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Regularity for elliptic systems of differential forms and applications

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Abstract We prove existence and up to the boundary regularity estimates in L^p and Hölder spaces for weak solutions of the linear system

$$\delta\left(Ad\omega\right)+B^{T}d\delta\left(B\omega\right)=\lambda B\omega+f \text{ in } \Omega,$$

with either $\nu\wedge\omega$ and $\nu\wedge\delta\left(B\omega\right)$ or $\nu\lrcorner B\omega$ and $\nu\lrcorner\left(A(x)d\omega\right)$ prescribed on $\partial\Omega$. The proofs are in the spirit of 'Campanato method' and thus avoid potential theory and do not require a verification of Agmon-Douglis-Nirenberg or Lopatinski-Shapiro type conditions. Applications to a number of related problems, such as general versions of the time-harmonic Maxwell system, stationary Stokes problem and the 'div-curl' systems, are included.

Keywords Boundary regularity, elliptic system, camapanato method, Hodge laplacian, Maxwell system, Stokes system, div-curl system, Gaffney-Friedrichs inequality, tangential and normal boundary condition

Mathematics Subject Classification (2000) 35J57 · 35D10

1 Introduction

Second order linear elliptic systems in divergence form with Dirichlet or conormal derivative type boundary conditions are well-known. However, a large class of elliptic systems can be written more crisply in the language of differential forms and often, for such systems, neither of those are the relevant boundary conditions. A typical example is the Poisson problem for the Hodge Laplacian

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with prescribed 'tangential part' and 'normal part' on the boundary respectively, namely the systems,

$$\begin{cases} \delta d\omega + d\delta\omega = f \text{ in } \Omega, \\ \nu \wedge \omega = 0 \text{ on } \partial\Omega. \\ \nu \wedge \delta\omega = 0 \text{ on } \partial\Omega. \end{cases} \qquad \text{or} \qquad \begin{cases} \delta d\omega + d\delta\omega = f \text{ in } \Omega, \\ \nu \lrcorner \omega = 0 \text{ on } \partial\Omega. \\ \nu \lrcorner d\omega = 0 \text{ on } \partial\Omega. \end{cases}$$

The regularity results for these two systems are known since Morrey[14] (see also [15]). However, regularity results for more general linear elliptic systems with these boundary conditions do not exist in the literature. The reason for this surprising absence probably lies in the available proofs of this result. The original proof of Morrey relied on potential theory and depends on the fact that $\delta d + d\delta$, i.e the Hodge Laplacian is precisely the componentwise scalar Laplacian. Also, after flattening the boundary, the condition $\nu \wedge \omega = 0$ and $\nu \wedge \delta\omega = 0$ imply that as far as the principal order terms are concerned, the whole system decouples and gets reduced to $\binom{n}{k}$ scalar Poisson problem with lower order terms, out of which $\binom{n-1}{k}$ number of equations has zero Dirichlet boundary conditions and the other $\binom{n-1}{k-1}$ number of equations has zero Neumann boundary conditions. Other available proofs verifies the Lopatinski-Shapiro, henceforth LS, (see Schwarz[17]) or the Agmon-Douglis-Nirenberg complementing condition, henceforth ADN, (see Csato [4]) and these verifications too rely on the fact that the principal symbol of the operator is rather 'simple'.

Deriving the regularity results for the system $\delta(Ad\omega) + B^T d\delta(B\omega) = f$ or even the simpler system $\delta(Ad\omega) + d\delta\omega = f$ with these type of boundary conditions calls for different methods, as the verification of the ADN or LS conditions for these systems looks algebraically tedious. On the other hand, the boundary conditions simply arise out of an integration by parts formula and in principle, verifying ADN or LS should be an avoidable overkill.

For linear elliptic systems with Dirichlet boundary conditions, the classical Campanato method (see Campanato[2], also Giaquinta-Martinazzi[13] and references therein) of deriving estimates in Morrey and Campanato spaces, avoids potential theory and yields at the same time both the Schauder estimates and via an interpolation theorem of Stampacchia[19], also the L^p estimates. This approach has also been adopted to elliptic systems with conormal derivative type condition, for example in Giaquinta-Modica[12]. The crux of the present article is to adopt this approach to these kind of boundary conditions. This, as a particular case yields a new proof for the regularity of the Hodge Laplacian system as well.

The main result of the present article (Theorem 17) is the existence and up to the boundary regularity results in Hölder and L^p spaces for the system

$$\begin{cases} \delta(Ad\omega) + B^T d\delta\,(B\omega) = \lambda B\omega + f \text{ in } \Omega, \\ \nu \wedge \omega = \nu \wedge \omega_0 \text{ on } \partial\Omega. \\ \nu \wedge \delta\,(B\omega) = \nu \wedge \delta\,(B\omega_0) \text{ on } \partial\Omega, \end{cases} \tag{1}$$

and also its counterpart with the 'normal condition'

$$\begin{cases} \delta(Ad\omega) + B^T d\delta\,(B\omega) = \lambda B\omega + f \text{ in } \Omega, \\ \nu \lrcorner\,(B\omega) = \nu \lrcorner\,(B\omega_0) \text{ on } \partial\Omega. \\ \nu \lrcorner\,(Ad\omega) = \nu \lrcorner\,(Ad\omega_0) \text{ on } \partial\Omega, \end{cases} \tag{2}$$

where A and B are matrix fields and ω is a k-form. Note that unlike the case of the Hodge Laplacian, where solving the tangential boundary value problem for k-forms is equivalent to solving the normal boundary value problem for $(n-k)$-forms by Hodge duality, in this case the two problems are not dual to each other.

The results for (1) and (2) yield also the existence and regularity results for a number of related problems. The time-harmonic Maxwell's equation in bounded domain in $\mathbb{R}^3$ is

$$\begin{cases} \operatorname{curl} H = i\omega\varepsilon E + J_e & \text{in } \Omega, \\ \operatorname{curl} E = -i\omega\mu H + J_m & \text{in } \Omega, \\ \nu \times E = \nu \times E_0 & \text{on } \partial\Omega. \end{cases}$$

Eliminating H and writing as a second order system in E, we obtain,

$$\begin{cases} \operatorname{curl}(\mu^{-1} \operatorname{curl} E) = \omega^2 \varepsilon E - i\omega J_e + \operatorname{curl}\left(\mu^{-1} J_m\right) & \text{in } \Omega, \\ \operatorname{div}(\varepsilon E) = \dfrac{i}{\omega} \operatorname{div} J_e & \text{in } \Omega, \\ \nu \times E = \nu \times E_0 & \text{on } \partial\Omega. \end{cases}$$

H also satisfies similar kind of second order system with normal boundary conditions. Writing in the language of differential form, the systems for E and H are special cases of the general systems

$$\begin{cases} \delta(Adu) = \lambda Bu + f & \text{in } \Omega, \\ \delta\,(Bu) = g & \text{in } \Omega, \\ \nu \wedge u = \nu \wedge u_0 & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} \delta(Adu) = \lambda Bu + f & \text{in } \Omega, \\ \delta\,(Bu) = g & \text{in } \Omega, \\ \nu \lrcorner Bu = \nu \lrcorner u_0 & \text{on } \partial\Omega, \\ \nu \lrcorner\,(Adu) = \nu \lrcorner\,(Adu_0) & \text{on } \partial\Omega, \end{cases}$$

respectively, for k-forms u. When B is the identity matrix, these two systems are respectively related to the systems

$$\begin{cases} \delta(Adu) + dp = \lambda u + f \text{ in } \Omega, \\ \delta u = 0 \text{ in } \Omega, \\ \nu \wedge u = 0 \text{ on } \partial\Omega, \\ p = p_0 \text{ on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} \delta(Adu) + dp = \lambda u + f \text{ in } \Omega, \\ \delta u = 0 \text{ in } \Omega, \\ \nu \lrcorner u = 0 \text{ on } \partial\Omega, \\ \nu \lrcorner (Adu) = 0 \text{ on } \partial\Omega, \end{cases}$$

both of which generalizes the vorticity-velocity-pressure formulation of Stationary Stokes system with different type of boundary conditions, both of which are studied in the context of Stokes flow in three dimensions (see [1], [3], [8]).

On our way to prove the above results, we also show existence and apriori regularity estimates for the generalized 'div-curl' systems

$$\begin{cases} d(A(x)u) = f & \text{in } \Omega, \\ \delta(B(x)u) = g & \text{in } \Omega, \\ \nu \wedge A(x)u = \nu \wedge u_0 & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} d(A(x)u) = f & \text{in } \Omega, \\ \delta(B(x)u) = g & \text{in } \Omega, \\ \nu \lrcorner B(x)u = \nu \lrcorner u_0 & \text{on } \partial\Omega. \end{cases}$$

These systems have been studied in various degrees of generality, see for example [16], also [6],[7]. This yields a few generalized Gaffney-Friedrichs type inequalities, generalizing the work of Friendrichs[9](see also Gaffney[10],[11]). We also prove the existence of a solution with optimal regularity for the Dirichlet boundary value problem

$$\begin{cases} \delta(A(x)du) = f \text{ in } \Omega, \\ u = u_0 \text{ on } \partial\Omega, \end{cases}$$

for k-forms u. Note that the problem is just the Dirichlet problem for the elliptic equation

$$\operatorname{div}(A\nabla u) = f$$

when $k = 0$ and is not elliptic and the space of solutions is infinite dimensional as soon as $k \geq 1$.

Many of these results are already known in the special cases when u is a vector field (i.e $k = 1$) and $n = 3$. But those proofs use somewhat ad hoc approaches, based on using scalar elliptic theory componentwise, using the fact that for vector fields in three dimensions, both the boundary conditions and the systems are simple enough to find explicitly, by direct calculation, the equations satisfied by each component. The advantage of our approach is that it brings out the core of the structural features of these systems that is ultimately responsible for regularity.

2 Notations

We now fix the notations, for further details we refer to [6]. Let $n \geq 2$ and $0 \leq k \leq n$ be an integer.

- We write $\Lambda^k(\mathbb{R}^n)$ (or simply Λ^k) to denote the vector space of all alternating $k-$linear maps $f : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k-\text{times}} \to \mathbb{R}$. For $k = 0$, we set $\Lambda^0(\mathbb{R}^n) = \mathbb{R}$. Note that $\Lambda^k(\mathbb{R}^n) = \{0\}$ for $k > n$ and, for $k \leq n$, $\dim\left(\Lambda^k(\mathbb{R}^n)\right) = \binom{n}{k}$.

- $\wedge$, $\lrcorner$, $\langle\ ;\ \rangle$ and, respectively, $*$ denote the exterior product, the interior product, the scalar product and, respectively, the Hodge star operator.
- If $\left\{e^1, \cdots, e^n\right\}$ is a basis of $\mathbb{R}^n$, then, identifying Λ^1 with $\mathbb{R}^n$,

$$\left\{e^{i_1} \wedge \cdots \wedge e^{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\right\}$$

is a basis of Λ^k. An element $\xi \in \Lambda^k\left(\mathbb{R}^n\right)$ will therefore be written as

$$\xi = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \xi_{i_1 i_2 \cdots i_k}\, e^{i_1} \wedge \cdots \wedge e^{i_k} = \sum_{I \in \mathcal{T}^k} \xi_I\, e^I$$

where

$$\mathcal{T}^k = \left\{I = (i_1, \cdots, i_k) \in \mathbb{N}^k : 1 \leq i_1 < \cdots < i_k \leq n\right\}.$$

We shall identify exterior 1-forms with vectors freely and shall refrain from using the musical notation to denote these identifications, in order not to burden our notations further. Also, we shall often write an exterior k-form as a vector in $\mathbb{R}^{\binom{n}{k}}$, when the alternating structure is not important for our concern. In a similar vein, we shall identify $m \times n$ matrices with the space $\mathbb{R}^{m \times n}$.

Let $0 \leqslant k \leqslant n$ and let $\Omega \subset \mathbb{R}^n$ be open, bounded and smooth.

- A differential k-form ω is a measurable function $\omega : \Omega \to \Lambda^k$.
- The usual Sobolev, Hölder, Morrey and Campanato spaces are defined componentwise and are denoted by their usual symbols.
- Two special differential operators on differential forms will have a special significance for us. A differential $(k+1)$-form $\varphi \in L^1_{\mathrm{loc}}(\Omega; \Lambda^{k+1})$ is called the exterior derivative of $\omega \in L^1_{\mathrm{loc}}\left(\Omega; \Lambda^k\right)$, denoted by $d\omega$, if

$$\int_\Omega \eta \wedge \varphi = (-1)^{n-k} \int_\Omega d\eta \wedge \omega,$$

for all $\eta \in C_0^\infty\left(\Omega; \Lambda^{n-k-1}\right)$. The Hodge codifferential of $\omega \in L^1_{\mathrm{loc}}\left(\Omega; \Lambda^k\right)$ is a $(k-1)$-form, denoted $\delta\omega \in L^1_{\mathrm{loc}}\left(\Omega; \Lambda^{k-1}\right)$ defined as

$$\delta\omega := (-1)^{nk+1} * d * \omega.$$

See [6] for the properties and the integration by parts formula regarding these operators.
- Let $1 \leq p \leq \infty$ and let ν be the outward unit normal to $\partial\Omega$, identified with the 1-form $\nu = \sum_{i=1}^n \nu_i dx^i$. The spaces $W^{1,p}_T\left(\Omega; \Lambda^k\right)$ and $W^{1,p}_N\left(\Omega; \Lambda^k\right)$ are defined as

$$W^{1,p}_T\left(\Omega; \Lambda^k\right) = \left\{\omega \in W^{1,p}\left(\Omega; \Lambda^k\right) : \nu \wedge \omega = 0 \text{ on } \partial\Omega\right\}$$

$$W^{1,p}_N\left(\Omega; \Lambda^k\right) = \left\{\omega \in W^{1,p}\left(\Omega; \Lambda^k\right) : \nu \lrcorner \omega = 0 \text{ on } \partial\Omega\right\}.$$

Also, we define,

$$W^{1,p}_{\delta,T}(\Omega;\Lambda^k) = \left\{\omega \in W^{1,p}_T(\Omega;\Lambda^k) : \delta\omega = 0 \text{ in } \Omega\right\}$$

and $W^{1,p}_{\delta,N}(\Omega;\Lambda^k)$ is defined similarly. The space $\mathcal{H}_T\left(\Omega;\Lambda^k\right)$ is defined as

$$\mathcal{H}_T\left(\Omega;\Lambda^k\right) = \left\{\omega \in W^{1,2}_T\left(\Omega;\Lambda^k\right) : d\omega = 0 \text{ and } \delta\omega = 0 \text{ in } \Omega\right\}$$

The spaces $\mathcal{H}_N\left(\Omega;\Lambda^k\right)$ is defined similarly.

– Let B^+_R denote the half-ball centered around 0 in the half space, i.e

$$B^+_R = \{x \in \mathbb{R}^n : |x| < R, x_n > 0\}.$$

Let Γ_R and C_R denote the flat part and the curved part, respectively, of the boundary of the half ball B^+_R. We define the following subspace of $W^{1,2}(B^+_R;\Lambda^k)$,

$$W^{1,2}_{T,flat}(B^+_R;\Lambda^k) = \left\{\psi \in W^{1,2} : e_n \wedge \psi = 0 \text{ on } \Gamma_R,\ \psi = 0 \text{ on } C_R\right\}.$$

The space $W^{1,2}_{N,flat}(B^+_{R_0};\Lambda^k)$ is defined similarly. We use the notation $(\cdot)_s$ to denote the average over half-balls, i.e

$$\left(\theta^I\right)_s = \frac{1}{\text{meas}(B^+_s)}\int_{B^+_s}\theta^I.$$

We shall frequently use two ellipticity conditions for matrix fields.

