrm{k}_2, \ldots, \mathrm{k}_{\mathrm{d}}\rangle = \tfrac{1}{\sqrt{\mathrm{k}_1! \cdots \mathrm{k}_{\mathrm{d}}!}} (\mathrm{a}_1^{\dagger})^{\mathrm{k}_1} \cdots (\mathrm{a}_{\mathrm{d}}^{\dagger})^{\mathrm{k}_{\mathrm{d}}} |0\rangle$$

$$\mathcal{F}_{\mathrm{p,q,1}} := \mathcal{F}(\mathbb{C}^{\mathrm{p}} \oplus \mathbb{C}^{\mathrm{q}}) \qquad \mathcal{F}_{\mathrm{p,q,n}} := \mathcal{F}(\mathbb{C}^{\mathrm{p}} \oplus \mathbb{C}^{\mathrm{q}})^{\otimes \mathrm{n}} \cong \mathcal{F}(\mathbb{C}^{\mathrm{np}} \oplus \mathbb{C}^{\mathrm{nq}})$$

$\mathcal{F}_{\mathrm{p,q,n}}$ is an $\mathrm{n}(\mathrm{p}+\mathrm{q})$-mode Fock space.

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Segal-Bargmann representation: $\mathcal{F}$ as a space of holomorphic functions

- $|\psi\rangle \in \mathcal{F}_{p,q,n}$ represented by a *holomorphic function* of $n(p+q)$ variables:

$$|\psi\rangle \quad \leftrightarrow \quad \psi : \mathbb{C}^{n\times p} \times \mathbb{C}^{n\times q} \to \mathbb{C}$$

- with norm: $\|\psi\|^2 := \frac{1}{\pi^{n(p+q)}} \int |\psi(X,Y)|^2 \exp(-\|X\|_2^2 - \|Y\|_2^2) dX dY < \infty$

Examples

- Glauber coherent state: $|\alpha\rangle = \sum_{k=0}^{\infty} \frac{\alpha^k}{\sqrt{k!}} |k\rangle = e^{\hat{a}^\dagger}|0\rangle \quad \leftrightarrow \quad e^{\alpha x} \quad (\alpha \in \mathbb{C})$

- Two-mode squeezed vacuum state:

$$\sum_{k=0}^{\infty} \lambda^k |k,k\rangle = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \hat{a}^{\dagger k} \hat{b}^{\dagger k} = e^{\lambda \hat{a}^\dagger \hat{b}^\dagger}|0\rangle \quad \leftrightarrow \quad e^{\lambda xy} \quad (|\lambda| < 1)$$

- n 2-mode squeezed vacuum states:

$$\bigotimes_{i=1}^{n} \left(\sum_{k=0}^{\infty} \lambda^k |k,k\rangle \right) = e^{\lambda(\hat{a}_1^\dagger \hat{b}_1^\dagger + \cdots + \hat{a}_n^\dagger \hat{b}_n^\dagger)}|0\rangle \quad \leftrightarrow \quad e^{\lambda(x_1 y_1 + \cdots + x_n y_n)} \quad (|\lambda| < 1)$$

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Action of the unitary group on $\mathcal{F}_{\mathrm{p,q,n}}$

Mathematically: as a change of variables

$\mathcal{F}_{\mathrm{p,q,n}}$ carries a representation $(\mathrm{u} \mapsto \mathrm{W_u})$ of the unitary group $\mathcal{U}(\mathrm{n})$:

$$\mathrm{W_u}\psi(\mathrm{X},\mathrm{Y}) := \psi(\mathrm{uX},\overline{\mathrm{u}}\mathrm{Y}), \quad \text{for } \mathrm{u} \in \mathcal{U}(\mathrm{n})$$

Physically: as a passive interferometer

Any n-mode passive interferometer (= *linear optical network*) consisting of beamsplitters and phase-shifters is described by a unitary $\mathrm{u} \in \mathcal{U}(\mathrm{n})$:

- each of the p blocks of n modes of X is processed by the interferometer u
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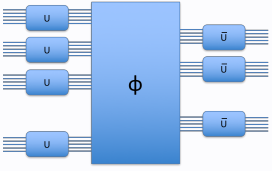

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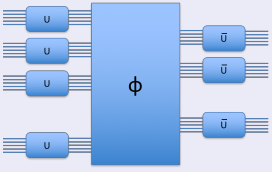

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Restriction to pure states (for permutation-invariance)

Lemma (Christandl, König, Mitchison, Renner, CMP 2007)

Let ρ be a permutation-invariant density matrix on $\mathcal{H}^{\otimes n}$. Then there exists a purification $|\psi\rangle$ of ρ in $\mathcal{K}^{\otimes n} \otimes \mathcal{H}^{\otimes n}$ (with $\mathcal{K} \cong \mathcal{H}$) such that

$$\pi|\psi\rangle = |\psi\rangle, \qquad \forall \pi \in \mathfrak{S}_n.$$

Proof.

