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Stackelberg Games of Water Extraction

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We analyze a simple problem of water extraction.

- Myopic agents harvesting a common water resource
- A regulator has some control on the cost function
- The regulator takes into account rainfall and strategic interaction between agents
Ideas of the paper (ctd.)

- Groundwater extraction: marginal cost depends on the level of the aquifer
- In general, resources with accessibility problems: cost depends on scarcity
- The main ingredient: make the cost depend on the projected evolution of the resource: before or after the extraction or rainfall
- The goal: deduce its economical and environmental consequences
- The method: a Stackelberg Game linear-quadratic static game between agents, different kinds of optimization for the regulator.
The model of Provencher and Burt

We consider the extraction of groundwater by $K$ players.

Dynamics of groundwater:

$$G_{t+1} = G_t + R - \sum_{k=1}^{K} u^k_t, \quad G_0, \quad \text{given.}$$

We suppose $R$ is a constant.

Instantaneous profit:

$$\pi^i(u^i_t, G_t) = F_i(u^i_t) - C_i(G_t) \times u^i_t.$$  

Marginal extraction cost ($C_i(.)$): depends on the current level of the groundwater.
The extended model

Introduce the more general instantaneous profit function:

$$\pi^i(u_t) = F_i(u^i_t) - C_i(G_t + mR - n \sum_j u^j_t) u^i_t$$

where $n, m \in [0, 1]$.

The extreme cases:

- $n = 0, m = 0$ (the standard case): cost based on current resource
- $n = 1, m = 1$: cost based on the state of the resource in the following period.

When $n \neq 0$ the profit function of player $i$ depends on the action of the other player: strategic interaction not just through the dynamics.
In a previous work, we developed the non-cooperative, Nash-feedback solution for agents.

Here, we develop the supervised setting

- myopic followers, static Stackelberg leader
- myopic followers, dynamic Stackelberg leader

→ an exercise in sensitivity analysis of LQ dynamic games, and non-LQ, non-concave optimal control problems.

Mostly work in progress...
Contents

1 Introduction

2 Non-cooperative setting
   • The myopic case

3 Supervised setting with myopic followers

4 Supervised setting with non-myopic followers
The case of myopic agents

Assume agents play Nash with their instantaneous profit:

\[
\max_{u^i_t} \left\{ F_i(u^i_t) - C_i(G_t + mR - n \sum_{k=1}^{K} u^k_t) u^i_t \right\}.
\]

For convenience, we continue with the particular linear-quadratic functional form:

\[
F_i(u) = a_i u - \frac{b_i}{2} u^2, \quad C_i(x) = z_i - c_i x > 0.
\]
Myopic agent reaction

In the symmetric case we find:

\[
\begin{align*}
    u(G) &= \frac{c}{b + (K + 1)cn} \cdot G + \frac{a - z + cmR}{b + (K + 1)cn} \\
    \pi(G_0) &= \frac{(cG_0 + a - z + cmR)^2}{(b + (K + 1)cn)^2} \cdot \frac{b + 2cn}{2}.
\end{align*}
\]
Myopic stock dynamics

The stock dynamics is:

\[ G_{t+1} = G_t + R - K(\alpha G_t + \gamma) \]

\[ = (1 - K\alpha)^t G_0 + (R - K\gamma) \frac{1 - (1 - K\alpha)^t}{K\alpha} \]

The asymptotic stock is:

\[ G_{\infty} = \frac{R - K\gamma}{K\alpha} = \frac{Rb}{Kc} + \frac{K + 1}{K} \frac{Rn - Rm - \frac{a - z}{c}}{c} \]
1. Introduction
   - The model
   - Analysis

2. Non-cooperative setting
   - The myopic case
     - Myopic Reaction
     - Stock Dynamics

3. Supervised setting with myopic followers
   - Static optimization
   - Stackelberg game

4. Supervised setting with non-myopic followers
A regulator is in charge of selecting the cost model, by choosing the cost parameters $n$ and $m$. The choice is announced to players, who play Nash. Followers are myopic:

$$u_t^i = \frac{c}{b + (K + 1)cn} G_t + \frac{a - z + cmR}{b + (K + 1)cn}$$

The supervisor optimizes her own criterion, taking this reaction into account.
Optimal static choice

The supervisor gets to choose $n$ and $m$ once and for all.

