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Stackelberg Games of Water Extraction

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We analyze a simple problem of water extraction.

- Myopic agents harvesting a common water resource
- A regulator has some control on the cost function
- The regulator takes into account rainfall and strategic interaction between agents
Ideas of the paper (ctd.)

- Groundwater extraction: marginal cost depends on the level of the aquifer
- In general, resources with accessibility problems: cost depends on scarcity
- The main ingredient: make the cost depend on the projected evolution of the resource: before or after the extraction or rainfall
- The goal: deduce its economical and environmental consequences
- The method: a Stackelberg Game linear-quadratic static game between agents, different kinds of optimization for the regulator.
The model of Provencher and Burt

We consider the extraction of groundwater by $K$ players. Dynamics of groundwater:

\[ G_{t+1} = G_t + R - \sum_{k=1}^{K} u_t^k, \quad G_0, \quad \text{given}. \]

We suppose $R$ is a constant.

Instantaneous profit:

\[ \pi^i(u_t^i, G_t) = F_i(u_t^i) - C_i(G_t) \times u_t^i. \]

Marginal extraction cost ($C_i(.)$): depends on the current level of the groundwater.
The extended model

Introduce the more general instantaneous profit function:

$$\pi^i(u_t) = F_i(u^i_t) - C_i(G_t + mR - n \sum_j u^j_t) u^i_t$$

where $n, m \in [0, 1]$.

The extreme cases:

- $n = 0, m = 0$ (the standard case): cost based on current resource
- $n = 1, m = 1$: cost based on the state of the resource in the following period.

When $n \neq 0$ the profit function of player $i$ depends on the action of the other player: strategic interaction not just through the dynamics.
Angles of analysis

In a previous work, we developed the non-cooperative, Nash-feedback solution for agents.

Here, we develop the supervised setting

- myopic followers, static Stackelberg leader
- myopic followers, dynamic Stackelberg leader

→ an exercise in sensitivity analysis of LQ dynamic games, and non-LQ, non-concave optimal control problems.

Mostly work in progress...
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   • The myopic case

3 Supervised setting with myopic followers

4 Supervised setting with non-myopic followers
The case of myopic agents

Assume agents play Nash with their instantaneous profit:

\[
\max_{u^i} \left\{ F_i(u^i_t) - C_i(G_t + mR - n \sum_{k=1}^{K} u^k_t) u^i_t \right\}.
\]

For convenience, we continue with the particular linear-quadratic functional form:

\[
F_i(u) = a_i u - \frac{b_i}{2} u^2, \quad C_i(x) = z_i - c_i x > 0.
\]
Myopic agent reaction

In the **symmetric** case we find:

\[
 u(G) = \frac{c}{b + (K + 1)cn} G + \frac{a - z + cmR}{b + (K + 1)cn}.
\]

The value function of each player is:

\[
 \pi(G_0) = \frac{(cG_0 + a - z + cmR)^2}{(b + (K + 1)cn)^2} \frac{b + 2cn}{2}.
\]
Myopic stock dynamics

The stock dynamics is:

\[
G_{t+1} = G_t + R - K(\alpha G_t + \gamma) \\
= (1 - K\alpha)^t G_0 + (R - K\gamma) \frac{1 - (1 - K\alpha)^t}{K\alpha}.
\]

The asymptotic stock is:

\[
G_\infty = \frac{R - K\gamma}{K\alpha} = \frac{Rb}{Kc} + \frac{K + 1}{K} \frac{Rn - Rm - \frac{a - z}{c}}{Kc}.
\]
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4 Supervised setting with non-myopic followers
A regulator is in charge of selecting the cost model, by choosing the cost parameters $n$ and $m$. The choice is announced to players, who play Nash. Followers are myopic:

$$u^i_t = \frac{c}{b + (K + 1)cn} G_t + \frac{a - z + cmR}{b + (K + 1)cn}$$

The supervisor optimizes her own criterion, taking this reaction into account.
Optimal static choice

The supervisor gets to choose $n$ and $m$ once and for all.

