Bispindle in strongly connected digraphs with large chromatic number

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Abstract

A $(k_1 + k_2)$-*bispindle* is the union of k_1 (x, y)-dipaths and k_2 (y, x)-dipaths, all these dipaths being pairwise internally disjoint. Recently, Cohen et al. showed that for every $(2 + 0)$- bispindle B, there exists an integer k such that every strongly connected digraph with chromatic number greater than k contains a subdivision of B. We investigate generalisations of this result by first showing constructions of strongly connected digraphs with large chromatic number without any $(3 + 0)$-bispindle or $(2+2)$-bispindle. Then we show that for any k, there exists γ_k such that every strongly connected digraph with chromatic number greater than γ_k contains

a $(2+1)$-bispindle with the (y,x)-dipath and one of the (x,y)-dipaths of length at least k.

1 Introduction

Throughout this paper, the *chromatic number* of a digraph D, denoted by $\chi(D)$, is the chromatic number of its underlying graph. In a digraph D, a *directed path*, or *dipath*, is an oriented path where all the arcs are oriented form the initial vertex towards the terminal vertex. A *k-spindle* is the union of k internally disjoint (x,y)-dipaths for some vertices x and y. Vertex x is said to be the *tail* of the spindle and y its *head*. A *(k_1+k_2)-bispindle* is the internally disjoint union of a k_1-spindle with tail x and head y and a k_2-spindle with tail y and head x. In other words, it is the union of k_1 (x,y)-dipaths and k_2 (y,x)-dipaths, all of these dipaths being pairwise internally disjoint.

A classical result due to Gallai, Hasse, Roy and Vitaver is the following.

Theorem 1.1 (Gallai [8], Hasse [9], Roy [11], Vitaver [12])
If $\chi(D) \geq k$, then D contains a dipath of length $k-1$.

This raises the question of which digraphs are subdigraphs of all digraphs with large chromatic number.

A classical theorem by Erdős [6] implies that if H is a digraph containing a cycle, there exist digraphs with arbitrarily high chromatic number with no subdigraph isomorphic to H. Thus the only possible candidates to generalise Theorem 1.1 are the *oriented trees* that are orientations of trees. Burr[3] proved that every $(k-1)^2$-chromatic digraph contains every oriented tree of order k and conjectured an upper bound of $2k-2$. The best known upper bound, due to Addario-Berry et al. [1], is in $(k/2)^2$.

However the following celebrated theorem of Bondy shows that the story does not stop there.

Theorem 1.2 (Bondy [2]) *Every strongly connected digraph with chromatic number at least k contains a directed cycle of length at least k.*

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The strong connectivity assumption is indeed necessary, as transitive tournaments contain no directed cycle but can have arbitrarily high chromatic number.

Observe that a directed cycle of length at least k can be seen as a subdivision of $\vec{C}_k$, the directed cycle of length k. Recall that a *subdivision* of a digraph F is a digraph that can be obtained from F by replacing each arc uv by a dipath from u to v.

Conjecture 1.3 (Cohen et al. [5]) *For every cycle C, there exists a constant $f(C)$ such that every strongly connected digraph with chromatic number at least $f(C)$ contains a subdivision of C.*

The strong connectivity assumption is also necessary in Conjecture 1.3 as shown by Cohen et al. in [5]. In the same paper, Conjecture 1.3 was confirmed for cycles with two blocks (i.e. maximal subdipaths of the cycle) and the antidirected cycle of length 4. More precisely, denoting by $C(k, \ell)$ the cycle on two blocks, one of length k and the other of length ℓ, Cohen et al. [5] proved the following.

Theorem 1.4 *Every strongly connected digraph with chromatic number at least $O((k + \ell)^4)$ contains a subdivision of $C(k, \ell)$.*

The bound has recently been improved to $O((k + \ell)^2)$ by Kim et al. [10].

