

Minimum density of identifying codes of king grids

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Abstract

A set $C \subseteq V(G)$ is an *identifying code* in a graph G if for all $v \in V(G)$, $C[v] \neq \emptyset$, and for all distinct $u, v \in V(G)$, $C[u] \neq C[v]$, where $C[v] = N[v] \cap C$ and $N[v]$ denotes the closed neighbourhood of v in G . The minimum density of an identifying code in G is denoted by $d^*(G)$. In this paper, we study the density of king grids which are strong product of two paths. We show that for every king grid G , $d^*(G) \geq 2/9$. In addition, we show this bound is attained only for king grids which are strong products of two infinite paths. Given $k \geq 3$, we denote by \mathcal{K}_k the (infinite) king strip with k rows. We prove that $d^*(\mathcal{K}_3) = 1/3$, $d^*(\mathcal{K}_4) = 5/16$, $d^*(\mathcal{K}_5) = 4/15$ and $d^*(\mathcal{K}_6) = 5/18$. We also prove that $\frac{2}{9} + \frac{8}{81k} \leq d^*(\mathcal{K}_k) \leq \frac{2}{9} + \frac{4}{9k}$ for every $k \geq 7$.

Keywords: Identifying code, King grid, Discharging Method.

1 Introduction

Let G be a graph. The *neighbourhood* of a vertex v of G , denoted by $N(v)$, is the set of vertices adjacent to v in G , and the *closed neighbourhood* of v is the set $N[v] = N(v) \cup \{v\}$. Given a set $C \subseteq V(G)$, let $C[v] = N[v] \cap C$. We say that C is an *identifying code* of G if $C[v] \neq \emptyset$ for all $v \in V(G)$, and $C[u] \neq C[v]$ for all distinct $u, v \in V(G)$. Clearly, a graph G has an identifying code if and only if it contains no *twins* (vertices $u, v \in V(G)$ with $N[u] = N[v]$).

Let G be a (finite or infinite) graph with bounded maximum degree. For any non-negative integer r and vertex v , we denote by $B_r(v)$ the ball of radius r in G centered at v , that is $B_r(v) = \{x \mid \text{dist}(v, x) \leq r\}$. For any set of vertices $C \subseteq V(G)$, the *density* of C in G , denoted by $d(C, G)$, is defined by

$$d(C, G) = \limsup_{r \rightarrow +\infty} \frac{|C \cap B_r(v_0)|}{|B_r(v_0)|},$$

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where v_0 is an arbitrary vertex in G . The infimum of the density of an identifying code in G is denoted by $d^*(G)$. Observe that if G is finite, then $d^*(G) = |C^*|/|V(G)|$, where C^* is a minimum-size identifying code of G .

The problem of finding low-density identifying codes was introduced in [12] in relation to fault diagnosis in arrays of processors. Particular interest was dedicated to grids as many processor networks have a grid topology. Many results have been obtained on square grids [4,1,9,2,11], triangular grids [12,10], and hexagonal grids [5,7,8]. In this paper, we study *king grids*, which are strong products of two paths. The *strong product* of two graphs G and H , denoted by $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H)$ and edge set :

$$\begin{aligned} E(G \boxtimes H) = & \{(a, b)(a, b') \mid a \in V(G) \text{ and } bb' \in E(H)\} \\ & \cup \{(a, b)(a', b) \mid aa' \in E(G) \text{ and } b \in V(H)\} \\ & \cup \{(a, b)(a', b') \mid aa' \in E(G) \text{ and } bb' \in E(H)\}. \end{aligned}$$

The *two-way infinite path*, denoted by $P_{\mathbb{Z}}$, is the graph with vertex set \mathbb{Z} and edge set $\{\{i, i + 1\} \mid i \in \mathbb{Z}\}$, and the *one-way infinite path*, denoted by $P_{\mathbb{N}}$, is the graph with vertex set \mathbb{N} and edge set $\{\{i, i + 1\} \mid i \in \mathbb{N}\}$. A *path* is a connected subgraph of $P_{\mathbb{Z}}$. For every positive integer k , P_k is the subgraph of $P_{\mathbb{Z}}$ induced by $\{1, 2, \dots, k\}$. A *king grid* is the strong product of two (finite or infinite) paths. The *plane king grid* is $\mathcal{G}_K = P_{\mathbb{Z}} \boxtimes P_{\mathbb{Z}}$, the *half-plane king grid* is $\mathcal{H}_K = P_{\mathbb{Z}} \boxtimes P_{\mathbb{N}}$, the *quater-plane king grid* is $\mathcal{Q}_K = P_{\mathbb{N}} \boxtimes P_{\mathbb{N}}$, and the *king strip of height k* is $\mathcal{K}_k = P_{\mathbb{Z}} \boxtimes P_k$.

