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Stackelberg Games of Water Extraction

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We analyze a simple problem of water extraction.

- Myopic agents harvesting a common water resource
- A regulator has some control on the cost function
- The regulator takes into account rainfall and strategic interaction between agents
Ideas of the paper (ctd.)

- Groundwater extraction: marginal cost depends on the level of the aquifer
- In general, resources with accessibility problems: cost depends on scarcity
- The main ingredient: make the cost depend on the projected evolution of the resource: before or after the extraction or rainfall
- The goal: deduce its economical and environmental consequences
- The method: a Stackelberg Game linear-quadratic static game between agents, different kinds of optimization for the regulator.
The model of Provencher and Burt

We consider the extraction of groundwater by $K$ players. Dynamics of groundwater:

$$G_{t+1} = G_t + R - \sum_{k=1}^{K} u_t^k, \quad G_0, \text{ given.}$$

We suppose $R$ is a constant.

Instantaneous profit:

$$\pi^i(u_t^i, G_t) = F_i(u_t^i) - C_i(G_t) \times u_t^i.$$ 

Marginal extraction cost ($C_i(.)$): depends on the current level of the groundwater.
The extended model

Introduce the more general instantaneous profit function:

\[ \pi^i(u_t) = F_i(u^i_t) - C_i(G_t + mR - n \sum_j u^j_t) u^i_t \]

where \( n, m \in [0, 1] \).

The extreme cases:

- \( n = 0, m = 0 \) (the standard case): cost based on current resource
- \( n = 1, m = 1 \): cost based on the state of the resource in the following period.

When \( n \neq 0 \) the profit function of player \( i \) depends on the action of the other player: strategic interaction not just through the dynamics.
Angles of analysis

In a previous work, we developed the non-cooperative, Nash-feedback solution for agents.

Here, we develop the supervised setting

- myopic followers, static Stackelberg leader
- myopic followers, dynamic Stackelberg leader

→ an exercise in sensitivity analysis of LQ dynamic games, and non-LQ, non-concave optimal control problems.

Mostly work in progress...
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The case of myopic agents

Assume agents play Nash with their instantaneous profit:

$$\max_{u^i} \left\{ F_i(u^i_t) - C_i(G_t + mR - n \sum_{k=1}^{K} u^k_t) u^i_t \right\}.$$ 

For convenience, we continue with the particular linear-quadratic functional form:

$$F_i(u) = a_i u - \frac{b_i}{2} u^2, \quad C_i(x) = z_i - c_i x > 0.$$
Myopic agent reaction

In the **symmetric** case we find:

\[ u(G) = \frac{c}{b + (K + 1)cn} G + \frac{a - z + cmR}{b + (K + 1)cn}. \]

The value function of each player is:

\[ \pi(G_0) = \frac{(cG_0 + a - z + cmR)^2}{(b + (K + 1)cn)^2} \frac{b + 2cn}{2}. \]
Myopic stock dynamics

The stock dynamics is:

\[ G_{t+1} = G_t + R - K(\alpha G_t + \gamma) \]
\[ = (1 - K\alpha)^t G_0 + (R - K\gamma) \frac{1 - (1 - K\alpha)^t}{K\alpha} \]

The asymptotic stock is:

\[ G_\infty = \frac{R - K\gamma}{K\alpha} = \frac{Rb}{Kc} + \frac{K + 1}{K} \left( \frac{Rn - Rm - a - z}{c} \right) \]
1 Introduction
   - The model
   - Analysis

2 Non-cooperative setting
   - The myopic case
     - Myopic Reaction
     - Stock Dynamics

3 Supervised setting with myopic followers
   - Static optimization
   - Stackelberg game

4 Supervised setting with non-myopic followers
Supervised setting

A regulator is in charge of selecting the cost model, by choosing the cost parameters $n$ and $m$. The choice is announced to players, who play Nash. Followers are myopic:

$$u_t^i = \frac{c}{b + (K + 1)cn} G_t + \frac{a - z + cmR}{b + (K + 1)cn}$$

The supervisor optimizes her own criterion, taking this reaction into account.
Optimal static choice

The supervisor gets to choose $n$ and $m$ once and for all.

