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# Normalizing constants of log-concave densities

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**Abstract:** We derive explicit bounds for the computation of normalizing constants  $Z$  for log-concave densities  $\pi = e^{-U}/Z$  w.r.t. the Lebesgue measure on  $\mathbb{R}^d$ . Our approach relies on a Gaussian annealing combined with recent and precise bounds on the Unadjusted Langevin Algorithm [15]. Polynomial bounds in the dimension  $d$  are obtained with an exponent that depends on the assumptions made on  $U$ . The algorithm also provides a theoretically grounded choice of the annealing sequence of variances. A numerical experiment supports our findings. Results of independent interest on the mean squared error of the empirical average of locally Lipschitz functions are established.

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## 1. Introduction

Let  $U : \mathbb{R}^d \rightarrow \mathbb{R}$  be a continuously differentiable convex function such that  $Z = \int_{\mathbb{R}^d} e^{-U(x)} dx < +\infty$ .  $Z$  is the normalizing constant of the probability density  $\pi$  associated with the potential  $U$ , defined for  $x \in \mathbb{R}^d$  by  $\pi(x) = Z^{-1}e^{-U(x)}$ . We discuss in this paper a method to estimate  $Z$  with polynomial complexity in the dimension  $d$ .

Computing the normalizing constant is a challenge which has applications in Bayesian inference and statistical physics in particular. In statistical physics,  $Z$  is better known under the name of partition function or free energy [3], [30]. Free energy differences allow to quantify the relative likelihood of different states (microscopic configurations) and are linked to thermodynamic work and

heat exchanges. In Bayesian inference, the models can be compared by the computation of the Bayes factor which is the ratio of two normalizing constants (see e.g. [43, chapter 7]). This problem has consequently attracted a wealth of contribution; see for example [9, chapter 5], [31], [20], [2], [16], [29], [49] and, for a more specific molecular simulations flavor, [30]. Compared to the large number of proposed methods to estimate  $Z$ , few theoretical guarantees have been obtained on the output of these algorithms; see below for further references and comments. Our algorithm relies on a sequence of Gaussian densities with increasing variances, combined with the precise bounds of [15].

The paper is organized as follows. The outline of the algorithm is first described, followed by the assumptions made on  $U$ . Our main results are then stated and compared to previous works on the subject. The theoretical analysis of the algorithm is detailed in Section 2. In Section 3, a numerical experiment is provided to support our theoretical claims. Finally, the proofs are gathered in Section 5. In Section 4, a result of independent interest on the mean squared error of the empirical average of locally Lipschitz functions is established.

### Notations and conventions

Denote by  $\mathcal{B}(\mathbb{R}^d)$  the Borel  $\sigma$ -field of  $\mathbb{R}^d$ . For  $\mu$  a probability measure on  $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$  and  $f$  a  $\mu$ -integrable function, denote by  $\mu(f)$  the integral of  $f$  w.r.t.  $\mu$ . We say that  $\zeta$  is a transference plan of  $\mu$  and  $\nu$  if it is a probability measure on  $(\mathbb{R}^d \times \mathbb{R}^d, \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d))$  such that for all measurable sets  $A$  of  $\mathbb{R}^d$ ,  $\zeta(A \times \mathbb{R}^d) = \mu(A)$  and  $\zeta(\mathbb{R}^d \times A) = \nu(A)$ . We denote by  $\Pi(\mu, \nu)$  the set of transference plans of  $\mu$  and  $\nu$ . Furthermore, we say that a couple of  $\mathbb{R}^d$ -random variables  $(X, Y)$  is a coupling of  $\mu$  and  $\nu$  if there exists  $\zeta \in \Pi(\mu, \nu)$  such that  $(X, Y)$  are distributed according to  $\zeta$ . For two probability measures  $\mu$  and  $\nu$ , we define the Wasserstein distance of order  $p \geq 1$  as

$$W_p(\mu, \nu) \stackrel{\text{def}}{=} \left( \inf_{\zeta \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \|x - y\|^p d\zeta(x, y) \right)^{1/p}. \quad (1)$$

By [46, Theorem 4.1], for all  $\mu, \nu$  probability measure on  $\mathbb{R}^d$ , there exists a transference plan  $\zeta^* \in \Pi(\mu, \nu)$  such that the infimum in (1) is reached in  $\zeta^*$ .  $\zeta^*$  is called an optimal transference plan associated with  $W_p$ .

$f : \mathbb{R}^d \rightarrow \mathbb{R}$  is a Lipschitz function if there exists  $C \geq 0$  such that for all  $x, y \in \mathbb{R}^d$ ,  $|f(x) - f(y)| \leq C \|x - y\|$ . Then we denote

$$\|f\|_{\text{Lip}} = \sup\{|f(x) - f(y)| \|x - y\|^{-1} \mid x, y \in \mathbb{R}^d, x \neq y\}.$$

For  $k \in \mathbb{N}$ ,  $\mathcal{C}^k(\mathbb{R}^d)$  denotes the set of  $k$ -continuously differentiable functions  $\mathbb{R}^d \rightarrow \mathbb{R}$ , with the convention that  $\mathcal{C}^0(\mathbb{R}^d)$  is the set of continuous functions. Let  $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ ,  $n, m \in \mathbb{N}^*$  and  $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a twice continuously differentiable function. Denote by  $\nabla F$  and  $\nabla^2 F$  the Jacobian and the Hessian of  $F$  respectively. For  $m = 1$ , the Laplacian is defined by  $\Delta F = \text{Tr} \nabla^2 F$  where  $\text{Tr}$  is the trace operator. In the sequel, we take the convention that for  $n, p \in \mathbb{N}$ ,

$n < p$  then  $\sum_p^n = 0$  and  $\prod_p^n = 1$ . By convention,  $\inf \{\emptyset\} = +\infty$ ,  $\sup \{\emptyset\} = -\infty$  and for  $j > i$  in  $\mathbb{Z}$ ,  $\{j, \dots, i\} = \emptyset$ . For a finite set  $E$ ,  $|E|$  denotes the cardinality of  $E$ . For  $a, b \in \mathbb{R}$ ,  $a \wedge b = \min(a, b)$  and  $a \vee b = \max(a, b)$ . Let  $\psi, \phi : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$ . We write  $\psi = \tilde{O}(\phi)$  if there exists  $t_0 > 0$ ,  $C, c > 0$  such that  $\psi(t) \leq C\phi(t) |\log t|^c$  for all  $t \in (0, t_0]$ . Denote by  $B(x, r) = \{y \in \mathbb{R}^d : \|y - x\| \leq r\}$ .

**Presentation of the algorithm**

Since  $Z < +\infty$  and  $U$  is convex, by [7, Lemma 2.2.1], there exist constants  $\rho_1 > 0$  and  $\rho_2 \in \mathbb{R}$  such that  $U(x) \geq \rho_1 \|x\| - \rho_2$ . Therefore, by continuity,  $U$  has a minimum  $x^*$ . Without loss of generality, it is assumed in the sequel that  $x^* = 0$  and  $U(x^*) = 0$ .

Let  $M \in \mathbb{N}^*$ ,  $\{\sigma_i^2\}_{i=0}^M$  be a positive increasing sequence of real numbers and set  $\sigma_M^2 = +\infty$ . Consider the sequence of functions  $\{U_i\}_{i=0}^M$  defined for all  $i \in \{0, \dots, M\}$  and  $x \in \mathbb{R}^d$  by

$$U_i(x) = \frac{\|x\|^2}{2\sigma_i^2} + U(x) , \tag{2}$$

with the convention  $1/\infty = 0$ . We define a sequence of probability densities  $\{\pi_i\}_{i=0}^M$  for  $i \in \{0, \dots, M\}$  and  $x \in \mathbb{R}^d$  by

$$\pi_i(x) = Z_i^{-1} e^{-U_i(x)} , \quad Z_i = \int_{\mathbb{R}^d} e^{-U_i(y)} dy . \tag{3}$$

The dependence of  $Z_i$  in  $\sigma_i^2$  is implicit. By definition, note that  $U_M = U$ ,  $Z_M = Z$  and  $\pi_M = \pi$ . As in the multistage sampling method [22, Section 3.3], we use the following decomposition

$$\frac{Z}{Z_0} = \prod_{i=0}^{M-1} \frac{Z_{i+1}}{Z_i} . \tag{4}$$

$Z_0$  is estimated by choosing  $\sigma_0^2$  small enough so that  $\pi_0$  is sufficiently close to a Gaussian distribution of mean 0 and covariance  $\sigma_0^2 \text{Id}$ . For  $i \in \{0, \dots, M-1\}$ , the ratio  $Z_{i+1}/Z_i$  may be expressed as

$$\frac{Z_{i+1}}{Z_i} = \int_{\mathbb{R}^d} g_i(x) \pi_i(x) dx = \pi_i(g_i) , \tag{5}$$

where  $g_i : \mathbb{R}^d \rightarrow \mathbb{R}_+$  is defined for all  $x \in \mathbb{R}^d$  by

$$g_i(x) = \exp\left(a_i \|x\|^2\right) , \quad a_i = \frac{1}{2} \left( \frac{1}{\sigma_i^2} - \frac{1}{\sigma_{i+1}^2} \right) . \tag{6}$$

The quantity  $\pi_i(g_i)$  is estimated by the Unadjusted Langevin Algorithm (ULA) targeting  $\pi_i$ . Introduced in [18] and [38] (see also [44]), the ULA algorithm can

be described as follows. For  $i \in \{0, \dots, M-1\}$ , the (overdamped) Langevin stochastic differential equation (SDE) is given by

$$dY_{i,t} = -\nabla U_i(Y_{i,t})dt + \sqrt{2}dB_{i,t}, \quad Y_{i,0} = 0, \quad (7)$$

where  $\{(B_{i,t})_{t \geq 0}\}_{i=0}^{M-1}$  are independent  $d$ -dimensional Brownian motions. The sampling method is based on the Euler discretization of the Langevin diffusion, which defines a discrete-time Markov chain, for  $i \in \{0, \dots, M-1\}$  and  $k \in \mathbb{N}$

$$X_{i,k+1} = X_{i,k} - \gamma_i \nabla U_i(X_{i,k}) + \sqrt{2\gamma_i} W_{i,k+1}, \quad X_{i,0} = 0, \quad (8)$$

where  $\{(W_{i,k})_{k \in \mathbb{N}^*}\}_{i=0}^{M-1}$  are independent i.i.d. sequences of standard Gaussian random variables and  $\gamma_i > 0$  is the stepsize. For  $i \in \{0, \dots, M-1\}$ , consider the following estimator of  $Z_{i+1}/Z_i$ ,

$$\hat{\pi}_i(g_i) = \frac{1}{n_i} \sum_{k=N_i+1}^{N_i+n_i} g_i(X_{i,k}), \quad (9)$$

where  $n_i \geq 1$  is the sample size and  $N_i \geq 0$  the burn-in period. We introduce the following assumptions on  $U$ .

**H1.**  $U : \mathbb{R}^d \rightarrow \mathbb{R}$  is continuously differentiable and  $L$ -gradient Lipschitz, i.e. there exists  $L \geq 0$  such that for all  $x, y \in \mathbb{R}^d$ ,

$$\|\nabla U(x) - \nabla U(y)\| \leq L \|x - y\|. \quad (10)$$

**H2(m).**  $U : \mathbb{R}^d \rightarrow \mathbb{R}$  is continuously differentiable and satisfies for all  $x, y \in \mathbb{R}^d$ ,

$$U(y) \geq U(x) + \langle \nabla U(x), y - x \rangle + (m/2) \|x - y\|^2. \quad (11)$$

**H3.** The function  $U$  is three times continuously differentiable and there exists  $\tilde{L} \geq 0$  such that for all  $x, y \in \mathbb{R}^d$

$$\|\nabla^2 U(x) - \nabla^2 U(y)\| \leq \tilde{L} \|x - y\|. \quad (12)$$

The strongly convex case (**H2(m)** with  $m > 0$ ) is considered in Section 2.1 and the convex case (**H2(m)** with  $m = 0$ ) is dealt with in Section 2.2. Assuming **H1** and **H2(m)** for  $m \geq 0$ , for  $i \in \{0, \dots, M\}$ ,  $U_i$  defined in (2) is  $L_i$ -gradient Lipschitz and  $m_i$ -strongly convex if  $m_i > 0$  (and convex if  $m_i = 0$ ) where

$$L_i = L + \frac{1}{\sigma_i^2}, \quad m_i = m + \frac{1}{\sigma_i^2}. \quad (13)$$

Define also the following useful quantities,

$$\kappa = \frac{2mL}{m+L}, \quad \kappa_i = \frac{2m_i L_i}{m_i + L_i}. \quad (14)$$

**H3** enables to have tighter bounds on the mean squared error of  $\hat{\pi}_i(g_i)$  defined in (9). Under **H3**, for all  $i \in \{0, \dots, M\}$ ,  $U_i$  satisfies (12) with  $\tilde{L}$ . Finally, since

$Z < +\infty$  and by [7, Lemma 2.2.1], there exist  $\rho_1 > 0$  and  $\rho_2 \in \mathbb{R}$  such that for all  $x \in \mathbb{R}^d$ ,

$$U(x) \geq \rho_1 \|x\| - \rho_2 . \quad (15)$$

Denote by  $\mathcal{S}$  the set of simulation parameters,

$$\mathcal{S} = \{M, \{\sigma_i^2\}_{i=0}^{M-1}, \{\gamma_i\}_{i=0}^{M-1}, \{n_i\}_{i=0}^{M-1}, \{N_i\}_{i=0}^{M-1}\} , \quad (16)$$

and by  $\hat{Z}$  the following estimator of  $Z$ ,

$$\hat{Z} = (2\pi\sigma_0^2)^{d/2}(1 + \sigma_0^2 m)^{-d/2} \left\{ \prod_{i=0}^{M-1} \hat{\pi}_i(g_i) \right\} , \quad (17)$$

where  $\hat{\pi}_i(g_i)$  is defined in (9). The dependence of  $\hat{Z}$  in  $\mathcal{S}$  is implicit. Note that  $\hat{Z}$  is a biased estimator of  $Z$  because  $Z_0$  is approximated by  $(2\pi\sigma_0^2)^{d/2}(1 + \sigma_0^2 m)^{-d/2}$ . We define the cost of the algorithm by the total number of iterations performed by the  $M$  Markov chains  $(X_{i,n})_{n \geq 0}$  for  $i \in \{0, \dots, M-1\}$ , *i.e.*

$$\text{cost} = \sum_{i=0}^{M-1} \{N_i + n_i\} . \quad (18)$$

Observe that each step of the Markov chain takes time linear in  $d$ . We state below a simplified version of our results; explicit bounds are given in Theorems 5, 6, 12 and 13.

**Theorem 1.** *Assume **H1**, **H2**( $m$ ) for  $m \geq 0$ . Let  $\mu, \epsilon \in (0, 1)$ . There exists an explicit choice of the simulation parameters  $\mathcal{S}$  such that the estimator  $\hat{Z}$  defined in (17) satisfies*

$$\mathbb{P} \left( \left| \hat{Z}/Z - 1 \right| > \epsilon \right) \leq \mu . \quad (19)$$

Moreover, the cost of the algorithm (18) is upper-bounded by,

	<b>H1, H2</b> ( $m$ ) for $m > 0$
cost	$\frac{L^3}{\mu^2 m^3} \log(d) d^3 \times \tilde{\mathcal{O}}(\epsilon^{-4})$
	<b>H1, H2</b> ( $m$ ) for $m > 0, \mathbf{H3}$
cost	$\left( \frac{\tilde{L}}{\mu^{3/2} m^{3/2}} + \frac{L^2}{\mu^{3/2} m^2} \right) \log(d) d^{5/2} \times \tilde{\mathcal{O}}(\epsilon^{-3})$
	<b>H1, H2</b> ( $m$ ) for $m \geq 0$
cost	$\frac{L^2}{\mu^2 \rho_1^4} (d + \rho_2)^4 \log(d) d^3 \times \tilde{\mathcal{O}}(\epsilon^{-4})$
	<b>H1, H2</b> ( $m$ ) for $m \geq 0, \mathbf{H3}$
cost	$\left( \frac{L^2}{\mu^{3/2} \rho_1^4} + \frac{\tilde{L}}{\mu^{3/2} (d + \rho_2) \rho_1^3} \right) (d + \rho_2)^4 \log(d) d^{5/2} \times \tilde{\mathcal{O}}(\epsilon^{-3})$

By the median trick (see e.g. [27, Lemma 6.1] or [36]), the dependence in  $\mu$  of the cost can be reduced to a logarithmic factor, see Corollaries 7 and 14.

It is interesting to compare these complexity bounds with previously reported results. In [33] and [5] (see also [12]), the authors propose to use sequential Monte Carlo (SMC) samplers to estimate the normalizing constant  $Z$  of a probability distribution  $\pi$ . In [5],  $\pi$  is supported on a compact set  $K$  included in  $\mathbb{R}^d$  and satisfies for  $x = (x_1, \dots, x_d) \in K$ ,  $\pi(x) = Z^{-1} \prod_{i=1}^d \exp(g(x_i))$ . [5, Theorem 3.2] states that there exists an estimator  $\hat{Z}$  of  $Z$  such that  $\lim_{d \rightarrow +\infty} \mathbb{E}[|\hat{Z}/Z - 1|^2] = C/N$  where  $N$  is the number of particles and  $C$  depends on  $g$  and on the parameters of the SMC (choice of the Markov kernel and of the annealing schedule). With our definition (18), the computational cost of the SMC algorithm is  $\mathcal{O}(Nd)$  (there are  $d$  phases and  $N$  particles for each phase). To obtain an estimator  $\hat{Z}$  satisfying (19) implies a cost of  $d\mu^{-1}\mathcal{O}(\epsilon^{-2})$ . However, the product form of the density  $\pi$  is restrictive, the result is only asymptotic in  $d$  and the state space is assumed to be compact. [13] combines SMC with a multilevel approach and [26] establishes results on a multilevel particle filter.

[23] deals with the case where  $\pi(x) = \exp(-\beta H(x))/Z(\beta)$  where  $x \in \Omega$ , a finite state space,  $\beta \geq 0$  and  $H(x) \in \{0, \dots, n\}$ . These distributions known as Gibbs distributions include in particular the Ising model. To compute  $Z(\beta)$ , [23] relies on an annealing process on the parameter  $\beta$ , starting from  $Z(0)$ . Let  $q = \log(Z(0))/\log(Z(\beta))$ . [23, Theorem 1.1] states that there exists an estimator  $\hat{Z}(\beta)$  of  $Z(\beta)$  such that (19) is satisfied with  $\mu = 1/4$  and  $q \log(n)\tilde{\mathcal{O}}(\epsilon^{-2})$  draws from the Gibbs distribution.

Our complexity results can also be related to the computation of the volume of a convex body  $K$  (compact convex set with non-empty interior) on  $\mathbb{R}^d$ . This problem has attracted a lot of attention in the field of computer science, starting with the breakthrough of [17] until the most recent results of [10]. Define for  $x \in \mathbb{R}^d$ ,  $\pi(x) = 1_K(x)/\text{Vol}(K)$ . Under the assumptions  $B(0, 1) \subset K$  and  $\int_{\mathbb{R}^d} \|x\|^2 \pi(x) dx = \mathcal{O}(d)$ , [10, Theorem 1.1] states that there exists an estimator  $\hat{Z}$  of  $Z = \text{Vol}(K)$  such that (19) is satisfied with  $\mu = 1/4$  and a cost of  $\log(d)d^3\tilde{\mathcal{O}}(\epsilon^{-2})$ .

Nonequilibrium methods have been recently developed and studied in order to compute free energy differences or Bayes factors, see [24] and [30, Chapter 4]. They are based on an inhomogeneous diffusion evolving (for example) from  $t = 0$  to  $t = 1$  such that  $\pi_0$  and  $\pi_1$  are the stationary distributions respectively for  $t = 0$  and  $t = 1$ . Recently, [1] provided an asymptotic and non-asymptotic analysis of the bias and variance for estimators associated with this methodology. The main aim of this paper is to obtain polynomial complexity and inspection of their results suggests a cost of order  $d^{15}$  at most to compute an estimator  $\hat{Z}$  satisfying (19). However, this cost may be due to the strategy of proofs.