Optimal pricing problem

The supervisor’s problem is:

$$\max_{(n,m) \in [0,1]^2} \left\{ \sum_{t=0}^{\infty} \beta_L^t \sum_i \pi^i(u_t, G_t) \right\}$$

with the myopic agent reactions and stock dynamics

$$G_{t+1} = G_t + R - \sum_i u_t^i.$$
Optimal static choice (ctd.)

Thanks to the explicit solution for the dynamics:

\[
V_{nm}(G_0) = \frac{b + 2cn}{2} \left[ \frac{\alpha^2(G_0 - G_\infty)^2}{1 - \beta(1 - K\alpha)^2} + \frac{2R}{K} \frac{\alpha(G_0 - G_\infty)}{1 - \beta(1 - K\alpha)} + \frac{R^2}{K^2(1 - \beta)} \right]
\]

with

\[
G_\infty = \frac{R - K\gamma}{K\alpha}.
\]

\[
\alpha = \frac{c}{(K + 1)cn + b}, \quad \gamma = \frac{mRc + a - z}{(K + 1)cn + b}.
\]
→ optimization wrt \((n, m)\) for each \(G_0\).

Results:

- \(m^* = 1\) is always optimal if \(K\alpha < 1\) (monotonous traj.)
- the optimal \(n\) is:

\[
    n^* = \begin{cases} 
        0 & \text{if } \beta_L \leq \beta(G_0) \\
        1 & \text{if } \beta_L \geq \beta(G_0) \\
        n^*(\beta_L) & \text{otherwise}
    \end{cases}
\]

but \(n^*(\beta_L)\) is the root of a 4th degree polynomial.
Optimal static choice (ctd.)

A situation where $n^*(\beta_L) \notin \{0, 1\}$:
Stackelberg with myopic followers

Followers being myopic, they apply the Nash controls. If symmetric:

\[ u^i(G) = \frac{c}{b + (K + 1)cn} G + \frac{a - z + cmR}{b + (K + 1)cn}. \]

The benevolent regulator maximizes the total discounted profit on their behalf:

\[
\max_{\{n_t,m_t\}} \left\{ \sum_{t=0}^{\infty} \beta^t L \sum_i F(u^i(G_t)) - C(G_t + m_t R - n_t U(G_t))u^i(G_t) \right\}
\]

with the dynamics:

\[ G_{t+1} = G_t + R - Ku^i(G_t). \]

\[ \Rightarrow \] an optimal control problem.
First-order conditions

From the maximum principle:

\[
0 = \beta_t^L cU(G_t, n_t, m_t)(b + 2cn_t) + (b + (K + 1)cn)(q_t - q_{t-1}) - Kcq_t
\]

\[
0 = \beta_t^L cU(G_t, n_t, m_t)^2(b - (K + 1)(b + cn_t)) \\
+ (b + (K + 1)cn_t)(\lambda_t^{(n)} - \mu_t^{(n)}) \\
+ Kcq_t(K + 1)U(G_t, n_t, m_t)
\]

\[
0 = \beta_t^L cRU(G_t, n_t, m_t)(b + 2cn_t) + (b + (K + 1)cn_t)(\lambda_t^{(m)} - \mu_t^{(m)}) \\
- KcRq_t.
\]

where:

- \(\lambda_t^{(n)}\), \(\mu_t^{(n)}\) multipliers for \(n_t \geq 0\) and \(n_t \leq 1\), id for \(\lambda_t^{(m)}\), \(\mu_t^{(m)}\).
- \(U(\ldots)\) is the feedback function of followers.
Stationary situations

If a stationary state and control exists, then:

- optimal $m$: $m^* = 1$,
- optimal $n$:

\[ n^* = \begin{cases} 
0 & \text{if } \beta_L \leq \frac{b}{b + c} \\
1 & \text{if } \beta_L \geq \frac{bK(b + c) + c}{K(b + 2c) + c} \\
\frac{K}{K + 1} \frac{\beta(b + c) - b}{(1 - \beta)c} & \text{otw.}
\end{cases} \]

But we don’t know if a stationary state exists.
Stationary situations (ctd.)