Optimal pricing problem

The supervisor’s problem is:

$$\max_{(n,m) \in [0,1]^2} \left\{ \sum_{t=0}^{\infty} \beta^t \sum_i \pi^i(u_t, G_t) \right\}$$

with the myopic agent reactions and stock dynamics

$$G_{t+1} = G_t + R - \sum_i u_t^i.$$
Optimal static choice (ctd.)

Thanks to the explicit solution for the dynamics:

\[
V_{nm}(G_0) = \frac{b + 2cn}{2} \left[ \frac{\alpha^2 (G_0 - G_\infty)^2}{1 - \beta (1 - K\alpha)^2} + \frac{2R}{K} \frac{\alpha (G_0 - G_\infty)}{1 - \beta (1 - K\alpha)} + \frac{R^2}{K^2 (1 - \beta)} \right]
\]

with

\[
G_\infty = \frac{R - K\gamma}{K\alpha}.
\]

\[
\alpha = \frac{c}{(K + 1)cn + b}, \quad \gamma = \frac{mRc + a - z}{(K + 1)cn + b}.
\]
Optimal static choice (ctd.)

\[ \rightarrow \text{optimization wrt } (n, m) \text{ for each } G_0. \]

Results:
- \( m^* = 1 \) is always optimal if \( K\alpha < 1 \) (monotonous traj.)
- the optimal \( n \) is:

\[
n^* = \begin{cases} 
0 & \text{if } \beta_L \leq \overline{\beta}(G_0) \\
1 & \text{if } \beta_L \geq \overline{\beta}(G_0) \\
n^*(\beta_L) & \text{otw.}
\end{cases}
\]

but \( n^*(\beta_L) \) is the root of a 4\textsuperscript{th} degree polynomial.
Optimal static choice (ctd.)

A situation where $n^*(\beta_L) \not\in \{0, 1\}$:
Stackelberg with myopic followers

Followers being myopic, they apply the Nash controls. If symmetric:

\[ u^i(G) = \frac{c}{b + (K + 1)cn} G + \frac{a - z + cmR}{b + (K + 1)cn} . \]

The benevolent regulator maximizes the total discounted profit on their behalf:

\[
\max_{\{n_t, m_t\}} \left\{ \sum_{t=0}^{\infty} \beta_t^L \sum_i F(u^i(G_t)) - C(G_t + m_t R - n_t U(G_t))u^i(G_t) \right\}
\]

with the dynamics:

\[ G_{t+1} = G_t + R - Ku^i(G_t) . \]

\[ \rightarrow \text{ an optimal control problem.} \]
First-order conditions

From the maximum principle:

\[ 0 = \beta^t_c U(G_t, n_t, m_t)(b + 2cn_t) + (b + (K + 1)cn)(q_t - q_{t-1}) - Kcq_t \]

\[ 0 = \beta^t_c U(G_t, n_t, m_t)^2(b - (K + 1)(b + cn_t)) \]
\[ + (b + (K + 1)cn_t)(\lambda_t^{(n)} - \mu_t^{(n)}) \]
\[ + Kcq_t(K + 1)U(G_t, n_t, m_t) \]

\[ 0 = \beta^t_c RU(G_t, n_t, m_t)(b + 2cn_t) + (b + (K + 1)cn_t)(\lambda_t^{(m)} - \mu_t^{(m)}) - KcRq_t. \]

where:

- \( \lambda_t^{(n)} \), \( \mu_t^{(n)} \) multipliers for \( n_t \geq 0 \) and \( n_t \leq 1 \), \text{id} for \( \lambda_t^{(m)} \), \( \mu_t^{(m)} \).
- \( U(\ldots) \) is the feedback function of followers.
Stationary situations

If a stationary state and control exists, then:

- optimal $m$: $m^* = 1$,
- optimal $n$:

$$
\begin{align*}
    n^* &= \begin{cases} 
        0 & \text{if } \beta_L \leq \frac{b}{b + c} \\
        1 & \text{if } \beta_L \geq \frac{K(b + c) + c}{K(b + 2c) + c} \\
        \frac{K}{K + 1} \frac{\beta(b + c) - b}{(1 - \beta)c} & \text{otw.}
    \end{cases}
\end{align*}
$$

But we don’t know if a stationary state exists.
Stationary situations (ctd.)