Definition 1 A map $A : \Omega \to L(\Lambda^k,\Lambda^k)$ is said to satisfy the **Legendre-Hadamard condition** if A satisfies, for all $x \in \Omega$,

$$\langle A(x)(a\wedge b)\ ;\ a\wedge b\rangle \geq \gamma\left|a\wedge b\right|^2, \qquad \text{for every } a \in \Lambda^1, b \in \Lambda^{k-1}$$

for some constant $\gamma > 0$. A is said to satisfy the **Legendre condition** if for some constant $\gamma > 0$ we have , for all $x \in \Omega$,

$$\langle A(x)\xi\ ;\ \xi\rangle \geq \gamma\left|\xi\right|^2, \qquad \text{for every } \xi \in \Lambda^k.$$

3 Boundary estimates

Our starting point is the classical Gaffney inequality, which can be proved from a simple integration by parts formula (see [5]). Using this inequality, we show that the usual 'Campanato method' for regularity estimates (see for example, Giaquinta-Martinazzi[13]) can be adopted in this setting to yield regularity estimates up to the boundary for the systems (1) and (2) in the special case when $B \equiv \mathbf{I}$, the identity matrix. As a consequence, we can prove a general version of the Gaffney-Friedrichs inequality which in turn yields the estimates for the general case.

Our strategy is to flatten the boundary and freeze the coefficients and derive estimates in the Campanato spaces for the resulting constant operators on half-balls.

3.1 Preliminary lemmas

We start with a few easy results that we shall use.

Lemma 1 (ellipticity lemma) *Let $B : \Lambda^k \to \Lambda^k$ satisfy*

$$\langle B\xi;\xi\rangle \geq \gamma_B|\xi|^2 \qquad \text{for all } \xi \in \Lambda^k,$$

for some constant $\gamma_B > 0$. Then for every $\gamma_A > 0$, there exists a constant $c > 0$, depending only on γ_A and γ_B, such that

$$\gamma_A|e_n \wedge \xi|^2 + |e_n \lrcorner B\xi|^2 \geq c\,|\xi|^2 \qquad \text{for all } \xi \in \Lambda^k.$$

Proof If the inequality is false then there exists a sequence $\{\xi_s\}$ such that for every $s \geq 1$, we have $|\xi_s| = 1$ and $\gamma_A|e_n \wedge \xi_s|^2 + |e_n \lrcorner B\xi_s|^2 < \frac{1}{s}$. This implies, passing to a subsequence which do not relabel, $\xi_s \to \xi$ for some $\xi \in \Lambda^k$ such that $|\xi| = 1$, $e_n \wedge \xi = 0$ and $e_n \lrcorner B\xi = 0$. Using the identity $\xi = e_n \lrcorner (e_n \wedge \xi) + e_n \wedge (e_n \lrcorner \xi)$, this in particular implies,

$$0 = \langle e_n \lrcorner B\xi; e_n \lrcorner \xi\rangle = \langle B\xi; e_n \wedge (e_n \lrcorner \xi)\rangle = \langle B\xi;\xi\rangle \geq \gamma_B|\xi|^2 = \gamma_B > 0.$$

This contradiction proves the lemma. □

The next result is a variant of classical Gårding inequality, which can be proved in the standard way.

Lemma 2 (Gårding inequality) *Let $A \in C\left(\overline{\Omega}; L(\Lambda^{k+1}, \Lambda^{k+1})\right)$ satisfy the Legendre-Hadamard condition. Then there exist constants $\lambda_0 \geq 0$ and λ_1 such that for all $u \in W^{1,2}_T(\Omega, \Lambda^k)$, we have,*

$$\int_\Omega \langle A(x)du, du\rangle \geq \lambda_0 \|du\|^2_{L^2} - \lambda_1 \|u\|^2_{L^2}. \tag{3}$$

Moreover, if A has constant coefficients or $A \in L^\infty\left(\Omega; L(\Lambda^{k+1}, \Lambda^{k+1})\right)$ satisfies the Legendre condition, then the inequality is true with $\lambda_1 = 0$.

We now need a Poincaré inequality which can be proved by a simple contradiction argument.

Lemma 3 *There exist a constants $c_1, c_2 > 0$ such that*

$$\int_{B_R^+} |u|^2 \leq c_1 R^2 \int_{B_R^+} |\nabla u|^2, \qquad \text{for all } u \in W^{1,2}_T(B_R^+; \Lambda^k) \cap W^{1,2}_N(B_R^+; \Lambda^k)$$

and

$$\int_{B_R^+} |u - (u)_R|^2 \leq c_2 R^2 \int_{B_R^+} |\nabla u|^2, \qquad \text{for all } u \in W^{1,2}(B_R^+; \Lambda^k).$$

3.2 Estimates for constant coefficient operator

We begin with deriving estimates on half-balls for constant coefficient operators.

3.2.1 Boundary Caccioppoli inequality

The most crucial ingredient for these estimates are the Caccioppoli type inequalities on half-balls.

Theorem 1 (Caccioppoli inequality) *Let $A : \Lambda^{k+1} \to \Lambda^{k+1}$ satisfy the Legendre-Hadamard condition. Let $u \in W^{1,2}(B_R^+; \Lambda^k)$ with $\nu \wedge u = 0$ on Γ_R satisfy, for all $\psi \in W^{1,2}_{T,flat}(B_R^+; \Lambda^k)$,*

$$\int_{B_R^+} \langle A(du); d\psi\rangle + \int_{B_R^+} \langle \delta u; \delta\psi\rangle = 0. \tag{4}$$

Then there exists a constant $c > 0$ such that for every $0 < \rho < R$, we have the following boundary Caccioppoli inequality,

$$\int_{B_\rho^+} |\nabla u|^2 \leq \frac{c}{(R-\rho)^2} \left\{ \sum_{\substack{I\in\mathcal{T}^k \\ n\notin I}} \int_{B_R^+\setminus B_\rho^+} \left|u^I\right|^2 + \sum_{\substack{I\in\mathcal{T}^k \\ n\in I}} \int_{B_R^+\setminus B_\rho^+} \left|u^I - \xi^I\right|^2 \right\}, \tag{5}$$

for any collection of constants $\xi^I \in \mathbb{R}$ for each $I \in \mathcal{T}^k$ with $n \in I$.

Proof Choose $\eta \in C_c^\infty(B_R)$ such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ in B_ρ and $|D\eta| \leq \frac{2}{R-\rho}$ and define $\widetilde{u} : B_R^+ \to \Lambda^k$ as,

$$\widetilde{u}^I = \begin{cases} u^I & \text{if } n \notin I, \\ u^I - \xi^I & \text{if } n \in I, \end{cases} \qquad \text{for every } I \in \mathcal{T}^k.$$

Substituting $\eta^2\widetilde{u} \in W^{1,2}_{T,flat}(B_R^+; \Lambda^k)$ for the test function ψ in (4), standard calculations and Young's inequality with $\varepsilon > 0$ implies the estimate,

$$\int_{B_R^+} \langle A(d(\eta\widetilde{u})); d(\eta\widetilde{u})\rangle + \int_{B_R^+} \langle \delta(\eta\widetilde{u}); \delta(\eta\widetilde{u})\rangle \leq c\varepsilon \int_{B_R^+} |\eta|^2 |\nabla\widetilde{u}|^2 + c\int_{B_R^+} |D\eta|^2 |\widetilde{u}|^2.$$

Since the half ball B_R^+ is contractible, combining this Gaffney inequality and Gårding inequality

$$\begin{aligned} &\int_{B_R^+} |\eta|^2 |\nabla\widetilde{u}|^2 = \int_{B_R^+} |\nabla(\eta\widetilde{u})|^2 - \int_{B_R^+} \langle 2\eta\nabla\widetilde{u}; D\eta\otimes\widetilde{u}\rangle \\ &\leq c\left\{ \int_{B_R^+} \langle A(d(\eta\widetilde{u})); d(\eta\widetilde{u})\rangle + \int_{B_R^+} \langle \delta(\eta\widetilde{u}); \delta(\eta\widetilde{u})\rangle \right\} - \int_{B_R^+} \langle 2\eta\nabla\widetilde{u}; D\eta\otimes\widetilde{u}\rangle \\ &\leq c\varepsilon \int_{B_R^+} |\eta|^2 |\nabla\widetilde{u}|^2 + c\int_{B_R^+} |D\eta|^2 |\widetilde{u}|^2 \end{aligned}$$

Choosing ε small and using the facts that $\eta \equiv 1$ on B_ρ, $D\eta$ vanishes outside $B_R^+ \setminus B_\rho^+$ and $|D\eta| \leq \frac{2}{R-\rho}$, we get

$$\int_{B_\rho^+} |\nabla \widetilde{u}|^2 \leq \int_{B_R^+} |\eta|^2 |\nabla \widetilde{u}|^2 \leq \frac{c}{(R-\rho)^2} \int_{B_R^+ \setminus B_\rho^+} |\widetilde{u}|^2 \leq \frac{c}{(R-\rho)^2} \int_{B_R^+ \setminus B_\rho^+} |\widetilde{u}|^2.$$

This finishes the proof. □

Since the Gårding inequality we used is not true for $W^{1,2}_N$, we need stronger ellipticity assumption on A for a Caccioppoli inequality in that space. If A satisfy the Legendre condition, then similar arguments show that for any $u \in W^{1,2}(B_R^+; \Lambda^k)$ with $\nu \lrcorner u = 0$ on Γ_R satisfying (4) for all $\psi \in W^{1,2}_{N,flat}(B_R^+; \Lambda^k)$ and for every $0 < \rho < R$, we have

$$\int_{B_\rho^+} |\nabla u|^2 \leq \frac{c}{(R-\rho)^2} \left\{ \sum_{\substack{I \in \mathcal{T}^k \\ n \notin I}} \int_{B_R^+ \setminus B_\rho^+} \left| u^I - \xi^I \right|^2 + \sum_{\substack{I \in \mathcal{T}^k \\ n \in I}} \int_{B_R^+ \setminus B_\rho^+} \left| u^I \right|^2 \right\},$$

for any collection of constants $\xi^I \in \mathbb{R}$ for each $I \in \mathcal{T}^k$ with $n \notin I$.

3.2.2 L^2 estimates

Caccioppoli inequalities lead to L^2 estimates.

Theorem 2 (L^2 estimates for constant coefficients) *Let $A : \Lambda^{k+1} \to \Lambda^{k+1}$ satisfy the Legendre-Hadamard condition. Let $u \in W^{1,2}(B_R^+; \Lambda^k)$ with $\nu \wedge u = 0$ on Γ_R satisfy, for all $\psi \in W^{1,2}_{T,flat}(B_R^+; \Lambda^k)$,*

$$\int_{B_R^+} \langle A(du); d\psi \rangle + \int_{B_R^+} \langle \delta u; \delta \psi \rangle = 0. \tag{6}$$

Then for every $m \geq 2$, there exist constants $c_m, \widetilde{c}_m > 0$ such that we have the estimates

$$\int_{B^+_{R/2^m}} |D^m u|^2 \leq \frac{\widetilde{c}_m}{(R)^{2^{m-1}}} \int_{B_R^+} |\nabla u|^2 \leq \frac{c_m}{(R)^{2^m}} \int_{B_R^+} |u|^2 . \tag{7}$$

Remark 1 Same estimates hold true for $u \in W^{1,2}(B_R^+; \Lambda^k)$ with $\nu \lrcorner u = 0$ on Γ_R, satisfying (6) for every $\psi \in W^{1,2}_{N,flat}(B_R^+; \Lambda^k)$ if A satisfies the Legendre condition.

Proof We only show (7) for $m = 2$. Using the difference quotient $\tau_{h,s} u(x) = \frac{1}{h} \{u(x + he_s) - u(x)\}$, for every $s = 1, \ldots, n-1$, and the Caccioppoli inequality for $\tau_{h,s} u$ and u, we obtain,

$$\int_{B^+_{\frac{R}{2^2}}} |\nabla (\tau_{h,s} u)|^2 \leq \frac{c}{(R)^2} \int_{B^+_{R/2}} |\tau_{h,s} u|^2 \leq \frac{c}{(R)^2} \int_{B^+_{R/2}} |\nabla u|^2 \leq \frac{c}{(R)^4} \int_{B_R^+} |u|^2 .$$

This implies,

$$\int_{B^+_{R/2^2}} |D_{ij}u|^2 \leq \frac{c}{(R)^2}\int_{B^+_{R/2}} |\nabla u|^2 \leq \frac{c}{(R)^{2^2}}\int_{B^+_R} |u|^2,$$

for all $i,j = 1,\ldots,n$ such that $(i,j) \neq (n,n)$. To estimate the D_{nn} derivative, first we rewrite the system (6) as a system in terms of gradients. For every $\psi \in W^{1,2}_{T,flat}(B^+_R;\Lambda^k)$, we have,

$$\int_{B^+_R} \langle A(du); d\psi\rangle + \int_{B^+_R} \langle \delta u; \delta\psi\rangle = \int_{B^+_R} \langle \widetilde{A}\nabla u; \nabla\psi\rangle, \tag{8}$$

where $\widetilde{A} : \mathbb{R}^{\binom{n}{k}\times n} \to \mathbb{R}^{\binom{n}{k}\times n}$ is the linear map, defined by the pointwise algebraic identities

$$\langle \widetilde{A}a_1 \otimes b_1; a_2 \otimes b_2\rangle = \langle A\left(a_1 \wedge b_1\right); a_2 \wedge b_2\rangle + \langle a_1 \lrcorner b_1; a_2 \lrcorner b_2\rangle,$$

for every $a_1, a_2 \in \Lambda^1$, viewed also as vectors in $\mathbb{R}^n$ and $b_1, b_2 \in \Lambda^k$, viewed also as vectors in $\mathbb{R}^{\binom{n}{k}}$. We also define the maps $\widetilde{A}^{pq} : \mathbb{R}^{\binom{n}{k}} \to \mathbb{R}^{\binom{n}{k}}$ for every $p,q = 1,\ldots,n$, by the identities,

$$\langle \widetilde{A}^{pq}\xi;\xi\rangle = \sum_{\alpha,\beta\in\mathcal{T}^k} \widetilde{A}^{pq}_{\alpha\beta}\xi^\alpha\xi^\beta = \langle \widetilde{A}(e_p\otimes\xi); e_q\otimes\xi\rangle \quad \text{for every } \xi \in \mathbb{R}^{\binom{n}{k}}.$$

Now, for every $\xi \in \mathbb{R}^{\binom{n}{k}}$, by lemma 1, we get,

$$\begin{aligned}\langle \widetilde{A}^{nn}\xi;\xi\rangle &= \langle \widetilde{A}(e_n\otimes\xi); e_n\otimes\xi\rangle = \langle A\left(e_n\wedge\xi\right); e_n\wedge\xi\rangle + \langle e_n\lrcorner\xi; e_n\lrcorner\xi\rangle\\ &\geq \gamma_A|e_n\wedge\xi|^2 + |e_n\lrcorner\xi|^2 \geq c\,|\xi|^2,\end{aligned}$$

proving that $\widetilde{A}^{nn}$ is invertible. Choosing $\psi \in C^\infty_c(B^+_{R/2^2};\Lambda^k)$ in (6), using (8) and integrating by parts, $u \in W^{1,2}(B^+_R;\Lambda^k)$ satisfy, for all $\psi \in C^\infty_c(B^+_{R/2^2};\Lambda^k)$,

$$\begin{aligned}\int_{B^+_{R/2^2}} \sum_{\alpha,\beta\in\mathcal{T}^k} \widetilde{A}^{nn}_{\alpha\beta}\frac{\partial u^\alpha}{\partial x_n}\frac{\partial\psi^\beta}{\partial x_n} &= -\sum_{\substack{p,q=1,\ldots,n\\(p,q)\neq(n,n)\\ \alpha,\beta\in\mathcal{T}^k}} \int_{B^+_{R/2^2}} \widetilde{A}^{pq}_{\alpha\beta}\frac{\partial u^\alpha}{\partial x_p}\frac{\partial\psi^\beta}{\partial x_q}\\ &= \sum_{\substack{p,q=1,\ldots,n\\(p,q)\neq(n,n)\\ \alpha,\beta\in\mathcal{T}^k}} \int_{B^+_{R/2^2}} \frac{\partial}{\partial x_q}\left(\widetilde{A}^{pq}_{\alpha\beta}\frac{\partial u^\alpha}{\partial x_p}\right)\psi^\beta.\end{aligned}$$

