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# Stackelberg Games of Water Extraction

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# Ideas of the paper

We analyze a simple problem of water extraction.

- Myopic agents harvesting a common water resource
- A regulator has some control on the cost function
- The regulator takes into account rainfall and strategic interaction between agents

## Ideas of the paper (ctd.)

- Groundwater extraction: marginal cost depends on the level of the aquifer
- In general, resources with accessibility problems: cost depends on scarcity
- **The main ingredient: make the cost depend on the projected evolution of the resource: before or after the extraction or rainfall**
- The goal: deduce its economical and environmental consequences
- The method: a Stackelberg Game  
linear-quadratic static game between agents,  
different kinds of optimization for the regulator.

# The model of Provencher and Burt

We consider the extraction of groundwater by  $K$  players.  
Dynamics of groundwater:

$$G_{t+1} = G_t + R - \sum_{k=1}^K u_t^k, \quad G_0, \quad \text{given.}$$

We suppose  $R$  is a constant.

Instantaneous profit:

$$\pi^i(u_t^i, G_t) = F_i(u_t^i) - C_i(G_t) \times u_t^i.$$

Marginal extraction cost ( $C_i(\cdot)$ ): depends on the **current** level of the groundwater.

# The extended model

Introduce the more general instantaneous profit function:

$$\pi^i(u_t) = F_i(u_t^i) - C_i(G_t + mR - n \sum_j u_t^j) u_t^i$$

where  $n, m \in [0, 1]$ .

The extreme cases:

- $n = 0, m = 0$  (the standard case): cost based on current resource
- $n = 1, m = 1$ : cost based on the state of the resource in the following period.

When  $n \neq 0$  the profit function of player  $i$  depends on the action of the other player: strategic interaction not just through the dynamics.

# Angles of analysis

In a previous work, we developed the non-cooperative, Nash-feedback solution for agents.

Here, we develop the supervised setting

- myopic followers, static Stackelberg leader
- myopic followers, dynamic Stackelberg leader

→ an exercise in sensitivity analysis of LQ dynamic games, and non-LQ, non-concave optimal control problems.

Mostly work in progress...

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# The case of myopic agents

Assume agents play Nash with their instantaneous profit:

$$\max_{u^i} \left\{ F_i(u_t^i) - C_i(G_t + mR - n \sum_{k=1}^K u_t^k) u_t^i \right\}.$$

For convenience, we continue with the particular linear-quadratic functional form:

$$F_i(u) = a_i u - \frac{b_i}{2} u^2, \quad C_i(x) = z_i - c_i x > 0.$$

# Myopic agent reaction

In the **symmetric** case we find:

$$u(G) = \underbrace{\frac{c}{b + (K + 1)cn}}_{\alpha} G + \underbrace{\frac{a - z + cmR}{b + (K + 1)cn}}_{\gamma}.$$

The value function of each player is:

$$\pi(G_0) = \frac{(cG_0 + a - z + cmR)^2}{(b + (K + 1)cn)^2} \frac{b + 2cn}{2}.$$

# Myopic stock dynamics

The stock dynamics is:

$$\begin{aligned} G_{t+1} &= G_t + R - K(\alpha G_t + \gamma) \\ &= (1 - K\alpha)^t G_0 + (R - K\gamma) \frac{1 - (1 - K\alpha)^t}{K\alpha}. \end{aligned}$$

The asymptotic stock is:

$$G_\infty = \frac{R - K\gamma}{K\alpha} = \frac{Rb}{Kc} + \frac{K+1}{K} Rn - Rm - \frac{a-z}{c}.$$

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## Supervised setting

A **regulator** is in charge of selecting the cost model, by choosing the cost parameters  $n$  and  $m$ .

The choice is announced to players, who play Nash.

Followers are myopic:

$$u_t^i = \frac{c}{b + (K + 1)cn} G_t + \frac{a - z + cmR}{b + (K + 1)cn}$$

The supervisor optimizes her own criterion, taking this reaction into account.

# Optimal static choice

The supervisor gets to choose  $n$  and  $m$  once and for all.