Example with corner solution:

\[ a = 1, \quad b = 1, \quad c = \frac{1}{10}, \quad z = \frac{9}{10}, \quad R = 1, \quad \beta_L = \frac{95}{100} \]

gives a candidate optimal stationary solution:

\[ G^\infty = \frac{9}{2}, \quad n^* = 1. \]

Satisfies all first-order conditions.
Example with interior solution:

\[ a = 1, \quad b = 1, \quad c = \frac{1}{10}, \quad z = \frac{9}{10}, \quad R = 1, \quad \beta_L = 0.919 \]

gives a candidate optimal stationary solution:

\[ G^\infty \simeq 4.35, \quad n^* = \frac{218}{243}. \]

Satisfies all first-order conditions.
Stationary situations (ctd.)

The constant-policy value (starting from the expected steady state) is maximum at $n^*$:

![Graph showing the constant-policy value as a function of n]
Non-stationary situations

However, convexity plays us tricks.
Example:

\[ \begin{align*}
    a &= 1, \quad b = 1, \quad c = \frac{6}{10}, \quad z = \frac{9}{10}, \quad R = 1, \quad \beta_L = \frac{68}{100}
\end{align*} \]

gives a candidate optimal stationary solution:

\[ \begin{align*}
    G^\infty &= \frac{1}{8}, \quad n^* = \frac{11}{36}.
\end{align*} \]

But...
the optimal constant policy is not there!
And the Hamiltonian has actually a *local minimum* there wrt \( n \).
Non-stationary situations (ctd.)

Hamiltonian is for constant trajectory from the expected steady state.
Non-stationary situations (ctd.)
Non-stationary situations (ctd.)

Same values except $R = 50$, $\beta = 7/10$: 

![Graph showing stocks, consumptions, and cumulated gains with different lines representing various scenarios.](image)
A heuristic policy

These periodic policies were identified by using the following heuristic:

\[
\begin{align*}
m(G) &= 1 \\
n(G) &= \arg\max\{V_{n_1}(G) \mid 0 \leq n \leq 1\}.
\end{align*}
\]

* i.e. use the best static policy... but as state feedback.
Stackelberg with non-myopic followers

Followers being non-myopic, at time $t$ they react to the sequence of “announced” regulations:

$$\{(n_t, m_t), (n_{t+1}, m_{t+1}), \ldots\}$$

while playing Nash!

$\Rightarrow$ very complicated control law...

$\Rightarrow$ not time-consistent?
Stackelberg with non-myopic followers (ctd)

A reasonable formulation: the supervisor announces feedback laws

\[ G \mapsto n(G), \quad G \mapsto m(G). \]

Followers play Nash Feedback with criterion:

\[
\max_{\{u_t^i\}} \sum_0^\infty \beta_F^t [F_i(u_t^i) - C_i(G_t + m(G_t)R - n(G_t) \sum_k u_t^k) u_t^i],
\]

and dynamics

\[ G_{t+1} = G_t + R - \sum_k u_t^k, \quad G_0, \quad \text{given}. \]

Again an optimal control problem... to be studied.
Conclusions and extensions

Conclusions:

- regulator charges users in function of their behavior, not just in function of the level of resource
- introduce strategic interaction where there was none, in case of myopic agents ($n \neq 0$)
- difficult optimal control problem
Extensions under investigation

- simulations for the optimal control problem of the regulator
  → numerical methods, value iteration, policy iteration...
- non-myopic followers in the case of constant regulator policies
- stochastic case
  → learning algorithms under test
  → importance of $m$