Multistage sampling type algorithms are widely used and known under different names: multistage sampling [45], (extended) bridge sampling [22], annealed importance sampling (AIS) [34], thermodynamic integration [37], power posterior [4]. For the stability and accuracy of the method, the choice of the parameters (in our case  $\{\sigma_i^2\}_{i=0}^{M-1}$ ) is crucial and is known to be difficult. Indeed, the

issue has been pointed out in several articles under the names of tuning tempered transitions [4], temperature placement [21], annealing sequence [5, Sections 3.2.1, 4.1], temperature ladder [37, Section 3.3.2], effects of grid size [16], cooling schedule [10]. In Sections 2.1 and 2.2, we explicitly define the sequence  $\{\sigma_i^2\}_{i=0}^{M-1}$ .

## 2. Theoretical analysis of the algorithm

In this Section, we analyse the algorithm outlined in Section 1. The strongly convex and convex cases are considered in Sections 2.1 and 2.2, respectively. The choice of the simulation parameters  $\mathcal{S}$  explicitly depends on the (strong) convexity of  $U$ . Throughout this Section, we assume that  $L > m$ ; note that if  $L = m$ ,  $\pi$  is a Gaussian density and  $Z$  is known. For  $M \in \mathbb{N}^*$  and  $i \in \{0, \dots, M-1\}$ , we first provide an upper bound on the mean squared error  $\text{MSE}_i$  of  $\hat{\pi}_i(g_i)$  defined by

$$\text{MSE}_i = \mathbb{E} \left[ \{\hat{\pi}_i(g_i) - \pi_i(g_i)\}^2 \right], \quad (20)$$

where  $\pi_i(g_i)$  and  $\hat{\pi}_i(g_i)$  are given by (5) and (9) respectively. The  $\text{MSE}_i$  can be decomposed as a sum of the squared bias and variance,

$$\text{MSE}_i = \{\mathbb{E}[\hat{\pi}_i(g_i)] - \pi_i(g_i)\}^2 + \text{Var}[\hat{\pi}_i(g_i)]. \quad (21)$$

Propositions 2 and 3 give upper bounds on the squared bias and Proposition 4 on the variance. The results are based on the non-asymptotic bounds of the Wasserstein distance for a strongly convex potential obtained in [15] (see also [11], [14]). We introduce the following conditions on the stepsize  $\gamma_i$  used in the Euler discretization and the variance  $\sigma_{i+1}^2$

$$\gamma_i \in \left( 0, \frac{1}{m + L + 2/\sigma_i^2} \right], \quad \sigma_{i+1}^2 \leq 2(d+4) \left( \frac{2d+7}{\sigma_i^2} - m \right)_+^{-1}, \quad (22)$$

where by convention  $1/0 = +\infty$ . Note that the condition on  $\sigma_{i+1}^2$  is equivalent to  $a_i \in [0, m_i/\{4(d+4)\} \wedge (2\sigma_i^2)^{-1}]$  where  $a_i$  is defined in (6) and  $m_i$  in (13). Assuming that  $\gamma_i$  and  $\sigma_{i+1}^2$  satisfy (22), we define the positive quantities

$$C_{i,0} = \exp\left(\frac{4a_i(d+2)}{\kappa_i - 8a_i}\right), \quad C_{i,1} = 2d \frac{1 - 8a_i\gamma_i}{\kappa_i - 8a_i}, \quad C_{i,2} = 4 \frac{d}{m_i}, \quad (23)$$

where  $m_i$ ,  $L_i$  and  $\kappa_i$  are defined in (13) and (14), respectively. Denote by,

$$A_{i,0} = 2L_i^2 \kappa_i^{-1} d, \quad (24)$$

$$A_{i,1} = 2dL_i^2 + dL_i^4(\kappa_i^{-1} + (m_i + L_i)^{-1})(m_i^{-1} + 6^{-1}(m_i + L_i)^{-1}). \quad (25)$$

**Proposition 2.** Assume **H1** and **H2**( $m$ ) for some  $m \geq 0$ . For  $N_i \in \mathbb{N}$ ,  $n_i \in \mathbb{N}^*$  and  $\gamma_i, \sigma_{i+1}^2$  satisfying (22), we have

$$\begin{aligned} \{\mathbb{E}[\hat{\pi}_i(g_i)] - \pi_i(g_i)\}^2 &\leq 4a_i^2(C_{i,2} + C_{i,0}C_{i,1}) \\ &\times \left\{ \frac{4d}{n_i m_i \kappa_i \gamma_i} \exp\left(-N_i \frac{\kappa_i \gamma_i}{2}\right) + 2\kappa_i^{-1}(A_{i,0}\gamma_i + A_{i,1}\gamma_i^2) \right\}. \end{aligned}$$



*Proof.* The proof is postponed to Section 5.1.  $\square$

The squared bias can thus be controlled by adjusting the parameters  $\gamma_i, n_i$  and  $N_i$ . If  $U$  satisfies **H3**, the bound on the squared bias can be improved. Define,

$$B_{i,0} = d \left( 2L_i^2 + \kappa_i^{-1} \{ (d\tilde{L}^2)/3 + 4L_i^4/(3m_i) \} \right), \quad (26)$$

$$B_{i,1} = dL_i^4 \left( \kappa_i^{-1} + \{ 6(m_i + L_i) \}^{-1} + m_i^{-1} \right). \quad (27)$$

**Proposition 3.** *Assume **H1**, **H2**( $m$ ) for some  $m \geq 0$ , and **H3**. For  $N_i \in \mathbb{N}$ ,  $n_i \in \mathbb{N}^*$  and  $\gamma_i, \sigma_{i+1}^2$  satisfying (22), we have*

$$\begin{aligned} \{ \mathbb{E}[\hat{\pi}_i(g_i)] - \pi_i(g_i) \}^2 &\leq 4a_i^2 (C_{i,2} + C_{i,0}C_{i,1}) \\ &\times \left\{ \frac{4d}{n_i m_i \kappa_i \gamma_i} \exp\left(-N_i \frac{\kappa_i \gamma_i}{2}\right) + 2\kappa_i^{-1} (B_{i,0} \gamma_i^2 + B_{i,1} \gamma_i^3) \right\}. \end{aligned}$$

*Proof.* The proof is postponed to Section 5.1.  $\square$

Note that the leading term is of order  $\gamma_i^2$  instead of  $\gamma_i$ . We consider now the variance term in (21).

**Proposition 4.** *Assume **H1** and **H2**( $m$ ) for some  $m \geq 0$ . For  $N_i \in \mathbb{N}$ ,  $n_i \in \mathbb{N}^*$  and  $\gamma_i, \sigma_{i+1}^2$  satisfying (22), we have*

$$\text{Var} [\hat{\pi}_i(g_i)] \leq \frac{32a_i^2 C_{i,0} C_{i,1}}{\kappa_i^2 n_i \gamma_i} \left( 1 + \frac{2}{\kappa_i n_i \gamma_i} \right).$$

*Proof.* The proof is postponed to Section 5.1.  $\square$

### 2.1. Strongly convex potential $U$

**Theorem 5.** *Assume **H1** and **H2**( $m$ ) for  $m > 0$  and let  $\mu, \epsilon \in (0, 1)$ . There exists an explicit choice of the simulation parameters  $\mathcal{S}$  (16) such that the estimator  $\hat{Z}$  defined in (17) satisfies with probability at least  $1 - \mu$*

$$(1 - \epsilon)Z \leq \hat{Z} \leq (1 + \epsilon)Z,$$

and the cost (18) of the algorithm is upper-bounded by

$$\text{cost} \leq \left( \frac{6272C}{\epsilon^2 \mu} + \log(5Cd^2) \right) \frac{(1088C)^2 d^2 (d+4)}{\epsilon^2 \mu} \left( \frac{m+L}{2m} \right)^3 (C+3), \quad (28)$$

with

$$C = \left\lceil \frac{1}{\log(2)} \log \left( d \left( d + \frac{7}{2} \right) \left( \frac{L}{m} - 1 \right) \frac{1}{\log(1 + \epsilon/3)} \right) \right\rceil. \quad (29)$$

*Proof.* The proof is postponed to Section 5.3.3.  $\square$

**Theorem 6.** Assume **H1**, **H2**( $m$ ) for  $m > 0$ , **H3** and let  $\mu, \epsilon \in (0, 1)$ . There exists an explicit choice of the simulation parameters  $\mathcal{S}$  (16) such that the estimator  $\hat{Z}$  defined in (17) satisfies with probability at least  $1 - \mu$

$$(1 - \epsilon)Z \leq \hat{Z} \leq (1 + \epsilon)Z ,$$

and the cost (18) of the algorithm is upper-bounded by

$$\text{cost} \leq \left( \frac{6272C}{\epsilon^2\mu} + \log(5Cd^2) \right) \sqrt{\frac{7}{3}} \frac{512Cd^{3/2}}{\epsilon\sqrt{\mu}} (d+4)(C+3) \times \left\{ \tilde{L} \frac{2^{3/2}}{m^{3/2}} + \sqrt{10} \left( \frac{m+L}{2m} \right)^2 \right\} , \quad (30)$$

with  $C$  defined in (29).

*Proof.* The proof is postponed to Section 5.3.3. □

Using the median trick (see e.g. [27, Lemma 6.1] or [36]), we have the following corollary,

**Corollary 7.** Let  $\epsilon, \tilde{\mu} \in (0, 1)$ . Repeat  $2 \lceil 4 \log(\tilde{\mu}^{-1}) \rceil + 1$  times the algorithm of Theorems 5 and 6 with  $\mu = 1/4$  and denote by  $\hat{Z}$  the median of the output values. We have with probability at least  $1 - \tilde{\mu}$ ,

$$(1 - \epsilon)Z \leq \hat{Z} \leq (1 + \epsilon)Z .$$

*Proof.* The proof is postponed to Section 5.3.3. □

The proof of Theorems 5 and 6 and corollary 7 relies on several lemmas which are stated below. These lemmas explain how the simulation parameters  $\mathcal{S}$  must be chosen. The details of the proofs are gathered in Section 5.3. Set

$$\sigma_0^2 = \{2 \log(1 + \epsilon/3)\} / \{d(L - m)\} . \quad (31)$$

This choice of  $\sigma_0^2$  is justified by the following result,

**Lemma 8.** Under **H1** and **H2**( $m$ ) for  $m \geq 0$ , we have

$$Z_0 \leq (2\pi\sigma_0^2)^{d/2} / (1 + \sigma_0^2 m)^{d/2} \leq Z_0 (1 + \epsilon/3) . \quad (32)$$

*Proof.* The proof is postponed to Section 5.3.1. □

Given a choice of  $\mathcal{S}$ , define the event

$$A_{\mathcal{S},\epsilon} = \left\{ \left| \prod_{i=0}^{M-1} \hat{\pi}_i(g_i) - \prod_{i=0}^{M-1} \pi_i(g_i) \right| \leq \prod_{i=0}^{M-1} \pi_i(g_i) \frac{\epsilon}{2} \right\} . \quad (33)$$

On  $A_{\mathcal{S},\epsilon}$ , using Lemma 8, (4) and (17), we have:

$$Z (1 - \epsilon/2) \leq \hat{Z} \leq Z (1 + \epsilon) .$$

It remains to choose  $\mathcal{S}$  to minimize approximately the cost defined in (18) under the constraint  $\mathbb{P}(A_{\mathcal{S},\epsilon}) \geq 1 - \mu$ . We define the positive increasing sequence  $\{\sigma_i^2\}_{i=0}^{M-1}$  recursively, starting from  $i = 0$ . For  $i \in \mathbb{N}$ , set

$$\sigma_{i+1}^2 = \varsigma_s(\sigma_i^2), \quad (34)$$

where  $\varsigma_s : \mathbb{R}_+^* \rightarrow \mathbb{R}$  is defined for  $t \in (0, (2d+7)/m)$  by

$$\varsigma_s(t) = \left( \frac{1}{t} - \frac{m + (2^{k(t)+1}\sigma_0^2)^{-1}}{2(d+4)} \right)^{-1}, \quad k(t) = \left\lfloor \frac{\log(t/\sigma_0^2)}{\log(2)} \right\rfloor \quad (35)$$

and  $\varsigma_s(t) = +\infty$  otherwise. The subscript  $s$  in  $\varsigma_s$  stresses that this choice is valid for the strongly convex case and will be different for the convex case. With this choice of  $(\sigma_i^2)_{i \geq 0}$ , the number of phases  $M$  is defined by

$$M = \inf \{i \geq 1 : \sigma_{i-1}^2 \geq (2d+7)/m\}. \quad (36)$$

By (35), for  $t \in [\sigma_0^2, (2d+7)/m)$ ,  $\varsigma_s(t) \geq t(4d+16)/(4d+15)$ , which implies  $M < +\infty$ . With this definition of  $\varsigma_s$ , for  $i \in \{0, \dots, M-2\}$ , we have

$$a_i = \frac{1}{2} \left( \frac{1}{\sigma_i^2} - \frac{1}{\sigma_{i+1}^2} \right) = \frac{m + (2^{k+1}\sigma_0^2)^{-1}}{4(d+4)}, \quad \text{if } 2^k \sigma_0^2 \leq \sigma_i^2 < 2^{k+1} \sigma_0^2, \quad (37)$$

and  $a_{M-1} = (2\sigma_{M-1}^2)^{-1}$ . Define  $\mathcal{I}_k \subset \mathbb{N}$  for  $k \in \mathbb{N}$  and  $K \in \mathbb{N}$  by,

$$\mathcal{I}_k = \{i \in \{0, \dots, M-2\} : 2^k \sigma_0^2 \leq \sigma_i^2 < 2^{k+1} \sigma_0^2\}, \quad (38)$$

$$K = \inf \{k \geq 0 : \mathcal{I}_k = \emptyset\} < +\infty. \quad (39)$$

The number of phases  $M$  and variances  $\{\sigma_i^2\}_{i=0}^{M-1}$  being defined, we now proceed with the choice of the stepsize  $\gamma_i$ , the number of samples  $n_i$  and the burn-in period  $N_i$  for  $i \in \{0, \dots, M-1\}$ .

**Lemma 9.** *Set  $\eta = (\epsilon\sqrt{\mu})/8$ . Assume that there exists a choice of the simulation parameters  $\{N_i\}_{i=0}^{M-1}$ ,  $\{n_i\}_{i=0}^{M-1}$  and  $\{\gamma_i\}_{i=0}^{M-1}$  satisfying,*

i) *For all  $k \in \{0, \dots, K-1\}$ ,  $i \in \mathcal{I}_k$ ,*

$$|\mathbb{E}[\hat{\pi}_i(g_i)] - \pi_i(g_i)| \leq \frac{\eta}{K|\mathcal{I}_k|}, \quad \text{Var}[\hat{\pi}_i(g_i)] \leq \frac{\eta^2}{K|\mathcal{I}_k|},$$

ii)  $|\mathbb{E}[\hat{\pi}_{M-1}(g_{M-1})] - \pi_{M-1}(g_{M-1})| \leq \eta, \quad \text{Var}[\hat{\pi}_{M-1}(g_{M-1})] \leq \eta^2,$

where  $\pi_i(g_i)$  is defined in (5) and  $\hat{\pi}_i(g_i)$  in (9). Then  $\mathbb{P}(A_{\mathcal{S},\epsilon}) \geq 1 - \mu$ , where  $A_{\mathcal{S},\epsilon}$  is defined in (33).

*Proof.* The proof is postponed to Section 5.2 □

To show the existence of  $\gamma_i, n_i, N_i$  satisfying the conditions of Lemma 9, we apply Propositions 2 to 4 for each  $i \in \{0, \dots, M-1\}$ . We then have the following lemmas,

**Lemma 10.** Set  $\eta = (\epsilon\sqrt{\mu})/8$ . Assume **H1**, **H2**( $m$ ) for  $m > 0$  and,

i) for all  $k \in \{0, \dots, K-1\}$ ,  $i \in \mathcal{I}_k$ ,

$$\gamma_i \leq \frac{1}{2285} \frac{\eta^2 \kappa_i^2 \sigma_i^4 m_i}{K^2 d^2 L_i^2} \leq \frac{1}{m_i + L_i}, \quad (40)$$

$$n_i \geq \frac{196K}{\eta^2} \frac{\sqrt{m_i}}{\kappa_i \sigma_i} \frac{1}{\kappa_i \gamma_i}, \quad (41)$$

$$N_i \geq 2(\kappa_i \gamma_i)^{-1} \log(5Kd^2), \quad (42)$$

ii)

$$\gamma_{M-1} \leq 40^{-1} \eta^2 L_{M-1}^{-2} m_{M-1} \leq (m_{M-1} + L_{M-1})^{-1}, \quad (43)$$

$$n_{M-1} \geq 19(\kappa_{M-1} \gamma_{M-1})^{-1} \eta^{-2}, \quad (44)$$

$$N_{M-1} \geq (\kappa_{M-1} \gamma_{M-1})^{-1}. \quad (45)$$

Then, the conditions i)-ii) of Lemma 9 are satisfied.

*Proof.* The proof is postponed to Section 5.3.2.  $\square$

We have a similar result under the additional assumption **H3**.

**Lemma 11.** Set  $\eta = (\epsilon\sqrt{\mu})/8$ . Assume **H1**, **H2**( $m$ ) for  $m > 0$ , **H3** and,

i) for all  $k \in \{0, \dots, K-1\}$ ,  $i \in \mathcal{I}_k$ ,

$$\gamma_i \leq \sqrt{\frac{3}{7}} \frac{\eta \kappa_i m_i^{1/2} \sigma_i^2}{8Kd} \left( d\tilde{L}^2 + 10L_i^4 m_i^{-1} \right)^{-1/2} \leq \frac{1}{m_i + L_i}, \quad (46)$$

and  $n_i, N_i$  as in (41), (42),

ii)

$$\gamma_{M-1} \leq \sqrt{\frac{3}{7}} \frac{\eta \kappa_{M-1} m_{M-1}^{-1/2}}{4} \left( d\tilde{L}^2 + 10L_{M-1}^4 m_{M-1}^{-1} \right)^{-1/2} \leq \frac{1}{m_{M-1} + L_{M-1}}, \quad (47)$$

and  $n_{M-1}, N_{M-1}$  as in (44), (45).

Then, the conditions i)-ii) of Lemma 9 are satisfied.

*Proof.* The proof is postponed to the supplementary material [8, Appendix A.2].  $\square$

## 2.2. Convex potential $U$

We now consider the convex case. The annealing process on the variances  $\{\sigma_i^2\}_{i=0}^{M-1}$  is different from the strongly convex case and is defined in (54). In particular, the stopping criteria for the annealing process is distinct from the case where  $U$  is strongly convex and relies on a truncation argument. More precisely,

a concentration theorem for log-concave functions [39, Theorem 3.1] states that for  $\alpha \in (0, 1)$ ,

$$\int_{\mathbb{R}^d} \mathbb{1}_{\{U \geq d(\tau_\alpha + 1)\}}(x) \pi(x) dx \leq \alpha, \quad \tau_\alpha = \left( \frac{16 \log(3/\alpha)}{d} \right)^{1/2}.$$

Let  $\epsilon \in (0, 1)$ ,  $\tau = \tau_{\epsilon/2}$  and  $D = \rho_1^{-1} \{d(\tau + 1) + \rho_2\}$ . By (15), we have

$$\int_{\mathbb{R}^d} \mathbb{1}_{B(0, D)}(x) \pi(x) dx \geq 1 - \epsilon/2. \quad (48)$$

Given a choice of  $M$  and  $\sigma_{M-1}^2$ , define  $\bar{g}_{M-1} : \mathbb{R}^d \rightarrow \mathbb{R}_+$  for all  $x \in \mathbb{R}^d$  by

$$\bar{g}_{M-1}(x) = \exp \left\{ \frac{1}{2\sigma_{M-1}^2} (\|x\|^2 \wedge D^2) \right\}, \quad (49)$$

and  $J$  by,

$$J = \int_{\mathbb{R}^d} e^{-U(x)} dx \Big/ \int_{\mathbb{R}^d} e^{-U(x) - (\|x\|^2 - D^2)_+ / (2\sigma_{M-1}^2)} dx.$$

Note that  $Z/Z_{M-1} = J \times \pi_{M-1}(\bar{g}_{M-1})$  and by (48),

$$J(1 - \epsilon/2) \leq 1 \leq J. \quad (50)$$

On the event  $A_{S, \epsilon}$  defined in (33) with  $g_{M-1}$  replaced by  $\bar{g}_{M-1}$  and by (32) (with  $m = 0$ ), (50), we get

$$Z(1 - \epsilon/2)^2 \leq \hat{Z} \leq Z(1 + \epsilon),$$

where  $\hat{Z}$  is defined in (17) with  $g_{M-1}$  replaced by  $\bar{g}_{M-1}$ . We now state our results in the convex case.