## Optimal pricing problem

The supervisor's problem is:

$$\max_{(n,m) \in [0,1]^2} \left\{ \sum_{t=0}^{\infty} \beta_L^t \sum_i \pi^i(u_t, G_t) \right\}$$

with the myopic agent reactions and stock dynamics

$$G_{t+1} = G_t + R - \sum_i u_t^i .$$

## Optimal static choice (ctd.)

Thanks to the explicit solution for the dynamics:

$$V_{nm}(G_0) = \frac{b + 2cn}{2} \left[ \frac{\alpha^2(G_0 - G_\infty)^2}{1 - \beta(1 - K\alpha)^2} + \frac{2R}{K} \frac{\alpha(G_0 - G_\infty)}{1 - \beta(1 - K\alpha)} + \frac{R^2}{K^2(1 - \beta)} \right]$$

with

$$G_\infty = \frac{R - K\gamma}{K\alpha} \cdot$$

$$\alpha = \frac{c}{(K + 1)cn + b} \quad \gamma = \frac{mRc + a - z}{(K + 1)cn + b} \cdot$$

## Optimal static choice (ctd.)

→ optimization wrt  $(n, m)$  for each  $G_0$ .

Results:

- $m^* = 1$  is always optimal if  $K\alpha < 1$  (monotonous traj.)
- the optimal  $n$  is:

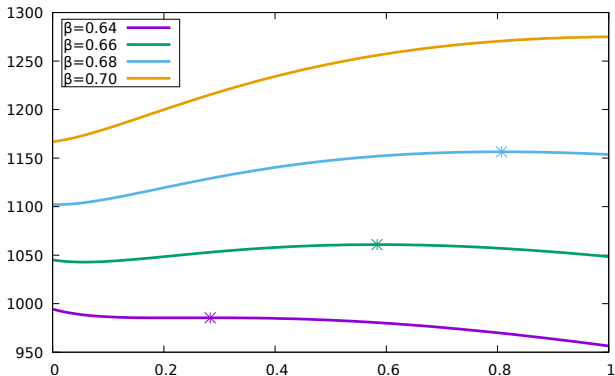
$$n^* = \begin{cases} 0 & \text{if } \beta_L \leq \underline{\beta}(G_0) \\ 1 & \text{if } \beta_L \geq \overline{\beta}(G_0) \\ n^*(\beta_L) & \text{otw.} \end{cases}$$

but  $n^*(\beta_L)$  is the root of a 4<sup>th</sup> degree polynomial.



## Optimal static choice (ctd.)

A situation where  $n^*(\beta_L) \notin \{0, 1\}$ :



## Stackelberg with myopic followers

Followers being myopic, they apply the Nash controls. If symmetric:

$$u^i(G) = \frac{c}{b + (K + 1)cn} G + \frac{a - z + cmR}{b + (K + 1)cn}.$$

The benevolent regulator maximizes the total discounted profit on their behalf:

$$\max_{\{n_t, m_t\}} \left\{ \sum_{t=0}^{\infty} \beta_L^t \sum_i F(u^i(G_t)) - C(G_t + m_t R - n_t U(G_t)) u^i(G_t) \right\}$$

with the dynamics:

$$G_{t+1} = G_t + R - Ku^i(G_t).$$

⇒ an optimal control problem.

# First-order conditions

From the maximum principle:

$$0 = \beta_L^t c U(G_t, n_t, m_t) (b + 2cn_t) + (b + (K + 1)cn) (q_t - q_{t-1}) - Kcq_t$$

$$0 = \beta_L^t c U(G_t, n_t, m_t)^2 (b - (K + 1)(b + cn_t)) + (b + (K + 1)cn_t) (\lambda_t^{(n)} - \mu_t^{(n)}) + Kcq_t (K + 1) U(G_t, n_t, m_t)$$

$$0 = \beta_L^t c R U(G_t, n_t, m_t) (b + 2cn_t) + (b + (K + 1)cn_t) (\lambda_t^{(m)} - \mu_t^{(m)}) - KcRq_t.$$

where:

- $\lambda_t^{(n)}, \mu_t^{(n)}$  multipliers for  $n_t \geq 0$  and  $n_t \leq 1$ , *id* for  $\lambda_t^{(m)}, \mu_t^{(m)}$ .
- $U(\dots)$  is the feedback function of followers.