In 2002, Charon et al. [3] proved that $d^*(\mathcal{G}_K)$ is $2/9$. They provided the tile depicted in Figure 1, which generates a periodic tiling of the plane with periods $(0, 6)$ and $(6, 0)$, yielding an identifying code C_{∞} of the bidimensional infinite king grid with density $\frac{2}{9}$.

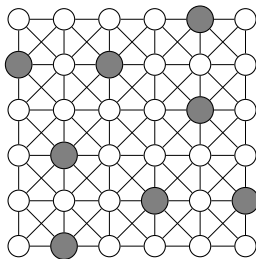


Fig. 1. Tile generating an optimal identifying code of the bidimensional infinite grid. Black vertices are those of the code.

In this paper, using the Discharging Method (see Section 3 of [10] for a detailed presentation of this technique for identifying codes), we provide the following tight general lower bound on the minimum density of identifying

codes of king grids.

Theorem 1.1 *If G is a (finite or infinite) king grid, then $d^*(G) \geq \frac{2}{9}$.*

Keeping on, we prove the following.

Theorem 1.2 *If G is a finite king grid, then $d^*(G) > \frac{2}{9}$.*

Finally, we give some bounds for king strips. Pushing further the proof of Theorem 1.1, we prove the following.

Theorem 1.3 *For every $k \geq 6$, $d^*(\mathcal{K}_k) \geq \frac{2}{9} + \frac{8}{81k}$.*

Modifying C_∞ , we construct identifying codes of \mathcal{K}_k yielding the following upper bounds.

Theorem 1.4 *For every $k \geq 5$,*

$$d^*(\mathcal{K}_k) \leq \begin{cases} \frac{2}{9} + \frac{6}{18k}, & \text{if } k \equiv 0 \pmod{3}, \\ \frac{2}{9} + \frac{8}{18k}, & \text{if } k \equiv 1 \pmod{3}, \\ \frac{2}{9} + \frac{7}{18k}, & \text{if } k \equiv 2 \pmod{3}. \end{cases}$$

Finally, we show some identifying codes of \mathcal{K}_3 , \mathcal{K}_4 , \mathcal{K}_5 and \mathcal{K}_6 (see Figures 2, 3, 4, and 5.) and prove that they are optimal. This yields the following.

Theorem 1.5 $d^*(\mathcal{K}_3) = 1/3 = 0.333\dots$ $d^*(\mathcal{K}_4) = 5/16 = 0.3125$
 $d^*(\mathcal{K}_5) = 4/15 = 0.2666\dots$ $d^*(\mathcal{K}_6) = 5/18 = 0.2777\dots$

Clearly $d^*(\mathcal{K}_1) = 1/2$ (as $\mathcal{K}_1 = \mathcal{S}_1 = P_{\mathbb{Z}}$) and \mathcal{K}_2 has no identifying code because it has twins. All these results imply that \mathcal{G}_K , \mathcal{H}_K and \mathcal{Q}_K are the unique king grids having an identifying code with density $2/9$. (One can easily derive from C_∞ identifying codes with density $2/9$ of \mathcal{H}_K and \mathcal{Q}_K).

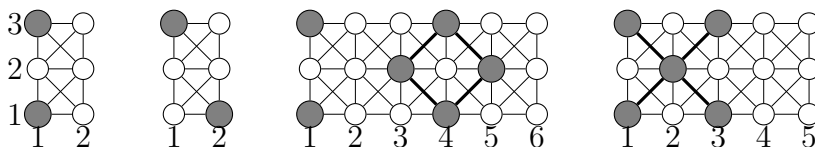


Fig. 2. Four tiles generating optimal identifying codes of \mathcal{K}_3 (density $1/3$)

2 Sketches of proofs

Sketch of proof of Theorem 1.1. Let G be a king grid and C an identifying code of G . We shall prove that $d(C, G) \geq 2/9$. For this, we use the Discharging

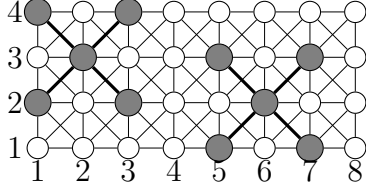


Fig. 3. Tile generating an optimal identifying code of \mathcal{K}_4 (density $5/16$)



Fig. 4. Two tiles generating optimal identifying codes of \mathcal{K}_5 (density $4/15$)



Fig. 5. Two tiles generating optimal identifying codes of \mathcal{K}_6 (density $5/18$)

Method. The initial charge of a vertex v is 1 if $v \in C$ and 0 otherwise. We then apply some local discharging rules. We shall prove that the final charge of every vertex in C is at least $2/9$. This would imply the result.