Define $|\psi\rangle = (\mathbb{1} \otimes \sqrt{\rho})|\Phi\rangle$ with $|\Phi\rangle = \left(\sum_i |i\rangle_{\mathcal{K}}|i\rangle_{\mathcal{H}}\right)^{\otimes n}$

- $\sqrt{\rho}$ and $|\Phi\rangle$ are permutation-invariant $\implies$ $|\psi\rangle$ is permutation-invariant
- $\mathrm{tr}_{\mathcal{K}^{\otimes n}}\left((\mathbb{1} \otimes \sqrt{\rho})|\Phi\rangle\langle\Phi|(\mathbb{1} \otimes \sqrt{\rho})^\dagger\right) = \sqrt{\rho}\mathbb{1}\sqrt{\rho}^\dagger = \rho$

□

$\implies$ it is sufficient to consider pure states $|\psi\rangle$ such that $\pi|\psi\rangle = |\psi\rangle, \quad \forall \pi \in \mathfrak{S}_n$

i.e. $|\psi\rangle$ belongs to the *symmetric subspace* $\mathrm{Sym}^n(\mathcal{H})$

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Restriction to pure states (for unitarily-invariance)

Lemma (arXiv:1612.05080)

Let ρ be a *unitarily-invariant* density matrix on $\mathcal{F}_{\mathrm{p,q,n}}$. Then there exists a purification $|\psi\rangle$ of ρ in $\mathcal{F}_{\mathrm{p+q,q+p,n}}$ such that

$$\mathrm{W_u} \otimes \mathrm{W_{\bar{u}}} |\psi\rangle = |\psi\rangle, \qquad \forall \mathrm{u} \in \mathcal{U}(\mathrm{n}).$$

Proof.

Similar argument with $|\psi\rangle = (\mathbb{1} \otimes \sqrt{\rho})|\mathrm{MES}\rangle$ with $|\mathrm{MES}\rangle =$ "maximally entangled state" between two copies of $\mathcal{F}_{\mathrm{p,q,n}} \cong \mathcal{F}_{\mathrm{q,p,n}}$ (doesn't really exist ...)

- $\sqrt{\rho}$ and $|\mathrm{MES}\rangle$ are unitarily-invariant $\implies$ $|\psi\rangle$ is unitarily-invariant
- $\mathrm{tr}_{\mathcal{F}_{\mathrm{q,p,n}}} \left((\mathbb{1} \otimes \sqrt{\rho})|\mathrm{MES}\rangle\langle\mathrm{MES}|(\mathbb{1} \otimes \sqrt{\rho})^\dagger \right) = \sqrt{\rho}\mathbb{1}\sqrt{\rho}^\dagger = \rho$

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$\implies$ it is sufficient to consider pure states $|\psi\rangle$ such that $\mathrm{W_u}|\psi\rangle = |\psi\rangle, \quad \forall \mathrm{u} \in \mathcal{U}_\mathrm{n}$

i.e. $|\psi\rangle$ belongs to the *symmetric subspace* $\mathcal{F}^{\mathcal{U}(\mathrm{n})}_{\mathrm{p+q,p+q,n}}$

Restriction to pure states (for unitarily-invariance)

Lemma (arXiv:1612.05080)

Let ρ be a *unitarily-invariant* density matrix on $\mathcal{F}_{\mathrm{p,q,n}}$. Then there exists a purification $|\psi\rangle$ of ρ in $\mathcal{F}_{\mathrm{p+q,q+p,n}}$ such that

$$\mathrm{W_u} \otimes \mathrm{W_{\bar{u}}} |\psi\rangle = |\psi\rangle, \qquad \forall \mathrm{u} \in \mathcal{U}(\mathrm{n}).$$

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Symmetric subspaces

Permutation invariance

$$\mathrm{Sym}^{n}(\mathbb{C}^{d}) := \left\{ |\psi\rangle \in (\mathbb{C}^{d})^{\otimes n} \ : \ \pi|\psi\rangle = |\psi\rangle, \forall \pi \in \mathrm{S}_{n} \right\}$$

Properties:

- $\mathrm{Sym}^{n}(\mathbb{C}^{d}) = \mathrm{Span}\left\{ |\phi\rangle^{\otimes n} \quad : \quad |\phi\rangle \in \mathbb{C}^{d} \right\}$
- $|\phi\rangle^{\otimes n}$: $\mathrm{SU}(d)$ coherent states
- $\dim\left(\mathrm{Sym}^{n}(\mathbb{C}^{d})\right) = \mathrm{O}(n^{d})$ (for $d \ll n$)

Unitarily invariance

$$\mathcal{F}_{p,q,n}^{\mathcal{U}(n)} := \{ |\psi\rangle \in \mathcal{F}_{p,q,n} \ : \ \mathrm{W}_{u}|\psi\rangle = |\psi\rangle, \forall u \in \mathfrak{S}_{n} \}$$

Theorem (arXiv:1612.05080)

$\mathcal{F}_{p,q,n}^{\mathcal{U}(n)}$ is spanned by $\mathrm{SU}(p,q)$ *generalized coherent states*.