## Stationary situations

If a stationary state and control exists, then:

- optimal  $m$ :  $m^* = 1$ ,
- optimal  $n$ :

$$n^* = \begin{cases} 0 & \text{if } \beta_L \leq \frac{b}{b+c} \\ 1 & \text{if } \beta_L \geq \frac{K(b+c) + c}{K(b+2c) + c} \\ \frac{K}{K+1} \frac{\beta(b+c) - b}{(1-\beta)c} & \text{otw.} \end{cases}$$

But we don't know if a stationary state exists.

## Stationary situations (ctd.)

Example with corner solution:

$$a = 1, \quad b = 1, \quad c = \frac{1}{10}, \quad z = \frac{9}{10}, \quad R = 1, \quad \beta_L = \frac{95}{100}$$

gives a candidate optimal stationary solution:

$$G^\infty = \frac{9}{2}, \quad n^* = 1.$$

Satisfies all first-order conditions.

## Stationary situations (ctd.)

Example with interior solution:

$$a = 1, \quad b = 1, \quad c = \frac{1}{10}, \quad z = \frac{9}{10}, \quad R = 1, \quad \beta_L = 0.919$$

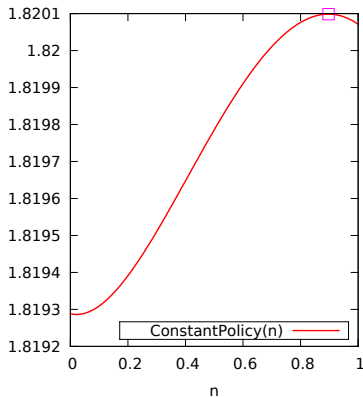
gives a candidate optimal stationary solution:

$$G^\infty \simeq 4.35, \quad n^* = \frac{218}{243}.$$

Satisfies all first-order conditions.

## Stationary situations (ctd.)

The constant-policy value (starting from the expected steady state) is maximum at  $n^*$ :



## Non-stationary situations

However, convexity plays us tricks.

Example:

$$a = 1, \quad b = 1, \quad c = \frac{6}{10}, \quad z = \frac{9}{10}, \quad R = 1, \quad \beta_L = \frac{68}{100}$$

gives a candidate optimal stationary solution:

$$G^\infty = \frac{1}{8}, \quad n^* = \frac{11}{36}.$$

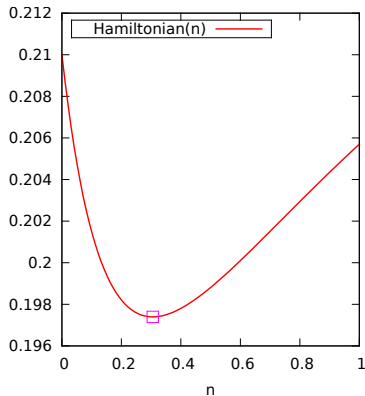
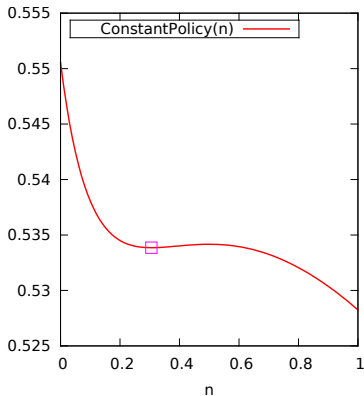
But...

the optimal constant policy is not there!

And the Hamiltonian has actually a *local minimum* there wrt  $n$ .