We set $U = V(G) \setminus C$. Given $X \subseteq V(G)$ and $1 \leq i \leq 9$, we denote by X_i (resp. $X_{\geq i}$) the set of vertices in X having exactly i vertices (resp. at least i vertices) in their identifier. An X -vertex is a vertex in X . A vertex is *full* if its eight neighbours in \mathcal{G}_K are in G ; otherwise it is a *side vertex*.

We first establish the following properties of C .

- (i) Two C_2 -vertices are not adjacent.
- (ii) Every C -vertex has at most one neighbour in U_1 .
- (iii) Every full C_2 -vertex has at least three neighbours in $U_{\geq 3}$.
- (iv) Every full C_3 -vertex has a neighbour in $U_{\geq 3}$.

- (v) Every C_1 -vertex (a, b) has no neighbour in U_1 and at most six neighbours in U_2 . Furthermore, if it has six neighbours in U_2 , then either $\{(a-1, b-2), (a-2, b-1), (a+2, b+1), (a+1, b+2)\} \subseteq C$ or $\{(a+1, b-2), (a+2, b-1), (a-2, b+1), (a-1, b+2)\} \subseteq C$.

A *defective vertex* is a vertex in C_1 with six neighbours in U_2 . Let $v = (a, b)$ be a defective vertex. The *team* of v is a set among $\{(a-1, b-2), (a-2, b-1), (a+2, b+1), (a+1, b+2)\}$ and $\{(a+1, b-2), (a+2, b-1), (a-2, b+1), (a-1, b+2)\}$ which is included in C . By Property (v), the team exists. Moreover, by Property (i), at least two vertices of the team are in $C_{\geq 3}$. Those vertices are the *partners* of v .

We apply the following discharging rules.

- (R1) Every C -vertex sends $\frac{2}{9i}$ to each of its neighbours in U_i .
(R2) Every defective vertex receives $\frac{1}{54}$ from each of its partners.

Using the above properties, we then prove that the final charge of every vertex v is at least $2/9$. \square

Sketch of proof of Theorem 1.2. We only need to prove that, at the end of the proof of Theorem 1.1, one vertex has final charge greater than $2/9$. To do so we shall prove that there is a side C -vertex or a $C_{\geq 3}$ -vertex and check that such a vertex has final charge at least $\frac{2}{9} + \frac{1}{27}$. \square

Sketch of proof of Theorem 1.3. Using the Discharging Method, we prove that in average, for every column, there is an extra charge of at least $\frac{4}{81}$ on the three top vertices and an extra charge of at least $\frac{4}{81}$ on the three top vertices. \square

Sketch of proof of Theorem 1.5. The b th row of \mathcal{K}_k is $R_b = \{(a, b) \mid a \in \mathbb{Z}\}$. We have $d(C, \mathcal{K}_k) = \frac{1}{k} \sum_{i=1}^k d(C, R_i)$. We show that if C is an identifying code of \mathcal{K}_k ($k \geq 3$), then $d(C, R_1) + d(C, R_2) \geq 1/2$, $d(C, R_k) + d(C, R_{k-1}) \geq 1/2$, $d(C, R_3) \geq 1/3$ and $d(C, R_{k-2}) \geq 1/3$. One easily derives that if C is an identifying code of \mathcal{K}_5 (resp. \mathcal{K}_6), then $d(C, \mathcal{K}_5) \geq 4/15$. (resp. $d(C, \mathcal{K}_6) \leq 5/18$.)

To prove lower bounds on $d^*(\mathcal{K}_k)$ for $k \in \{3, 4\}$, we use the Discharging Method on the columns $Q_a = \{(a, b) \mid 1 \leq b \leq k\}$. Let C be an identifying code of \mathcal{K}_k . We set the initial charge of every integer $a \in \mathbb{Z}$ to $\text{chrg}_0(a) = |Q_a \cap C|$. We say that $a \in \mathbb{Z}$ is *satisfied* if its charge is least q_k and *unsatisfied* otherwise, where $q_3 = 1$ and $q_4 = 5/4$. We apply five discharging rules, Rule i for $i = 1$ to 5 one after another. We denote by $\text{chrg}_i(a)$ the charge of a after applying Rule i .

Rule i : every unsatisfied $a \in \mathbb{Z}$ receives $\min\{\text{chrg}_{i-1}(a-i) - q_k, q_k - \text{chrg}_{i-1}(a)\}$

from $a - i$, if $a - i$ is satisfied (before Rule i).

Finally, we prove that, after these rules, every integer $a \in \mathbb{Z}$ is satisfied. This implies $d(C, \mathcal{K}_k) \geq q_k/k$. \square

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