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Generalized coherent states

The states $|\phi\rangle^{\otimes n}$ with $|\phi\rangle \in \mathbb{C}^d$ are an example of generalized CS, associated to $\mathrm{SU}(d)$.

An example of Perelomov generalized CS construction for $\mathcal{H}^{\otimes n} \cong (\mathbb{C}^d)^{\otimes n}$

- a *Lie group* G, e.g $\mathrm{SU}(d)$, and a representation $(g \mapsto T_g)$ of G on $\mathcal{H}^{\otimes n}$

$$u \in \mathrm{SU}(d) \quad \mapsto u^{\otimes n} \quad \text{on} \quad (\mathbb{C}^d)^{\otimes n}$$

- a distinguished *vector* $\psi_0 \in \mathcal{H}^{\otimes n}$, e.g. $|0\rangle^{\otimes n}$
- generalized *G-coherent states*: $\{|\psi_g\rangle = T_g|\psi_0\rangle, g \in G\}$, e.g. $|\phi_u\rangle^{\otimes n} = u^{\otimes n}|0\rangle^{\otimes n}$
- CS labeled by elements of G/H, e.g. $\phi_u \in \mathrm{SU}(d)/\mathrm{SU}(d-1) \cong \mathcal{S}_1(\mathbb{C}^d)$
- with H: stationary subgroup $\left\{g \in G \,:\, T_g|\psi_0\rangle = e^{i\theta}|\psi_0\rangle\right\}$

$\mathrm{SU}(p,q)$ *coherent states*

The quadratic form $Z_{i,j} = \sum_{k=1}^{n} x_{k,i} y_{k,j}$ is invariant under $X \to uX$, $Y \to \overline{u}Y$:

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Resolution of the identity

$$\begin{aligned}\phi_{\Lambda,\mathrm{n}} \leftrightarrow |\Lambda,\mathrm{n}\rangle &= \det(1-\Lambda\Lambda^\dagger)^{\mathrm{n}/2}\exp(\lambda_{11}\mathrm{Z}_{11}+\cdots+\lambda_{\mathrm{p,q}}\mathrm{Z}_{\mathrm{pq}})|0\rangle \\ &= |\Lambda,1\rangle^{\otimes \mathrm{n}}\end{aligned}$$

with $\mathrm{Z}_{\mathrm{ij}} := \sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{a}_{\mathrm{ki}}^\dagger \mathrm{b}_{\mathrm{kj}}^\dagger$

Defining property of coherent states

For $\mathrm{n} \geq \mathrm{p}+\mathrm{q}$,

$$\int_{\mathcal{D}} \left(|\Lambda,1\rangle\langle\Lambda,1|\right)^{\otimes \mathrm{n}} \mathrm{d}\mu_{\mathrm{n}}(\Lambda) = \mathbb{1}_{\mathrm{F}_{\mathrm{p,q,n}}^{\mathrm{U(n)}}},$$

with the invariant measure on $\mathcal{D} = \{\mathbb{C}^{\mathrm{p}\times\mathrm{q}} : \Lambda\Lambda^\dagger < \mathbb{1}_{\mathrm{p}}\}$

$$\mathrm{d}\mu_{\mathrm{n}}(\Lambda) = \mathrm{C}_{\mathrm{n}}[\det(\mathbb{1}_{\mathrm{p}}-\Lambda\Lambda^\dagger)]^{-(\mathrm{p}+\mathrm{q})} \prod_{\mathrm{i,j}}^{\mathrm{p}} \mathrm{d}\Lambda_{\mathrm{i,j}}$$

and

$$\mathrm{C}_{\mathrm{n}} := \frac{1}{\pi^{\mathrm{pq}}} \prod_{\mathrm{i}=0}^{\mathrm{q}-1} \frac{(\mathrm{n}-\mathrm{q}+\mathrm{i})!}{(\mathrm{n}-\mathrm{p}-\mathrm{q}+\mathrm{i})!}$$

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$$d\mu_n(\Lambda) = C_n[\det(\mathbb{1}_p - \Lambda\Lambda^\dagger)]^{-(p+q)} \prod_{i,j}^{p} d\Lambda_{i,j}$$

and

$$C_n := \frac{1}{\pi^{pq}} \prod_{i=0}^{q-1} \frac{(n-q+i)!}{(n-p-q+i)!}$$

de Finetti Theorem (arXiv:1612.05080)