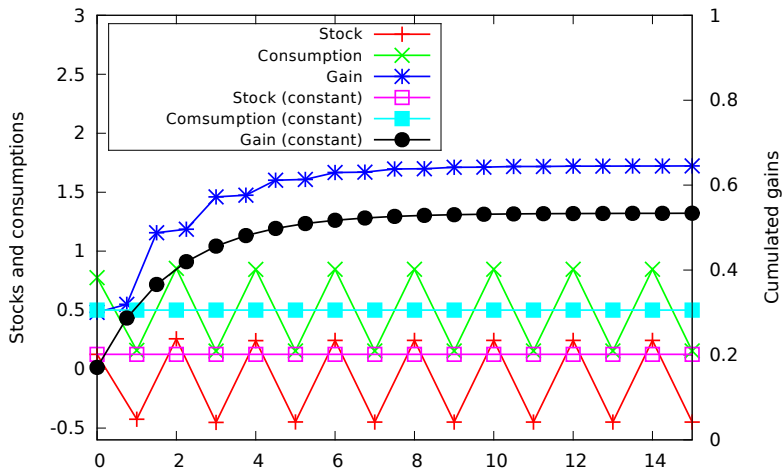


## Non-stationary situations (ctd.)



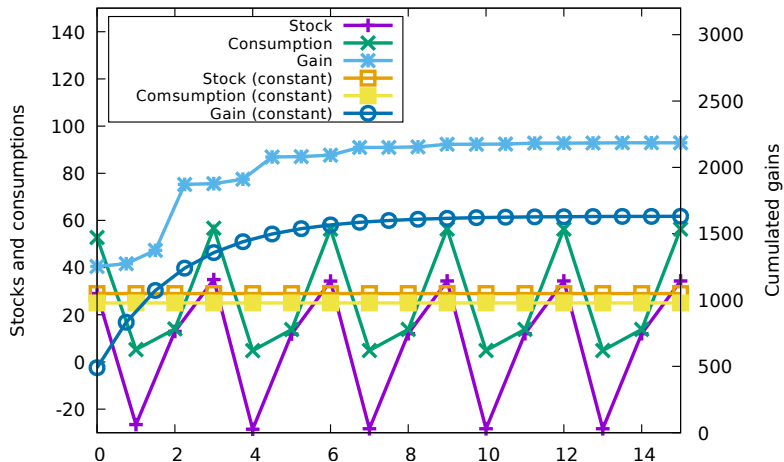
Hamiltonian is for constant trajectory from the expected steady state.

# Non-stationary situations (ctd.)



# Non-stationary situations (ctd.)

Same values except  $R = 50$ ,  $\beta = 7/10$ :



## A heuristic policy

These periodic policies were identified by using the following heuristic:

$$m(G) = 1$$

$$n(G) = \operatorname{argmax}\{V_{n1}(G) \mid 0 \leq n \leq 1\}.$$

*i.e.* use the best static policy... but as state feedback.

## Stackelberg with non-myopic followers

Followers being non-myopic, at time  $t$  they react to the sequence of “announced” regulations:

$$\{(n_t, m_t), (n_{t+1}, m_{t+1}), \dots\}$$

while playing Nash!

⇒ very complicated control law...

⇒ not time-consistent?

## Stackelberg with non-myopic followers (ctd)

A reasonable formulation: the supervisor announces feedback laws

$$G \mapsto n(G), \quad G \mapsto m(G).$$

Followers play Nash Feedback with criterion:

$$\max_{\{u_t^i\}} \sum_0^{\infty} \beta_F^t [F_i(u_t^i) - C_i(G_t + m(G_t)R - n(G_t) \sum_k u_t^k) u_t^i],$$

and dynamics

$$G_{t+1} = G_t + R - \sum_k u_t^k, \quad G_0, \quad \text{given.}$$

Again an optimal control problem... to be studied.

# Conclusions and extensions

## Conclusions:

- regulator charges users in function of their behavior, not just in function of the level of resource
- introduce strategic interaction where there was none, in case of myopic agents ( $n \neq 0$ )
- difficult optimal control problem

# Extensions under investigation

- simulations for the optimal control problem of the regulator
  - numerical methods, value iteration, policy iteration...
- non-myopic followers in the case of constant regulator policies
- stochastic case
  - learning algorithms under test
  - importance of  $m$