**Theorem 12.** *Assume **H1**, **H2**( $m$ ) for  $m \geq 0$ . Let  $\epsilon, \mu \in (0, 1)$ . There exists an explicit choice of the simulation parameters  $\mathcal{S}$  (16) such that the estimator  $\hat{Z}$  defined in (17) (with  $g_{M-1}$  replaced by  $\bar{g}_{M-1}$  defined in (49)) satisfies with probability at least  $1 - \mu$*

$$(1 - \epsilon)Z \leq \hat{Z} \leq (1 + \epsilon)Z,$$

and the cost (18) of the algorithm is upper-bounded by

$$\begin{aligned} \text{cost} &\leq \left( \frac{17728C}{\epsilon^2 \mu} + \log(Cd^2) \right) \frac{(487C)^2 d^2 (d+4)}{\epsilon^2 \mu} \\ &\quad \times \left( C + \frac{6L \{d(\tau+1) + \rho_2\}^2}{\rho_1^2} + \frac{8L^2 \{d(\tau+1) + \rho_2\}^4}{3\rho_1^4} \right), \end{aligned} \quad (51)$$

where  $\rho_1, \rho_2$  are defined in (15),  $\tau = 4d^{-1/2} \{\log(6/\epsilon)\}^{1/2}$  and

$$C = \left\lceil \frac{1}{\log(2)} \log \left( \frac{dL \{d(\tau+1) + \rho_2\}^2}{2\rho_1^2 \log(1 + \epsilon/3)} \right) \right\rceil. \quad (52)$$

**Theorem 13.** Assume **H 1**, **H 2**( $m$ ) for  $m \geq 0$  and **H 3**. Let  $\epsilon, \mu \in (0, 1)$ . There exists an explicit choice of the simulation parameters  $\mathcal{S}$  (16) such that the estimator  $\hat{Z}$  defined in (17) (with  $g_{M-1}$  replaced by  $\bar{g}_{M-1}$  defined in (49)) satisfies with probability at least  $1 - \mu$

$$(1 - \epsilon)Z \leq \hat{Z} \leq (1 + \epsilon)Z ,$$

and the cost (18) of the algorithm is upper-bounded by

$$\begin{aligned} \text{cost} \leq & 2474 \left( \frac{17728C}{\epsilon^2 \mu} + \log(Cd^2) \right) \frac{(C+1)d(d+4)}{\epsilon \sqrt{\mu}} \left\{ \frac{8L^2 \{d(\tau+1) + \rho_2\}^4}{3\rho_1^4} \right. \\ & + \frac{d^{1/2} \tilde{L} \{d(\tau+1) + \rho_2\}^3}{\sqrt{10}\rho_1^3} \max \left( \frac{5\rho_1}{d(\tau+1) + \rho_2}, \left( \frac{5}{9} + \frac{\rho_1^2}{\{d(\tau+1) + \rho_2\}^2 L} \right)^2 \right) \\ & \left. + \frac{6L \{d(\tau+1) + \rho_2\}^2}{\rho_1^2} + C \right\}, \quad (53) \end{aligned}$$

where  $\rho_1, \rho_2, C$  are defined in (15), (52) respectively and  $\tau = 4d^{-1/2} \{\log(6/\epsilon)\}^{1/2}$ .

**Corollary 14.** Let  $\epsilon, \tilde{\mu} \in (0, 1)$ . Repeat  $2 \lceil 4 \log(\tilde{\mu}^{-1}) \rceil + 1$  times the algorithm of Theorems 12 and 13 with  $\mu = 1/4$  and denote by  $\hat{Z}$  the median of the output values. We have with probability at least  $1 - \tilde{\mu}$ ,

$$(1 - \epsilon)Z \leq \hat{Z} \leq (1 + \epsilon)Z .$$

The proofs follow the same arguments as Theorems 5 and 6 and corollary 7 and are detailed in the supplementary material [8, Appendix B.3].

Note that  $\bar{g}_{M-1}$  (49) is a  $\|\bar{g}_{M-1}\|_{\text{Lip}}$ -Lipschitz function where,

$$\|\bar{g}_{M-1}\|_{\text{Lip}} = \frac{D}{\sigma_{M-1}^2} \exp \left( \frac{D^2}{2\sigma_{M-1}^2} \right) .$$

The results of Section 4 give an upper bound on  $\text{MSE}_{M-1}$  which is polynomial in the parameters if  $\sigma_{M-1}^2$  is approximately equal to  $D^2$ . For  $i \in \mathbb{N}^*$ , we define  $(\sigma_i^2)_{i \geq 0}$  recursively. Set  $\sigma_0^2$  as in (31) and

$$\sigma_{i+1}^2 = \varsigma_c(\sigma_i^2) , \quad (54)$$

where  $\varsigma_c : \mathbb{R}_+^* \rightarrow \mathbb{R}$  is defined for  $t \in (0, D^2)$  by,

$$\varsigma_c(t) = \left( \frac{1}{t} - \frac{1}{2(d+4)(2^{k(t)+1}\sigma_0^2)} \right)^{-1}, \quad k(t) = \left\lfloor \frac{\log(t/\sigma_0^2)}{\log(2)} \right\rfloor, \quad (55)$$

and  $\varsigma_c(t) = +\infty$  otherwise. Define  $M$  in this Section by,

$$M = \inf \{ i \geq 1 : \sigma_{i-1}^2 \geq D^2 \} . \quad (56)$$

By (55), for  $t \in [\sigma_0^2, D^2)$ ,  $\varsigma_c(t) \geq \{(4d+16)/(4d+15)\}t$ , which implies  $M < +\infty$ . The following lemmas are the counterparts of Lemmas 10 and 11. They specify the choice of  $\{\gamma_i\}_{i=0}^{M-1}$ ,  $\{n_i\}_{i=0}^{M-1}$ ,  $\{N_i\}_{i=0}^{M-1}$  to satisfy the conditions of Lemma 9.

**Lemma 15.** Set  $\eta = (\epsilon\sqrt{\mu})/8$ . Assume **H1**, **H2**( $m$ ) for  $m \geq 0$  and,

i) for all  $k \in \{0, \dots, K-1\}$ ,  $i \in \mathcal{I}_k$ ,

$$\gamma_i \leq \frac{1}{462} \frac{\eta^2 L_i^{-2} \sigma_i^{-2}}{K^2 d^2} \leq \frac{1}{m_i + L_i}, \quad (57)$$

$$n_i \geq \frac{453K}{\eta^2} \frac{1}{\kappa_i \gamma_i}, \quad (58)$$

$$N_i \geq 2(\kappa_i \gamma_i)^{-1} \log(Kd^2), \quad (59)$$

ii)

$$\gamma_{M-1} \leq (1/26)\eta^2 d^{-1} L_{M-1}^{-2} \kappa_{M-1} \leq (m_{M-1} + L_{M-1})^{-1}, \quad (60)$$

$$n_{M-1} \geq 29\eta^{-2} (\kappa_{M-1} \gamma_{M-1})^{-1}, \quad (61)$$

$$N_{M-1} \geq 2(\kappa_{M-1} \gamma_{M-1})^{-1} \log(d). \quad (62)$$

Then, the conditions i)-ii) of Lemma 9 are satisfied, with  $g_{M-1}$  replaced by  $\bar{g}_{M-1}$ .

*Proof.* The proof is postponed to the supplementary material [8, Appendix B.1].  $\square$

We have a similar result under the additional assumption **H3**.

**Lemma 16.** Set  $\eta = (\epsilon\sqrt{\mu})/8$ . Assume **H1**, **H2**( $m$ ) for  $m \geq 0$ , **H3** and,

i) for all  $k \in \{0, \dots, K-1\}$ ,  $i \in \mathcal{I}_k$ ,

$$\gamma_i \leq \sqrt{\frac{3}{7}} \frac{\eta \sigma_i^{-1}}{8Kd} \left( d\tilde{L}^2 + 10L_i^4 \sigma_i^2 \right)^{-1/2} \leq \frac{1}{m_i + L_i}, \quad (63)$$

$n_i, N_i$  as in (58), (59) and,

ii)

$$\gamma_{M-1} \leq \sqrt{\frac{3}{8e}} \frac{\eta \kappa_{M-1} \sigma_{M-1}}{\sqrt{d}} \left( d\tilde{L}^2 + 10L_{M-1}^4 \sigma_{M-1}^2 \right)^{-1/2} \leq \frac{1}{m_{M-1} + L_{M-1}}, \quad (64)$$

$n_{M-1}, N_{M-1}$  as in (61), (62).

Then, the conditions i)-ii) of Lemma 9 are satisfied, with  $g_{M-1}$  replaced by  $\bar{g}_{M-1}$ .

*Proof.* The proof is postponed to the supplementary material [8, Appendix B.2].  $\square$

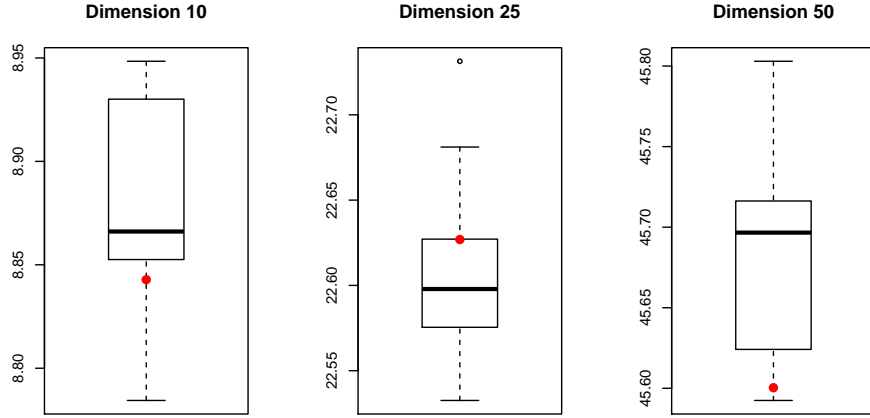


FIG 1. Boxplots of the logarithm of the normalizing constants of a multivariate Gaussian distribution in dimension  $d \in \{10, 25, 50\}$ .

### 3. Numerical experiments

For the following numerical experiments, the code and data are available at <https://github.com/nbrosse/normalizingconstant>. We first experiment our algorithm to compute the logarithm of the normalizing constant of a multivariate Gaussian distribution in dimension  $d \in \{10, 25, 50\}$ , of mean 0 and inverse covariance matrix  $\text{diag}(2, 1^{\otimes(d-1)})$ . We set  $\epsilon = \mu = 0.1$ . The number of phases  $M$  of the algorithm and the variances  $\{\sigma_i^2\}_{i=0}^{M-1}$  are chosen accordingly to the formulas (34) and (36). For each phase of the algorithm, the step size  $\gamma_i$  is set equal to  $10^{-2}(m_i + L_i)^{-1}$ , the burn-in period  $N_i$  to  $10^4$  and the number of samples  $n_i$  to  $10^5$  where  $m_i, L_i$  are defined in (13). We carry out 10 independent runs of the algorithm and compute the boxplots in Figure 1. The true values of the logarithm of the normalizing constants are known and displayed by the red points in Figure 1.

We illustrate then our methodology to compute Bayesian model evidence; see [20] and the references therein. Let  $y \in \mathbb{R}^p$  be a vector of observations and  $\mathcal{M}_1, \dots, \mathcal{M}_l$  be a collection of competing models. Let  $\{p(\mathcal{M}_i)\}_{i=1}^l$  be a prior distribution on the collection of models. For  $i \in \{0, \dots, l\}$ , denote by  $p(y|\theta^{(\mathcal{M}_i)}, \mathcal{M}_i)$  the likelihood of the model  $\mathcal{M}_i$ . The dominating measure is implicitly considered to be the Lebesgue measure on  $\mathbb{R}^p$ . Similarly, for  $i \in \{0, \dots, l\}$ , denote by  $p(\theta^{(\mathcal{M}_i)}|\mathcal{M}_i)$  the prior density on the parameters  $\theta^{(\mathcal{M}_i)}$  under the model  $\mathcal{M}_i$  where the dominating measure is implicitly considered to be the Lebesgue measure on  $\mathbb{R}^{d^{(\mathcal{M}_i)}}$ . The posterior distribution of interest is



then for  $i \in \{0, \dots, l\}$ ,

$$p(\theta^{(\mathcal{M}_i)}, \mathcal{M}_i | y) \propto p(y | \theta^{(\mathcal{M}_i)}, \mathcal{M}_i) p(\theta^{(\mathcal{M}_i)} | \mathcal{M}_i) p(\mathcal{M}_i)$$

The posterior distribution conditional on model  $\mathcal{M}_i$  can also be considered

$$p(\theta^{(\mathcal{M}_i)} | \mathcal{M}_i, y) \propto p(y | \theta^{(\mathcal{M}_i)}, \mathcal{M}_i) p(\theta^{(\mathcal{M}_i)} | \mathcal{M}_i) \quad (65)$$

For  $i \in \{0, \dots, l\}$ , the evidence  $p(y | \mathcal{M}_i)$  of the model  $\mathcal{M}_i$  is defined by the normalizing constant for the posterior distribution (65)

$$p(y | \mathcal{M}_i) = \int_{\mathbb{R}^{d(\mathcal{M}_i)}} p(y | \theta^{(\mathcal{M}_i)}, \mathcal{M}_i) p(\theta^{(\mathcal{M}_i)} | \mathcal{M}_i) d\theta^{(\mathcal{M}_i)} .$$

The Bayes factor  $BF_{12}$  between two models  $\mathcal{M}_i$  and  $\mathcal{M}_j$  is then defined by the ratio of evidences [43, Section 7.2.2],  $BF_{ij} = p(y | \mathcal{M}_i) / p(y | \mathcal{M}_j)$ . In the following experiments, we estimate the log evidence  $\log(p(y | \mathcal{M}_i))$ . For ease of notation, the dependence on the model  $\mathcal{M}$  of the parameters  $\theta$  and the dimension  $d$  of the state space is implicit in the sequel.

Define  $\ell^{(\mathcal{M})} : \mathbb{R}^d \rightarrow \mathbb{R}$  by  $\ell^{(\mathcal{M})}(\theta) = -\log(p(y | \theta, \mathcal{M}) p(\theta | \mathcal{M}))$  for  $\theta \in \mathbb{R}^d$ . In the examples we consider,  $\ell^{(\mathcal{M})}$  satisfies **H1**, **H2**, **H3** and has a unique minimum  $\theta_\star^{(\mathcal{M})}$ . Define then  $U^{(\mathcal{M})} : \mathbb{R}^d \rightarrow \mathbb{R}_+$  by  $U^{(\mathcal{M})}(\theta) = \ell^{(\mathcal{M})}(\theta + \theta_\star^{(\mathcal{M})}) - \ell^{(\mathcal{M})}(\theta_\star^{(\mathcal{M})})$  for  $\theta \in \mathbb{R}^d$ . The algorithm described in Section 2 can be applied to  $U^{(\mathcal{M})}$ . For each example, two different models will be considered and  $U^{(\mathcal{M})}$  will be written as  $U^{(k)}$  for  $k = 1, 2$ .

The numerical experiments are carried out on a Gaussian linear and logistic regression following the experimental setup of [20, Section 4], which is now considered as a classical benchmark. The linear regression is conducted on  $p = 42$  specimens of radiata pine [47]. The response variable  $y \in \mathbb{R}^p$  is the maximum compression strength parallel to the grain. The explanatory variables are  $x \in \mathbb{R}^p$  the density and  $z \in \mathbb{R}^p$  the density adjusted for resin content.  $x$  and  $z$  are centered. The covariates of the first model  $\mathcal{M}_1$ ,  $X^{(1)} \in \mathbb{R}^{p \times 2}$ , are composed of an intercept and  $x$ , while the covariates of the second model  $\mathcal{M}_2$ ,  $X^{(2)} \in \mathbb{R}^{p \times 2}$ , are composed of an intercept and  $z$ . For  $k = 1, 2$ , the likelihood is defined by,

$$p(y | \theta, \mathcal{M}_k) = \left( \frac{\lambda}{2\pi} \right)^{d/2} \exp \left( -(\lambda/2) \left\| y - X^{(k)} \theta \right\|^2 \right) ,$$

where  $\lambda = 10^{-5}$ . For the two models, the parameter  $\theta$  follows the same Gaussian prior of mean (3000, 185) and inverse covariance matrix  $\lambda Q_0 = \lambda \text{diag}(0.06, 6)$  where  $\text{diag}$  denotes a diagonal matrix. These values are taken from [20, section 4.1]. For  $k = 1, 2$ ,  $U^{(k)}$  is  $m^{(k)}$ -strictly convex and  $L^{(k)}$ -gradient Lipschitz, where  $m^{(k)}$  (resp.  $L^{(k)}$ ) is the minimal (resp. maximal) eigenvalue of  $\lambda([X^{(k)}]^T X^{(k)} + Q_0)$ . We set  $\epsilon = \mu = 0.1$ . The number of phases  $M$  of the algorithm and the variances  $\{\sigma_i^2\}_{i=0}^{M-1}$  are chosen accordingly to the formulas (34) and (36). For each phase, the step size  $\gamma_i$  is set equal to  $10^{-2}(\kappa_i \sigma_i^2 m_i) / (dL_i^2)$ , the burn-in period  $N_i$  to  $10^3(\kappa_i \gamma_i)^{-1}$  and the number of samples  $n_i$  to  $10^4 m_i^{1/2} / (\kappa_i^2 \sigma_i \gamma_i)$

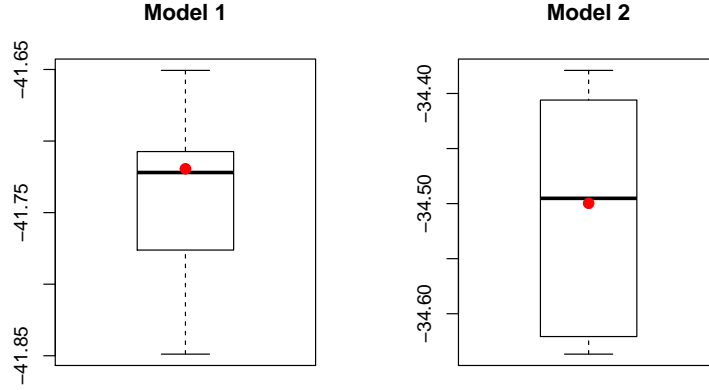


FIG. 2. Boxplots of the log evidence for the two models on the Gaussian regression.

where  $m_i, L_i, \kappa_i$  are defined in (13) and (14). The experiments are repeated 10 times and the boxplots for each model  $\mathcal{M}$  are plotted in Figure 2. Note that for this Gaussian model, the log evidence is known and displayed by the red points in Figure 2.

With the same parameters for the algorithm, we run 10 independent runs at each phase to measure the variability of each estimator  $\hat{\pi}_i(g_i)$  defined in (9). The result is plotted in Figure 3 for the model  $\mathcal{M}_1$ . The last estimator  $\hat{\pi}_{M-1}(g_{M-1})$  is much higher, which underlines the specificity of the last phase in the algorithm.