Since $\frac{\partial}{\partial x_q}\left(\widetilde{A}^{pq}_{\alpha\beta}\frac{\partial u^\alpha}{\partial x_p}\right)$ are L^2 for all choice of $p,q = 1,\ldots,n, (p,q) \neq (n,n)$ and for all $\alpha,\beta \in \mathcal{T}^k$, and $\widetilde{A}^{nn}$ is invertible, this implies the estimate for $D_{nn}u$. □

3.2.3 Decay estimates for the constant coefficient operator

Theorem 3 *Let $A : \Lambda^{k+1} \to \Lambda^{k+1}$ satisfy the Legendre-Hadamard condition. Let $u \in W^{1,2}_{T,flat}(B^+_R; \Lambda^k)$ satisfy, for all $\psi \in W^{1,2}_{T,flat}(B^+_R; \Lambda^k)$,*

$$\int_{B^+_R} \langle A(du); d\psi\rangle + \int_{B^+_R} \langle \delta u; \delta\psi\rangle = 0. \tag{9}$$

Then, for every $0 < \rho < R$, the following decay estimates hold true.

$$\int_{B^+_\rho} |u|^2 \le c\left(\frac{\rho}{R}\right)^n \int_{B^+_R} |u|^2, \tag{10}$$

$$\int_{B^+_\rho} |\nabla u|^2 \le c\left(\frac{\rho}{R}\right)^n \int_{B^+_R} |\nabla u|^2, \tag{11}$$

$$\int_{B_\rho} \left|\nabla u - (\nabla u)_\rho\right|^2 \le c\left(\frac{\rho}{R}\right)^{n+2} \int_{B_R} |\nabla u - (\nabla u)_R|^2. \tag{12}$$

Remark 2 Analogous estimates hold for $u \in W^{1,2}_{N,flat}$ satisfying (9) for all $\psi \in W^{1,2}_{N,flat}(B^+_R; \Lambda^k)$, when A satisfies the Legendre condition.

Proof Using the rescaled function $\widetilde{u}(x) = u(Rx)$, we can replace ρ ρ/R and R by 1. Since the inequalities are nontrivial only for $\frac{\rho}{R}$ small, we assume $\frac{\rho}{R} < \frac{1}{2^m}$ for some integer m. Now by Sobolev embedding and L^2 regularity results give,

$$\int_{B^+_{\rho/R}} |u|^2 \le c\left(\frac{\rho}{R}\right)^n \sup_{B^+_{\rho/R}} |u|^2 \le c\left(\frac{\rho}{R}\right)^n \|u\|^2_{W^{m_0,2}(B^+_{\frac{1}{2^{m_0}}})} \le c\left(\frac{\rho}{R}\right)^n \int_{B^+_1} |u|^2.$$

Same argument for ∇u and Poincaré inequality give (11). Now we define $\widetilde{u} \in W^{1,2}(B^+_R; \Lambda^k)$,

$$\widetilde{u}^I = \begin{cases} u^I & \text{if } n \notin I, \\ u^I - \displaystyle\sum_{i=1}^n x_i \left(\frac{\partial u^I}{\partial x_i}\right)_{B^+_1} & \text{if } n \in I, \end{cases} \qquad \text{for every } I \in \mathcal{T}^k.$$

Note that $D^2\widetilde{u} = D^2 u$, $\widetilde{u}$ satisfies the same PDE as u and $\nu \wedge \widetilde{u} = 0$ on Γ_R. Thus by L^2 regularity, Poincaré inequality and Sobolev embedding gives,

$$\begin{aligned}\int_{B_{\rho/R}} \left|\nabla u - (\nabla u)_{\rho/R}\right|^2 &\le c\left(\frac{\rho}{R}\right)^2 \int_{B^+_{\rho/R}} \left|D^2 u\right|^2 \le c\left(\frac{\rho}{R}\right)^{n+2} \sup_{B^+_{1/2^{m_1}}} \left|D^2 u\right|^2 \\ &= c\left(\frac{\rho}{R}\right)^{n+2} \sup_{B^+_{1/2^{m_1}}} \left|D^2\widetilde{u}\right|^2 \le c\left(\frac{\rho}{R}\right)^{n+2} \|D\widetilde{u}\|^2_{L^2(B^+_1)}.\end{aligned}$$

But we have

$$\|D\widetilde{u}\|^2_{L^2(B^+_1)} = \int_{B^+_1} |\nabla u - (\nabla u)_1|^2,$$

since $(\nabla u_I)_1 = 0$ for all $I \in \mathcal{T}^k$ with $n \notin I$. □

3.2.4 $\mathcal{L}^{2,\mu}$ estimates

Now we proceed to estimates in the Campanato spaces.

Theorem 4 ($\mathcal{L}^{2,\mu}$ estimates) *Let $0 \leq \mu \leq n+2\gamma$ be a real number. Let the matrix $A : \Lambda^{k+1} \to \Lambda^{k+1}$ satisfy the Legendre-Hadamard condition. If $u \in W^{1,2}_{T,flat}(B^+_R;\Lambda^k)$ satisfy, for all $\psi \in W^{1,2}_{T,flat}(B^+_R;\Lambda^k)$,*

$$\int_{B^+_R} \langle A(du); d\psi\rangle + \int_{B^+_R} \langle \delta u; \delta\psi\rangle + \int_{B^+_R} \langle \mathcal{P}; \psi\rangle - \int_{B^+_R} \langle \mathcal{Q}; \nabla\psi\rangle = 0. \tag{13}$$

Then if $\mathcal{P} \in \mathrm{L}^{2,\mu-2}(B^+_R;\Lambda^k)$, and $\mathcal{Q} \in \mathcal{L}^{2,\mu}(B^+_R;\mathbb{R}^{n\times\binom{n}{k}})$, this implies $Du \in \mathcal{L}^{2,\mu}\left(B^+_{R/2};\mathbb{R}^{n\times\binom{n}{k}}\right)$, with the estimate

$$[Du]_{\mathcal{L}^{2,\mu}\left(B^+_{R/2}\right)} \leq c\left\{\|Du\|_{L^2(B^+_R)} + [\mathcal{P}]_{\mathrm{L}^{2,\mu-2}(B^+_R)} + [\mathcal{Q}]_{\mathcal{L}^{2,\mu}(B^+_R)}\right\}.$$

Moreover, if we have $\mathcal{P} \in \mathcal{L}^{2,\mu}(B^+_R;\Lambda^k)$ and $\mathcal{Q} \in \mathcal{L}^{2,\mu+2}(B^+_R;\mathbb{R}^{n\times\binom{n}{k}})$, then we also have $D^2u \in \mathcal{L}^{2,\mu}\left(B^+_{R/2};\mathbb{R}^{n^2\times\binom{n}{k}}\right)$, with the estimate

$$\left[D^2u\right]_{\mathcal{L}^{2,\mu}\left(B^+_{R/2}\right)} \leq c\left\{\|Du\|_{L^2} + [\mathcal{P}]_{\mathcal{L}^{2,\mu}(B^+_R)} + [\mathcal{Q}]_{\mathcal{L}^{2,\mu+2}(B^+_R)}\right\}.$$

Remark 3 The same estimates hold as well if A satisfy the Legendre condition and $u \in W^{1,2}_{N,flat}(B^+_R;\Lambda^k)$ satisfy (13) for all $\psi \in W^{1,2}_{N,flat}(B^+_R;\Lambda^k)$.

Proof We just show the estimate of

$$\sup_{0<\rho<\delta R/2} \frac{1}{\rho^\mu}\int_{B^+_\rho} |Du - (Du)_\rho|^2,$$

with $0 < \delta < \frac{1}{2}$ fixed. We write $u = v + w$ where $v \in W^{1,2}(B^+_R;\Lambda^k)$ solves

$$\begin{cases} \delta(Adv) + d\delta v = 0 & \text{in } B^+_R, \\ v = u & \text{on } \partial B^+_R. \end{cases}$$

We have, by standard arguments using theorem 3 for v,

$$\int_{B^+_\rho} |Du - (Du)_\rho|^2 \leq c\left(\frac{\rho}{R}\right)^{n+2}\int_{B^+_R} |Du - (Du)_R|^2 + c_1\int_{B^+_R} |\nabla w|^2. \tag{14}$$

Note that w satisfies

$$\int_{B^+_R} \langle A(dw); dw\rangle + \int_{B^+_R} \langle \delta w; \delta w\rangle + \int_{B^+_R} \langle \mathcal{P}; w\rangle - \int_{B^+_R} \langle \mathcal{Q}; \nabla w\rangle = 0.$$

Using Hölder inequality, Poincaré inequality and Young's inequality with $\varepsilon > 0$,

$$\left|\int_{B_R^+}\langle\mathcal{P};w\rangle\right| \leq \left(\int_{B_R^+}|\mathcal{P}|^2\right)^{\frac{1}{2}}\left(cR^2\int_{B_R^+}|\nabla w|^2\right)^{\frac{1}{2}} \leq \varepsilon\int_{B_R^+}|\nabla w|^2 + c[\mathcal{P}]^2_{\mathrm{L}^{2,\mu-2}}R^\mu.$$

Thus, using Young's inequality with $\varepsilon > 0$ and $w = 0$ on all of ∂B_R^+, we deduce,

$$\left|\int_{B_R^+}\langle\mathcal{Q};\nabla w\rangle\right| = \left|\int_{B_R^+}\langle\mathcal{Q}-(\mathcal{Q})_R\,;\nabla w\rangle\right| \leq \varepsilon\int_{B_R^+}|\nabla w|^2 + c\,[\mathcal{Q}]^2_{\mathcal{L}^{2,\mu}(B_R^+)}\,R^\mu.$$

Now, by Gaffney and Gårding inequality, we have,

$$\int_{B_R^+}|\nabla w|^2 \leq c\left\{\int_{B_R^+}|dw|^2+\int_{B_R^+}|\delta w|^2\right\} \leq c\left\{\int_{B_R^+}\langle A(dw);dw\rangle+\int_{B_R^+}|\delta w|^2\right\}.$$

Combining this with the above estimates and choosing ε to be small enough, we obtain an estimate for $\int_{B_R^+}|\nabla w|^2$. Combining with (14) gives

$$\int_{B_\rho^+}|Du-(Du)_\rho|^2 \leq c\left(\frac{\rho}{R}\right)^{n+2}\int_{B_R^+}|Du-(Du)_R|^2 + cR^\mu\left([\mathcal{P}]^2_{\mathrm{L}^{2,\mu-2}}+[\mathcal{Q}]^2_{\mathcal{L}^{2,\mu}}\right).$$

By usual arguments utilizing the scaling lemma (see lemma 5.13 in [13]),

$$\frac{1}{\rho^\mu}\int_{B_\rho^+}|Du-(Du)_\rho|^2 \leq c\left(\|Du\|^2_{L^2(B_+^R)}+[\mathcal{P}]^2_{\mathrm{L}^{2,\mu-2}}+[\mathcal{Q}]^2_{\mathcal{L}^{2,\mu}}\right).$$

This yields the desired estimate. □

3.2.5 L^p estimates

Theorem 5 (L^p estimates) *Let $1 < p < \infty$ and $1 < q < \infty$ be such that $q^* = \frac{nq}{n-q} \geq p$ if $q < n$. Let $A : \Lambda^{k+1} \to \Lambda^{k+1}$ satisfy the Legendre-Hadamard condition. Let $u \in W^{1,2}_{T,flat}(B_R^+;\Lambda^k)$ satisfy, for all $\psi \in W^{1,2}_{T,flat}(B_R^+;\Lambda^k) \cap W^{1,p'}(B_R^+;\Lambda^k)$,*

$$\int_{B_R^+}\langle A(du);d\psi\rangle+\int_{B_R^+}\langle\delta u;\delta\psi\rangle+\int_{B_R^+}\langle\mathcal{P};\psi\rangle-\int_{B_R^+}\langle\mathcal{Q};\nabla\psi\rangle = 0. \qquad (15)$$

If $\mathcal{P} \in L^q(B_R^+; \Lambda^k)$ *and* $\mathcal{Q} \in L^p(B_R^+; \mathbb{R}^{n\times\binom{n}{k}})$ *then* $Du \in L^p\left(B_R^+; \mathbb{R}^{n\times\binom{n}{k}}\right)$, *and we have the estimate*

$$\|Du\|_{L^p\left(B_R^+\right)} \leq c\left\{\|\mathcal{P}\|_{L^q(B_r^+)} + \|\mathcal{Q}\|_{L^p(B_r^+)}\right\}.$$

Moreover, if $\mathcal{P} \in L^p(B_R^+; \Lambda^k)$ *and* $\mathcal{Q} \in W^{1,p}(B_r^+; \mathbb{R}^{n\times\binom{n}{k}})$, *then we also have* $D^2u \in L^p\left(B_{r/2}^+; \mathbb{R}^{n^2\times\binom{n}{k}}\right)$ *with the estimate*

$$\left\|D^2u\right\|_{L^p\left(B_R^+\right)} \leq c\left\{\|\mathcal{P}\|_{L^p(B_r^+)} + \|\mathcal{Q}\|_{W^{1,p}(B_r^+)}\right\}.$$

Proof The only part which is different from standard methods is the argument to show that $\mathcal{P}$ can be absorbed in the $\mathcal{Q}$ term. So we show that we can assume $\mathcal{P} = 0$.