A subdivision of $C(k, \ell)$ can be seen as a 2-spindle made of two internally disjoint dipaths, one of length at least k and one of length at least ℓ. In this paper, we generalize this and study the existence of subdivision of spindles and bispindles in strongly connected digraphs with large chromatic number. Our first result is to give constructions for the following theorem:

Theorem 1.5 *For every integer k, there exists a strongly connected digraph D with $\chi(D) > k$ that contains no 3-spindle and no $(2 + 2)$-bispindle.*

Therefore, the most we can expect in all strongly connected digraphs with large chromatic number are $(2 + 1)$-bispindle. Let $B(k_1, k_2; k_3)$ denote the $(2+1)$-bispindle formed by three internally disjoint paths between two vertices x, y, two (x, y)-dipaths, one of size k_1, the other of size k_2, and one (y, x)-dipath of size k_3. We conjecture the following.

Conjecture 1.6 *There is a function $g : \mathbb{N}^3 \to \mathbb{N}$ such that every strongly connected digraph with chromatic number at least $g(k_1, k_2, k_3)$ contains a subdivision of $B(k_1, k_2; k_3)$.*

As an evidence, we prove the following theorem:

Theorem 1.7 *For every positive integer k, there is a constant γ_k such that every strongly connected digraph witch chromatic number greater than γ_k contains a subdivsion of $B(k, 1; k)$.*

The value of γ_k is the above theorem is huge, and certainly not best possible. We get a better bound for subdivision of $B(k, 1; k)$.

Theorem 1.8 *Let $k \geq 3$ be an integer and let D be a strong digraph. If $\chi(D) > (2k-2)(2k-3)$, then D contains a subdivision of $B(k, 1; 1)$.*

2 Proof of Theorem 1.7

We prove Theorem 1.7 by the contrapositive. We consider a digraph D without any subdivision of $B(k, 1; k)$. We shall prove that $\chi(D) \leq \gamma_k$.

The general idea is to use the following easy lemma.

Lemma 2.1 *Let D be a digraph, $D_1 \ldots D_l$ be disjoint subdigraphs of D and D' the digraph obtained by contracting each D_i into one vertex d_i. Then $\chi(D) \leq \chi(D') \cdot \max\{\chi(D_i) \mid 1 \leq i \leq l\}$.*

The key is to find appropriate subdigraphs D_i. To do so, we consider some particular collections of directed cycles : a collection $\mathcal{C}$ of directed cycles is *k-suitable* if all cycles of $\mathcal{C}$ have length at least $8k$, and any two distinct cycles $C_i, C_j \in \mathcal{C}$ intersect on a subpath of order at most k. A *component* of $\mathcal{C}$ is a connected component of the underlying graph of the digraph $\bigcup \mathcal{C}$ which is the union of cycles of $\mathcal{C}$.

Consider $\mathcal{C}$ be a maximal k-suitable collection of cycles in D. Let D' be the digraph obtained by contracting every strong component S of $\bigcup \mathcal{C}$ (which is $\bigcup \mathcal{S}$ for some component $\mathcal{S}$ of $\mathcal{C}$) into one vertex. For each connected component $\mathcal{S}_i$ we call s_i the new vertex created. To apply Lemma 2.1, we shall prove in the next two lemmas that for every component $\mathcal{S}$ of $\mathcal{C}$, the digraph $D[\mathcal{S}]$ induced by D on the vertices of $\bigcup \mathcal{S}$ has bounded chromatic number and that $\chi(D') \leq 8k$.

Lemma 2.2 *Let $\mathcal{C}$ be a k-suitable collection of directed cycles in a $B(k, 1; k)$-free digraph D. There exists a constant β_k such that, for every component $\mathcal{S}$ of $\mathcal{C}$, we have $\chi(D[\mathcal{S}]) \leq \beta_k$.*

Sketch of proof: We first consider $\bigcup \mathcal{S}$ which is a subdigraph of $D[\mathcal{S}]$. We prove by induction on the number of cycles in $\mathcal{S}$ that this digraph admits a proper colouring ϕ with $\alpha_k = 2 \cdot (6k^2)^{3k} + 14k$ colours satisfying the following

additional property, called *rainbow property* : the vertices of each subpath of length at most $7k$ of each cycle of $\mathcal{S}$ get different colours.