The logistic regression is performed on the Pima Indians dataset<sup>1</sup>. In this case,  $y \in \{0, 1\}^p$  is a vector of diabetes indicators for  $p = 532$  Pima Indian women and the potential predictors for diabetes are: number of pregnancies  $NP \in \mathbb{R}^p$ , plasma glucose concentration  $PGC \in \mathbb{R}^p$ , diastolic blood pressure  $BP \in \mathbb{R}^p$ , triceps skin fold thickness  $TST \in \mathbb{R}^p$ , body mass index  $BMI \in \mathbb{R}^p$ , diabetes pedigree function  $DP \in \mathbb{R}^p$  and age  $AGE \in \mathbb{R}^p$ . These variates are centered and standardized. The covariates of the first model  $\mathcal{M}_1$  are  $X^{(1)} = (\text{intercept}, NP, PGC, BMI, DP) \in \mathbb{R}^{p \times 5}$  and the covariates of the second model  $\mathcal{M}_2$  are  $X^{(2)} = (\text{intercept}, NP, PGC, BMI, DP, AGE) \in \mathbb{R}^{p \times 6}$ , where intercept is the intercept of the regressions. The likelihood is defined for  $k = 1, 2$  by,

$$p(y|\theta, \mathcal{M}_k) = \exp \left( \sum_{i=1}^p \left\{ y_i \theta^T X_i^{(k)} - \log \left( 1 + e^{\theta^T X_i^{(k)}} \right) \right\} \right),$$

where  $X_i^{(k)}$  denotes the  $i^{\text{th}}$  row of  $X^{(k)}$ . For the two models, the prior on  $\theta$

<sup>1</sup><http://archive.ics.uci.edu/ml/datasets/Pima+Indians+Diabetes>

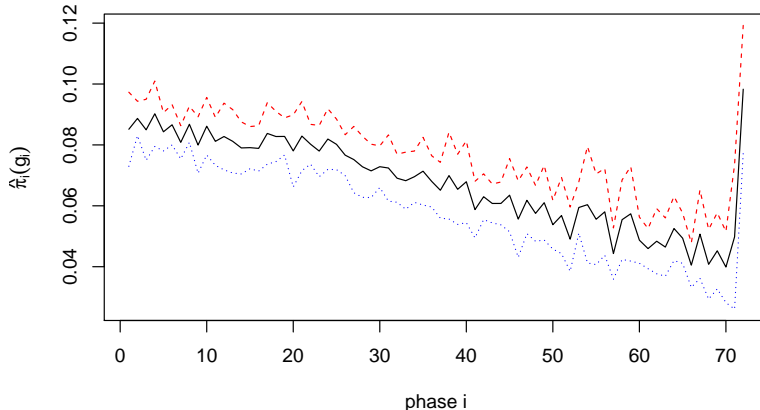


FIG 3. Error plot of  $\hat{\pi}_i(g_i)$  for  $i \in \{0, \dots, M - 1\}$  in the example of the Gaussian regression (model  $\mathcal{M}_1$ ). The mean of  $\hat{\pi}_i(g_i)$  is displayed in black and is spaced apart from the other two curves by the standard deviation of  $\hat{\pi}_i(g_i)$ .

is Gaussian, of mean 0 and inverse covariance matrix  $\tau \text{Id}$  where  $\tau = 0.01$ . For  $k = 1, 2$ ,  $U^{(k)}$  is  $\tau$ -strongly convex and  $L^{(k)}$ -gradient Lipschitz, where  $L^{(k)} = \lambda_{\max}([X^{(k)}]^T X^{(k)})/4 + \tau$  and  $\lambda_{\max}([X^{(k)}]^T X^{(k)})$  is the maximal eigenvalue of  $[X^{(k)}]^T X^{(k)}$ . We set  $\epsilon = \mu = 0.1$ . The algorithm to estimate  $\log(p(y|\mathcal{M}))$  described in Section 2.1 is applied with the following modifications. The number of phases is decreased and the recurrence for the variances  $\{\sigma_i^2\}_{i=0}^{M-1}$  is thus redefined by  $\sigma_{i+1}^2 = \varsigma_s^5(\sigma_i^2)$  as long as the stopping condition (36) is not fulfilled. For  $i \in \{1, \dots, 30\}$ , the burn-in period  $N_i$  is set equal to  $10^4$ , the number of samples  $n_i$  to  $10^6$  and the step size  $\gamma_i$  to  $10^{-2}(m_i + L_i)^{-1}$  where  $m_i, L_i$  are defined in (13); for  $i > 30$ , the number of samples  $n_i$  is set equal to  $10^5$  and the step size  $\gamma_i$  to  $10^{-1}(m_i + L_i)^{-1}$ . We compare our results with different methods reviewed in [20] and implemented in [48]. These are the Laplace method (L), Laplace at the Maximum a Posteriori (L-MAP), Chib’s method (C) Annealed Importance Sampling (AIS) and Power Posterior (PP). The experiments are repeated 10 times and the boxplots for each model  $\mathcal{M}$  and each method are plotted in Figure 4.

With the same parameters for the algorithm, we run 10 independent runs at each phase to measure the variability of each estimator  $\hat{\pi}_i(g_i)$  defined in (9) and display the result in Figure 5 for the model  $\mathcal{M}_1$ .

The final example we address is a Bayesian analysis of a finite mixture of Gaussian distributions, see [33, Section 4.2] and we aim at estimating the log evidence of the posterior distribution. Note that this model does not fit into our assumptions because the potential  $U$  is not continuously differentiable on  $\mathbb{R}^d$

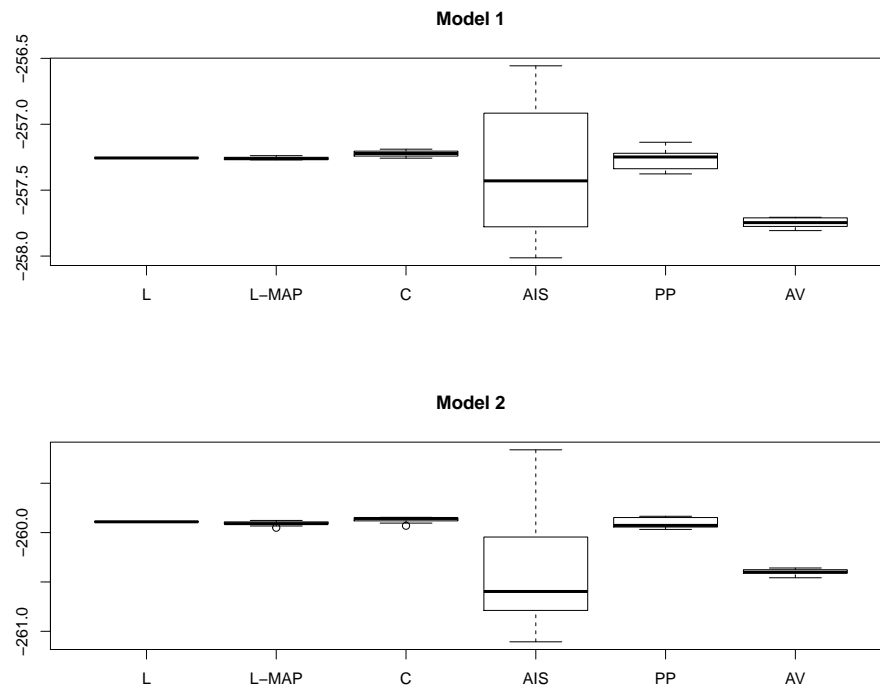


FIG 4. Boxplots of the log evidence for the two models on the logistic regression. The methods are the Laplace method (L), Laplace at the Maximum a Posteriori (L-MAP), Chib's method (C), Annealed Importance Sampling (AIS), Power Posterior (PP) and our method (AV).

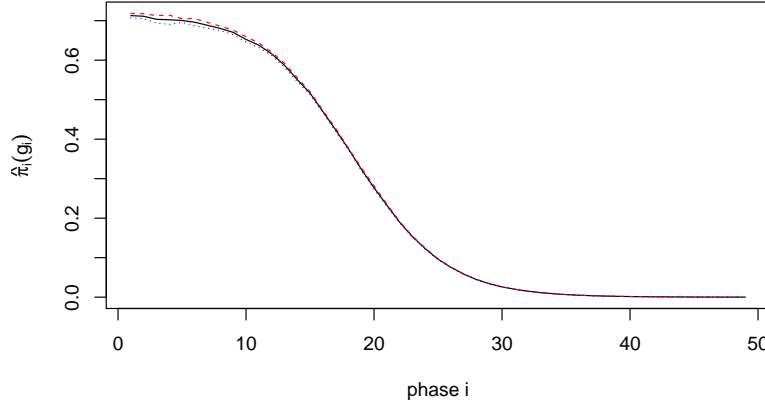


FIG 5. Error plot of  $\hat{\pi}_i(g_i)$  for  $i \in \{0, \dots, M-1\}$  in the example of the logistic regression (model  $\mathcal{M}_1$ ). The mean of  $\hat{\pi}_i(g_i)$  is displayed in black and is spaced apart from the other two curves by the standard deviation of  $\hat{\pi}_i(g_i)$ .

and neither convex. Nevertheless, we experiment heuristically our algorithm on a close model given by its likelihood

$$p(y|\{\theta_j\}_{j=1}^4) = \prod_{i=1}^p \left[ \frac{1}{4} \left( \frac{\lambda}{2\pi} \right)^2 \left\{ \sum_{j=1}^4 \exp(-(\lambda/2)(y_i - \theta_j)^2) \right\} \right]$$

for  $y = (y_1, \dots, y_p) \in \mathbb{R}^p$  a vector of observations. The prior distributions are set following the recommendations of [33, Section 4.2.1] and [42]. For  $j \in \{1, \dots, 4\}$ ,  $\theta_j$  is drawn from a Gaussian distribution of mean  $\xi = 1.35$  and inverse variance  $\varsigma = 7.6 \times 10^{-3}$ .  $\lambda$  is set equal to 0.03. The observations  $y \in \mathbb{R}^{100}$  are 100 simulated data points from an equally weighted mixture of four Gaussian densities with means  $(-3, 0, 3, 6)$  and standard deviations 0.55, taken from [25]. Define for  $\theta = (\theta_1, \dots, \theta_4) \in \mathbb{R}^4$ ,  $\ell: \mathbb{R}^4 \rightarrow \mathbb{R}$  by  $\ell(\theta) = -\log(p(y|\theta)p(\theta))$ . The `optim` function of R [41] gives a local minimum at  $\theta^* \approx (1.76562^{\otimes 4})$ . Define then the potential  $U: \mathbb{R}^4 \rightarrow \mathbb{R}$  for  $\theta \in \mathbb{R}^4$  by  $U(\theta) = \ell(\theta + \theta^*) - \ell(\theta^*)$ . Set  $\epsilon = \mu = 0.1$ ,  $m = \varsigma$  and  $L = 1$ . Similarly to the logistic regression, to decrease the running time of the algorithm, the recurrence for the variances  $\{\sigma_i^2\}_{i=0}^{M-1}$  is defined by  $\sigma_{i+1}^2 = \varsigma_s^5(\sigma_i^2)$  as long as the stopping condition (36) is not fulfilled. For each phase, the step size  $\gamma_i$  is set equal to  $10^{-1}(\kappa_i \sigma_i^2 m_i)/(dL_i^2)$ , the burn-in period  $N_i$  to  $10^4$  and the number of samples  $n_i$  to  $10^5$  where  $m_i, L_i, \kappa_i$  are defined in (13) and (14). For comparison purposes, we run the same algorithm using the Metropolis Adjusted Langevin Algorithm (MALA) instead of ULA to estimate  $\hat{\pi}_i(g_i)$  at each phase. The step size  $\gamma_i$  is set equal to  $(\kappa_i \sigma_i^2 m_i)/(dL_i^2)$  and the number of samples  $n_i$  to

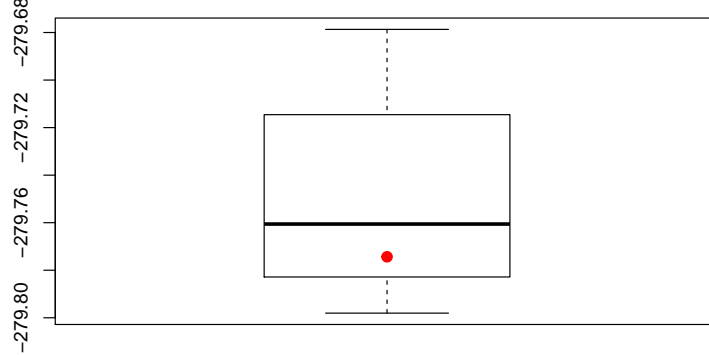


FIG 6. *Boxplot of the log evidence for the mixture of Gaussian distributions.*

$10^6$ . The experiments are repeated 10 times. The boxplot is plotted in Figure 6 and the red point indicates the mean of our algorithm using MALA.

#### 4. Mean squared error for locally Lipschitz functions

In this Section, we extend the results of [15, Section 3] to locally Lipschitz functions. This Section is of independent interest and only Propositions 17 and 20 are used in Section 5. Let  $U : \mathbb{R}^d \rightarrow \mathbb{R}$  be a continuously differentiable function. Consider the target distribution  $\pi$  with density  $x \mapsto e^{-U(x)} / \int_{\mathbb{R}^d} e^{-U(y)} dy$  w.r.t. the Lebesgue measure. We deal with the problem of estimating  $\int_{\mathbb{R}^d} f(x) d\pi(x)$  for locally Lipschitz  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  by the ULA algorithm defined for  $k \in \mathbb{N}$  by,

$$X_{k+1} = X_k - \gamma_{k+1} \nabla U(X_k) + \sqrt{2\gamma_{k+1}} Z_{k+1}, \tag{66}$$

where  $(Z_k)_{k \geq 1}$  is an i.i.d. sequence of  $d$ -dimensional Gaussian vectors with zero mean, identity covariance and  $(\gamma_k)_{k \geq 1}$  is a sequence of positive step sizes, which can either be held constant or be chosen to decrease to 0. For  $n, p \in \mathbb{N}$ , denote by

$$\Gamma_{n,p} \stackrel{\text{def}}{=} \sum_{k=n}^p \gamma_k, \quad \Gamma_n = \Gamma_{1,n}, \tag{67}$$

and consider the Markov kernel  $R_\gamma$  given for all  $A \in \mathcal{B}(\mathbb{R}^d)$  and  $x \in \mathbb{R}^d$  by

$$R_\gamma(x, A) = \int_A (4\pi\gamma)^{-d/2} \exp\left(- (4\gamma)^{-1} \|y - x + \gamma \nabla U(x)\|^2\right) dy. \tag{68}$$

Define

$$Q_\gamma^{n,p} = R_{\gamma_n} \cdots R_{\gamma_p}, \quad Q_\gamma^n = Q_\gamma^{1,n}, \quad (69)$$

with the convention that for  $n, p \geq 0$ ,  $n < p$ ,  $Q_\gamma^{p,n}$  and  $Q_\gamma^{0,0}$  are the identity operator.

For all initial distribution  $\mu_0$  on  $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ ,  $\mathbb{P}_{\mu_0}$  and  $\mathbb{E}_{\mu_0}$  denote the probability and the expectation respectively associated with the sequence of Markov kernels (68) and the initial distribution  $\mu_0$  on the canonical space  $(\mathbb{R}^d)^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^d)^{\otimes \mathbb{N}}$  and  $(X_k)_{k \in \mathbb{N}}$  denotes the canonical process. Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  and consider the following assumption,

- L1.** 1. There exists  $L_f : \mathbb{R}^d \rightarrow [0, +\infty)$  a continuous function such that for all  $x, y \in \mathbb{R}^d$ ,  $|f(y) - f(x)| \leq \|y - x\| \max\{L_f(x), L_f(y)\}$ .  
 2. There exist  $\varepsilon > 0$ ,  $C_\pi > 0$  and continuous functions  $C_Q, C_{Q,\varepsilon} : \mathbb{R}^d \rightarrow [0, +\infty)$  such that for all  $x \in \mathbb{R}^d$ ,

$$\begin{aligned} \pi(L_f^2) &\leq C_\pi, \quad \sup_{p \geq n \geq 0} \delta_x Q_\gamma^{n,p}(L_f^2) \leq C_Q(x), \quad (70) \\ \sup_{p \geq n \geq 0} \delta_x Q_\gamma^{n,p}(L_f^{2(1+\varepsilon)}) &\leq C_{Q,\varepsilon}(x) \end{aligned}$$

Under **L1**, we study the approximation of  $\int_{\mathbb{R}^d} f(y)\pi(dy)$  by the weighted average estimator

$$\hat{\pi}_n^N(f) = \sum_{k=N+1}^{N+n} \omega_{k,n}^N f(X_k), \quad \omega_{k,n}^N = \gamma_{k+1} \Gamma_{N+2, N+n+1}^{-1}, \quad (71)$$

where  $N \geq 0$  is the length of the burn-in period and  $n \geq 1$  is the number of samples. The Mean Squared Error (MSE) of  $\hat{\pi}_n^N(f)$  is defined by:

$$\text{MSE}_f(x, N, n) = \mathbb{E}_x \left[ \left\{ \hat{\pi}_n^N(f) - \pi(f) \right\}^2 \right], \quad (72)$$

and can be decomposed as,

$$\text{MSE}_f(x, N, n) = \left\{ \mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f) \right\}^2 + \text{Var}_x \left[ \hat{\pi}_n^N(f) \right]. \quad (73)$$

The analysis of  $\text{MSE}_f(x, N, n)$  is similar to [15, Section 3]. First, the squared bias in (73) is bounded. Denote by,

$$A_0 = 2L^2 \kappa^{-1} d, \quad (74)$$

$$A_1 = 2dL^2 + dL^4(\kappa^{-1} + (m+L)^{-1})(m^{-1} + 6^{-1}(m+L)^{-1}), \quad (75)$$

$$B_0 = d \left( 2L^2 + \kappa^{-1} \{d\tilde{L}^2/3 + 4L^4/(3m)\} \right), \quad (76)$$

$$B_1 = dL^4 (\kappa^{-1} + \{6(m+L)\}^{-1} + m^{-1}), \quad (77)$$

where  $\kappa$  is given by (14). Define then for  $n \in \mathbb{N}^*$ ,

$$u_n^{(1)}(\gamma) = \prod_{k=1}^n (1 - \kappa\gamma_k/2), \quad (78)$$

$$u_n^{(2)}(\gamma) = \sum_{i=1}^n (A_0\gamma_i^2 + A_1\gamma_i^3) \prod_{k=i+1}^n (1 - \kappa\gamma_k/2), \quad (79)$$

$$u_n^{(3)}(\gamma) = \sum_{i=1}^n (B_0\gamma_i^3 + B_1\gamma_i^4) \prod_{k=i+1}^n (1 - \kappa\gamma_k/2). \quad (80)$$

**Proposition 17.** *Assume **H1** and **H2**( $m$ ) for  $m > 0$ . Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  satisfying **L1**. Let  $(\gamma_k)_{k \geq 1}$  be a nonincreasing sequence with  $\gamma_1 \leq 1/(m+L)$ . Let  $x^*$  be the unique minimizer of  $U$ . Let  $(X_n)_{n \geq 0}$  be given by (66) and started at  $x \in \mathbb{R}^d$ . Then for all  $N \geq 0$ ,  $n \geq 1$ :*

$$\begin{aligned} \{\mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f)\}^2 &\leq \{C_\pi + C_Q(x)\} \\ &\times \sum_{k=N+1}^{N+n} \omega_{k,n}^N \left\{ 2(\|x - x^*\|^2 + d/m)u_k^{(1)}(\gamma) + w_k(\gamma) \right\}, \quad (81) \end{aligned}$$

where  $u_n^{(1)}(\gamma)$  is given in (78) and  $w_n(\gamma)$  is equal to  $u_n^{(2)}(\gamma)$  defined by (79) and to  $u_n^{(3)}(\gamma)$ , defined by (80), if **H3** holds.