For every $I \in \mathcal{T}^k$ such that $n \in I$, we solve the following Neumann boundary value problem for the scalar Laplacian,

$$\begin{cases} -\Delta\theta_I = \mathcal{P}_I & \text{in } B_R^+, \\ \dfrac{\partial\theta_I}{\partial\nu} = g_I & \text{in } \partial B_R^+, \end{cases}$$

where the functions g_Is are such that

$$\int_{\partial B_R^+} g = \int_{B_R^+} \mathcal{P}_I \qquad \text{and} \qquad g_I = 0 \text{ on } \Gamma_R^+.$$

For every $I \in \mathcal{T}^k$ such that $n \notin I$, we solve the following Dirichlet boundary value problem for the scalar Laplacian,

$$\begin{cases} -\Delta\theta_I = \mathcal{P}_I & \text{in } B_R^+, \\ \theta_I = 0 & \text{in } \partial B_R^+. \end{cases}$$

Since $1 < q < \infty$, by regularity results for the scalar Laplacian, we deduce that $\theta_I \in W^{2,q}\left(B_R^+\right)$ along with the estimate

$$\|\theta_I\|_{W^{2,q}} \leq \|\mathcal{P}_I\|_{L^q}.$$

Now, using integrating by parts, we can rewrite the term $\int_{B_R^+}\langle\mathcal{P};\psi\rangle$ as,

$$\begin{aligned} \int_{B_R^+}\langle\mathcal{P};\psi\rangle &= \int_{B_R^+}\sum_{I\in\mathcal{T}^k}\mathcal{P}_I\psi_I = -\int_{B_R^+}\sum_{I\in\mathcal{T}^k}\Delta\theta_I\psi_I \\ &= \int_{B_R^+}\sum_{I\in\mathcal{T}^k}\langle\nabla\theta_I;\nabla\psi_I\rangle - \int_{\partial B_R^+}\sum_{I\in\mathcal{T}^k}\frac{\partial\theta_I}{\partial\nu}\psi_I \\ &= \int_{B_R^+}\sum_{I\in\mathcal{T}^k}\langle\nabla\theta_I;\nabla\psi_I\rangle - \int_{\Gamma_R^+}\sum_{\substack{n\in I\\ I\in\mathcal{T}^k}}\frac{\partial\theta_I}{\partial\nu}\psi_I \\ &= \int_{B_R^+}\sum_{I\in\mathcal{T}^k}\langle\nabla\theta_I;\nabla\psi_I\rangle. \end{aligned}$$

But this means this term can be absorbed in the $\mathcal{Q}$ term, since $\nabla\theta_I \in L^p$ as a consequence of Sobolev embedding and the fact that $\theta_I \in W^{2,q}$ with $q^* \geq p$. The rest is standard. Using theorem 4, we prove that the linear map $\mathcal{Q} \mapsto T(\mathcal{Q}) = \nabla u$, can be extended as a bounded linear map from $L^2(B_R^+; \mathbb{R}^{n\times\binom{n}{k}})$ to $L^2(B_R^+; \mathbb{R}^{n\times\binom{n}{k}})$ and from $L^\infty(B_R^+; \mathbb{R}^{n\times\binom{n}{k}})$ to $BMO(B_R^+; \mathbb{R}^{n\times\binom{n}{k}})$. By Stampacchia's interpolation theorem, this implies that the map extends as a bounded linear operator from $L^p(B_R^+; \mathbb{R}^{n\times\binom{n}{k}})$ to $L^p(B_R^+; \mathbb{R}^{n\times\binom{n}{k}})$, for all $2 \leq p < \infty$. The case $1 < p < 2$ follows by usual duality arguments. □

3.3 Schauder and L^p estimates

We are now ready to derive the global estimates. We use lemma 4 to flatten the boundary, whose proof has been relegated to the appendix. By usual patching arguments, everything boils down to proving the estimates for $u \in W_T^{1,2}(B_R^+; \Lambda^k)$ satisfying, for all $\psi \in W_T^{1,2}(B_R^+; \Lambda^k)$,

$$\int_{B_R^+} \langle \bar{A}(du); d\psi\rangle + \int_{B_R^+} \langle \delta u; \delta\psi\rangle + \int_{B_R^+} \langle \mathcal{P}; \psi\rangle - \int_{B_R^+} \langle \mathcal{Q}; \nabla\psi\rangle + \int_{B_R^+} \langle S\nabla u; \nabla\psi\rangle = 0,$$

where $\bar{A}$ satisfies the Legendre-Hadamard condition, $\mathcal{P} = \widetilde{f} + Pu + R\nabla u$ and $\mathcal{Q} = \widetilde{F} - Qu$, for $R > 0$ suitably small.

Theorem 6 ($C^{r+2,\gamma}$ regularity) *Let $1 \leq k \leq n-1$, $r \geq 0$ be integers and let $0 < \gamma < 1$ be a real number. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $\partial\Omega$ be of class $C^{r+2,\gamma}$. Let $A \in C^{r,\gamma}(\overline{\Omega}; L(\Lambda^{k+1}, \Lambda^{k+1}))$ satisfy either the Legendre-Hadamard or the Legendre condition. Also let $f \in C^{r,\gamma}(\overline{\Omega}, \Lambda^k)$, $F \in C^{r,\gamma}(\overline{\Omega}, \Lambda^{k+1})$ and $\lambda \in \mathbb{R}$. Let $\omega \in W_T^{1,2}(\Omega, \Lambda^k)$ be a weak solution of the following,*

$$\int_\Omega \langle A(x)d\omega, d\phi\rangle + \int_\Omega \langle \delta\omega, \delta\phi\rangle + \lambda\int_\Omega \langle \omega, \phi\rangle + \int_\Omega \langle f, \phi\rangle - \int_\Omega \langle F, d\phi\rangle = 0, \quad (16)$$

for all $\phi \in W_T^{1,2}(\Omega, \Lambda^k)$. Then $\omega \in C^{r+1,\gamma}(\overline{\Omega}, \Lambda^k)$, and satisfies the estimate,

$$\|\omega\|_{C^{r+1,\gamma}(\overline{\Omega};\Lambda^k)} \leq c\left\{\|\omega\|_{C^{0,\gamma}(\overline{\Omega};\Lambda^k)} + \|f\|_{C^{r,\gamma}(\overline{\Omega};\Lambda^k)} + \|F\|_{C^{r,\gamma}(\overline{\Omega};\Lambda^{k+1})}\right\},$$

where the constant $c > 0$ depends only on A, λ, γ_0 and Ω. Moreover, if $A \in C^{r+1,\gamma}(\overline{\Omega}; L(\Lambda^{k+1}, \Lambda^{k+1}))$ and $F \in C^{r+1,\gamma}(\overline{\Omega}, \Lambda^{k+1})$, then $\omega \in C^{r+2,\gamma}(\overline{\Omega}, \Lambda^k)$, and satisfies the estimate,

$$\|\omega\|_{C^{r+2,\gamma}(\overline{\Omega};\Lambda^k)} \leq c\left\{\|\omega\|_{C^{0,\gamma}(\overline{\Omega};\Lambda^k)} + \|f\|_{C^{r,\gamma}(\overline{\Omega};\Lambda^k)} + \|F\|_{C^{r+1,\gamma}(\overline{\Omega};\Lambda^{k+1})}\right\},$$

where the constant $c > 0$ depends, once again, only on A, λ, γ_0 and Ω.

Proof Writing $u = v + w$, where $v \in W^{1,2}(B_R^+; \Lambda^k)$ is the solution of the homogeneous equation for $\bar{A}$ coinciding with u on the boundary and noting that

$$\left| \int_{B_R^+} \langle S\nabla u; \nabla w \rangle \right| \leq \varepsilon \int_{B_R^+} |\nabla w|^2 + c\left(\mathcal{C}_A^2(R) + \mathcal{C}_B^2(R) + \mathcal{C}_\Phi^2(R)\right) \int_{B_R^+} |\nabla u|^2,$$

the Schauder estimates follow from theorem 4 by standard techniques (see [13]) after absorbing the lower order terms by interpolation. □

Theorem 7 ($W^{r+2,p}$ regularity) *Let $1 \leq k \leq n-1$, $r \geq 0$ be integers and let $1 < p < \infty$ and $1 < q < \infty$ be real numbers such that $q^* = \frac{nq}{n-q} \geq p$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $\partial\Omega$ be of class $C^{r+2,\gamma}$. Let $A \in C^r(\overline{\Omega}; L(\Lambda^{k+1}, \Lambda^{k+1}))$ satisfy either the Legendre-Hadamard or the Legendre condition. Also let $f \in W^{r,q}(\Omega, \Lambda^k)$, $F \in W^{r,p}(\Omega, \Lambda^{k+1})$ and $\lambda \in \mathbb{R}$. Let $\omega \in W_T^{1,2}(\Omega, \Lambda^k)$ be a weak solution of the following,*

$$\int_\Omega \langle A(x)d\omega, d\phi \rangle + \int_\Omega \langle \delta\omega, \delta\phi \rangle + \lambda \int_\Omega \langle \omega, \phi \rangle + \int_\Omega \langle f, \phi \rangle - \int_\Omega \langle F, d\phi \rangle = 0, \quad (17)$$

for all $\phi \in W_T^{1,2}(\Omega, \Lambda^k) \cap W^{1,p'}(\Omega, \Lambda^k)$. Then $\omega \in W^{r+1,p}(\Omega, \Lambda^k)$, and satisfies the estimate,

$$\|\omega\|_{W^{r+1,p}(\Omega;\Lambda^k)} \leq c\left\{\|\omega\|_{L^p(\Omega;\Lambda^k)} + \|f\|_{W^{r,q}(\Omega;\Lambda^k)} + \|F\|_{W^{r,p}(\Omega;\Lambda^{k+1})}\right\},$$

where the constant $c > 0$ depends only on A, λ, γ_0 and Ω. Moreover, if $A \in C^{r+1}(\overline{\Omega}; L(\Lambda^{k+1}, \Lambda^{k+1}))$, $f \in W^{r,p}(\Omega, \Lambda^k)$ and $F \in W^{r+1,p}(\Omega, \Lambda^{k+1})$, then $\omega \in W^{r+2,p}(\Omega, \Lambda^k)$, and satisfies the estimate,

$$\|\omega\|_{W^{r+2,p}(\Omega;\Lambda^k)} \leq c\left\{\|\omega\|_{L^p(\Omega;\Lambda^k)} + \|f\|_{W^{r,p}(\Omega;\Lambda^k)} + \|F\|_{W^{r+1,p}(\Omega;\Lambda^{k+1})}\right\},$$

where the constant $c > 0$ depends, once again, only on A, λ, γ_0 and Ω.

Proof Theorem 5 yields the theorem for $2 \leq p < \infty$ coupled with standard arguments. The case $1 < p < 2$ follows by duality arguments. The crucial step is to show that if $u \in W^{1,m}$, for some $2 \leq m \leq p$, then for every $V \in W_{T,flat}^{1,s}$, there is a unique solution $v \in W_{T,flat}^{1,s}$ satisfying, for every $\psi \in W_{T,flat}^{1,2}(B_R^+; \Lambda^k) \cap W^{1,p'}(B_R^+; \Lambda^k)$,

$$\int_{B_R^+} \langle \bar{A}(dv); d\psi \rangle + \int_{B_R^+} \langle \delta v; \delta\psi \rangle + \int_{B_R^+} \langle \widetilde{f} + Pu + R\nabla u; \psi \rangle$$
$$- \int_{B_R^+} \langle \widetilde{F} - Qu; \nabla\psi \rangle + \int_{B_R^+} \langle S\nabla V; \nabla\psi \rangle = 0,$$

where $s = \min(p, m^*)$. Then we show that the map $T : W_{T,flat}^{1,s} \to W_{T,flat}^{1,s}$ that maps V to v has a fixed point if R is chosen small enough. But by Poincaré inequality and theorem 5,

$$\|v_1 - v_2\|_{W^{1,s}} \leq c\left[\mathcal{C}_A(R) + \mathcal{C}_B(R) + \mathcal{C}_\Phi(R)\right] \|\nabla V_1 - \nabla V_2\|_{L^s}.$$

This means for R small enough, T is contraction and thus has an unique fixed point in $W_{T,flat}^{1,s}$. Lower order terms can again be absorbed by interpolation.

□

4 Applications

Now we show how the regularity results derived earlier can be utilized to deduce a number of results concerning a wide variety of problems. Although we state the theorems for Hessian estimates, gradient estimate versions of all the theorems in the subsections 4.1. 4.2, 4.3 and 4.4 hold as well.

4.1 Second order Hodge type systems: A special case

First such problem is the Hodge-type boundary value problem for a second order elliptic system, the prototype being the Poisson problem for the Hodge Laplacian with absolute and relative boundary conditions.

Theorem 8 *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and let $\partial\Omega$ be of class $C^{r+2,\gamma}$. Let $1 \leq k \leq n-1$, $r \geq 0$ be integers and $0 < \alpha \leq \gamma < 1$ and $1 < p < \infty$ be real numbers. Let $A \in C^{r+1}(\overline{\Omega}; L(\Lambda^{k+1}, \Lambda^{k+1}))$, respectively $C^{r+1,\alpha}(\overline{\Omega}; L(\Lambda^{k+1}, \Lambda^{k+1}))$, satisfy the Legendre-Hadamard condition. Then the following holds.*

1. *There exists a constant $\rho \in \mathbb{R}$ and an at most countable set $\sigma \subset (-\infty, \rho)$, with no limit points except possibly $-\infty$.*
2. *The following boundary value problem,*

$$\begin{cases} \delta(A(x)d\alpha) + d\delta\alpha = \sigma_i\alpha \text{ in } \Omega, \\ \nu \wedge \alpha = 0 \text{ on } \partial\Omega, \\ \nu \wedge \delta\alpha = 0 \text{ on } \partial\Omega. \end{cases} \tag{EP}$$

   admits non-trivial solutions $\alpha \in C^\infty(\overline{\Omega}, \Lambda^k)$ if and only if $\sigma_i \in \sigma$ and the space of solutions to (EP) *is finite-dimensional for any $\sigma_i \in \sigma$.*
3. *If $\lambda \notin \sigma$, then for any $f \in W^{r,p}(\Omega, \Lambda^k)$, respectively $C^{r,\alpha}(\overline{\Omega}, \Lambda^k)$, and any $\omega_0 \in W^{r+2,p}(\Omega, \Lambda^k)$, respectively $C^{r+2,\alpha}(\overline{\Omega}, \Lambda^k)$, there exists a unique solution $\omega \in W^{r+2,p}(\Omega, \Lambda^k)$, respectively $C^{r+2,\alpha}(\overline{\Omega}, \Lambda^k)$, to the following boundary value problem:*

$$\begin{cases} \delta(A(x)d\omega) + d\delta\omega = \lambda\omega + f \text{ in } \Omega, \\ \nu \wedge \omega = \nu \wedge \omega_0 \text{ on } \partial\Omega. \\ \nu \wedge \delta\omega = \nu \wedge \delta\omega_0 \text{ on } \partial\Omega, \end{cases} \tag{P}$$

   which satisfies the estimate

$$\|\omega\|_{W^{r+2,p}} \leq c\left(\|\omega\|_{W^{r,p}} + \|f\|_{W^{r,p}} + \|\omega_0\|_{W^{r+2,p}}\right),$$

   respectively,

$$\|\omega\|_{C^{r+2,\alpha}} \leq c\left(\|\omega\|_{C^{r,\alpha}} + \|f\|_{C^{r,\alpha}} + \|\omega_0\|_{C^{r+2,\alpha}}\right).$$

Proof We divide the proof in two steps.

Step 1 (Existence in L^2): We first show existence assuming $f \in L^p$ with $p \geq 2$. Clearly, this is not a restriction if $f \in C^{0,\alpha}$. For a given $\lambda \in \mathbb{R}$, the bilinear operator $a_\lambda : W^{1,2}_T(\Omega; \Lambda^k) \times W^{1,2}_T(\Omega; \Lambda^k) \to \mathbb{R}$ defined by,

$$a_\lambda(u, v) = \int_\Omega \langle A(x)du, dv\rangle + \int_\Omega \langle \delta u, \delta v\rangle + \lambda \int_\Omega \langle u, v\rangle,$$

is continuous and coercive by virtue of Gaffney and Gårding inequality, for any λ large enough. Thus Lax-Milgram theorem yields existence for λ large. Since $W^{1,2}_T(\Omega; \Lambda^k)$ embeds compactly in $L^2(\Omega; \Lambda^k)$, Fredholm theory holds and we conclude the existence of the spectrum σ, the claimed properties of the spectrum, the existence of $\overline{\omega} \in W^{1,2}_T(\Omega; \Lambda^k)$ solving

$$\int_\Omega \langle A(x)d\overline{\omega}, d\theta\rangle + \langle \delta\overline{\omega}, \delta\theta\rangle + \lambda \int_\Omega \langle \overline{\omega}, \theta\rangle + \int_\Omega \langle g, \theta\rangle = 0, \tag{18}$$

for any $g \in L^2(\Omega, \Lambda^k)$ when $\lambda \notin \sigma$ and the existence of $\alpha \in W^{1,2}_T(\Omega; \Lambda^k)$ solving

$$\int_\Omega \langle A(x)d\alpha, d\theta\rangle + \langle \delta\alpha, \delta\theta\rangle + \sigma_i \int_\Omega \langle \alpha, \theta\rangle = 0, \tag{19}$$

for every $\sigma_i \in \sigma$, along with the finite-dimensionality of the eigenspaces.