Let n be an arbitrary integer and $k \geq p + q$. Let $\rho = |\psi\rangle\langle\psi|$ be a symmetric (pure) state in $\mathcal{F}^{U(n+k)}_{p,q,n+k}$. Then the state obtained after tracing out over $k(p+q)$ modes can be well approximated by a mixture of generalized coherent states:

$$\|\mathrm{tr}_k(\rho) - C_k \int \nu(\Lambda)(|\Lambda, 1\rangle\langle\Lambda, 1|)^{\otimes n} d\mu_{p,q}(\Lambda)\|_{tr} \leq \frac{3npq}{2(n+k-p-q)}.$$

special case of a result by König, Mitchison (JMP 2009)

proof adapted from Christandl, König, Mitchison, Renner (CMP 2007):

- relies on resolution of the identity and

$$1 - \frac{C_k}{C_{n+k}} \leq \frac{npq}{n+k-p-q}$$

de Finetti Theorem (arXiv:1612.05080)

Let n be an arbitrary integer and $k \geq p+q$. Let $\rho = |\psi\rangle\langle\psi|$ be a symmetric (pure) state in $\mathcal{F}^{U(n+k)}_{p,q,n+k}$. Then the state obtained after tracing out over $k(p+q)$ modes can be well approximated by a mixture of generalized coherent states:

$$\|\mathrm{tr}_k(\rho) - C_k \int \nu(\Lambda)(|\Lambda,1\rangle\langle\Lambda,1|)^{\otimes n} d\mu_{p,q}(\Lambda)\|_{tr} \leq \frac{3npq}{2(n+k-p-q)}.$$

special case of a result by König, Mitchison (JMP 2009)

proof adapted from Christandl, König, Mitchison, Renner (CMP 2007):

- relies on resolution of the identity and

$$1 - \frac{C_k}{C_{n+k}} \leq \frac{npq}{n+k-p-q}$$

Outline of the talk

- de Finetti theorems: classical and quantum
- Representation of the unitary group on Fock spaces
- Which density matrices are unitarily-invariant?
- Truncation of Haar random unitaries

Back to the initial problem of truncation of Haar random unitaries

Example: truncation of random Haar matrices

Let $\mathcal{H}_{k,k}$ be the distribution over $k \times k$ complex matrices obtained by drawing a random unitary $U \sim \mathsf{Haar}(\mathcal{U}(n))$ and outputting $\sqrt{n}U_{k,k}$ where $U_{k,k}$ is the $k \times k$ upper-left submatrix of U.

Let $\mathcal{G}^{k\times k}$ be the distribution over $k \times k$ complex matrices where each entry $\sim \mathcal{N}_{\mathbb{C}}(0,1)$.

- this work (with a quantum approach):

$$\|\mathcal{H}_{k,k} - \mathcal{G}^{k\times k}\|_{TV} \leq \frac{2k^3}{n-k}$$

idea:

- find a quantum state and a measurement that yields a distribution arbitrarily close to $\mathrm{Haar}(\mathcal{U}(n))$
- looking at the k marginal yields $\mathcal{H}_{k,k}$
- use Gaussian de Finetti to argue that it is close to measuring a Gaussian state

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Truncation of Haar random unitaries

- ρ_α^1 is the kn-mode pure state $\exp(\alpha x_{1,1} + \cdots + \alpha x_{k,k})$, a product of standard coherent states: measuring with heterodyne detection and renormalizing yields $\approx (e_1, \ldots, e_k)$
- $\rho_\alpha^2 = \int V_u \rho_\alpha^1 V_u^\dagger du$ (twirling $\implies$ unitarily invariant)
- trace out k rows, yields ρ_α^3 and measure with heterodyne detection (Husimi function):

$$\implies \text{approximates } \mathcal{H}_{k,k}$$

- purification of ρ_α^2 in $\mathcal{F}_{k,k,n}^{U(n)}$ (2nk modes)

$$\implies \text{Gaussian de Finetti applies with } p = q = k$$

proof skteck

- taking $\alpha \to \infty$ and renormalizing yields $\mathcal{H}_{k,k}$
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Conclusion

- de Finetti theorems are ubiquitous for studying large permutation-invariant multipartite systems / protocols
- but they fail to address infinite-dimensional systems (continuous variables)
- for some problems, a stronger invariance under U(n) is satisfied
  $\Longrightarrow$ ex: continuous-variable quantum cryptography
- open question: other applications?

Dualities

- Schur-Weyl duality:

$$\mathrm{SU(d)} \quad \leftrightarrow \quad \mathrm{S_n} \quad \text{on} \quad (\mathbb{C}^d)^{\otimes n} = \mathbb{C}^d \otimes \cdots \otimes \mathbb{C}^d$$

- this work:

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