Optimal pricing problem

The supervisor’s problem is:

$$\max_{(n,m)\in[0,1]^2} \left\{ \sum_{t=0}^{\infty} \beta^t \sum_i \pi^i(u_t, G_t) \right\}$$

with the myopic agent reactions and stock dynamics

$$G_{t+1} = G_t + R - \sum_i u^i_t.$$
Optimal static choice (ctd.)

Thanks to the explicit solution for the dynamics:

$$V_{nm}(G_0) = \frac{b + 2cn}{2} \left[ \frac{\alpha^2(G_0 - G_\infty)^2}{1 - \beta(1 - K\alpha)^2} + \frac{2R}{K} \frac{\alpha(G_0 - G_\infty)}{1 - \beta(1 - K\alpha)} \right]$$

$$+ \frac{R^2}{K^2(1 - \beta)}$$

with

$$G_\infty = \frac{R - K\gamma}{K\alpha}.$$

$$\alpha = \frac{c}{(K + 1)cn + b} \quad \gamma = \frac{mRc + a - z}{(K + 1)cn + b}.$$
Optimal static choice (ctd.)

→ optimization wrt \((n, m)\) for each \(G_0\).

Results:

- \(m^* = 1\) is always optimal if \(K\alpha < 1\) (monotonous traj.)
- the optimal \(n\) is:

\[
n^* = \begin{cases} 
0 & \text{if } \beta_L \leq \beta(G_0) \\
1 & \text{if } \beta_L \geq \beta(G_0) \\
n^*(\beta_L) & \text{otw.}
\end{cases}
\]

but \(n^*(\beta_L)\) is the root of a 4th degree polynomial.
Optimal static choice (ctd.)

A situation where \( n^*(\beta_L) \not\in \{0, 1\} : \)

\[
\begin{align*}
\beta = 0.64 & \quad \text{\textcolor{purple}{purple}} \\
\beta = 0.66 & \quad \text{\textcolor{cyan}{cyan}} \\
\beta = 0.68 & \quad \text{\textcolor{teal}{teal}} \\
\beta = 0.70 & \quad \text{\textcolor{orange}{orange}}
\end{align*}
\]
Stackelberg with myopic followers

Followers being myopic, they apply the Nash controls. If symmetric:

\[ u^i(G) = \frac{c}{b + (K + 1)cn} G + \frac{a - z + cmR}{b + (K + 1)cn}. \]

The benevolent regulator maximizes the total discounted profit on their behalf:

\[
\max_{\{n_t, m_t\}} \left\{ \sum_{t=0}^{\infty} \sum_i \beta^t_i F(u^i(G_t)) - C(G_t + m_t R - n_t U(G_t))u^i(G_t) \right\}
\]

with the dynamics:

\[ G_{t+1} = G_t + R - Ku^i(G_t). \]

\[ \Rightarrow \] an optimal control problem.
First-order conditions

From the maximum principle:

\[ 0 = \beta_t^L cU(G_t, n_t, m_t)(b + 2cn_t) + (b + (K + 1)cn)(q_t - q_{t-1}) - Kcq_t \]

\[ 0 = \beta_t^L cU(G_t, n_t, m_t)^2(b - (K + 1)(b + cn_t)) + (b + (K + 1)cn_t)(\lambda_t^{(n)} - \mu_t^{(n)}) + Kcq_t(K + 1)U(G_t, n_t, m_t) \]

\[ 0 = \beta_t^L cRU(G_t, n_t, m_t)(b + 2cn_t) + (b + (K + 1)cn_t)(\lambda_t^{(m)} - \mu_t^{(m)}) - KcRq_t. \]

where:

- \( \lambda_t^{(n)}, \mu_t^{(n)} \) multipliers for \( n_t \geq 0 \) and \( n_t \leq 1 \), id for \( \lambda_t^{(m)}, \mu_t^{(m)} \).
- \( U(\ldots) \) is the feedback function of followers.
Stationary situations