Step 2 (Regularity): Now theorem 7, respectively theorem 6 in the Hölder case, gives us the desired regularity. The estimates for $1 < p < 2$ also extends the existence theory to the case $1 < p < 2$ by usual density arguments. Integrating by parts, we immediately obtain from (19) that α is a C^∞ solution to (EP), i.e an eigenfunction. From (18), integrating by parts, we obtain,

$$\int_\Omega \langle \delta(A(x)d\overline{\omega}) + d\delta\overline{\omega}; \phi\rangle - \int_{\partial\Omega} (\langle A(x)d\overline{\omega}; \nu \wedge \phi\rangle + \langle \nu \wedge \delta\omega; \phi\rangle) = \int_\Omega \langle \lambda\omega + g; \phi\rangle,$$

for all $\phi \in W^{1,2}_T(\Omega, \Lambda^k)$. Thus taking $\phi \in C^\infty_c(\Omega, \Lambda^k)$ we have,

$$\delta(A(x)d\overline{\omega}) + d\delta\overline{\omega} = \lambda\overline{\omega} + g \qquad \text{in } \Omega.$$

But this implies that the integral on the boundary vanish separately and thus,

$$\int_{\partial\Omega} \langle \nu \wedge \delta\overline{\omega}; \phi\rangle = 0 \qquad \text{for any } \phi \in W^{1,2}_T(\Omega; \Lambda^k).$$

We now show that this identity is valid for any $u \in W^{1,2}(\Omega; \Lambda^k)$ as well. Extending ν as a C^1 function inside Ω and using the identity $u = \nu \wedge (\nu \lrcorner u) + \nu \lrcorner (\nu \wedge u)$, we deduce, for any $u \in W^{1,2}(\Omega; \Lambda^k)$,

$$\int_{\partial\Omega} \langle \nu \wedge \delta\overline{\omega}; u\rangle = \int_{\partial\Omega} \langle \nu \wedge \delta\overline{\omega}; \nu \wedge (\nu \lrcorner u)\rangle = 0,$$

since $\nu \wedge (\nu \lrcorner u) \in W^{1,2}_T(\Omega, \Lambda^k)$. Since $u \in W^{1,2}(\Omega; \Lambda^k)$ is arbitrary, this implies $\nu \wedge \delta\overline{\omega} = 0$ on $\partial\Omega$. Now taking $g = f + \lambda\omega_0 - \delta(A(x)d\omega_0) - d\delta\omega_0$ and setting $\omega = \overline{\omega} + \omega_0$, we immediately see that ω is a solution to (P) with the desired regularity. This finishes the proof. □

Remark 4 The possibility of solving (P) when $\lambda = 0$ is clearly the most important case. However, if there are non-trivial harmonic fields with vanishing tangential component, i.e $\mathcal{H}_T\left(\Omega;\Lambda^k\right) \neq \{0\}$, then each non-trivial $h \in \mathcal{H}_T\left(\Omega;\Lambda^k\right)$ is a non-trivial solution for (EP) with $\sigma_i = 0$. Thus $0 \in \sigma$ and (P) can not be solved for arbitrary f. However, when A is such that we have $\int_\Omega \langle A(x)du, du\rangle \geq c\int_\Omega |du|^2$ for some $c > 0$, (for example, when A is constant or satisfies the Legendre condition) it can be shown that every non-trivial solution of (EP) is a harmonic field. Consequently, (P) can be solved for all $f \in \left(\mathcal{H}_T\left(\Omega;\Lambda^k\right)\right)^\perp$, i.e all f satisfying the condition

$$\int_\Omega \langle f; h\rangle = 0 \quad \text{for every } h \in \mathcal{H}_T\left(\Omega;\Lambda^k\right).$$

However, the solution ω in that case would be unique only upto such harmonic fields. Since Harmonic fields are smooth, the regularity results would remain valid for all such solutions.

Similar arguments on $W^{1,2}_N$ for existence and the regularity estimates yield analogous statements for the system

$$\begin{cases} \delta(A(x)d\omega) + d\delta\omega = \lambda\omega + f \text{ in } \Omega, \\ \nu \lrcorner \omega = \nu \lrcorner \omega_0 \text{ on } \partial\Omega. \\ \nu \lrcorner (A(x)d\omega) = \nu \lrcorner (A(x)d\omega_0) \text{ on } \partial\Omega, \end{cases} \tag{P}$$

when A satisfies the Legendre condition. Remarks similar to remark 4 about the solvability of (P) with $\lambda = 0$ applies here as well, with the modification that since A is assumed to satisfy the Legendre condition, (P) can be solved with $\lambda = 0$ for all $f \in \left(\mathcal{H}_N\left(\Omega;\Lambda^k\right)\right)^\perp$. As a corollary, these theorems imply the Hodge decomposition theorem (see e.g Theorem 6.21 in [6] for the different versions of the theorem).

4.2 Maxwell operator: A simple case

Theorem 9 *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and let $\partial\Omega$ be of class $C^{r+2,\gamma}$. Let $1 \leq k \leq n-1$, $r \geq 0$ be integers and $0 < \alpha \leq \gamma < 1$ and $1 < p < \infty$ be real numbers. Let $A \in C^{r+1}(\overline{\Omega}; L(\Lambda^{k+1}, \Lambda^{k+1}))$, respectively $C^{r+1,\alpha}(\overline{\Omega}; L(\Lambda^{k+1}, \Lambda^{k+1}))$, satisfy the Legendre-Hadamard condition. Then the following holds.*

1. *There exists a constant $\rho \in \mathbb{R}$ and an at most countable set $\sigma \subset (-\infty, \rho)$, with no limit points except possibly $-\infty$.*
2. *The following boundary value problem,*

$$\begin{cases} \delta(A(x)d\alpha) = \sigma_i\alpha \text{ in } \Omega, \\ \delta\alpha = 0 \text{ in } \Omega, \\ \nu \wedge \alpha = 0 \text{ on } \partial\Omega, \end{cases} \tag{EVM}$$

admits non-trivial solutions $\alpha \in C^\infty(\overline{\Omega}, \Lambda^k)$ if and only if $\sigma_i \in \sigma$ and the space of solutions to (EVM) *is finite-dimensional for any $\sigma_i \in \sigma$.*

3. *If $\lambda \notin \sigma$, then for any $f \in W^{r,p}(\Omega, \Lambda^k)$, respectively $C^{r,\alpha}(\overline{\Omega}, \Lambda^k)$, satisfying $\delta f = 0$ in the sense of distributions in Ω, there exists a unique solution $\omega \in W^{r+2,p}(\Omega, \Lambda^k)$, respectively $C^{r+2,\alpha}(\overline{\Omega}, \Lambda^k)$, to the following boundary value problem:*

$$\begin{cases} \delta(A(x)d\omega) = \lambda\omega + f \text{ in } \Omega, \\ \delta\omega = 0 \text{ in } \Omega, \\ \nu \wedge \omega = 0 \text{ on } \partial\Omega. \end{cases} \tag{PM}$$

which satisfies the estimate

$$\|\omega\|_{W^{r+2,p}} \leq c\left(\|\omega\|_{W^{r,p}} + \|f\|_{W^{r,p}}\right),$$

respectively,

$$\|\omega\|_{C^{r+2,\alpha}} \leq c\left(\|\omega\|_{C^{r,\alpha}} + \|f\|_{C^{r,\alpha}}\right).$$

Proof **(of theorem** (9)**)** By theorem 8, we only need to show is that if $\delta f = 0$, then every solution to (P) with $\lambda \in \mathbb{R}$ gives rise to a solution of (PM) with the same λ and that every solution of (EP) for $\sigma_i \in \mathbb{R}$ yields a solution of (EVM) with the same σ_i. We divide the proof into several substeps.

Step 1 We start with the second. Taking δ of the equation, we deduce from (EP),

$$(\delta d + d\delta)(\delta\alpha) = \delta d\delta\alpha = \delta\delta(A(x)d\alpha) + \delta d\delta\alpha = \delta(\sigma_i\alpha) = \sigma_i(\delta\alpha) \qquad \text{in } \Omega,$$

and $\nu \wedge \delta\alpha = 0$ and $\nu \wedge \delta\delta\alpha = 0$ on $\partial\Omega$. Thus, σ_i is an eigenvalue for the Hodge Laplacian on $(k-1)$-forms with $\delta\alpha$ as an eigenform. Now if $\sigma_i = 0$, then we see that $\delta\alpha = h \in \mathcal{H}_T\left(\Omega; \Lambda^{k-1}\right)$. But no non-zero harmonic field can be coexact. Thus $\delta\alpha = 0$ and α itself solves (EVM). If $\sigma_i \neq 0$, then it is easy to check that $\bar{\alpha} = \alpha - \frac{1}{\sigma_i} d\delta\alpha$ solves (EVM).

Step 2 The first one is exactly the same. Once again, taking δ of (P) and using the fact that $\delta f = 0$, we see similarly that if $\lambda = 0$, ω itself and if $\lambda \neq 0$, $\bar{\omega} = \omega - \frac{1}{\lambda} d\delta\omega$ solves (PM). □

Remark 5 In this case too, remarks similar to remark 4 applies.

By similar arguments, we also have

Theorem 10 *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and let $\partial\Omega$ be of class $C^{r+2,\gamma}$. Let $1 \leq k \leq n-1$, $r \geq 0$ be integers and $0 < \alpha \leq \gamma < 1$ and $1 < p < \infty$ be real numbers. Let $A \in C^{r+1}(\overline{\Omega}; L(\Lambda^{k+1}, \Lambda^{k+1}))$, respectively $C^{r+1,\alpha}(\overline{\Omega}; L(\Lambda^{k+1}, \Lambda^{k+1}))$, satisfy the Legendre condition. Then the following holds.*

1. *There exists a constant $\rho \in \mathbb{R}$ and an at most countable set $\sigma \subset (-\infty, \rho)$, with no limit points except possibly $-\infty$.*

2. *The following boundary value problem,*

$$\begin{cases} \delta(A(x)d\alpha) = \sigma_i \alpha \text{ in } \Omega, \\ \delta\alpha = 0 \text{ in } \Omega, \\ \nu \lrcorner \alpha = 0 \text{ on } \partial\Omega, \\ \nu \lrcorner (Ad\alpha) = 0 \text{ on } \partial\Omega, \end{cases} \tag{EVM}$$

admits non-trivial solutions $\alpha \in C^\infty(\overline{\Omega}, \Lambda^k)$ if and only if $\sigma_i \in \sigma$ and the space of solutions to (EVM) *is finite-dimensional for any $\sigma_i \in \sigma$.*

3. *If $\lambda \notin \sigma$, then for any $f \in W^{r,p}(\Omega, \Lambda^k)$, respectively $C^{r,\alpha}(\overline{\Omega}, \Lambda^k)$, satisfying $\delta f = 0$ in the sense of distributions in Ω and $\nu \lrcorner f = 0$ on $\partial\Omega$, there exists a solution $\omega \in W^{r+2,p}(\Omega, \Lambda^k)$, respectively $C^{r+2,\alpha}(\overline{\Omega}, \Lambda^k)$, unique upto harmonic fields, to the following boundary value problem:*

$$\begin{cases} \delta(A(x)d\omega) = \lambda\omega + f \text{ in } \Omega, \\ \delta\omega = 0 \text{ in } \Omega, \\ \nu \lrcorner \omega = 0 \text{ on } \partial\Omega, \\ \nu \lrcorner (Ad\omega) = 0 \text{ on } \partial\Omega. \end{cases} \tag{PM}$$

which satisfies the estimate

$$\|\omega\|_{W^{r+2,p}} \leq c \left\{ \|\omega\|_{W^{r,p}} + \|f\|_{W^{r,p}} \right\},$$

respectively,

$$\|\omega\|_{C^{r+2,\alpha}} \leq c \left\{ \|\omega\|_{C^{r,\alpha}} + \|f\|_{C^{r,\alpha}} \right\}.$$

Remark 6 (i) When $r = 0$, the condition $\nu \lrcorner f = 0$ on $\partial\Omega$ in the Sobolev case is to be interpreted as

$$\int_\Omega \langle f; d\psi \rangle = 0 \qquad \text{for any } \psi \in C^\infty\left(\overline{\Omega}; \Lambda^{k-1}\right).$$

Note that when ψ is compactly supported, the integral vanishes anyway due to the condition $\delta f = 0$. This extra condition, which can be easily seen to be necessary one, is analogous the integral compatibility condition for the Neumann boundary value problem for the Laplacian in the scalar case.

(ii) Remarks similar to Remark 5 applies here as well.

4.3 Stokes type operator

An easy corollary of the results in the last subsection gives us regularity results for some stationary Stokes type problem. We just use Hodge decomposition with vanishing tangential component to write the decomposition $f = d\phi + \tilde{f}$, with $\delta\tilde{f} = 0$ in Ω. Then we solve the boundary value problem

$$dp = -d\phi \text{ in } \Omega, \qquad p = p_0 \text{ on } \partial\Omega.$$

This reduces the next theorem to Theorem 9.

Theorem 11 *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and let $\partial\Omega$ be of class $C^{r+2,\gamma}$. Let $1 \leq k \leq n-1$, $r \geq 0$ be integers and $0 < \alpha \leq \gamma < 1$ and $1 < p < \infty$ be real numbers. Let $A \in C^{r+1}(\overline{\Omega}; L(\Lambda^{k+1}, \Lambda^{k+1}))$, respectively $C^{r+1,\alpha}(\overline{\Omega}; L(\Lambda^{k+1}, \Lambda^{k+1}))$, satisfy the Legendre-Hadamard condition. Let σ be the spectrum, as given by theorem 9. Then for any $\lambda \notin \sigma$ and for any $f \in W^{r,p}(\Omega, \Lambda^k)$, respectively $C^{r,\alpha}(\overline{\Omega}, \Lambda^k)$, any $p_0 \in W^{r+1,p}(\Omega, \Lambda^{k-1})$, respectively $C^{r+1,\alpha}(\overline{\Omega}, \Lambda^{k-1})$, with $\nu \wedge dp_0 = 0$ on $\partial\Omega$, there exists a unique pair (ω, p) such that $\omega \in W^{r+2,p}(\Omega, \Lambda^k)$, respectively $C^{r+2,\alpha}(\overline{\Omega}, \Lambda^k)$, $p \in W^{r+1,p}(\Omega, \Lambda^{k-1})$, respectively $C^{r+1,\alpha}(\overline{\Omega}, \Lambda^{k-1})$, solve the following boundary value problem:*

$$\begin{cases} \delta(A(x)d\omega) + dp = \lambda\omega + f \text{ in } \Omega, \\ \delta\omega = 0 \text{ in } \Omega, \\ \nu \wedge \omega = 0 \text{ on } \partial\Omega \\ p = p_0 \text{ on } \partial\Omega. \end{cases} \tag{PST}$$

which satisfies the estimates

$$\|\omega\|_{W^{r+2,p}} \leq c\left(\|\omega\|_{W^{r,p}} + \|f\|_{W^{r,p}}\right)$$

and

$$\|p\|_{W^{r+1,p}} \leq c\left(\|f\|_{W^{r,p}} + \|p_0\|_{W^{r+1,p}}\right),$$

respectively,

$$\|\omega\|_{C^{r+2,\alpha}} \leq c\left(\|\omega\|_{C^{r,\alpha}} + \|f\|_{C^{r,\alpha}}\right)$$

and

$$\|p\|_{C^{r+1,\alpha}} \leq c\left(\|f\|_{C^{r,\alpha}} + \|p_0\|_{C^{r+1,\alpha}}\right).$$

By using Hodge decomposition with vanishing normal component to write the decomposition $f = -dp + \tilde{f}$, with $\delta\tilde{f} = 0$ in Ω and $\nu \lrcorner \tilde{f} = 0$ on $\partial\Omega$, we also have the following theorem.