We then define a sort of Breadth-First-Search for $\mathcal{S}$. Let C_0 be a cycle of $\mathcal{S}$ and set $L_0 = \{C_0\}$. We build the levels L_i inductively until all cycles of $\mathcal{S}$ are put in a level : L_{i+1} consists of every cycle C_l not in $\bigcup_{j \le i} L_j$ such that there exists a cycle in L_i intersecting C_l. For every $C_l \in L_{i+1}$, we choose one of the cycles L_i intersecting it to be its *father*. For a vertex x of $\bigcup \mathcal{S}$, we say that x *belongs to* level L_i if i is the smallest integer such that there exists a cycle in L_i containing x.

We partition the arc set of $D[\mathcal{S}]$ in (A_0, A_1, A_2), where

- A_0 is the set of arcs of $D[\mathcal{S}]$ which ends belong to the same level, and
- A_1 is the set of arcs of $D[\mathcal{S}]$ which ends belong to different levels i and j with $|i - j| < k$.
- A_2 is the set of arcs of $D[\mathcal{S}]$ which ends belong to different levels i and j with $|i - j| \ge k$.

For $i \in \{0, 1, 2\}$, let D_i be the spanning subdigraph of $D[\mathcal{S}]$ with arc set A_i. It is well-known that $\chi(D[\mathcal{S}]) \le \chi(D_0) \times \chi(D_1) \times \chi(D_2)$.

Clearly, $\chi(D_1) \le k$, and we show that $\chi(D_2) \le 4k^2 + 2$. To bound $\chi(D_0)$ we partition the vertex set according to the above-mention colouring ϕ of $\bigcup \mathcal{S}$. Using the rainbow property, we prove that the subdigraph of D_0 induced by the vertices of colour c has chromatic number at most $2 \cdot (4k)^{4k} + 1$ for all colour c. Hence $\chi(D_0) \le (2 \cdot (4k)^{4k} + 1)\alpha_k$. This gives the result for $\beta_k = k(4k^2 + 2)(2 \cdot (4k)^{4k} + 1)\alpha_k$. $\qquad\square$

Lemma 2.3 $\chi(D') \le 8k$.

Proof. First note that since D is strongly connected so is D'.

Suppose for a contradiction that $\chi(D') > 8k$. By Theorem 1.2, there exists a directed cycle $C' = (x_1, x_2, \ldots, x_l, x_1)$ of length at least $8k$. For each vertex x_j that corresponds to a $\mathcal{S}_i$ in D, the arc $x_{j-1}x_j$ corresponds in D to an arc whose head is a vertex p_i of $\mathcal{S}_i$ and the arc $x_j x_{j+1}$ corresponds to an arc whose tail is a vertex l_i of $\mathcal{S}_i$. Let P_j be the dipath from p_i to l_i in $\bigcup \mathcal{C}$. Note that this path intersects the elements of $\mathcal{S}_i$ only along a subdipath. Let C be the cycle obtained from C' where we replace all contracted vertices x_j by the path P_j. First note that C has length at least $8k$. Moreover, a cycle of $\mathcal{C}$ can intersect C only along one P_j, because they all correspond to different strong components of $\bigcup \mathcal{C}$. Thus C intersects each cycle of $\mathcal{C}$ on a subdipath. Moreover this subdipath has length smaller than k for otherwise D would contain a subdivision of $B(k, 1; k)$. So C is a directed cycle of length at least

$8k$ which intersects every cycle of $\mathcal{C}$ along a subdipath of length less than k. This contradicts the maximality of $\mathcal{C}$. $\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Using Lemma 2.1 with Claim 2.3 and Lemma 2.2, we get that $\chi(D) \leq 8k \cdot \beta_k$. This proves Theorem 1.7 for $\gamma_k = 8k \cdot \beta_k$.

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