If a stationary state and control exists, then:

- optimal $m$: $m^* = 1$,
- optimal $n$:

$$n^* = \begin{cases} 
0 & \text{if } \beta_L \leq \frac{b}{b+c} \\
1 & \text{if } \beta_L \geq \frac{K(b+c)+c}{K(b+2c)+c} \\
\frac{K}{K+1} \frac{\beta(b+c) - b}{(1-\beta)c} & \text{otw.}
\end{cases}$$

But we don’t know if a stationary state exists.
Example with corner solution:

\[ a = 1, \quad b = 1, \quad c = \frac{1}{10}, \quad z = \frac{9}{10}, \quad R = 1, \quad \beta_L = \frac{95}{100} \]

gives a candidate optimal stationary solution:

\[ G^\infty = \frac{9}{2}, \quad n^* = 1. \]

Satisfies all first-order conditions.
Example with interior solution:

\[ a = 1, \quad b = 1, \quad c = \frac{1}{10}, \quad z = \frac{9}{10}, \quad R = 1, \quad \beta_L = 0.919 \]

gives a candidate optimal stationary solution:

\[ G^\infty \simeq 4.35, \quad n^* = \frac{218}{243}. \]

Satisfies all first-order conditions.
Stationary situations (ctd.)

The constant-policy value (starting from the expected steady state) is maximum at $n^*$:
Non-stationary situations

However, convexity plays us tricks. Example:

\[ a = 1, \quad b = 1, \quad c = \frac{6}{10}, \quad z = \frac{9}{10}, \quad R = 1, \quad \beta_L = \frac{68}{100} \]

gives a candidate optimal stationary solution:

\[ G^\infty = \frac{1}{8}, \quad n^* = \frac{11}{36}. \]

But...
the optimal constant policy is not there!
And the Hamiltonian has actually a local minimum there wrt \( n \).
Non-stationary situations (ctd.)

Hamiltonian is for constant trajectory from the expected steady state.
Non-stationary situations (ctd.)
Non-stationary situations (ctd.)

Same values except $R = 50$, $\beta = 7/10$: 

![Graph showing stocks, consumptions, and gains over time with legend indicating different lines for Stock, Consumption, Gain, Stock (constant), Consumption (constant), and Gain (constant).]
These periodic policies were identified by using the following heuristic:

$$m(G) = 1$$
$$n(G) = \arg\max\{ V_{n1}(G) \mid 0 \leq n \leq 1 \}.$$

i.e. use the best static policy... but as state feedback.
Stackelberg with non-myopic followers

Followers being non-myopic, at time $t$ they react to the sequence of “announced” regulations:

$$\{(n_t, m_t), (n_{t+1}, m_{t+1}), \ldots \}$$

While playing Nash!

$\implies$ very complicated control law...

$\implies$ not time-consistent?
Stackelberg with non-myopic followers (ctd)

A reasonable formulation: the supervisor announces feedback laws

\[ G \mapsto n(G), \quad G \mapsto m(G). \]

Followers play Nash Feedback with criterion:

\[
\max_{\{u_t^i\}} \sum_0^\infty \beta_F^t [F_i(u_t^i) - C_i(G_t + m(G_t)R - n(G_t) \sum_k u_t^k) u_t^i],
\]

and dynamics

\[ G_{t+1} = G_t + R - \sum_k u_t^k, \quad G_0, \quad \text{given}. \]

Again an optimal control problem... to be studied.
Conclusions and extensions

Conclusions:
- regulator charges users in function of their behavior, not just in function of the level of resource
- introduce strategic interaction where there was none, in case of myopic agents ($n \neq 0$)
- difficult optimal control problem
Extensions under investigation

- simulations for the optimal control problem of the regulator
  → numerical methods, value iteration, policy iteration...
- non-myopic followers in the case of constant regulator policies
- stochastic case
  → learning algorithms under test
  → importance of $m$