Theorem 12 *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and let $\partial\Omega$ be of class $C^{r+2,\gamma}$. Let $1 \leq k \leq n-1$, $r \geq 0$ be integers and $0 < \alpha \leq \gamma < 1$ and $1 < p < \infty$ be real numbers. Let $A \in C^{r+1}(\overline{\Omega}; L(\Lambda^{k+1}, \Lambda^{k+1}))$, respectively $C^{r+1,\alpha}(\overline{\Omega}; L(\Lambda^{k+1}, \Lambda^{k+1}))$, satisfy the Legendre condition. Let σ be the spectrum, as given by theorem 10. Then for any $\lambda \notin \sigma$, for any $f \in W^{r,p}(\Omega, \Lambda^k)$, respectively $C^{r,\alpha}(\overline{\Omega}, \Lambda^k)$, there exists a unique pair (ω, p) such that $\omega \in W^{r+2,p}(\Omega, \Lambda^k)$, respectively $C^{r+2,\alpha}(\overline{\Omega}, \Lambda^k)$, $p \in W^{r+1,p}(\Omega, \Lambda^{k-1})$, respectively $C^{r+1,\alpha}(\overline{\Omega}, \Lambda^{k-1})$, solve the following boundary value problem:*

$$\begin{cases} \delta(A(x)d\omega) + dp = \lambda\omega + f \text{ in } \Omega, \\ \delta\omega = 0 \text{ in } \Omega, \\ \nu \lrcorner \omega = 0 \text{ on } \partial\Omega, \\ \nu \lrcorner (A(x)d\omega) = 0 \text{ on } \partial\Omega. \end{cases} \tag{PS}$$

and satisfies the estimate

$$\|\omega\|_{W^{r+2,p}} \leq c\,(\|\omega\|_{W^{r,p}} + \|f\|_{W^{r,p}}) \qquad \textit{and} \qquad \|p\|_{W^{r+1,p}} \leq c\,\|f\|_{W^{r,p}}\,,$$

respectively,

$$\|\omega\|_{C^{r+2,\alpha}} \leq c\,(\|\omega\|_{C^{r,\alpha}} + \|f\|_{C^{r,\alpha}}) \qquad \textit{and} \qquad \|p\|_{C^{r+1,\alpha}} \leq c\,\|f\|_{C^{r,\alpha}}\,.$$

Remark 7 For $k = 1$ and $n = 3$, the above two problems reduce to

$$\begin{gathered}\operatorname{curl}(A(x)\operatorname{curl} u) + \nabla p = \lambda u + f \text{ in } \Omega,\\ \operatorname{div} u = 0 \text{ in } \Omega,\end{gathered}$$

$$\text{and either} \quad \begin{matrix}\nu \times u = 0,\\ p = p_0,\end{matrix} \quad \text{or} \quad \begin{matrix}\nu \cdot u = 0,\\ \nu \times (A(x)\operatorname{curl} u) = 0,\end{matrix} \quad \text{on } \partial\Omega.$$

This is generally called the *vorticity-velocity-pressure formulation of the incompressible stationary Stokes problem.*

4.4 A non-elliptic Dirichlet problem

Theorem 13 *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and let $\partial\Omega$ be of class $C^{r+2,\gamma}$. Let $1 \leq k \leq n-1$, $r \geq 0$ be integers and $0 < \alpha \leq \gamma < 1$ and $1 < p < \infty$ be real numbers. Let $A \in C^{r+1}(\overline{\Omega}; L(\Lambda^{k+1}, \Lambda^{k+1}))$, respectively $C^{r+1,\alpha}(\overline{\Omega}; L(\Lambda^{k+1}, \Lambda^{k+1}))$, satisfy either the Legendre condition or satisfies the Legendre-Hadamard condition and has constant coefficients. Then for any $\omega_0 \in W^{r+2,p}\left(\Omega, \Lambda^k\right)$, respectively $C^{r+2,\alpha}(\overline{\Omega}, \Lambda^k)$, and for any $f \in W^{r,p}\left(\Omega, \Lambda^k\right) \cap \left(\mathcal{H}_T\left(\Omega, \Lambda^k\right)\right)^{\perp}$, respectively $C^{r,\alpha}(\overline{\Omega}, \Lambda^k) \cap \left(\mathcal{H}_T\left(\Omega, \Lambda^k\right)\right)^{\perp}$, such that $\delta f = 0$ in the sense of distributions, there exists a solution $\omega \in W^{r+p,2}\left(\Omega, \Lambda^k\right)$, respectively $C^{r+2,\alpha}(\overline{\Omega}, \Lambda^k)$, to the following boundary value problem,*

$$\begin{cases}\delta(A(x)d\omega) = f \text{ in } \Omega,\\ \omega = \omega_0 \text{ on } \partial\Omega,\end{cases} \tag{$\mathcal{P}_D$}$$

which satisfies the estimate

$$\|\omega\|_{W^{r+2,p}} \leq c\,(\|f\|_{W^{r,p}} + \|\omega_0\|_{W^{r+2,p}})\,,$$

respectively,

$$\|\omega\|_{C^{r+2,\alpha}} \leq c\,(\|f\|_{C^{r,\alpha}} + \|\omega_0\|_{C^{r+2,\alpha}})\,.$$

Remark 8 (i) Note that there is no uniqueness and the claimed estimate is not an a priori estimate. The conclusion of the theorem is that it is possible to find a solution which enjoys the claimed regularity and satisfies these estimates. All solutions of the equation need neither satisfy such estimates nor have the claimed regularity. Indeed, for any $\beta \in W_0^{1,2}\left(\Omega, \Lambda^{k-1}\right)$, adding $d\beta$ to any distributional solution of $(\mathcal{P}_D)$ yields another distributional solution which can be merely L^2.

(ii) A dual version of this theorem holds and can be easily obtained from this by Hodge duality.

Proof The hypothesis on A implies that by theorem PM, we can find an unique solution $\overline{\omega}$, with the expected regularity, to the system

$$\begin{cases} \delta(A(x)d\overline{\omega}) = f - \delta(A(x)d\omega_0) \text{ in } \Omega, \\ \delta\overline{\omega} = 0 \text{ in } \Omega, \\ \nu \wedge \overline{\omega} = 0 \text{ on } \partial\Omega, \end{cases}$$

Now since $\nu \wedge (-\overline{\omega}) = 0$ on $\partial\Omega$, we can find $v \in W^{r+3,2}(\Omega, \Lambda^{k-1})$ (see lemma 8.11 in [6]) such that $dv = -\overline{\omega}$ on $\partial\Omega$. Setting $\omega = \omega_0 + \overline{\omega} + dv$, we see that ω solves $(\mathcal{P}_D)$. □

4.5 First order div-curl type systems and Gaffney inequalities

Theorem 14 *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and smooth. Let ν be the outward unit normal to the boundary $\partial\Omega$. Let $1 \leq k \leq n-1$ and $r \geq 0$ be integers and $0 < \alpha \leq \gamma < 1$ and $1 < p < \infty$ be real numbers. Given two maps $A, B \in C^{r+1}(\overline{\Omega}; L(\Lambda^k, \Lambda^k))$, respectively $C^{r+1,\gamma}(\overline{\Omega}; L(\Lambda^k, \Lambda^k))$, satisfy the Legendre condition. Let $\omega_0 \in W^{r+1,p}(\Omega, \Lambda^k)$, respectively $C^{r+1,\alpha}(\overline{\Omega}; \Lambda^k)$, $f \in W^{r,p}(\Omega, \Lambda^{k+1})$, respectively $C^{r,\alpha}(\overline{\Omega}; \Lambda^{k+1})$ and $g \in W^{r,p}(\Omega, \Lambda^{k-1})$, respectively $C^{r,\alpha}(\overline{\Omega}; \Lambda^{k-1})$. Then the following hold true.*

(i) *Suppose f and g satisfy $df = 0$, $\delta g = 0$ in Ω and $\nu \wedge d\omega_0 = \nu \wedge f$ on $\partial\Omega$, and for every $\chi \in \mathcal{H}_T(\Omega; \Lambda^{k+1})$ and $\psi \in \mathcal{H}_T(\Omega; \Lambda^{k-1})$,*

$$\int_\Omega \langle f; \chi\rangle - \int_{\partial\Omega} \langle \nu \wedge \omega_0; \chi\rangle = 0 \text{ and } \int_\Omega \langle g; \psi\rangle = 0.$$

Then there exists a solution $\omega \in W^{r+1,p}(\Omega, \Lambda^k)$, respectively $C^{r+1,\alpha}(\overline{\Omega}; \Lambda^k)$, to the following boundary value problem,

$$\begin{cases} d(A(x)\omega) = f \quad \text{and} \quad \delta(B(x)\omega) = g & \text{in } \Omega, \\ \nu \wedge A(x)\omega = \nu \wedge \omega_0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{P}_1$}$$

satisfying the estimates

$$\|\omega\|_{W^{r+1,p}} \leq c\left(\|\omega\|_{L^p} + \|f\|_{W^{r,p}} + \|g\|_{W^{r,p}} + \|\omega_0\|_{W^{r+1,p}}\right),$$

respectively,

$$\|\omega\|_{C^{r+1,\alpha}} \leq c\left(\|\omega\|_{C^{0,\alpha}} + \|f\|_{C^{r,\alpha}} + \|g\|_{C^{r,\alpha}} + \|\omega_0\|_{C^{r+1,\alpha}}\right).$$

Furthermore, these estimates are apriori estimates, i.e any weak solution $\omega \in L^p(\Omega, \Lambda^k)$, respectively $C^{0,\alpha}(\overline{\Omega}; \Lambda^k)$, satisfying $(\mathcal{P}_1)$ in the sense of distributions with the assumed conditions on f, g and ω_0 are actually $W^{r+1,p}(\Omega, \Lambda^k)$, respectively $C^{r+1,\alpha}(\overline{\Omega}; \Lambda^k)$, and the respective estimates hold.

(ii) *Suppose f and g satisfy $df = 0$, $\delta g = 0$ in Ω and $\nu \lrcorner g = \nu \lrcorner \delta\omega_0$ on $\partial\Omega$, and for every $\chi \in \mathscr{H}_N(\Omega; \Lambda^{k+1})$ and $\psi \in \mathscr{H}_N(\Omega; \Lambda^{k-1})$,*

$$\int_\Omega \langle g; \chi\rangle - \int_{\partial\Omega} \langle \nu \lrcorner \omega_0; \chi\rangle = 0 \text{ and } \int_\Omega \langle f; \psi\rangle = 0.$$

Then there exists a solution $\omega \in W^{r+1,p}(\Omega, \Lambda^k)$, *respectively* $C^{r+1,\alpha}(\overline{\Omega}; \Lambda^k)$, *to the following boundary value problem,*

$$\begin{cases} d(A(x)\omega) = f \quad \textit{and} \quad \delta(B(x)\omega) = g & \textit{in } \Omega, \\ \nu \lrcorner B(x)\omega = \nu \lrcorner \omega_0 & \textit{on } \partial\Omega, \end{cases} \tag{$\mathcal{P}_2$}$$

satisfying the estimates

$$\|\omega\|_{W^{r+1,p}} \leq c\left(\|\omega\|_{L^p} + \|f\|_{W^{r,p}} + \|g\|_{W^{r,p}} + \|\omega_0\|_{W^{r+1,p}}\right),$$

respectively,

$$\|\omega\|_{C^{r+1,\alpha}} \leq c\left(\|\omega\|_{C^{0,\alpha}} + \|f\|_{C^{r,\alpha}} + \|g\|_{C^{r,\alpha}} + \|\omega_0\|_{C^{r+1,\alpha}}\right).$$

Furthermore, these estimates are apriori estimates, i.e any weak solution $\omega \in L^p(\Omega, \Lambda^k)$, *respectively* $C^{0,\alpha}(\overline{\Omega}; \Lambda^k)$, *satisfying* $(\mathcal{P}_1)$ *in the sense of distributions with the conditions on* f, g *and* ω_0 *are actually* $W^{r+1,p}(\Omega, \Lambda^k)$, *respectively* $C^{r+1,\alpha}(\overline{\Omega}; \Lambda^k)$, *and the respective estimates hold.*

Remark 9 (i) When $r = 0$, the conditions $df = 0$, $\delta g = 0$ are understood in the sense of distributions. The conditions $\nu \wedge d\omega_0 = \nu \wedge f$ on $\partial\Omega$ and $\nu \lrcorner \delta\omega_0 = \nu \lrcorner g$ on $\partial\Omega$ are well defined in the Hölder case and are to be interpreted in the weak sense in the Sobolev case (see remark 7.3(iii) in [6]).

Proof We prove only part (i) and the Sobolev case and assume that $2 \leq k \leq n-1$. The Hölder case or Part (ii) are analogous and the case $k = 1$ is much easier. The hypotheses on f imply that there exists $F \in W^{r+1,p}(\Omega, \Lambda^k)$ such that

$$\begin{cases} dF = f & \text{in } \Omega, \\ F = \omega_0 & \text{on } \partial\Omega. \end{cases}$$

Now, we find a solution $\alpha \in W^{r+2,p}(\Omega, \Lambda^{k-1})$ such that

$$\begin{cases} \delta(BA^{-1}d\alpha) = g - \delta(BA^{-1}F) & \text{in } \Omega, \\ \delta\alpha = 0 & \text{in } \Omega, \\ \nu \wedge \alpha = 0 & \text{on } \partial\Omega. \end{cases}$$

Now setting

$$\omega = A^{-1}(d\alpha + F),$$

we easily verify ω solves $(\mathcal{P}_1)$. The estimates are actually apriori estimates, since the argument is essentially reversible. □

With the help of the previous results, we can deduce a new Gaffney type inequality (see [9], [10],[11]).

Theorem 15 (Gaffney type inequality) *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and C^2, respectively $C^{2,\gamma}$ for some $0 < \gamma < 1$. Let $1 \leq k \leq n-1$ be an integer and $1 < p < \infty$ and $0 < \alpha \leq \gamma$ be real numbers. Let $B \in C^1(\overline{\Omega}; L(\Lambda^k; \Lambda^k))$, respectively $C^{1,\alpha}(\overline{\Omega}; L(\Lambda^k; \Lambda^k))$, satisfy the Legendre condition. Let $u \in L^p(\Omega; \Lambda^k)$, respectively $u \in C^{0,\alpha}(\overline{\Omega}; \Lambda^k)$, $du \in L^p(\Omega; \Lambda^{k+1})$, respectively $du \in C^{0,\alpha}(\overline{\Omega}; \Lambda^{k+1})$, $\delta(Bu) \in L^p(\Omega; \Lambda^{k-1})$, respectively $\delta(Bu) \in C^{0,\alpha}(\overline{\Omega}; \Lambda^{k-1})$. Suppose either $\nu \wedge u = 0$ on $\partial\Omega$ or $\nu \lrcorner (B(x)u) = 0$ on $\partial\Omega$. Then $u \in W^{1,p}(\Omega; \Lambda^k)$, respectively $u \in C^{1,\alpha}(\overline{\Omega}; \Lambda^k)$ and there exists a constant $C_p = C(\Omega, B, \gamma_0, p) > 0$, respectively $C_\alpha = C(\Omega, B, \gamma_0, \alpha) > 0$, such that*

$$\|u\|_{W^{1,p}} \leq C_p \left(\|du\|_{L^p(\Omega;\Lambda^{k+1})} + \|\delta(Bu)\|_{L^p(\Omega;\Lambda^{k-1})} + \|u\|_{L^p(\Omega;\Lambda^k)} \right),$$

and respectively

$$\|u\|_{C^{1,\alpha}} \leq C_\alpha \left(\|du\|_{C^{0,\alpha}(\overline{\Omega};\Lambda^{k+1})} + \|\delta(Bu)\|_{C^{0,\alpha}(\overline{\Omega};\Lambda^{k-1})} + \|u\|_{C^{0,\alpha}(\overline{\Omega};\Lambda^k)} \right).$$

Remark 10 The other inequality

$$\|u\|_{W^{1,p}} \leq C_p \left(\|d\,(Au)\|_{L^p(\Omega;\Lambda^{k+1})} + \|\delta u\|_{L^p(\Omega;\Lambda^{k-1})} + \|u\|_{L^p(\Omega;\Lambda^k)} \right),$$

and its Hölder space versions can be deduced analogously if A satisfies the Legendre condition and $\nu \lrcorner u = 0$ or $\nu \wedge (A(x)u) = 0$ on $\partial\Omega$.