Example with corner solution:

\[ a = 1, \quad b = 1, \quad c = \frac{1}{10}, \quad z = \frac{9}{10}, \quad R = 1, \quad \beta_L = \frac{95}{100} \]

gives a candidate optimal stationary solution:

\[ G^\infty = \frac{9}{2}, \quad n^* = 1. \]

Satisfies all first-order conditions.
Stationary situations (ctd.)

Example with interior solution:

\[ a = 1, \quad b = 1, \quad c = \frac{1}{10}, \quad z = \frac{9}{10}, \quad R = 1, \quad \beta_L = 0.919 \]

gives a candidate optimal stationary solution:

\[ G^\infty \approx 4.35, \quad n^* = \frac{218}{243}. \]

Satisfies all first-order conditions.
Stationary situations (ctd.)

The constant-policy value (starting from the expected steady state) is maximum at $n^*$:

![Graph showing the constant-policy value as a function of $n$]

- ConstantPolicy(n)

Values at specific $n$ points:

- $n^* = 1.8192$
- $n^* = 1.8193$
- $n^* = 1.8194$
- $n^* = 1.8195$
- $n^* = 1.8196$
- $n^* = 1.8197$
- $n^* = 1.8198$
- $n^* = 1.8199$
- $n^* = 1.82$
- $n^* = 1.8201$
Non-stationary situations

However, convexity plays us tricks.
Example:

\[ a = 1, \quad b = 1, \quad c = \frac{6}{10}, \quad z = \frac{9}{10}, \quad R = 1, \quad \beta_L = \frac{68}{100} \]

gives a candidate optimal stationary solution:

\[ G^\infty = \frac{1}{8}, \quad n^* = \frac{11}{36}. \]

But...
the optimal constant policy is not there!
And the Hamiltonian has actually a *local minimum* there wrt \( n \).
Non-stationary situations (ctd.)

Hamiltonian is for constant trajectory from the expected steady state.
Non-stationary situations (ctd.)
Non-stationary situations (ctd.)

Same values except $R = 50$, $\beta = 7/10$:

<table>
<thead>
<tr>
<th>Stock (constant)</th>
<th>Consumption (constant)</th>
<th>Gain (constant)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
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<td>100</td>
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<td>100</td>
</tr>
</tbody>
</table>

The graph shows the trends of stocks, consumptions, and gains over time, with distinct markers for each category.
A heuristic policy

These periodic policies were identified by using the following heuristic:

\[ m(G) = 1 \]
\[ n(G) = \arg\max\{V_{n_1}(G) \mid 0 \leq n \leq 1\}. \]

\textit{i.e.} use the best static policy... but as state feedback.
Stackelberg with non-myopic followers

Followers being non-myopic, at time $t$ they react to the sequence of “announced” regulations:

$$\{(n_t, m_t), (n_{t+1}, m_{t+1}), \ldots\}$$

while playing Nash!

$\implies$ very complicated control law...

$\implies$ not time-consistent?
Stackelberg with non-myopic followers (ctd)

A reasonable formulation: the supervisor announces feedback laws

\[ G \mapsto n(G), \quad G \mapsto m(G). \]

Followers play Nash Feedback with criterion:

\[
\max_{\{u_i^t\}} \sum_{t=0}^{\infty} \beta_F^t [F_i(u_i^t) - C_i(G_t + m(G_t)R - n(G_t) \sum_k u_k^t) u_i^t],
\]

and dynamics

\[ G_{t+1} = G_t + R - \sum_k u_k^t, \quad G_0, \text{ given.} \]

Again an optimal control problem... to be studied.
Conclusions:

- regulator charges users in function of their behavior, not just in function of the level of resource
- introduce strategic interaction where there was none, in case of myopic agents ($n \neq 0$)
- difficult optimal control problem
Extensions under investigation

- simulations for the optimal control problem of the regulator
  → numerical methods, value iteration, policy iteration...
- non-myopic followers in the case of constant regulator policies
- stochastic case
  → learning algorithms under test
  → importance of $m$