*Proof.* For all  $k \in \{N+1, \dots, N+n\}$ , let  $\xi_k$  be the optimal transference plan between  $\delta_x Q_\gamma^k$  and  $\pi$  for  $W_2$ . By the Jensen and the Cauchy Schwarz inequalities, and **L1**, we have:

$$\begin{aligned} (\mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f))^2 &= \left( \sum_{k=N+1}^{N+n} \omega_{k,n}^N \int_{\mathbb{R}^d \times \mathbb{R}^d} \{f(z) - f(y)\} \xi_k(dz, dy) \right)^2 \\ &\leq \sum_{k=N+1}^{N+n} \omega_{k,n}^N \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} \|z - y\| \max\{L_f(z), L_f(y)\} \xi_k(dz, dy) \right)^2 \\ &\leq \{C_\pi + C_Q(x)\} \sum_{k=N+1}^{N+n} \omega_{k,n}^N \int_{\mathbb{R}^d \times \mathbb{R}^d} \|z - y\|^2 \xi_k(dz, dy). \end{aligned}$$

The proof follows from [15, Theorems 5 and 8].  $\square$

To deal with the variance term in (73), we adapt the proof of [28, Theorem 2] to our setting, where  $f$  is only locally Lipschitz and the Markov chain (66) is inhomogeneous. It is based on the Gaussian Poincaré inequality [6, Theorem 3.20]. Let  $Z = (Z_1, \dots, Z_d)$  be a Gaussian vector with identity covariance matrix and  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  be a locally Lipschitz function. Recall that by Rademacher's Theorem [19, Theorem 3.2], a locally Lipschitz function is almost everywhere differentiable w.r.t. Lebesgue measure on  $\mathbb{R}^d$ . The Gaussian Poincaré inequality



states that  $\text{Var}[f(Z)] \leq \mathbb{E}[\|\nabla f(Z)\|^2]$ . Noticing that for all  $x \in \mathbb{R}^d$ ,  $R_\gamma(x, \cdot)$  defined in (68) is a Gaussian distribution with mean  $x - \gamma \nabla U(x)$  and covariance matrix  $2\gamma I_d$ , the Gaussian Poincaré inequality implies:

$$0 \leq \int R_\gamma(x, dy) \{f(y) - R_\gamma f(x)\}^2 \leq 2\gamma \int R_\gamma(x, dy) \|\nabla f(y)\|^2. \quad (82)$$

First consider the following decomposition of  $\hat{\pi}_n^N(f) - \mathbb{E}_x[\hat{\pi}_n^N(f)]$  as the sum of martingale increments,

$$\begin{aligned} \hat{\pi}_n^N(f) - \mathbb{E}_x[\hat{\pi}_n^N(f)] &= \sum_{k=N}^{N+n-1} \{ \mathbb{E}_x^{\mathcal{G}_{k+1}}[\hat{\pi}_n^N(f)] - \mathbb{E}_x^{\mathcal{G}_k}[\hat{\pi}_n^N(f)] \} \\ &\quad + \mathbb{E}_x^{\mathcal{G}_N}[\hat{\pi}_n^N(f)] - \mathbb{E}_x[\hat{\pi}_n^N(f)], \end{aligned}$$

where  $(\mathcal{G}_n)_{n \geq 0}$  is the natural filtration associated with the Markov chain  $(X_n)_{n \geq 0}$ . This implies that the variance may be decomposed as the following sum

$$\begin{aligned} \text{Var}_x[\hat{\pi}_n^N(f)] &= \sum_{k=N}^{N+n-1} \mathbb{E}_x \left[ (\mathbb{E}_x^{\mathcal{G}_{k+1}}[\hat{\pi}_n^N(f)] - \mathbb{E}_x^{\mathcal{G}_k}[\hat{\pi}_n^N(f)])^2 \right] \\ &\quad + \mathbb{E}_x \left[ (\mathbb{E}_x^{\mathcal{G}_N}[\hat{\pi}_n^N(f)] - \mathbb{E}_x[\hat{\pi}_n^N(f)])^2 \right]. \quad (83) \end{aligned}$$

Because  $\hat{\pi}_n^N(f)$  is an additive functional, the martingale increment  $\mathbb{E}_x^{\mathcal{G}_{k+1}}[\hat{\pi}_n^N(f)] - \mathbb{E}_x^{\mathcal{G}_k}[\hat{\pi}_n^N(f)]$  has a simple expression. For  $k = N + n, \dots, N + 1$ , define backward in time the function

$$\Phi_{n,k}^N : x_k \mapsto \omega_{k,n}^N f(x_k) + R_{\gamma_{k+1}} \Phi_{n,k+1}^N(x_k), \quad (84)$$

with the convention  $\Phi_{n,N+n+1}^N = 0$ . Denote finally

$$\Psi_n^N : x_N \mapsto R_{\gamma_{N+1}} \Phi_{n,N+1}^N(x_N). \quad (85)$$

Note that for  $k \in \{N, \dots, N + n - 1\}$ , by the Markov property,

$$\Phi_{n,k+1}^N(X_{k+1}) - R_{\gamma_{k+1}} \Phi_{n,k+1}^N(X_k) = \mathbb{E}_x^{\mathcal{G}_{k+1}}[\hat{\pi}_n^N(f)] - \mathbb{E}_x^{\mathcal{G}_k}[\hat{\pi}_n^N(f)], \quad (86)$$

and  $\Psi_n^N(X_N) = \mathbb{E}_x^{\mathcal{G}_N}[\hat{\pi}_n^N(f)]$ . With these notations, (83) may be equivalently expressed as

$$\begin{aligned} \text{Var}_x[\hat{\pi}_n^N(f)] &= \sum_{k=N}^{N+n-1} \mathbb{E}_x \left[ R_{\gamma_{k+1}} \{ \Phi_{n,k+1}^N(\cdot) - R_{\gamma_{k+1}} \Phi_{n,k+1}^N(X_k) \}^2 (X_k) \right] \\ &\quad + \text{Var}_x[\Psi_n^N(X_N)]. \quad (87) \end{aligned}$$

Now for  $k = N + n, \dots, N + 1$ , we will use the Gaussian Poincaré inequality (82) to the sequence of function  $\Phi_{n,k}^N$ . It is required to prove that  $\Phi_{n,k}^N$  is locally Lipschitz (see Lemma 18). For the variance of  $\Psi_n^N(X_N)$ , similar arguments apply using Lemma 19.

**Lemma 18.** Assume **H1**, **H2**( $m$ ) for  $m > 0$  and let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  satisfying **L1**. Let  $(\gamma_k)_{k \geq 1}$  be a nonincreasing sequence with  $\gamma_1 \leq 2/(m + L)$ . Then for all  $\ell \geq n \geq 0$ ,  $Q_\gamma^{n,\ell} f$  is locally Lipschitz and differentiable for almost all  $x \in \mathbb{R}^d$ . Its gradient is bounded by,

$$\|\nabla Q_\gamma^{n,\ell} f(x)\| \leq \prod_{k=n}^{\ell} (1 - \kappa\gamma_k)^{1/2} (\delta_x Q_\gamma^{n,\ell} L_f^2)^{1/2}. \quad (88)$$

*Proof.* Let  $\xi_{x,y}$  be the optimal transference plan between  $\delta_x Q_\gamma^{n,\ell}$  and  $\delta_y Q_\gamma^{n,\ell}$  for  $W_2$ . By Rademacher's Theorem [19, Theorem 3.2],  $\nabla Q_\gamma^{n,\ell} f(x)$  exists for almost all  $x \in \mathbb{R}^d$ . For such  $x$ , using Cauchy-Schwarz's inequality and [15, Theorem 4], we have

$$\begin{aligned} \|\nabla Q_\gamma^{n,\ell} f(x)\| &= \sup_{\|u\| \leq 1} \lim_{t \rightarrow 0} |(Q_\gamma^{n,\ell} f(x + tu) - Q_\gamma^{n,\ell} f(x)) / t| \\ &= \sup_{\|u\| \leq 1} \lim_{t \rightarrow 0} \left| t^{-1} \int_{\mathbb{R}^d \times \mathbb{R}^d} \{f(z_2) - f(z_1)\} \xi_{x, x+tu}(dz_1, dz_2) \right| \\ &\leq \sup_{\|u\| \leq 1} \liminf_{t \rightarrow 0} t^{-1} W_2(\delta_x Q_\gamma^{n,\ell}, \delta_{x+tu} Q_\gamma^{n,\ell}) \\ &\quad \times \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} (L_f^2(z_1) \vee L_f^2(z_2)) \xi_{x, x+tu}(dz_1, dz_2) \right\}^{1/2} \\ &\leq \sup_{\|u\| \leq 1} \liminf_{t \rightarrow 0} \prod_{k=n}^{\ell} (1 - \kappa\gamma_k)^{1/2} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} (L_f^2(z_1) \vee L_f^2(z_2)) \xi_{x, x+tu}(dz_1, dz_2) \right\}^{1/2}. \end{aligned}$$

It is then sufficient to prove that,

$$\lim_{y \rightarrow x} \int_{\mathbb{R}^d \times \mathbb{R}^d} L_f^2(z_1) \vee L_f^2(z_2) \xi_{x,y}(dz_1, dz_2) = \int_{\mathbb{R}^d} L_f^2(z_1) \delta_x Q_\gamma^{n,\ell}(dz_1).$$

Let  $\varepsilon, \eta, R > 0$  and  $y \in \mathbb{R}^d$ . Since  $a \vee b - a = (b - a)_+$ , we have

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} (L_f^2(z_2) - L_f^2(z_1))_+ \xi_{x,y}(dz_1, dz_2) = E_1(y) + E_2(y) + E_3(y)$$

where,

$$\begin{aligned} E_1(y) &= \int_{\mathbb{R}^d \times \mathbb{R}^d} (L_f^2(z_2) - L_f^2(z_1))_+ \mathbb{1}_{\{\|z_1\| + \|z_2\| \geq 2R\}} \xi_{x,y}(dz_1, dz_2), \\ E_2(y) &= \int_{\mathbb{R}^d \times \mathbb{R}^d} (L_f^2(z_2) - L_f^2(z_1))_+ \mathbb{1}_{\{\|z_1\| + \|z_2\| \leq 2R\}} \mathbb{1}_{\{\|z_1 - z_2\| \leq \eta\}} \xi_{x,y}(dz_1, dz_2), \\ E_3(y) &= \int_{\mathbb{R}^d \times \mathbb{R}^d} (L_f^2(z_2) - L_f^2(z_1))_+ \mathbb{1}_{\{\|z_1\| + \|z_2\| \leq 2R\}} \mathbb{1}_{\{\|z_1 - z_2\| \geq \eta\}} \xi_{x,y}(dz_1, dz_2). \end{aligned}$$

Hölder's inequality gives for  $p, q > 1$ ,  $1/p + 1/q = 1$ ,

$$E_1(y) \leq \left( \int_{\mathbb{R}^d} L_f^{2q}(z_2) \delta_y Q_\gamma^{n,\ell}(dz_2) \right)^{1/q} \\ \times \left( \int_{\mathbb{R}^d} \mathbb{1}_{\{\|z_1\| \geq R\}} \delta_x Q_\gamma^{n,\ell}(dz_1) + \int_{\mathbb{R}^d} \mathbb{1}_{\{\|z_2\| \geq R\}} \delta_y Q_\gamma^{n,\ell}(dz_2) \right)^{1/p}.$$

Under **L1-2**, the first term on the right hand side is dominated by a constant for  $q$  small enough, and the second term tends to 0 for  $R$  large enough, uniformly for  $y$  in a compact neighborhood of  $x$  by [15, Theorem 3] and

$$\int_{\mathbb{R}^d} \mathbb{1}_{\{\|z_2\| \geq R\}} \delta_y Q_\gamma^{n,\ell}(dz_2) \leq R^{-2} \int_{\mathbb{R}^d} \|z_2\|^2 \delta_y Q_\gamma^{n,\ell}(dz_2).$$

We can then choose  $R$  such that  $E_1(y) \leq \varepsilon/3$ . We consider now  $E_2(y)$ .  $L_f^2$  is a continuous function, uniformly continuous on a compact set and we can then choose  $\eta$  such that  $E_2(y) \leq \varepsilon/3$ . We finally consider  $E_3(y)$ . By Markov's inequality and  $\lim_{y \rightarrow x} W_2^2(\delta_x Q_\gamma^{n,\ell}, \delta_y Q_\gamma^{n,\ell}) = 0$ , there exists a compact neighborhood  $\mathcal{V}(x)$  of  $x$  such that  $y \in \mathcal{V}(x)$  implies  $E_3(y) \leq \varepsilon/3$ . □

**Lemma 19.** *Assume **H1** and **H2**( $m$ ) for  $m > 0$ . Let  $(\gamma_k)_{k \geq 1}$  be a nonincreasing sequence with  $\gamma_1 \leq 2/(m+L)$  and  $N \geq 0$ . Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  be such that  $Q_\gamma^{k+1,N} f$  is locally Lipschitz for  $k \in \{1, \dots, N\}$ . Then for all  $x \in \mathbb{R}^d$ ,*

$$\int_{\mathbb{R}^d} Q_\gamma^N(x, dy) \{f(y) - Q_\gamma^N f(x)\}^2 \leq 2 \sum_{k=1}^N \gamma_k \int_{\mathbb{R}^d} Q_\gamma^k(x, dy) \|\nabla Q_\gamma^{k+1,N} f(y)\|^2.$$

*Proof.* Using  $\mathbb{E}_x^{\mathcal{G}_k} [f(X_N)] = Q_\gamma^{k+1,N} f(X_k)$ , we get

$$\text{Var}_x [f(X_N)] = \sum_{k=1}^N \mathbb{E}_x \left[ \mathbb{E}_x^{\mathcal{G}_{k-1}} \left[ \left( \mathbb{E}_x^{\mathcal{G}_k} [f(X_N)] - \mathbb{E}_x^{\mathcal{G}_{k-1}} [f(X_N)] \right)^2 \right] \right] \\ = \sum_{k=1}^N \mathbb{E}_x \left[ R_{\gamma_k} \left\{ Q_\gamma^{k+1,N} f(\cdot) - R_{\gamma_k} Q_\gamma^{k+1,N} f(X_{k-1}) \right\}^2 (X_{k-1}) \right].$$

Eq. (82) implies that

$$\text{Var}_x [f(X_N)] \leq 2 \sum_{k=1}^N \gamma_k \int_{\mathbb{R}^d} Q_\gamma^k(x, dy) \|\nabla Q_\gamma^{k+1,N} f(y)\|^2.$$

□

**Proposition 20.** *Assume **H1** and **H2**( $m$ ) for  $m > 0$ . Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  satisfying **L1** and  $(\gamma_k)_{k \geq 1}$  be a nonincreasing sequence with  $\gamma_1 \leq 2/(m+L)$ . Then for all  $N \geq 0$ ,  $n \geq 1$ , we get*

$$\text{Var}_x [\hat{\pi}_n^N(f)] \leq \frac{8C_Q(x)}{\kappa^2 \Gamma_{N+2, N+n+1}} \left\{ 1 + \Gamma_{N+2, N+n+1}^{-1} \left( \kappa^{-1} + \frac{2}{m+L} \right) \right\}. \quad (89)$$

*Proof.* For  $k \in \{N, \dots, N+n-1\}$  and for all  $y, x \in \mathbb{R}^d$ , we have

$$\begin{aligned} |\Phi_{n,k+1}^N(y) - \Phi_{n,k+1}^N(x)| &= \left| \omega_{k+1,n}^N \{f(y) - f(x)\} \right. \\ &\quad \left. + \sum_{i=k+2}^{N+n} \omega_{i,n}^N \{Q_\gamma^{k+2,i} f(y) - Q_\gamma^{k+2,i} f(x)\} \right|. \end{aligned} \quad (90)$$

By Lemma 18,  $\Phi_{n,k+1}^N$  is locally Lipschitz and for almost all  $x \in \mathbb{R}^d$ ,

$$\|\nabla \Phi_{n,k+1}^N(x)\| \leq \sum_{i=k+1}^{N+n} \omega_{i,n}^N \left\{ \prod_{\ell=k+2}^i (1 - \kappa\gamma\ell)^{1/2} \right\} (\delta_x Q_\gamma^{k+2,i} L_f^2)^{1/2}.$$

For  $k \in \{N, \dots, N+n-1\}$  and  $x \in \mathbb{R}^d$ , we have by (82) and the Cauchy-Schwarz inequality,

$$\begin{aligned} R_{\gamma_{k+1}} \left\{ \Phi_{n,k+1}^N(\cdot) - R_{\gamma_{k+1}} \Phi_{n,k+1}^N(x) \right\}^2(x) \\ \leq 2\gamma_{k+1} \Omega_{k,n}^N \left\{ \sum_{i=k+1}^{N+n} \omega_{i,n}^N \prod_{\ell=k+2}^i (1 - \kappa\gamma\ell)^{1/2} (\delta_x Q_\gamma^{k+1,i} L_f^2) \right\}, \end{aligned}$$

where,

$$\Omega_{k,n}^N = \sum_{i=k+1}^{N+n} \omega_{i,n}^N \prod_{\ell=k+2}^i (1 - \kappa\gamma\ell)^{1/2}. \quad (91)$$

By L1-2, we get for  $k \in \{N, \dots, N+n-1\}$

$$\begin{aligned} \mathbb{E}_x \left[ R_{\gamma_{k+1}} \left\{ \Phi_{n,k+1}^N(\cdot) - R_{\gamma_{k+1}} \Phi_{n,k+1}^N(X_k) \right\}^2(X_k) \right] \\ \leq 2\gamma_{k+1} \Omega_{k,n}^N \left\{ \sum_{i=k+1}^{N+n} \omega_{i,n}^N \prod_{\ell=k+2}^i (1 - \kappa\gamma\ell)^{1/2} (\delta_x Q_\gamma^i L_f^2) \right\} \leq 2\gamma_{k+1} C_Q(x) (\Omega_{k,n}^N)^2. \end{aligned}$$

Using  $(1-t)^{1/2} \leq (1-t/2)$  for  $t \in [0, 1]$ , we have

$$\Omega_{k,n}^N \leq (\kappa\Gamma_{N+2, N+n+1}/2)^{-1}. \quad (92)$$

Using this inequality, we get

$$\begin{aligned} \sum_{k=N}^{N+n-1} \mathbb{E}_x \left[ R_{\gamma_{k+1}} \left\{ \Phi_{n,k+1}^N(\cdot) - R_{\gamma_{k+1}} \Phi_{n,k+1}^N(X_k) \right\}^2(X_k) \right] \\ \leq 8C_Q(x) \Gamma_{N+1, N+n} / (\kappa\Gamma_{N+2, N+n+1})^2. \end{aligned} \quad (93)$$

We now bound  $\text{Var}_x [\Psi_n^N(X_N)]$ . Since for all  $x \in \mathbb{R}^d$ , we have

$$\Psi_n^N(x) = \sum_{i=N+1}^{N+n} \omega_{i,n}^N Q_\gamma^{N+1,i} f(x),$$

by Lemma 18,  $Q_\gamma^{k+1,N}\Psi_n^N$  is locally Lipschitz for  $k \in \{1, \dots, N\}$  with for almost all  $x \in \mathbb{R}^d$ ,

$$\|\nabla Q_\gamma^{k+1,N}\Psi_n^N(x)\| \leq \sum_{i=N+1}^{N+n} \omega_{i,n}^N \prod_{\ell=k+1}^i (1 - \kappa\gamma_\ell)^{1/2} (\delta_x Q_\gamma^{k+1,i} L_f^2)^{1/2} .$$

Isolating the term  $\prod_{\ell=k+1}^N (1 - \kappa\gamma_\ell)^{1/2}$  and since  $(1 - \kappa\gamma_{N+1})^{1/2} \leq 1$ , the Cauchy-Schwarz inequality implies

$$\begin{aligned} \|\nabla Q_\gamma^{k+1,N}\Psi_n^N(x)\|^2 &\leq \left\{ \prod_{\ell=k+1}^N (1 - \kappa\gamma_\ell) \right\} \\ &\quad \times \Omega_{N,n}^N \sum_{i=N+1}^{N+n} \omega_{i,n}^N \prod_{\ell=N+1}^i (1 - \kappa\gamma_\ell)^{1/2} \delta_x Q_\gamma^{k+1,i} L_f^2 . \end{aligned}$$