5 Second order Hodge type systems revisited

With the help of the results deduced in the last section, it is now possible to deduce the solvability of a more general second order Hodge type system, namely the the system

$$\begin{cases} \delta(A(x)d\omega) + (B(x))^T d\delta\,(B(x)\omega) = \lambda B(x)\omega + f \text{ in } \Omega, \\ \nu \wedge \omega = \nu \wedge \omega_0 \text{ on } \partial\Omega. \\ \nu \wedge \delta\,(B(x)\omega) = \nu \wedge \delta\,(B(x)\omega_0) \text{ on } \partial\Omega. \end{cases} \tag{P}$$

In the next subsection, we show that regularity results extends to this general situation as well. The regularity estimates follow essentially the same way as in Section 3, the only difference being that we shall now use the Gaffney type inequality given in theorem 15 instead of the usual Gaffney inequality.

5.1 Regularity estimates

Only the proof of L^2 estimates need slight modification.

Theorem 16 (L^2 estimates for constant coefficients) *Let $A : \Lambda^{k+1} \to \Lambda^{k+1}$ and $B : \Lambda^k \to \Lambda^k$ satisfy*

$$\langle A(a \wedge b); a \wedge b\rangle \geq \gamma_A |a \wedge b|^2 \qquad \text{for all } a \in \Lambda^1, b \in \Lambda^k,$$

and

$$\langle B\xi; \xi\rangle \geq \gamma_B |\xi|^2 \qquad \text{for all } \xi \in \Lambda^k,$$

for some constants $\gamma_A, \gamma_B > 0$. Let $u \in W^{1,2}(B_R^+; \Lambda^k)$ with $\nu \wedge u = 0$ on Γ_R satisfy, for all $\psi \in W^{1,2}_{T,flat}(B_R^+; \Lambda^k)$,

$$\int_{B_R^+} \langle A(du); d\psi\rangle + \int_{B_R^+} \langle \delta(Bu); \delta(B\psi)\rangle = 0. \tag{20}$$

Then for every $m \geq 2$, there exist constants $c_m, \widetilde{c}_m > 0$ such that we have the estimates

$$\int_{B^+_{R/2^m}} |D^m u|^2 \leq \frac{\widetilde{c}_m}{(R)^{2m-1}} \int_{B_R^+} |\nabla u|^2 \leq \frac{c_m}{(R)^{2m}} \int_{B_R^+} |u|^2. \tag{21}$$

Proof Exactly the same arguments as in Theorem 2 proves,

$$\int_{B^+_{R/2^2}} |D_{ij} u|^2 \leq \frac{c}{(R)^2} \int_{B^+_{R/2}} |\nabla u|^2 \leq \frac{c}{(R)^{2^2}} \int_{B_R^+} |u|^2,$$

for all $i, j = 1, \ldots, n$ such that $(i, j) \neq (n, n)$. The only difference is in estimating the D_{nn} derivative. Once again we rewrite the system (20) as a system in terms of gradients. For every $\psi \in W^{1,2}_{T,flat}(B_R^+; \Lambda^k)$, we have,

$$\int_{B_R^+} \langle A(du); d\psi\rangle + \int_{B_R^+} \langle \delta(Bu); \delta(B\psi)\rangle = \int_{B_R^+} \langle \widetilde{A}\nabla u; \nabla\psi\rangle, \tag{22}$$

where $\widetilde{A} : \mathbb{R}^{\binom{n}{k}\times n} \to \mathbb{R}^{\binom{n}{k}\times n}$ is the linear map, defined by the pointwise algebraic identities

$$\langle \widetilde{A}(a_1 \otimes b_1); a_2 \otimes b_2\rangle = \langle A(a_1 \wedge b_1); a_2 \wedge b_2\rangle + \langle a_1 \lrcorner B b_1; a_2 \lrcorner B b_2\rangle,$$

for every $a_1, a_2 \in \Lambda^1$, viewed also as vectors in $\mathbb{R}^n$ and $b_1, b_2 \in \Lambda^k$, viewed also as vectors in $\mathbb{R}^{\binom{n}{k}}$. We also define the maps $\widetilde{A}^{pq} : \mathbb{R}^{\binom{n}{k}} \to \mathbb{R}^{\binom{n}{k}}$ for every $p, q = 1, \ldots, n$, by the identities,

$$\langle \widetilde{A}^{pq}\xi; \xi\rangle = \sum_{\alpha,\beta\in\mathcal{T}^k} \widetilde{A}^{pq}_{\alpha\beta} \xi^\alpha \xi^\beta = \langle \widetilde{A}(e_p \otimes \xi); e_q \otimes \xi\rangle \quad \text{for every } \xi \in \mathbb{R}^{\binom{n}{k}}.$$

Also, note that by virtue of the identity $\xi = e_n \lrcorner (e_n \wedge \xi) + e_n \wedge (e_n \lrcorner \xi)$, this in particular implies, for every $\xi \in \mathbb{R}^{\binom{n}{k}}$,

$$\begin{aligned}\langle \widetilde{A}^{nn}\xi;\xi\rangle &= \langle \widetilde{A}(e_n\otimes\xi); e_n\otimes\xi\rangle = \langle A\,(e_n\wedge\xi)\,; e_n\wedge\xi\rangle + \langle e_n\lrcorner B\xi; e_n\lrcorner B\xi\rangle \\ &\geq \gamma_A|e_n\wedge\xi|^2 + |e_n\lrcorner B\xi|^2 \geq c\,|\xi|^2\,,\end{aligned}$$

where $c > 0$ is the constant given by lemma 1. This implies that $\widetilde{A}^{nn}$ is invertible and yields the estimate for $D_{nn}u$ the same way as in Theorem 2. □

5.2 Main theorems

5.2.1 General Hodge type systems

Theorem 17 *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and let $\partial\Omega$ be of class $C^{r+2,\gamma}$. Let $1 \leq k \leq n-1$, $r \geq 0$ be integers and $0 < \alpha \leq \gamma < 1$ and $1 < p < \infty$ be real numbers. Let $A \in C^{r+1}(\overline{\Omega}; L(\Lambda^{k+1}, \Lambda^{k+1}))$, respectively $C^{r+1,\alpha}(\overline{\Omega}; L(\Lambda^{k+1}, \Lambda^{k+1}))$, and $B \in C^{r+1}(\overline{\Omega}; L(\Lambda^k, \Lambda^k))$, respectively $C^{r+1,\alpha}(\overline{\Omega}; L(\Lambda^k, \Lambda^k))$, satisfy the Legendre condition. Then the following holds.*

1. *There exists a constant $\rho \in \mathbb{R}$ and an at most countable set $\sigma \subset (-\infty, \rho)$, with no limit points except possibly $-\infty$.*
2. *The following boundary value problem,*

$$\begin{cases} \delta(A(x)d\alpha) + (B(x))^T\, d\delta\,(B(x)\alpha) = \sigma_i B(x)\alpha \text{ in } \Omega, \\ \nu\wedge\alpha = 0 \text{ on } \partial\Omega, \\ \nu\wedge\delta\,(B(x)\alpha) = 0 \text{ on } \partial\Omega. \end{cases} \tag{EP}$$

   admits non-trivial solutions $\alpha \in C^\infty(\overline{\Omega}, \Lambda^k)$ if and only if $\sigma_i \in \sigma$ and the space of solutions to (EP) *is finite-dimensional for any $\sigma_i \in \sigma$.*
3. *If $\lambda \notin \sigma$, then for any $f \in W^{r,p}(\Omega, \Lambda^k)$, respectively $C^{r,\alpha}(\overline{\Omega}, \Lambda^k)$, and any $\omega_0 \in W^{r+2,p}(\Omega, \Lambda^k)$, respectively $C^{r+2,\alpha}(\overline{\Omega}, \Lambda^k)$, there exists a unique solution $\omega \in W^{r+2,p}(\Omega, \Lambda^k)$, respectively $C^{r+2,\alpha}(\overline{\Omega}, \Lambda^k)$, to the following boundary value problem:*

$$\begin{cases} \delta(A(x)d\omega) + (B(x))^T\, d\delta\,(B(x)\omega) = \lambda B(x)\omega + f \text{ in } \Omega, \\ \nu\wedge\omega = \nu\wedge\omega_0 \text{ on } \partial\Omega. \\ \nu\wedge\delta\,(B(x)\omega) = \nu\wedge\delta\,(B(x)\omega_0) \text{ on } \partial\Omega, \end{cases} \tag{P}$$

   which satisfies the estimate

$$\|\omega\|_{W^{r+2,p}} \leq c\,(\|\omega\|_{W^{r,p}} + \|f\|_{W^{r,p}} + \|\omega_0\|_{W^{r+2,p}})\,,$$

   respectively,

$$\|\omega\|_{C^{r+2,\alpha}} \leq c\,(\|\omega\|_{C^{r,\alpha}} + \|f\|_{C^{r,\alpha}} + \|\omega_0\|_{C^{r+2,\alpha}})\,.$$

Proof We divide the proof in two steps.

Step 1 (Existence in L^2): The arguments are similar to the previous case. For a given $\lambda \in \mathbb{R}$, the bilinear operator $a_\lambda : W^{1,2}_T(\Omega;\Lambda^k) \times W^{1,2}_T(\Omega;\Lambda^k) \to \mathbb{R}$ defined by,

$$a_\lambda(u,v) = \int_\Omega \langle A(x)du, dv\rangle + \int_\Omega \langle \delta\left(B(x)u\right), \delta\left(B(x)v\right)\rangle + \lambda \int_\Omega \langle B(x)u, v\rangle,$$

is continuous and coercive, by theorem 15 and Gårding inequality, for any λ large enough. Now Lax-Milgram theorem, together with the compact embedding of $W^{1,2}_T(\Omega;\Lambda^k)$ into $L^2(\Omega;\Lambda^k)$ implies that Fredholm theory holds. Thus, for any $g \in L^2(\Omega,\Lambda^k)$ and any $\lambda \notin \sigma$, there exists an unique solution $\overline{\omega} \in W^{1,2}_T(\Omega;\Lambda^k)$ solving

$$\int_\Omega \langle A(x)d\overline{\omega}, d\theta\rangle + \langle \delta\left(B(x)\overline{\omega}\right), \delta\left(B(x)\theta\right)\rangle + \lambda \int_\Omega \langle B(x)\overline{\omega}, \theta\rangle + \int_\Omega \langle g, \theta\rangle = 0, \tag{23}$$

for all $\theta \in W^{1,2}_T(\Omega;\Lambda^k)$.

Step 2 (Regularity): Now theorem 7, respectively theorem 6 in the Hölder case, gives us the desired regularity. The estimates for $1 < p < 2$ also extends the existence theory to the case $1 < p < 2$ by usual density arguments. From (23), integrating by parts, we obtain,

$$\begin{aligned}\int_\Omega \langle \delta(A(x)d\overline{\omega}) + \left(B(x)\right)^T d\delta\left(B(x)\overline{\omega}\right); \phi\rangle - \int_\Omega \langle \lambda B(x)\overline{\omega} + g; \phi\rangle \\ = \int_{\partial\Omega} \left(\langle A(x)d\overline{\omega}; \nu\wedge\phi\rangle + \langle \nu\wedge\delta\left(B(x)\overline{\omega}\right); \left(B(x)\phi\right)\rangle\right),\end{aligned}$$

for all $\phi \in W^{1,2}_T(\Omega,\Lambda^k)$. Thus taking $\phi \in C^\infty_c(\Omega,\Lambda^k)$ we have,

$$\delta(A(x)d\overline{\omega}) + \left(B(x)\right)^T d\delta\left(B(x)\overline{\omega}\right) = \lambda\left(B(x)\overline{\omega}\right) + g \qquad \text{in } \Omega.$$

But this implies that the integral on the boundary vanish separately. But since $\phi \in W^{1,2}_T(\Omega,\Lambda^k)$, $\nu\wedge\phi = 0$. Hence we obtain,

$$\int_{\partial\Omega} \langle \nu\wedge\delta\left(B(x)\overline{\omega}\right); \left(B(x)\phi\right)\rangle = 0 \qquad \text{for any } \phi \in W^{1,2}_T(\Omega;\Lambda^k).$$

We now show that, for any $u \in W^{1,2}(\Omega;\Lambda^k)$, we have,

$$\int_{\partial\Omega} \langle \nu\wedge\delta\left(B(x)\overline{\omega}\right); u\rangle = 0.$$

Extending ν as a C^1 function inside Ω and using the identity $u = \nu\wedge(\nu\lrcorner u) + \nu\lrcorner(\nu\wedge u)$, we deduce, for any $u \in W^{1,2}(\Omega;\Lambda^k)$,

$$\int_{\partial\Omega} \langle \nu\wedge\delta\left(B(x)\overline{\omega}\right); u\rangle = \int_{\partial\Omega} \langle \nu\wedge\delta\left(B(x)\overline{\omega}\right); \nu\wedge(\nu\lrcorner u)\rangle.$$

Since $\nu \wedge (\nu \lrcorner u) \in W^{1,2}_T(\Omega, \Lambda^k)$, setting $\phi = B^{-1}(x)\left(\nu \wedge (\nu \lrcorner u)\right)$, we obtain

$$\int_{\partial\Omega} \langle \nu \wedge \delta\left(B(x)\overline{\omega}\right); u\rangle = \int_{\partial\Omega} \langle \nu \wedge \delta\left(B(x)\overline{\omega}\right); B(x)\phi\rangle = 0.$$

Since $u \in W^{1,2}(\Omega; \Lambda^k)$ is arbitrary, this implies $\nu \wedge \delta\left(B(x)\overline{\omega}\right) = 0$ on $\partial\Omega$. Now taking $g = f + \lambda B(x)\omega_0 - \delta(A(x)d\omega_0) - \left(B(x)\right)^T d\delta\left(B(x)\omega_0\right)$ and setting $\omega = \overline{\omega} + \omega_0$, we immediately see that ω is a solution to (P) with the desired regularity. This finishes the proof. □

5.2.2 General Maxwell operators

Theorem 18 *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and let $\partial\Omega$ be of class $C^{r+2,\gamma}$. Let $1 \leq k \leq n-1$, $r \geq 0$ be integers and $0 < \alpha \leq \gamma < 1$ and $1 < p < \infty$ be real numbers. Let $A \in C^{r+1}(\overline{\Omega}; L(\Lambda^{k+1}, \Lambda^{k+1}))$, respectively $C^{r+1,\alpha}(\overline{\Omega}; L(\Lambda^{k+1}, \Lambda^{k+1}))$, and $B \in C^{r+1}(\overline{\Omega}; L(\Lambda^k, \Lambda^k))$, respectively $C^{r+1,\alpha}(\overline{\Omega}; L(\Lambda^k, \Lambda^k))$, satisfy the Legendre condition. Let $\omega_0 \in W^{r+2,p}(\Omega, \Lambda^k)$, respectively $C^{r+2,\alpha}(\overline{\Omega}; \Lambda^k)$, $f \in W^{r,p}(\Omega, \Lambda^{k+1})$, respectively $C^{r,\alpha}(\overline{\Omega}; \Lambda^{k+1})$ and $g \in W^{r+1,p}(\Omega, \Lambda^{k-1})$, respectively $C^{r+1,\alpha}(\overline{\Omega}; \Lambda^{k-1})$, and $\lambda \geq 0$. Suppose f, g and λ satisfy $\delta f + \lambda g = 0$, $\delta g = 0$ in Ω and $\int_\Omega \langle f; \phi\rangle = 0$ for every $\phi \in \mathcal{H}_T(\Omega; \Lambda^k)$ and $\int_\Omega \langle g; \psi\rangle = 0$ for every $\psi \in \mathcal{H}_T(\Omega; \Lambda^{k-1})$. Then there exists a unique solution $\omega \in W^{r+2,p}(\Omega, \Lambda^k)$, respectively $C^{r+2,\alpha}(\overline{\Omega}; \Lambda^k)$, to the following boundary value problem,*

$$\begin{cases} \delta(A(x)d\omega) = \lambda B(x)\omega + f \text{ in } \Omega, \\ \delta\left(B(x)\omega\right) = g \text{ in } \Omega, \\ \nu \wedge \omega = \nu \wedge \omega_0 \text{ on } \partial\Omega. \end{cases} \tag{PM}$$

satisfying the estimates

$$\|\omega\|_{W^{r+2,p}} \leq c\left(\|\omega\|_{L^p} + \|f\|_{W^{r,p}} + \|g\|_{W^{r,p}} + \|\omega_0\|_{W^{r+2,p}}\right),$$

respectively,

$$\|\omega\|_{C^{r+2,\alpha}} \leq c\left(\|\omega\|_{C^{0,\alpha}} + \|f\|_{C^{r,\alpha}} + \|g\|_{C^{r,\alpha}} + \|\omega_0\|_{C^{r+2,\alpha}}\right).$$

Proof We show only the Sobolev case, the other case is similar. We divide the proof into several steps.

Step 1 First we claim that we can assume $\omega_0 = 0$ and $g = 0$. Since $\delta g = 0$ in Ω and $\int_\Omega \langle g; \psi\rangle = 0$ for every $\psi \in \mathcal{H}_T(\Omega; \Lambda^{k-1})$, there exists a unique $G \in W^{r+2,p}(\Omega, \Lambda^k)$ such that

$$\begin{cases} dG = 0 \quad \text{and} \quad \delta(B(x)G) = g - \delta\left(B(x)\omega_0\right) & \text{in } \Omega, \\ \nu \wedge G = 0 & \text{on } \partial\Omega, \end{cases}$$

and we have $\int_\Omega \langle B(x)\left(G + \omega_0\right); \psi\rangle = 0$ for every $\psi \in \mathcal{H}_T(\Omega; \Lambda^k)$. Let $u \in W^{r+2,p}(\Omega, \Lambda^k)$ is a solution of

$$\begin{cases} \delta(A(x)du) = \lambda B(x)u + \widetilde{f} \text{ in } \Omega, \\ \delta\left(B(x)u\right) = 0 \text{ in } \Omega, \\ \nu \wedge u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\widetilde{f} = f + \lambda B(x)\omega_0 + \lambda B(x)G - \delta(A(x)d\omega_0)$. Note that $\delta\widetilde{f} = 0$ in Ω and also $\int_\Omega \langle \widetilde{f}; \phi\rangle = 0$ for every $\phi \in \mathcal{H}_T(\Omega; \Lambda^k)$ Now, setting $\omega = \omega_0 + u + G$, we obtain the claim.

Step 2 Now we assume $g = 0$ and $\omega_0 = 0$. Since $\lambda \geq 0$, λ is not in the spectrum and thus we can find $\omega \in W^{r+2,p}(\Omega, \Lambda^k)$, satisfying

$$\begin{cases} \delta(A(x)d\omega) + (B(x))^T d\delta\left(B(x)\omega\right) = \lambda B(x)\omega + f & \text{in } \Omega, \\ \nu \wedge \omega = 0 & \text{on } \partial\Omega. \\ \nu \wedge \delta\left(B(x)\omega\right) = 0 & \text{on } \partial\Omega. \end{cases}$$

But then this implies $v = \delta\left(B(x)\omega\right)$ is $W^{1,p}$ weak solution of

$$\begin{cases} \delta\left(B^T(x)dv\right) = \lambda v & \text{in } \Omega, \\ \delta v = 0 & \text{in } \Omega. \\ \nu \wedge v = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $\lambda \geq 0$ and B^T satisfies uniform Legendre condition, $v \in \mathcal{H}_T(\Omega, \Lambda^{k-1})$. But no non-zero harmonic field can be co-exact. Thus $\delta\left(B(x)\omega\right) = 0$ in Ω and ω solves (PM) with $g = 0$ and $\omega_0 = 0$. This completes the proof. □

6 Appendix: Flattening the boundary

Let $1 \leq k \leq n-1$ and $\Omega \subset \mathbb{R}^n$ be a bounded smooth open set. Let $A : \Omega \to L(\Lambda^{k+1}, \Lambda^{k+1})$ and $B : \Omega \to L(\Lambda^k, \Lambda^k)$ be measurable maps. Also let $f \in L^2(\Omega; \Lambda^k)$, $F \in L^2(\Omega; \Lambda^{k+1})$ and $\lambda \in \mathbb{R}$. Let $\omega \in W^{1,2}_T(\Omega, \Lambda^k)$, respectively, $W^{1,2}_N(\Omega, \Lambda^k)$, be a weak solution of the following,

$$\int_\Omega \langle A(x)d\omega, d\phi\rangle + \int_\Omega \langle \delta\left(B(x)\omega\right), \delta\left(B(x)\phi\right)\rangle + \lambda \int_\Omega \langle B(x)\omega, \phi\rangle + \int_\Omega \langle f, \phi\rangle - \int_\Omega \langle F, d\phi\rangle = 0, \quad (24)$$

for all $\phi \in W^{1,2}_T(\Omega; \Lambda^k)$, respectively, $W^{1,2}_N(\Omega, \Lambda^k)$.
Now we localize ω near a boundary point, flatten the boundary and transfer our problem to a half-ball. We only mention the essential points of the proof. See [18], lemma 6.23, for details.

Lemma 4 *Let $m \geq r \geq 0$ be integers and $0 < \gamma < 1$. Let $\partial\Omega$ is of class C^{m+2}, respectively $C^{m+2,\gamma}$. Let $f \in W^{r,p}(\Omega, \Lambda^k)$ and $F \in W^{r+1,p}(\Omega, \Lambda^{k+1})$ for some $1 \leq p \leq \infty$, respectively $f \in C^{s,\mu}(\Omega, \Lambda^k)$ and $F \in C^{s+1,\mu}(\Omega, \Lambda^{k+1})$ for some $0 < \mu \leq \gamma$. Let $\omega \in W^{1,2}_T(\Omega, \Lambda^k) \cap W^{r+1,2}(\Omega, \Lambda^k)$ satisfy*

$$\int_\Omega \langle A(x)d\omega, d\phi\rangle + \int_\Omega \langle \delta\left(B(x)\omega\right), \delta\left(B(x)\phi\right)\rangle + \lambda \int_\Omega \langle B(x)\omega, \phi\rangle + \int_\Omega \langle f, \phi\rangle - \int_\Omega \langle F, d\phi\rangle = 0, \quad (25)$$

for all $\phi \in W^{1,2}_T(\Omega;\Lambda^k)$, *for some* $\lambda \in \mathbb{R}$. *Let* $A \in C^{r+1}\left(\overline{\Omega}; L(\Lambda^{k+1},\Lambda^{k+1})\right)$, *respectively* $C^{r+1,\gamma}\left(\overline{\Omega}; L(\Lambda^{k+1},\Lambda^{k+1})\right)$, *and* $B \in C^{r+1}\left(\overline{\Omega}; L(\Lambda^{k},\Lambda^{k})\right)$, *respectively* $C^{r+1,\gamma}\left(\overline{\Omega}; L(\Lambda^{k},\Lambda^{k})\right)$, *satisfy the Legendre condition.*

Then for every given $R > 0$ *and for every* $x_0 \in \partial\Omega$, *there exist*

- *a neighborhood* W *of* x_0 *in* $\mathbb{R}^n$, *a function* $\theta \in C^\infty_c(W)$, *an admissible boundary coordinate system* $\Phi \in \mathrm{Diff}^{m+2}(\overline{B_R};\overline{W})$, *respectively* $\Phi \in \mathrm{Diff}^{m+2,\gamma}(\overline{B_R};\overline{W})$, *such that* $\Phi(0) = x_0$, $\Phi(B^+_R) = \Omega \cap W$ *and* $\Phi(\Gamma_R) = \partial\Omega \cap W$,
- *three matrix functions* $P = (p_{\alpha\beta})_{\alpha,\beta\in\mathcal{T}^k}$, $Q = \left(q^i_{\alpha\beta}\right)^{i=1,\dots,n}_{\alpha\beta\in\mathcal{T}^k}$ *and* $R = \left(r^i_{\alpha\beta}\right)^{i=1,\dots,n}_{\alpha\beta\in\mathcal{T}^k}$ *which are* C^r, *respectively* $C^{r,\gamma}$, *in* B^+_R *and a matrix field* $S = \left(s^{ij}_{\alpha\beta}\right)^{i,j=1,\dots,n}_{\alpha\beta\in\mathcal{T}^k}$ *which is* C^{r+1}, *respectively* $C^{r+1,\gamma}$, *in* B^+_R *with*

$$\mathcal{C}_S(R) \le c_0\left(\mathcal{C}_A(R) + \mathcal{C}_B(R) + \mathcal{C}_\Phi(R)\right),$$

  depending only on A, B, Φ, θ, W *and* R, *where* $c_0 > 0$ *is a constant independent of* R, $\mathcal{C}_S$, $\mathcal{C}_A, \mathcal{C}_B$ *and* $\mathcal{C}_\Phi$ *are the modulus of continuity of* S, A, B *and* Φ, *respectively,*
- *a constant matrix* $\bar{A} : \Lambda^{k+1} \to \Lambda^{k+1}$ *satisfying the Legendre-Hadamard condition and a constant matrix* $\bar{B} : \Lambda^k \to \Lambda^k$ *satisfying the Legendre condition,*
- *a form* $\widetilde{f} \in W^{r,p}(B^+_R;\Lambda^k)$ *and a matrix field* $\widetilde{F} \in W^{r+1,p}\left(B^+_R; \mathbb{R}^{\binom{n}{k}\times n}\right)$, *respectively* $\widetilde{f} \in C^{r,\mu}(\overline{B^+_R};\Lambda^k)$ *and* $\widetilde{F} \in C^{r+1,\mu}\left(\overline{B^+_R}; \mathbb{R}^{\binom{n}{k}\times n}\right)$, *with estimates on the norms by the norms of* f *and* F, *the constants in the estimates depending only on* Φ, θ, W *and* R,

such that $u = \Phi^*(\theta\omega) \in W^{1,2}_{T,flat}(B^+_R;\Lambda^k) \cap W^{r+1,2}(B^+_R;\Lambda^k)$ *satisfies, for all* $\psi \in W^{1,2}_{T,flat}(B^+_R;\Lambda^k)$,

$$\int_{B^+_R}\langle \bar{A}(du); d\psi\rangle + \int_{B^+_R}\langle \delta(\bar{B}u); \delta(\bar{B}\psi)\rangle + \int_{B^+_R}\langle \widetilde{f} + Pu + R\nabla u; \psi\rangle - \int_{B^+_R}\langle \widetilde{F} - Qu; \nabla\psi\rangle + \int_{B^+_R}\langle S\nabla u, \nabla\psi\rangle = 0. \quad (26)$$

Remark 11 The lemma essentially says that it is possible to flatten the boundary in such a way that by choosing a sufficiently small neighborhood, it is possible to get a constant coefficient system of the same form in the leading order coefficients, upto small perturbations.

Proof Given $R > 0$, there always exist a neighborhood W of x_0 in $\mathbb{R}^n$ and an admissible boundary co-ordinate system $\Phi \in \mathrm{Diff}^{r+2}(\overline{B_R};\overline{W})$, respectively,

$\mathrm{Diff}^{r+2,\gamma}(\overline{B_R};\overline{W})$ (see [4] or [15]). Also we can assume $D\Phi(0) \in \mathrm{SO(n)}$ and by a translation, $x_0 = 0$. Set

$$\widetilde{\Phi}(y) = (D\Phi(0))\, y \qquad \text{for every } y \in B_R.$$

Denote $T = (D\Phi(0))^{-1}$ and set

$$\bar{A} = \left(T^{-1}\right)^* \circ A(x_0) \circ (T)^* \text{ and } \bar{B} = \left(T^{-1}\right)^* \circ B(x_0) \circ (T)^*.$$

Then $\bar{A}$ satisfies a Legendre-Hadamard condition. Indeed, for any $a \in \Lambda^1, b \in \Lambda^k$, we have

$$\begin{aligned}\langle \bar{A}(a\wedge b); a\wedge b\rangle &= \langle \left(T^{-1}\right)^* \circ A(x_0) \circ (T)^* (a\wedge b); a\wedge b\rangle \\ &= \left(T^{-1}\right)^* \left(\langle A(x_0) \circ (T)^* (a\wedge b); (T)^* (a\wedge b)\rangle\right) \\ &= \left(T^{-1}\right)^* \left(\langle A(x_0)\,(T^*a\wedge T^*b)\,; (T^*a\wedge T^*b)\rangle\right) \geq \gamma_{\bar{A}} |a\wedge b|^2,\end{aligned}$$

for some $\gamma_{\bar{A}} > 0$, since $\left(T^{-1}\right)^*$, T^* are both bijective and $A(x_0)$ satisfies the Legendre-Hadamard condition. Clearly $\bar{B}$ satisfies a Legendre condition. Choosing $\phi = (\Phi^{-1})^*\psi$ and substituting $\theta\omega = (\Phi^{-1})^* u$ in the equation satisfied by $\theta\omega$, the conclusion of the lemma follows by applying change of variables and grouping the terms. The terms containing f and F, are grouped according as whether they are multiplied with ψ or derivatives of ψ, as $\widetilde{f}$ and $\widetilde{F}$, respectively. Since these terms are the components of f and F, multiplied with up to first order derivatives of θ and Φ, we have the estimates. It is also easy to see that the functions $p_{\alpha\beta}, q^i_{\alpha\beta}, r^i_{\alpha\beta}$ are made up of products and compositions of components of A, up to first order derivatives of θ and B and up to second order derivatives of Φ, implying their claimed regularity. On the other hand, the functions $s^{ij}_{\alpha\beta}$, the coefficients of the 'second order terms', i.e terms with derivatives of u and derivatives of ψ multiplied together, are products of components of A and B, derivatives of θ and up to first order derivatives of Φ. Thus the claimed regularity is established. Note also that the terms containing the functions $s^{ij}_{\alpha\beta}$s are of three types. One type of terms contain some component of the difference $A(x) - A(x_0)$, the second type contain some component of the difference $B(x) - B(x_0)$ and the other type of terms contain some components of the difference $D\Phi(y) - D\Phi(0)$. Thus $s^{ij}_{\alpha\beta}(0) = 0$ and the L^∞ norm of $s^{ij}_{\alpha\beta}(y) - s^{ij}_{\alpha\beta}(0)$ scales like the modulus of continuity of A, B or Φ. This finishes the proof. □

Remark 12 Note that if A satisfies a uniform Legendre condition, i.e if $A \in C^{r+1}\left(\overline{\Omega}; L(\Lambda^{k+1}, \Lambda^{k+1})\right)$, respectively $C^{r+1,\gamma}\left(\overline{\Omega}; L(\Lambda^{k+1}, \Lambda^{k+1})\right)$, satisfy,

$$\langle A(x)\xi\ ;\ \xi\rangle \geq \gamma_0 \left|\xi\right|^2, \qquad \text{for every } \xi \in \Lambda^{k+1}$$

for some constant $\gamma_0 > 0$ for all $x \in \Omega$, then $\bar{A}$ satisfies a uniform Legendre condition as well.

Analogous result holds true for solutions with vanishing normal part.

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