Plugging this inequality in Lemma 19, using **L1-2**,  $\sum_{k=1}^N \gamma_k \prod_{i=k+1}^N (1 - \kappa\gamma_i) \leq \kappa^{-1}$  and (92), we get

$$\text{Var}_x [\Psi_n^N(X_N)] \leq 2\kappa^{-1} C_Q(x) (\kappa/2)^{-2} \Gamma_{N+2, N+n+1}^{-2} . \quad (94)$$

Combining (93) and (94) in (83) concludes the proof.  $\square$

## 5. Proofs

### 5.1. Proofs of propositions 2, 3, 4

We assume in this Section that **H1** and **H2**( $m$ ) for some  $m \geq 0$  hold. The proofs rely on the results given in Section 4, Propositions 17 and 20 which establish bounds on the mean squared error for locally Lipschitz functions. For  $i \in \{0, \dots, M-1\}$ ,  $\sigma_i^2 > 0$  and  $\gamma_i > 0$ , consider the Markov chain  $(X_{i,n})_{n \geq 0}$  (8) and its associated Markov kernel  $R_i$  defined for all  $A \in \mathcal{B}(\mathbb{R}^d)$  and  $x \in \mathbb{R}^d$  by

$$R_i(x, A) = \int_A (4\pi\gamma_i)^{-d/2} \exp\left(- (4\gamma_i)^{-1} \|y - x + \gamma_i \nabla U_i(x)\|^2\right) dy . \quad (95)$$

Under **H1** and **H2**( $m$ ) for  $m \geq 0$ , [35, Theorems 2.1.12, 2.1.9] show the following useful inequalities for all  $x, y \in \mathbb{R}^d$ ,

$$\langle \nabla U_i(y) - \nabla U_i(x), y - x \rangle \geq \frac{\kappa_i}{2} \|y - x\|^2 + \frac{1}{m_i + L_i} \|\nabla U_i(y) - \nabla U_i(x)\|^2 , \quad (96)$$

$$\langle \nabla U_i(y) - \nabla U_i(x), y - x \rangle \geq m_i \|y - x\|^2 , \quad (97)$$

where  $L_i, m_i$  are defined in (13) and  $\kappa_i$  in (14). We then check **L1** for  $g_i$ , where  $g_i : \mathbb{R}^d \rightarrow \mathbb{R}$  is defined in (6). Note that  $g_i$  is continuously differentiable and for  $x \in \mathbb{R}^d$ ,  $\nabla g_i(x) = 2a_i x e^{a_i \|x\|^2}$ . Define  $L_{g_i} : \mathbb{R}^d \rightarrow \mathbb{R}_+$  for  $x \in \mathbb{R}^d$  by,

$$L_{g_i}(x) = 2a_i \|x\| e^{a_i \|x\|^2} \quad (98)$$

We have for all  $x, y \in \mathbb{R}^d$ :

$$\begin{aligned} |g_i(y) - g_i(x)| &= \left| \int_0^1 \langle \nabla g_i(ty + (1-t)x), y - x \rangle dt \right| \\ &\leq \|y - x\| \max(L_{g_i}(x), L_{g_i}(y)), \end{aligned} \quad (99)$$

which implies that **L1-1** holds for  $g_i$ . The following Lemmas 21 and 22 enable to check **L1-2** for  $g_i$ .

**Lemma 21.** *Assume **H1** and **H2**( $m$ ) for  $m \geq 0$ . For all  $\sigma_i^2 \in (0, +\infty)$ ,  $n \in \mathbb{N}$ ,  $\gamma_i \in (0, 2/(m_i + L_i)]$ ,  $a_i \in [0, \kappa_i/8 \wedge (2\sigma_i^2)^{-1}]$  and  $x \in \mathbb{R}^d$ , we have:*

$$\sup_{n \in \mathbb{N}} R_i^n(L_{g_i}^2)(x) \leq 4a_i^2 g_i^2(x) C_{i,0} \left\{ \|x\|^2 + C_{i,1} \right\},$$

where  $L_{g_i}$  is defined in (98) and  $C_{i,0}, C_{i,1}$  in (23).

*Proof.* In the proof, the subscript  $i$  is not specified for ease of notation. Let  $\gamma \in (0, 2/(m + L)]$ . Note that for all  $\alpha \in [0, (4\gamma)^{-1}]$ , we have

$$\begin{aligned} R_\gamma(e^{\alpha \|\cdot\|^2})(x) &= \frac{e^{-(4\gamma)^{-1} \|x - \gamma \nabla U(x)\|^2}}{(4\pi\gamma)^{d/2}} \int_{\mathbb{R}^d} e^{(\alpha - (4\gamma)^{-1}) \|y\|^2 + (2\gamma)^{-1} \langle y, x - \gamma \nabla U(x) \rangle} dy \\ &= \phi(x), \end{aligned}$$

where  $\phi(x) = (1 - 4\gamma\alpha)^{-d/2} \exp\{(\alpha/(1 - 4\gamma\alpha)) \|x - \gamma \nabla U(x)\|^2\}$ . By the Leibniz integral rule and (96), we obtain:

$$\begin{aligned} R_\gamma(\|\cdot\|^2 e^{\alpha \|\cdot\|^2})(x) &= \partial_\alpha R_\gamma(e^{\alpha \|\cdot\|^2})(x) \\ &= (1 - 4\gamma\alpha)^{-d/2-1} \left\{ 2\gamma d + \frac{\|x - \gamma \nabla U(x)\|^2}{1 - 4\gamma\alpha} \right\} \exp\left(\frac{\alpha}{1 - 4\gamma\alpha} \|x - \gamma \nabla U(x)\|^2\right) \\ &\leq (1 - 4\gamma\alpha)^{-d/2-1} \left\{ 2\gamma d + \frac{1 - \kappa\gamma}{1 - 4\gamma\alpha} \|x\|^2 \right\} \exp\left(\frac{\alpha(1 - \kappa\gamma)}{1 - 4\gamma\alpha} \|x\|^2\right). \end{aligned}$$

Let  $a \in [0, \kappa/8)$ . Since  $a < (4\gamma)^{-1}$ , by a straightforward induction we have

$$\begin{aligned} \delta_x R_\gamma^p(\|\cdot\|^2 e^{2a\|\cdot\|^2}) &\leq (1 - 4\gamma\alpha_0)^{-d/2-1} \exp\left(\alpha_p \|x\|^2\right) \\ &\quad \times \sum_{\ell=0}^{p-1} 2\gamma d \alpha_\ell \alpha_0^{-1} \left\{ \prod_{k=1}^{\ell} (1 - 4\gamma\alpha_k)^{-d/2-1} \right\} \left\{ \prod_{k=\ell+1}^{p-1} (1 - 4\gamma\alpha_k)^{-d/2} \right\} \\ &\quad + (1 - 4\gamma\alpha_0)^{-d/2-1} \left\{ \prod_{k=1}^{p-1} (1 - 4\gamma\alpha_k)^{-d/2-1} \right\} \alpha_p \alpha_0^{-1} \|x\|^2 \exp\left(\alpha_p \|x\|^2\right) \\ &\leq \frac{1}{\alpha_0} \exp\left(\alpha_p \|x\|^2\right) \left\{ \prod_{k=0}^{p-1} (1 - 4\gamma\alpha_k)^{-d/2-1} \right\} \left\{ \alpha_p \|x\|^2 + 2d\gamma \sum_{\ell=0}^{p-1} \alpha_\ell \right\}, \end{aligned} \quad (100)$$

where  $(\alpha_\ell)_{\ell \in \mathbb{N}}$  is the decreasing sequence defined for  $\ell \geq 1$  by:

$$\alpha_0 = 2a, \quad \alpha_\ell = \alpha_{\ell-1} \frac{(1 - \kappa\gamma)}{1 - 4\alpha_{\ell-1}\gamma}. \quad (101)$$

We now bound the right-hand-side of (100). First, by using the following inequality,

$$\log(1 - 4\gamma\alpha) = -4\alpha \int_0^\gamma (1 - 4\alpha t)^{-1} dt \geq -4\alpha\gamma(1 - 4\alpha\gamma)^{-1},$$

we have:

$$\begin{aligned} \prod_{k=0}^{p-1} (1 - 4\gamma\alpha_k)^{-d/2-1} &= \exp\left(-\left(\frac{d}{2} + 1\right) \sum_{k=0}^{p-1} \log(1 - 4\alpha_k\gamma)\right) \\ &\leq \exp\left(\left(\frac{d}{2} + 1\right) \frac{4\gamma}{1 - \kappa\gamma} \sum_{k=0}^{p-1} \alpha_k \frac{1 - \kappa\gamma}{1 - 4\alpha_k\gamma}\right). \end{aligned} \quad (102)$$

Second, by a straightforward induction we get for all  $\ell \geq 0$ ,  $\alpha_\ell \leq 2a\{(1 - \kappa\gamma)(1 - 8a\gamma)^{-1}\}^\ell$ . Using (101) and this result implies:

$$\sum_{k=0}^{p-1} \alpha_k \frac{1 - \kappa\gamma}{1 - 4\alpha_k\gamma} = \sum_{k=1}^p \alpha_k \leq 2a \frac{1 - \kappa\gamma}{\kappa\gamma - 8a\gamma}, \quad \sum_{\ell=0}^{p-1} \alpha_\ell \leq 2a \frac{1 - 8a\gamma}{\kappa\gamma - 8a\gamma}.$$

Combining these inequalities and (102) in (100) concludes the proof.  $\square$

**Lemma 22.** *Assume **H1** and **H2**( $m$ ) for  $m \geq 0$ . For all  $\sigma_i^2 \in (0, +\infty)$  and  $a_i \in [0, m_i/\{4(d+4)\} \wedge (2\sigma_i^2)^{-1}]$ , we have*

$$\pi(L_{g_i}^2) \leq 4a_i^2 C_{i,2},$$

where  $C_{i,2}$  is defined in (23).

*Proof.* In the proof, the subscript  $i$  is not specified for ease of notations. Recall that the generator of the Langevin diffusion (7) associated to  $U$  is defined for any  $f$  in  $\mathcal{C}^2(\mathbb{R}^d)$  by

$$\mathcal{A}f = -\langle \nabla U, \nabla f \rangle + \Delta f .$$

In particular, for  $f(x) = \|x\|^2 e^{2a\|x\|^2}$  and  $x \in \mathbb{R}^d$ , we have

$$\begin{aligned} \nabla f(x) &= 2(1 + 2a \|x\|^2) x e^{2a\|x\|^2} , \\ \Delta f(x) &= e^{2a\|x\|^2} \left\{ 16a^2 \|x\|^4 + 4a(d+4) \|x\|^2 + 2d \right\} . \end{aligned}$$

Using (97) and  $\nabla U(0) = 0$ , we get

$$\mathcal{A}(\|\cdot\|^2 e^{2a\|\cdot\|^2})(x) \leq e^{2a\|x\|^2} \left\{ 2d + 2(2a(d+4) - m) \|x\|^2 + 4a(4a - m) \|x\|^4 \right\} .$$

Using that  $a \in [0, m/(4(d+4))]$ , we have  $2a(4a - m) \leq -(8/5)am$ . Then an elementary study of  $t \mapsto e^{2at} \{2d + 4a(4a - m)t^2\}$  on  $\mathbb{R}_+$  shows that:

$$\sup_{x \in \mathbb{R}^d} e^{2a\|x\|^2} \left\{ 2d + 4a(4a - m) \|x\|^4 \right\} \leq 4d .$$

Therefore we get using  $2(2a(d+4) - m) \leq -m$ ,

$$\mathcal{A}(\|\cdot\|^2 e^{2a\|\cdot\|^2})(x) \leq -m \|x\|^2 e^{2a\|x\|^2} + 4d .$$

Finally applying [32, Theorem 4.3-(ii)] shows the result.  $\square$

*Proofs of Propositions 2 and 3.* Lemmas 21 and 22 and proposition 17 prove the result.  $\square$

*Proof of Proposition 4.* The proof follows from Lemma 21 and proposition 20.  $\square$

## 5.2. Proof of Lemma 9

The case  $K = 0$  being straightforward, assume  $K \in \mathbb{N}^*$ . Using Markov's inequality, we have

$$\mathbb{P}(A_{S,\epsilon}^c) \leq \frac{4}{\epsilon^2} \frac{\mathbb{E} \left[ \left( \prod_{i=0}^{M-1} \hat{\pi}_i(g_i) - \prod_{i=0}^{M-1} \pi_i(g_i) \right)^2 \right]}{\left( \prod_{i=0}^{M-1} \pi_i(g_i) \right)^2} . \quad (103)$$

Since  $\hat{\pi}_i(g_i)$  for  $i \in \{0, \dots, M-1\}$  are independent, we get

$$\frac{\mathbb{E} \left[ \left( \prod_{i=0}^{M-1} \hat{\pi}_i(g_i) - \prod_{i=0}^{M-1} \pi_i(g_i) \right)^2 \right]}{\left( \prod_{i=0}^{M-1} \pi_i(g_i) \right)^2} = F_1^2(F_2 - 1) + (F_1 - 1)^2 , \quad (104)$$



where

$$F_1 = \prod_{i=0}^{M-1} \mathbb{E} [\hat{\pi}_i(g_i)] / \pi_i(g_i), \quad F_2 = \prod_{i=0}^{M-1} \mathbb{E} [\{\hat{\pi}_i(g_i)\}^2] / \mathbb{E}^2 [\hat{\pi}_i(g_i)].$$

In addition, since  $\{0, \dots, M-2\} = \cup_{k=0}^{K-1} \mathcal{I}_k$ , we can consider the following decomposition

$$\begin{aligned} F_1 &= \prod_{k=0}^{K-1} \prod_{i \in \mathcal{I}_k} \left( 1 + \frac{\mathbb{E} [\hat{\pi}_i(g_i)] - \pi_i(g_i)}{\pi_i(g_i)} \right) \\ &\quad \times \left( 1 + \frac{\mathbb{E} [\hat{\pi}_{M-1}(g_{M-1})] - \pi_{M-1}(g_{M-1})}{\pi_{M-1}(g_{M-1})} \right), \\ F_2 &= \prod_{k=0}^{K-1} \prod_{i \in \mathcal{I}_k} \left( 1 + \frac{\text{Var} [\hat{\pi}_i(g_i)]}{\mathbb{E} [\hat{\pi}_i(g_i)]^2} \right) \left( 1 + \frac{\text{Var} [\hat{\pi}_{M-1}(g_{M-1})]}{\mathbb{E} [\hat{\pi}_{M-1}(g_{M-1})]^2} \right). \end{aligned}$$

We now bound  $F_1, F_2$  separately. Using  $1+t \leq \exp(t)$  for  $t \in \mathbb{R}$  with  $t = \eta/(K|\mathcal{I}_k|)$  and leaving the term  $i = M-1$  out, we get by conditions [i](#)-[ii](#))

$$F_1 \leq (1 + \eta) \exp(\eta). \quad (105)$$

Since  $\hat{\pi}_i(g_i) \geq 1$ , we have  $\text{Var} [\hat{\pi}_i(g_i)] / \mathbb{E} [\hat{\pi}_i(g_i)]^2 \leq \eta^2 / K|\mathcal{I}_k|$ . Therefore using  $1+t \leq \exp(t)$  for  $t \in \mathbb{R}$  with  $t = \eta^2/(K|\mathcal{I}_k|)$  leaving the term  $i = M-1$  out, we obtain by conditions [i](#)-[ii](#))

$$F_2 \leq (1 + \eta^2) \exp(\eta^2). \quad (106)$$

By combining [\(103\)](#), [\(104\)](#), [\(105\)](#) and [\(106\)](#), we get:

$$(\epsilon^2/4)\mathbb{P}(\mathcal{A}_{\mathcal{S},\epsilon}^c) \leq (1 + \eta)^2 e^{2\eta} \left( (1 + \eta^2)e^{\eta^2} - 1 \right) + ((1 + \eta)e^\eta - 1)^2.$$

With  $\eta \leq 1/8$  and  $e^t - 1 \leq te^t$  for  $t \geq 0$ , we have  $(\epsilon^2/4)\mathbb{P}(\mathcal{A}_{\mathcal{S},\epsilon}^c) \leq 9\eta^2$ .

### 5.3. Proofs of Section 2.1

We preface the proofs by a technical lemma which gathers useful bounds and inequalities. We recall that in this Section the number of phases  $M$  is defined by [\(36\)](#)

$$M = \inf \{i \geq 1 : \sigma_{i-1}^2 \geq (2d + 7)/m\}.$$

**Lemma 23.** *Assume [H1](#) and [H2](#)( $m$ ) for  $m > 0$ . Let  $\{\sigma_i^2\}_{i=0}^{M-1}$  defined by [\(34\)](#) for  $\sigma_0^2$  given in [\(31\)](#) and  $M$  in [\(36\)](#).*

1.  $K \leq \lceil (1/\log(2)) \log\{(2d + 7)/(m\sigma_0^2)\} \rceil$  where  $K$  is defined in [\(39\)](#).
2. For all  $k \in \{0, \dots, K-1\}$  and  $i \in \mathcal{I}_k$ ,  $2^{k+1}\sigma_0^2 a_i |\mathcal{I}_k| \leq 1$ , where  $a_i$  is defined in [\(37\)](#) and  $\mathcal{I}_k$  in [\(38\)](#).

3. For all  $i \in \{0, \dots, M-1\}$  and  $\gamma_i \leq 1/(m_i + L_i)$ , there exist  $\alpha_i \in [4, 14]$  and  $\beta_i \in [1, 10]$  such that  $C_{i,2} + C_{i,0}C_{i,1} = \alpha_i dm_i^{-1}$  and  $C_{i,0}C_{i,1} = \beta_i d\kappa_i^{-1}$  where  $C_{i,0}, C_{i,1}, C_{i,2}$  and  $\kappa_i$  are given in (23) and (14) respectively.
4. For all  $i \in \{0, \dots, M-1\}$ ,  $0 < A_{i,1} \leq 4dL_i^4\kappa_i^{-1}m_i^{-1}$ , where  $L_i, m_i$  and  $\kappa_i$  are given in (13) and (14) respectively.
5. For all  $i \in \{0, \dots, M-2\}$ ,  $\kappa_i\sigma_i^2 \leq 4d + 16$ .
6. For all  $i \in \{0, \dots, M-1\}$ ,  $\sqrt{m_i}/(\kappa_i\sigma_i) \leq 1$
7. For all  $i \in \{0, \dots, M-1\}$ ,

$$\frac{m_i + L_i}{2m_i} \leq \frac{m + L}{2m}, \quad \frac{L_i^2}{\kappa_i^3\sigma_i^4m_i} \leq \left(\frac{m + L}{2m}\right)^3.$$

8. For all  $k \in \{0, \dots, K-1\}$  and  $i \in \mathcal{I}_k$ ,

$$\kappa_i^{-2}m_i^{-1/2}\sigma_i^{-2} \leq \frac{(2^{k+1}\sigma_0^2)^{3/2}}{(1 + m2^k\sigma_0^2)^{5/2}}, \quad \frac{L_i^2m_i^{-1/2}}{\kappa_i^2\sigma_i^2m_i^{1/2}} \leq \left(\frac{m + L}{2m}\right)^2 \frac{1}{1 + m2^k\sigma_0^2}.$$

*Proof.* 1. By (36) and (38),

$$K \leq \inf \left\{ k \geq 0 : 2^k\sigma_0^2 \geq \frac{2d+7}{m} \right\} = \left\lceil \frac{1}{\log(2)} \log \left( \frac{2d+7}{m\sigma_0^2} \right) \right\rceil.$$

2. Let  $k \in \{0, \dots, K-1\}$ . Denote by  $i_0 = \inf \mathcal{I}_k$ . By (37) and (38),

$$|\mathcal{I}_k| a_i = \sum_{i \in \mathcal{I}_k} a_i \leq \frac{1}{2\sigma_{i_0}^2} \leq \frac{1}{2^{k+1}\sigma_0^2},$$

and the proof follows.

3. Let  $k \in \{0, \dots, K-1\}$  and  $i \in \mathcal{I}_k$ . Since  $m_i \geq m + (2^{k+1}\sigma_0^2)^{-1}$ ,  $a_i \leq m_i/\{4(d+4)\}$ . Therefore using in addition that  $\gamma_i \leq 1/(m_i + L_i)$ , we have  $\kappa_i - 8a_i \geq \kappa_i(d+2)/(d+4)$  and  $1 - 8a_i\gamma_i \geq (d+3)/(d+4)$ . The definition of  $C_{i,0}, C_{i,1}, C_{i,2}$  (23) completes the proof.
4. The upper bound is a straightforward consequence of  $(1 + \kappa_i(m_i + L_i)^{-1})(1 + 6^{-1}m_i(m_i + L_i)^{-1}) \leq 2$ .
5. The bound follows using that  $\sigma_{M-2}^2 \leq (2d+7)/m$  by (36) and the sequence  $\{\kappa_i\sigma_i^2\}_{i=0}^{M-2}$  is non-decreasing since

$$\kappa_i\sigma_i^2 = 2 \left\{ 1 + \frac{mL\sigma_i^2 - 1/\sigma_i^2}{m + L + 2/\sigma_i^2} \right\},$$

and  $\{\sigma_i^2\}_{i=0}^{M-2}$  is non-decreasing.

6. The proof is a direct consequence of the fact that the sequence  $i \mapsto \sqrt{m_i}/(\kappa_i\sigma_i)$  is non-increasing since  $m < L$ ,  $\{\sigma_i^2\}_{i=0}^{M-2}$  is non-decreasing and

$$\frac{\sqrt{m_i}}{\kappa_i\sigma_i} = \frac{1}{2} \frac{1}{\sqrt{1 + m\sigma_i^2}} \left\{ 1 + \frac{1 + m\sigma_i^2}{1 + L\sigma_i^2} \right\}.$$

7. Using that  $\{\sigma_i^2\}_{i=0}^{M-2}$  is non-decreasing and

$$\frac{m_i + L_i}{2m_i} = \frac{2 + (m + L)\sigma_i^2}{2 + 2m\sigma_i^2}, \quad \frac{L_i^2}{\kappa_i^3 \sigma_i^4 m_i} \leq \left( \frac{(m + L)\sigma_i^2 + 2}{(2m)\sigma_i^2 + 2} \right)^3,$$

concludes the proof.

8. Let  $k \in \{0, \dots, K - 1\}$  and  $i \in \mathcal{I}_k$ . Since  $\kappa_i \geq m_i$  and  $2^k \sigma_0^2 \leq \sigma_i^2 \leq 2^{k+1} \sigma_0^2$ , we have

$$\kappa_i^{-2} m_i^{-1/2} \sigma_i^{-2} \leq \frac{\sigma_i^3}{(m_i \sigma_i^2)^{5/2}} \leq \frac{(2^{k+1} \sigma_0^2)^{3/2}}{(1 + m 2^k \sigma_0^2)^{5/2}},$$

and

$$\frac{L_i^2}{\kappa_i^2} \leq \left( \frac{m + L}{2m} \right)^2, \quad \frac{1}{m_i \sigma_i^2} \leq \frac{1}{1 + m 2^k \sigma_0^2}.$$

□

### 5.3.1. Proof of Lemma 8

Because  $U$  satisfies **H1**, **H2**( $m$ ) for  $m \geq 0$  and  $U(0) = 0$ ,  $\nabla U(0) = 0$ , we have:

$$\exp(-(L/2) \|x\|^2) \leq \exp(-U(x)) \leq \exp(-(m/2) \|x\|^2),$$

which implies by integration that,

$$(2\pi\sigma_0^2)^{d/2}/(1 + \sigma_0^2 L)^{d/2} \leq Z_0 \leq (2\pi\sigma_0^2)^{d/2}/(1 + \sigma_0^2 m)^{d/2},$$

where  $Z_0 = \int_{\mathbb{R}^d} e^{-U_0}$  and  $U_0$  is defined in (2). The proof follows from the expression of  $\sigma_0^2$  and the bound,

$$\left( \frac{1 + L\sigma_0^2}{1 + m\sigma_0^2} \right)^{d/2} \leq \exp\left( \frac{d}{2} \sigma_0^2 (L - m) \right).$$

### 5.3.2. Proof of Lemma 10

Let  $k \in \{0, \dots, K - 1\}$  and  $i \in \mathcal{I}_k$ . Assume that  $\gamma_i \leq (m_i + L_i)^{-1}$ . By Proposition 2, Proposition 4, Lemma 23-2 and  $\sigma_i^2 \leq 2^{k+1} \sigma_0^2$ , to check condition-i) of Lemma 9, it is then sufficient for  $\gamma_i, n_i, N_i$  to satisfy,

$$\frac{4d}{n_i m_i \kappa_i \gamma_i} \exp\left(-N_i \frac{\kappa_i \gamma_i}{2}\right) + 2\kappa_i^{-1} (A_{i,0} \gamma_i + A_{i,1} \gamma_i^2) \leq \frac{\eta^2}{4K^2} \frac{\sigma_i^4}{C_{i,2} + C_{i,0} C_{i,1}}, \quad (107)$$

$$\frac{32a_i C_{i,0} C_{i,1}}{\kappa_i^2 n_i \gamma_i} \left(1 + \frac{2}{\kappa_i n_i \gamma_i}\right) \leq \frac{\sigma_i^2 \eta^2}{K}. \quad (108)$$

By (24), Lemma 23-3 and Lemma 23-4, there exist  $\alpha_i \in [4, 14]$  and  $\beta_i \in [1, 10]$  such that these two inequalities hold if  $\gamma_i, n_i, N_i$  satisfy

$$2L_i^2 \kappa_i^{-1} d \gamma_i + 4dL_i^4 \kappa_i^{-1} m_i^{-1} \gamma_i^2 \leq \frac{\eta^2}{16K^2} \frac{\kappa_i m_i \sigma_i^4}{\alpha_i d}, \quad (109)$$

$$\frac{1}{n_i} \left( 1 + \frac{2}{\kappa_i n_i \gamma_i} \right) \leq \frac{\eta^2 \sigma_i^2 \kappa_i^3 \gamma_i}{32K a_i \beta_i d}, \quad (110)$$

$$N_i \geq \bar{N}_i = -2(\kappa_i \gamma_i)^{-1} \log \left( \frac{\eta^2 \sigma_i^4 m_i^2 n_i \kappa_i \gamma_i}{32K^2 \alpha_i d^2} \right). \quad (111)$$

These inequalities are shown to be true successively for  $\gamma_i, n_i$  and  $N_i$  chosen as in the statement of the Lemma. Denote by  $\bar{\gamma}_i$  and  $\bar{n}_i^{-1}$  the positive roots associated to (109) and (110) seen as equalities and given by

$$\bar{\gamma}_i = 4^{-1} L_i^{-2} m_i \left( -1 + \sqrt{1 + \frac{\eta^2 \kappa_i^2 \sigma_i^4}{4\alpha_i K^2 d^2}} \right), \quad (112)$$

$$\bar{n}_i^{-1} = 4^{-1} \kappa_i \gamma_i \left( -1 + \sqrt{1 + \frac{\eta^2 \kappa_i^2 \sigma_i^2}{4K a_i \beta_i d}} \right). \quad (113)$$

Note that for (109) and (110) to hold, it suffices that  $\gamma_i \leq \bar{\gamma}_i$  and  $n_i \geq \bar{n}_i$ . We now lower bound  $\bar{\gamma}_i$  and upper bound  $\bar{n}_i$ .

Using that  $\sqrt{1+t} \geq 1 + 2^{-1}t(1+t)^{-1/2}$  for  $t = \eta^2 \kappa_i^2 \sigma_i^4 / (4\alpha_i K^2 d^2)$  and  $(\eta^2 \kappa_i^2 \sigma_i^4) / (4\alpha_i K^2 d^2) \leq 25$  by  $\alpha_i \geq 4$  and Lemma 23-5, concludes that if (40) holds then  $\gamma_i \leq \bar{\gamma}_i$ . The fact that  $\gamma_i \leq (m_i + L_i)^{-1}$  can be checked by simple algebra.

First, by (37) and the definition of  $\mathcal{I}_k$ ,  $a_i \leq m_i / \{4(d+4)\}$ ,

$$\bar{n}_i^{-1} \geq 4^{-1} \kappa_i \gamma_i \left( -1 + \sqrt{1 + \frac{\eta^2 \kappa_i^2 \sigma_i^2 (d+4)}{K m_i \beta_i d}} \right).$$

Then using that  $\sqrt{1+t} \geq 1 + 2^{-1}t(1+t)^{-1/2}$  for  $t = \eta^2 \kappa_i^2 \sigma_i^2 (d+4) / (K m_i \beta_i d)$  and  $\beta_i \geq 1$  concludes that if (41) holds then  $n_i \geq \bar{n}_i$ . Finally, we have by (41),  $(n_i \kappa_i \gamma_i)^{-1} \leq \eta^2 \kappa_i \sigma_i / (196 \sqrt{m_i} K)$ , which gives with  $\kappa_i \geq m_i$ ,

$$\begin{aligned} \bar{N}_i &\leq 2(\kappa_i \gamma_i)^{-1} \log \left\{ \frac{64\alpha_i}{196} K d^2 (1 + m\sigma_i^2)^{-3/2} \right\} \\ &\leq 2(\kappa_i \gamma_i)^{-1} \log (5K d^2), \end{aligned}$$

which concludes that (42) implies (111).

The same reasoning applies to check condition-ii) of Lemma 9. The details are gathered in the supplementary material [8, Appendix A.1].

5.3.3. Proof of Theorems 5 and 6 and corollary 7

For  $i \in \{0, \dots, M-1\}$ , set  $\gamma_i, n_i, N_i$  such that (40), (41), (42), (43), (44) and (45) are equalities. By (18), we consider the following decomposition for the cost =  $A + B$  where  $A = \sum_{i=0}^{M-2} \{N_i + n_i\}$  and  $B = n_{M-1} + N_{M-1}$ . We bound  $A$  and  $B$  separately.

First Lemma 23-6 implies that for all  $i \in \{0, \dots, M-2\}$ ,  $n_i \leq (196K)/(\eta^2 \kappa_i \gamma_i)$  and therefore using Lemma 23-7

$$A \leq \left( \frac{196K}{\eta^2} + 2 \log(5Kd^2) \right) \frac{2285K^2 d^2}{\eta^2} \left( \frac{m+L}{2m} \right)^3 (M-1). \quad (114)$$

We now give a bound on  $M-1$ . Define

$$K_{\text{int}} = \sup \{k \geq 1 : m2^k \sigma_0^2 \leq 1\} \wedge K \leq \left\lfloor -\frac{\log(m\sigma_0^2)}{\log(2)} \right\rfloor. \quad (115)$$

By Lemma 23-2 and (37), we have

$$\frac{M-1}{4(d+4)} = \sum_{k=0}^{K-1} \frac{|\mathcal{I}_k|}{4(d+4)} \leq K_{\text{int}} + 2. \quad (116)$$

Note that  $K_{\text{int}} \leq K \leq C$  by (115) and Lemma 23-1. Combining (114), Lemma 23-7 and (116), we get

$$\sum_{i=0}^{M-2} N_i + n_i \leq \left( \frac{98K}{\eta^2} + \log(5Kd^2) \right) \frac{4570K^2 d^2}{\eta^2} \left( \frac{m+L}{2m} \right)^3 4(d+4)(C+2). \quad (117)$$

Regarding the term  $i = M-1$ , we have

$$n_{M-1} + N_{M-1} \leq \left( \frac{19}{\eta^2} + 1 \right) \frac{40}{\eta^2} \frac{m+L}{2m} \frac{L}{m}. \quad (118)$$

Replacing  $\eta$  by  $(\epsilon\sqrt{\mu})/8$  and combining (117) and (118) gives (28).

Assume **H3**. We now prove Theorem 6 and use Lemma 11 instead of Lemma 10. For  $i \in \{0, \dots, M-1\}$ , set  $\gamma_i, n_i, N_i$  such that (46), (41), (42), (47), (44) and (45) are equalities. By (18), we have the decomposition cost =  $A + B$  where  $A = \sum_{i=0}^{M-2} \{N_i + n_i\}$  and  $B = n_{M-1} + N_{M-1}$ . Lemma 23-6 implies that for all  $i \in \{0, \dots, M-2\}$ ,  $n_i \leq (196K)/(\eta^2 \kappa_i \gamma_i)$ , and using that for  $a, b \geq 0$ ,  $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ , we have

$$A \leq \left( \frac{196K}{\eta^2} + 2 \log(5Kd^2) \right) \sqrt{\frac{7}{3}} \frac{8Kd}{\eta} \sum_{i=0}^{M-2} \frac{d^{1/2} \tilde{L} + \sqrt{10} L_i^2 m_i^{-1/2}}{\kappa_i^2 \sigma_i^2 m_i^{1/2}}.$$

Then, by Lemma 23-2 and Lemma 23-8, and splitting the sum in two parts

$k \leq K_{\text{int}}$  and  $k > K_{\text{int}}$ ,

$$\begin{aligned} \sum_{i=0}^{M-2} \frac{1}{\kappa_i^2 \sigma_i^2 m_i^{1/2}} &\leq \sum_{k=0}^{K-1} \sum_{i \in \mathcal{I}_k} \frac{(2^{k+1} \sigma_0^2)^{3/2}}{(1 + m2^k \sigma_0^2)^{5/2}} \\ &\leq 4(d+4) \frac{2^{3/2}}{m^{3/2}} \sum_{k=0}^{K-1} \frac{(m2^k \sigma_0^2)^{3/2}}{(1 + m2^k \sigma_0^2)^{7/2}} \\ &\leq 4(d+4) \frac{2^{3/2}}{m^{3/2}} \left( K_{\text{int}} + \sum_{k=K_{\text{int}}+1}^{K-1} (m2^k \sigma_0^2)^{-2} \right) \\ &\leq 4(d+4) \frac{2^{3/2}}{m^{3/2}} \left( K_{\text{int}} + \frac{4}{3} \right). \end{aligned}$$

We have similarly by Lemma 23-2 and Lemma 23-8,

$$\sum_{i=0}^{M-2} \frac{L_i^2 m_i^{-1/2}}{\kappa_i^2 \sigma_i^2 m_i^{1/2}} \leq 4(d+4) \left( \frac{m+L}{2m} \right)^2 (K_{\text{int}} + 2).$$

Combining these inequalities with

$$n_{M-1} + N_{M-1} \leq \left( \frac{19}{\eta^2} + 1 \right) \sqrt{\frac{7}{3}} \frac{4}{\eta} \left\{ \frac{d^{1/2} \tilde{L}}{m^{3/2}} + \sqrt{10} \left( \frac{m+L}{2m} \right)^2 \right\}, \quad (119)$$

and replacing  $\eta$  by  $(\epsilon\sqrt{\mu})/8$  establish (30).

*Proof of Corollary 7.* Let  $N = \lceil 4 \log(\tilde{\mu}^{-1}) \rceil$  and  $(\hat{Z}_i)_{i \in \{1, \dots, 2N+1\}}$  be  $2N+1$  independent outputs of the algorithms of Theorems 5 and 6 with  $\mu = 1/4$ , sorted by increasing order. Denote by  $\hat{Z} = \hat{Z}_{N+1}$  the median of  $(\hat{Z}_i)_{i \in \{1, \dots, 2N+1\}}$ . In addition, define the independent Bernoulli random variables  $(W_i)_{i \in \{1, \dots, 2N+1\}}$  by

$$W_i = \mathbb{1}_{A_i}, \text{ where } A_i = \left\{ \left| \hat{Z}_i / Z - 1 \right| \geq \epsilon \right\}.$$

Since  $\hat{Z}$  is the median of  $(\hat{Z}_i)_{i \in \{1, \dots, 2N+1\}}$ , we have

$$\mathbb{P} \left( \left| \hat{Z} / Z - 1 \right| > \epsilon \right) \leq \mathbb{P} \left( \sum_{i=1}^{2N+1} W_i \geq N+1 \right).$$

In addition since  $\mathbb{P}(W_i = 1) \leq 1/4$ , we have by [40, Corollary 5.2]

$$\mathbb{P} \left( \sum_{i=1}^{2N+1} W_i \geq N+1 \right) \leq \mathbb{P} \left( \sum_{i=1}^{2N+1} \tilde{W}_i \geq N+1 \right),$$

where  $(\tilde{W}_i)_{i \in \{1, \dots, 2N+1\}}$  are i.i.d. Bernoulli random variables with parameter  $1/4$ . Then by Hoeffding's inequality [6, Theorem 2.8] and using for all  $t \geq 1$ ,

$8(t/2 + 3/4)^2/\{t(2t + 1)\} \geq 1$ , we get

$$\begin{aligned} \mathbb{P}\left(\sum_{i=1}^{2N+1} \tilde{W}_i \geq N + 1\right) &\leq \mathbb{P}\left(\sum_{i=1}^{2N+1} \tilde{W}_i - (1/4)(2N + 1) \geq N/2 + 3/4\right) \\ &\leq \exp\left(\frac{-2(N/2 + 3/4)^2}{2N + 1}\right) \\ &\leq \exp(-N/4), \end{aligned}$$

which concludes the proof.  $\square$

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## Appendix A: Additional proofs of Section 2.1

### A.1. Proof of Lemma 10

In this Section, the proof for the case  $i = M - 1$  of Lemma 10 is dealt with. Note that  $a_{M-1} = (2\sigma_{M-1}^2)^{-1}$ . By Propositions 2 and 4, to check condition-ii) of Lemma 9, it is then sufficient for  $\gamma_{M-1}, n_{M-1}, N_{M-1}$  to satisfy,

$$\begin{aligned} &\frac{4d}{n_{M-1}m_{M-1}\kappa_{M-1}\gamma_{M-1}} \exp\left(-N_{M-1} \frac{\kappa_{M-1}\gamma_{M-1}}{2}\right) \\ &+ 2\kappa_{M-1}^{-1} (A_{M-1,0}\gamma_{M-1} + A_{M-1,1}\gamma_{M-1}^2) \leq \frac{\eta^2\sigma_{M-1}^4}{C_{M-1,2} + C_{M-1,0}C_{M-1,1}}, \end{aligned} \quad (120)$$

$$\frac{8C_{M-1,0}C_{M-1,1}}{\kappa_{M-1}^2 n_{M-1}\gamma_{M-1}} \left(1 + \frac{2}{\kappa_{M-1}n_{M-1}\gamma_{M-1}}\right) \leq \sigma_{M-1}^4 \eta^2. \quad (121)$$

Then (120) and (121) are satisfied if,

$$2L_{M-1}^2\kappa_{M-1}^{-1}d\gamma_{M-1} + 4dL_{M-1}^4\kappa_{M-1}^{-1}m_{M-1}^{-1}\gamma_{M-1}^2 \leq \frac{\eta^2}{4} \frac{\kappa_{M-1}m_{M-1}\sigma_{M-1}^4}{\alpha_{M-1}d}, \quad (122)$$

$$\frac{1}{n_{M-1}} \left(1 + \frac{2}{\kappa_{M-1}n_{M-1}\gamma_{M-1}}\right) \leq \frac{\eta^2\sigma_{M-1}^4\kappa_{M-1}^3\gamma_{M-1}}{8\beta_{M-1}d}, \quad (123)$$

$$-2(\kappa_{M-1}\gamma_{M-1})^{-1} \log\left(\frac{\eta^2\sigma_{M-1}^4m_{M-1}^2n_{M-1}\kappa_{M-1}\gamma_{M-1}}{8\alpha_{M-1}d^2}\right) = \bar{N}_{M-1} \leq N_{M-1}. \quad (124)$$

Denote by  $\bar{\gamma}_{M-1}$  and  $\bar{n}_{M-1}^{-1}$  the positive roots associated to (122) and (123) seen as equalities. We have:

$$\bar{\gamma}_{M-1} = 4^{-1} L_{M-1}^{-2} m_{M-1} \left( -1 + \sqrt{1 + \frac{\eta^2 \kappa_{M-1}^2 \sigma_{M-1}^4}{\alpha_{M-1} d^2}} \right), \quad (125)$$

$$\bar{n}_{M-1}^{-1} = 4^{-1} \kappa_{M-1} \gamma_{M-1} \left( -1 + \sqrt{1 + \frac{\eta^2 \kappa_{M-1}^2 \sigma_{M-1}^4}{\beta_{M-1} d}} \right). \quad (126)$$

Note that for (122) and (123) to hold, it suffices that  $\gamma_{M-1} \leq \bar{\gamma}_{M-1}$  and  $n_{M-1} \geq \bar{n}_{M-1}$ . We now lower bound  $\bar{\gamma}_{M-1}$  and upper bound  $\bar{n}_{M-1}$ .

Using that  $t \geq 0$ ,  $\sqrt{1+t} \geq 1+2^{-1}t(1+t)^{-1/2}$  for  $t = (\eta^2 \kappa_{M-1}^2 \sigma_{M-1}^4)/(\alpha_{M-1} d^2)$  and  $\kappa_{M-1} \sigma_{M-1}^2 d^{-1} \geq 2$ ,  $\alpha_{M-1} \geq 4$  concludes that if (43) holds then  $\gamma_{M-1} \leq \bar{\gamma}_{M-1}$ . The fact that  $\gamma_{M-1} \leq (m_{M-1} + L_{M-1})^{-1}$  can be checked by simple algebra.

Then using that  $\sqrt{1+t} \geq 1+2^{-1}t(1+t)^{-1/2}$  for  $t = (\eta^2 \kappa_{M-1}^2 \sigma_{M-1}^4)/(\beta_{M-1} d)$  and  $\kappa_{M-1} \sigma_{M-1}^2 \geq 10$ ,  $\beta_{M-1} \geq 1$  concludes that if (44) holds then  $n_{M-1} \geq \bar{n}_{M-1}$ . Finally, by (44), we get

$$\bar{N}_{M-1} \leq (\kappa_{M-1} \gamma_{M-1})^{-1} \log(7/3),$$

which concludes that (45) implies (124).

### A.2. Proof of Lemma 11

Let  $k \in \{0, \dots, K-1\}$  and  $i \in \mathcal{I}_k$ . Assume that  $\gamma_i \leq (m_i + L_i)^{-1}$ . The proof of Lemma 10 only needs to be slightly adapted. More precisely, Proposition 3 is applied instead of Proposition 2. By (26) and (27), we have

$$B_{i,0} \leq 3^{-1} d \kappa_i^{-1} (d \tilde{L}^2 + 10 L_i^4 m_i^{-1}), \quad B_{i,1} \leq (25/12) d L_i^4 m_i^{-1}. \quad (127)$$

It is sufficient for  $\gamma_i, n_i, N_i$  to satisfy (107) and (108) with  $A_{i,0} \gamma_i + A_{i,1} \gamma_i^2$  replaced by  $B_{i,0} \gamma_i^2 + B_{i,1} \gamma_i^3$ . The counterpart of (109) is then

$$\frac{1}{3 \kappa_i} \left( d \tilde{L}^2 + 10 L_i^4 m_i^{-1} \right) \gamma_i^2 + \frac{25}{12} L_i^4 m_i^{-1} \gamma_i^3 \leq \frac{\eta^2 \kappa_i m_i \sigma_i^4}{16 K^2 \alpha_i d^2}. \quad (128)$$

Since  $\gamma_i \leq 1/(m_i + L_i)$  and  $\kappa_i \leq L_i$ , we have

$$(3 \kappa_i)^{-1} \left( d \tilde{L}^2 + 10 L_i^4 m_i^{-1} \right) \geq (25/12) L_i^4 m_i^{-1} \gamma_i,$$

which establishes that if (46) holds, then (128) is satisfied.  $\gamma_i \leq (m_i + L_i)^{-1}$  can be checked by simple algebra. For  $i = M-1$ , the conclusion follows from  $m_{M-1} \sigma_{M-1}^2 d^{-1} \geq 2$  because  $\sigma_{M-1}^2 \geq (2d+7)/m$ .



## Appendix B: Additional proofs of Section 2.2

First, we state a technical lemma that gathers useful bounds. We recall that  $M$  is defined in this Section by (56),

$$M = \inf \{i \geq 1 : \sigma_{i-1}^2 \geq D^2\} .$$

**Lemma 24.** *Assume **H1** and **H2**( $m$ ) for  $m \geq 0$ . Let  $\{\sigma_i^2\}_{i=0}^{M-1}$  defined by (54) for  $\sigma_0^2$  given in (31) and  $M$  in (56).*

1.  $K \leq \lceil (1/\log(2)) \log(\sigma_0^{-2} \rho^{-2} d^2 (\tau + 1)^2) \rceil$  where  $K$  is defined in (39).
2. For  $k \in \{0, \dots, K-1\}$  and  $i \in \mathcal{I}_k$ ,  $2^{k+1} \sigma_0^2 a_i |\mathcal{I}_k| \leq 1$  where  $a_i$  is defined in (37) (with  $m=0$ ) and  $\mathcal{I}_k$  in (38). As a consequence,  $|\mathcal{I}_k| \leq 4(d+4)$ .
3. For  $i \in \{0, \dots, M-1\}$ ,  $\kappa_i \sigma_i^2 \in [1, 2]$ .
4.  $\sigma_{M-1}^2 \in [D^2, (10/9)D^2]$ .
5. For all  $i \in \{0, \dots, M-1\}$  and  $\gamma_i \leq 1/(m_i + L_i)$ , there exist  $\alpha_i \in [4, 14]$  and  $\beta_i \in [1, 10]$  such that  $C_{i,2} + C_{i,0}C_{i,1} = \alpha_i d m_i^{-1}$  and  $C_{i,0}C_{i,1} = \beta_i d \kappa_i^{-1}$  where  $C_{i,0}, C_{i,1}, C_{i,2}$  and  $\kappa_i$  are given in (23) and (14) respectively.
6. For all  $i \in \{0, \dots, M-1\}$ ,  $0 < A_{i,1} \leq 4dL_i^4 \kappa_i^{-1} m_i^{-1}$ , where  $L_i, m_i$  and  $\kappa_i$  are given in (13) and (14) respectively.

*Proof.* The proofs of 1,2,5,6 are identical to the ones of Lemma 23.

3.  $\kappa_i \sigma_i^2 = (2L_i)/(m_i + L_i)$ .
4. By definition of  $M$ ,  $\sigma_{M-2}^2 \leq D^2$  and  $a_{M-2} \leq \sigma_{M-2}^{-2}/\{4(d+4)\}$ . By (6), we get:

$$\sigma_{M-1}^{-2} = \sigma_{M-2}^{-2} - 2a_{M-2} \geq \sigma_{M-2}^{-2} \left(1 - \frac{1}{2(d+4)}\right)$$

$$\text{that is } \sigma_{M-1}^2 \leq (10/9)\sigma_{M-2}^2 \leq (10/9)D^2.$$

□

### B.1. Proof of Lemma 15

Let  $k \in \{0, \dots, K-1\}$  and  $i \in \mathcal{I}_k$ . Assume that  $\gamma_i \leq (m_i + L_i)^{-1}$ . The proof follows the same lines as the one in Section 5.3.2. By Lemma 24-5 and Lemma 24-6, to check condition-i) of Lemma 9, it suffices that  $\gamma_i \leq \bar{\gamma}_i$ ,  $n_i \geq \bar{n}_i$  and  $N_i$  satisfies (111), where  $\bar{\gamma}_i$  is defined in (112) and  $\bar{n}_i$  in (113).

Using that  $\sqrt{1+t} \geq 1 + 2^{-1}t(1+t)^{-1/2}$  for  $t = (\eta^2 \kappa_i^2 \sigma_i^4)/(4\alpha_i K^2 d^2)$  and by Lemma 24-3, concludes that if (57) holds then  $\gamma_i \leq \bar{\gamma}_i$ .  $\gamma_i \leq (m_i + L_i)^{-1}$  can be checked by simple algebra.

By (37) (with  $m=0$ ) and the definition of  $\mathcal{I}_k$ ,  $a_i \leq \sigma_i^{-2}/\{4(d+4)\}$ ,

$$\bar{n}_i^{-1} \geq 4^{-1} \kappa_i \gamma_i \left( -1 + \sqrt{1 + \frac{\eta^2 \kappa_i^2 \sigma_i^4 (d+4)}{K \beta_i d}} \right).$$

Using that  $\sqrt{1+t} \geq 1 + 2^{-1}t(1+t)^{-1/2}$  for  $t = (\eta^2 \kappa_i^2 \sigma_i^4)/(4\alpha_i K^2 d^2)$  and by Lemma 24-3, concludes that if (58) holds then  $n_i \geq \bar{n}_i$ . Finally, by (58), if (59) holds, (111) is satisfied.

The case  $i = M - 1$  is different because  $\bar{g}_{M-1}$  is Lipschitz. Assume  $\gamma_{M-1} \leq (m_{M-1} + L_{M-1})^{-1}$ . [15, section 2.1] entails that condition-ii) of Lemma 9 is satisfied if

$$\begin{aligned} \|\bar{g}_{M-1}\|_{\text{Lip}}^2 \left\{ \frac{4d}{n_{M-1} m_{M-1} \kappa_{M-1} \gamma_{M-1}} \exp\left(-N_{M-1} \frac{\kappa_{M-1} \gamma_{M-1}}{2}\right) \right. \\ \left. + 2\kappa_{M-1}^{-1} (A_{M-1,0} \gamma_{M-1} + A_{M-1,1} \gamma_{M-1}^2) \right\} \leq \eta^2, \quad (129) \\ \frac{8 \|\bar{g}_{M-1}\|_{\text{Lip}}^2}{\kappa_{M-1}^2 n_{M-1} \gamma_{M-1}} \left\{ 1 + \frac{2}{n_{M-1} \kappa_{M-1} \gamma_{M-1}} \right\} \leq \eta^2. \end{aligned}$$

Using  $\|\bar{g}_{M-1}\|_{\text{Lip}}^2 \leq (\sigma_{M-1}^{-2} e)$  and (24), Lemma 24-6 for  $i = M - 1$ , it is sufficient for  $\gamma_{M-1}, n_{M-1}, n_{M-1}$  to satisfy

$$2L_{M-1}^2 \kappa_{M-1}^{-1} d \gamma_{M-1} + 4dL_{M-1}^4 \kappa_{M-1}^{-1} m_{M-1}^{-1} \gamma_{M-1}^2 \leq (4e)^{-1} \kappa_{M-1} \eta^2 \sigma_{M-1}^2, \quad (130)$$

$$n_{M-1}^{-1} (1 + 2(\kappa_{M-1} \gamma_{M-1} n_{M-1})^{-1}) \leq \frac{\eta^2 \kappa_{M-1}^2 \sigma_{M-1}^2 \gamma_{M-1}}{8e}, \quad (131)$$

$$-2 \frac{\log((8ed)^{-1} n_{M-1} \kappa_{M-1} \gamma_{M-1} \eta^2)}{\kappa_{M-1} \gamma_{M-1}} = \bar{N}_{M-1} \leq N_{M-1}. \quad (132)$$

Denote by  $\bar{\gamma}_{M-1}, \bar{n}_{M-1}^{-1}$  the roots of (130), (131) seen as equalities. We have

$$\bar{\gamma}_{M-1} = 4^{-1} L_{M-1}^{-2} \sigma_{M-1}^{-2} \left\{ -1 + \sqrt{1 + \frac{\eta^2 \kappa_{M-1}^2 \sigma_{M-1}^4}{ed}} \right\}, \quad (133)$$

$$\bar{n}_{M-1}^{-1} = 4^{-1} \kappa_{M-1} \gamma_{M-1} \left\{ -1 + \sqrt{1 + e^{-1} \eta^2 \kappa_{M-1} \sigma_{M-1}^2} \right\}. \quad (134)$$

Using that  $\sqrt{1+t} \geq 1 + 2^{-1}t(1+t)^{-1/2}$  for  $t = (\eta^2 \kappa_{M-1}^2 \sigma_{M-1}^4)/(ed)$  and by Lemma 24-3, concludes that if (60) holds then  $\gamma_{M-1} \leq \bar{\gamma}_{M-1}$ .  $\gamma_{M-1} \leq (m_{M-1} + L_{M-1})^{-1}$  can be checked by simple algebra.

Using that  $\sqrt{1+t} \geq 1 + 2^{-1}t(1+t)^{-1/2}$  for  $t = e^{-1} \eta^2 \kappa_{M-1} \sigma_{M-1}^2$  and by Lemma 24-3, concludes that if (61) holds then  $n_{M-1} \geq \bar{n}_{M-1}$ .

Finally by (61), if (62) holds, (132) is satisfied.

## B.2. Proof of Lemma 16

The proof is identical to the one of Lemma 11. For  $k \in \{0, \dots, K - 1\}$  and  $i \in \mathcal{I}_k$ , it is sufficient for  $\gamma_i$  to satisfy (63) by (46) and Lemma 24-3.

Regarding the case  $i = M - 1$ , assuming that  $\gamma_{M-1} \leq (m_{M-1} + L_{M-1})^{-1}$ , it is sufficient for  $\gamma_{M-1}, n_{M-1}, N_{M-1}$  to satisfy (129) with  $A_{M-1,0}\gamma_{M-1} + A_{M-1,1}\gamma_{M-1}^2$  replaced by  $B_{M-1,0}\gamma_{M-1}^2 + B_{M-1,1}\gamma_{M-1}^3$ . The counterpart of (130) is then,

$$\frac{1}{3\kappa_{M-1}} \left( d\tilde{L}^2 + 10L_{M-1}^4 m_{M-1}^{-1} \right) \gamma_{M-1}^2 + \frac{25}{12} \frac{L_{M-1}^4}{m_{M-1}} \gamma_{M-1}^3 \leq \frac{\eta^2 \kappa_{M-1} \sigma_{M-1}^2}{4ed}.$$

This concludes the proof with the same argument as in Appendix A.2.

### B.3. Proof of Theorems 12 and 13 and corollary 14

For  $i \in \{0, \dots, M - 1\}$ , set  $\gamma_i, n_i, N_i$  such that (57), (58), (59), (60), (61) and (62) are equalities. By (18), we have

$$\text{cost} = \left( \frac{453K}{\eta^2} + 2 \log(Kd^2) \right) \frac{462K^2 d^2}{\eta^2} \sum_{i=0}^{M-2} \kappa_i^{-1} L_i^2 \sigma_i^2 + n_{M-1} + N_{M-1}.$$

Note that for  $i \in \{0, \dots, M - 2\}$ ,

$$\kappa_i^{-1} L_i^2 \sigma_i^2 = 1 + (3/2)L\sigma_i^2 + (L^2/2)\sigma_i^4.$$

By Lemma 24-2, for  $k \in \{0, \dots, K - 1\}$ ,  $|\mathcal{I}_k| \leq 4(d + 4)$  and for  $i \in \mathcal{I}_k$ ,  $\sigma_i^2 \leq 2^{k+1}\sigma_0^2$ . We then have

$$\begin{aligned} \sum_{i=0}^{M-2} \frac{L_i^2 \sigma_i^2}{\kappa_i} &\leq 4(d + 4) \sum_{k=0}^{K-1} \left\{ 1 + \frac{3L}{2} 2^{k+1} \sigma_0^2 + \frac{L^2}{2} (2^{k+1} \sigma_0^2)^2 \right\} \\ &\leq 4(d + 4) \left\{ K + 3L(2^K \sigma_0^2) + \frac{2L^2}{3} (2^K \sigma_0^2)^2 \right\}. \end{aligned}$$

By (56) and the definition of  $K$ , (39),  $2^K \sigma_0^2 \leq 2D^2$ . The expressions of  $\gamma_{M-1}, n_{M-1}, N_{M-1}$  give

$$n_{M-1} + N_{M-1} = \left( \frac{29}{\eta^2} + 2 \log(d) \right) \frac{26d}{\eta^2} \kappa_{M-1}^{-2} L_{M-1}^2,$$

with  $\kappa_{M-1}^{-2} L_{M-1}^2 = (1 + 2^{-1}L\sigma_{M-1}^2)^2$ . By Lemma 24-4, we then have

$$n_{M-1} + N_{M-1} \leq \left( \frac{29}{\eta^2} + 2 \log(d) \right) \frac{26d}{\eta^2} \left( 1 + \frac{5L}{9} D^2 \right)^2, \quad (135)$$

and (51) is established.

Assume **H 3**. We now prove Theorem 13 and use Lemma 11 instead of Lemma 10. For  $i \in \{0, \dots, M - 1\}$ , set  $\gamma_i, n_i, N_i$  such that (63), (58), (59), (64), (61) and (62) are equalities. By (18) and using that for  $a, b \geq 0$ ,  $\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}$ , we have

$$\text{cost} \leq \left( \frac{453K}{\eta^2} + 2 \log(Kd^2) \right) \sqrt{\frac{7}{3}} \frac{8Kd}{\eta} \sum_{i=0}^{M-2} \frac{\sigma_i}{\kappa_i} \left( d^{1/2} \tilde{L} + \sqrt{10} L_i^2 \sigma_i \right) + n_{M-1} + N_{M-1}.$$

For  $k \in \{0, \dots, K-1\}$  and  $i \in \mathcal{I}_k$ , note that

$$\kappa_i^{-1} \sigma_i \leq \sigma_i^3, \quad \kappa_i^{-1} \sigma_i^2 L_i^2 = 1 + \frac{3L}{2} \sigma_i^2 + \frac{L^2}{2} \sigma_i^4.$$

Using for  $k \in \{0, \dots, K-1\}$ ,  $|\mathcal{I}_k| \leq 4(d+4)$  by Lemma 24-2 and for  $i \in \mathcal{I}_k$ ,  $\sigma_i^2 \leq 2^{k+1} \sigma_0^2$ , we get

$$\begin{aligned} \sum_{i=0}^{M-2} \frac{\sigma_i}{\kappa_i} \left( d^{1/2} \tilde{L} + \sqrt{10} L_i^2 \sigma_i \right) &\leq 4(d+4) \sum_{k=0}^{K-1} \left\{ d^{1/2} \tilde{L} (2^{k+1} \sigma_0^2)^{3/2} \right. \\ &\quad \left. + \sqrt{10} \left( 1 + \frac{3L}{2} (2^{k+1} \sigma_0^2) + \frac{L^2}{2} (2^{k+1} \sigma_0^2)^2 \right) \right\} \\ &\leq 4(d+4) \left\{ 5d^{1/2} \tilde{L} D^3 + \sqrt{10} \left( K + 6LD^2 + \frac{8L^2}{3} D^4 \right) \right\}, \end{aligned}$$

with  $2^K \sigma_0^2 \leq 2D^2$ . The expressions of  $\gamma_{M-1}, n_{M-1}, N_{M-1}$  give

$$n_{M-1} + N_{M-1} \leq \left( 2 \log(d) + \frac{29}{\eta^2} \right) \sqrt{\frac{8e}{3}} \frac{\sqrt{d} d^{1/2} \tilde{L} + \sqrt{10} L_{M-1}^2 \sigma_{M-1}}{\kappa_{M-1}^2 \sigma_{M-1}}.$$

By Lemma 24-4,  $\sigma_{M-1}^2 \in [D^2, (10/9)D^2]$ . We get then

$$\kappa_{M-1}^{-2} \sigma_{M-1}^{-1} = \frac{\left( 1 + (L/2) \sigma_{M-1}^2 \right)^2}{L^2 \sigma_{M-1}} \leq \frac{1}{DL^2} \left( 1 + \frac{5L}{9} D^2 \right)^2,$$

and,

$$\kappa_{M-1}^{-2} L_{M-1}^2 = \left( 1 + \frac{L}{2} \sigma_{M-1}^2 \right)^2 \leq \left( 1 + \frac{5L}{9} D^2 \right)^2,$$

which gives,

$$n_{M-1} + N_{M-1} \leq \left( 2 \log(d) + \frac{29}{\eta^2} \right) \sqrt{\frac{8e}{3}} \frac{\sqrt{d}}{\eta} \left( 1 + \frac{5L}{9} D^2 \right)^2 \left( \frac{d^{1/2} \tilde{L}}{DL^2} + \sqrt{10} \right). \quad (136)$$

(53) is established. The proof of Corollary 14 is the same as the one of Corollary 7.

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