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#### 1 GENERICITY OF THE STRONG OBSERVABILITY FOR SAMPLED 2 SYSTEMS.

SABEUR AMMAR<sup>\*</sup>, MAJID MASSAOUD<sup>†</sup>, AND JEAN-CLAUDE VIVALDA <sup>‡</sup>

4 **Abstract.** In this paper we prove that, generically, a sampled data system is observable provided 5 that the number of outputs is greater than the number of inputs plus 1.

6 Key words. Observability, Sampled systems, Genericity, Transversality

7 AMS subject classifications. 93B07, 93C57

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8 **1. Introduction.** In this paper we deal with the genericity of the observability 9 of sampled data systems. Consider a controlled continuous time system written as

10 (1) 
$$\begin{cases} \dot{x} = f(x, u) \\ y = h(x). \end{cases}$$

11 Given a time T, to system (1), we relate the following continuous-discrete-time system

12 (2) 
$$\begin{cases} \dot{x}(t) = f(x(t), u_k), t \in [kT, (k+1)T) \\ y_k = h(x(kT)) \end{cases}$$

13 where the control u is maintained constant on the intervals [kT, (k+1)T) and the

14 measurements of the state are made only at each of the times  $0, T, 2T, \ldots$  System (2) 15 is called the sampled data system related to (1).

Many physical processes or industrial devices can be modeled by a system of continuous-time differential equations as (1). From a mathematical viewpoint, the 17 18 time and the state of this system can vary continuously but in practice, a controlled process is regulated by a digital computer which is not able to record a continuum 19 of data. This is why control decisions are restricted to be taken at fixed times 20 $0, T, 2T, \ldots$ ; here T is called the sampling time and is a (generally small) parame-21 ter which depends on the instrumentation of the process, on the computing power 22and other parameters. For a continuous time system, the resulting situation can be 2324 modeled through the restriction that the applied inputs are constant on the intervals  $[0, T), [T, 2T), \ldots$  and the state is (partially) measured only at those fixed times 25 $0, T, 2T, \ldots$ , that is to say we access to the values of the observation function only at 26times  $0, T, \ldots$ 27

For the sake of clarity, the precise assumptions that we make on these systems are stated in section 1.1 but we recall here the notion of observability. Regarding system (2), an input  $u^0$  is a sequence  $(u_k)_{k\geq 0}$  with  $u_k \in U$  (U, the input space). An input  $u^0$  being given, we denote by x(t) and  $\bar{x}(t)$  the solutions of (2) starting from  $x_0$  and  $\bar{x}_0$  respectively; we say that system (2) is observable for  $u^0$  if for every pair of initial conditions  $(x_0, \bar{x}_0)$ , there exists an integer k such that  $h(x(kT)) \neq h(\bar{x}(kT))$ .

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The aim of this paper is to prove that, under some conditions on the respective dimensions of the inputs and the output, generically, the sampled system obtained from a continuous time system is observable.

Two questions can be investigated about observability and sampled systems. The 37 first one is the problem of the preservation of the observability: if the continuous time 38 system (1) is observable for any inputs, is it also the case for the sampled system (2)? 39 The second question is the subject of this paper: given a sampling time T, how many 40 are the continuous time systems (1) such that the sampled system (2) is observable? 41 Concerning the first question, the answer for linear systems is well known (see 42 e.q. [18] and also [14] when the sampling time is not constant). If we deal with 43 nonlinear systems, one would think that the observability of the continuous time 44 45system involves the observability of the sampled one, at least if the sampling time is chosen small enough. Surprisingly, this is not the case: a counter-example can 46be found in [6]; in order to get the observability of the sampled system, to this 47 natural condition (the observability of the continuous time system), we have to add the 48 condition of infinitesimal observability (see [11]) together with a technical condition 4950bearing on the sequence of controls  $u_0, u_1, \ldots$ 

The aim of this paper lies on a more "philosophical plane". Due to the importance of the notion of observability, it is of interest to know "how many" continuous time 52systems give rise to observable sampled data systems. To be more precise, in [6] (systems given on a compact manifold) and in [3, 5] (systems given on  $\mathbf{R}^n$ ), we 54gave some natural sufficient conditions bearing on the continuous time system and 56 under which the sampled system is observable. So, these works have a practical interest: for a class of continuous time systems our result allows us to decide on 57 the observability of the sampled system. The present paper intends to prove that 58 the set of continuous time systems which admit an observable sampled system is everywhere dense. Knowing that the set of rational numbers is dense in the set of 60 real numbers does not permit us to decide if a particular given number is rational; in 61 62 the same way, the result proven in this paper does not permit to decide if a particular sampled system is observable. Moreover, this result cannot be deduced from the 63 abovementioned papers because, while the observability of the continuous time system 64 is generic (see [11]) the additional conditions in [6, 3, 5] are not; also, in these papers, 65 the observability is ensured only for sufficiently small sampling time T. 66

The genericity of the observability has been the subject of some researches in the 67 68 last decades. As regards continuous-time systems, the first paper on the subject was about the genericity of the observability for uncontrolled systems [9]; this work was 69 generalized to controlled systems by J.-P. Gauthier and I. Kupka, in [10] these authors 70 proved the genericity of differential observability for systems with more outputs than 7172 inputs. A reference book on this subject is [11]. A related issue is the problem of the identifiability, in [8], the authors deal with general nonlinear systems which contain 73 an unknown function, they prove that these (uncontrolled) systems are generically 74 identifiable if the number of observations is at least three. Regarding the discrete-75 time systems, the first paper on the subject was from Aeyels [2], we can cite also [20] 76for the uncontrolled case and [7, 4] for the controlled case. In all of these papers, 77 it is proved that the observability is a generic property provided that the number of 78 79 outputs is greater than the number of inputs. Surprisingly, this result is no more valid for the systems considered in this paper: in the next sections, we shall prove that if 80 the number of inputs is one and the number of outputs is two, the set of pairs (f, h)81 such that system (2) is observable is not dense. Concerning the subject of this paper, 82 we have also to cite [13], in this paper the authors prove also a result of genericity of 83

the observability for sampled data systems; the systems considered in this paper are uncontrolled and the sampling time is not constant but depends on the sate of the system.

The tools used to prove our main result are essentially the same (but applied to different situations) as the ones used in the above-mentioned papers, that is to say the major theorem of the transversality theory.

The paper is organized as follows: in the next section, we state the precise formula-90 tion of the problem we deal with and we recall some useful facts from the transversality 91 theory. In section 2, we state the main result of the paper and we introduce the lists  $L_{2n}$  and  $L_{2n}$ , which are built from two initial conditions  $x_0$  and  $\bar{x}_0$  and their images 93 under the iterates of f; we then introduce five possible configurations for these lists 9495(cf section 2.1). We then prove our main theorem for each of these configurations: the corresponding results are stated in the three propositions 5, 9 and 10; the conjunc-96 tion of these propositions give the main result. Finally, we give a counter-example in 97 order to prove that the observability of the sampled system is not generic in the case 98  $d_u = d_y + 1$  (*cf.* section 5). 99

**1.1. Problem formulation.** We consider two compact manifolds X and U; we 100 let  $n = \dim X$  and  $d_u = \dim U$ . As usual we denote by  $T_x X$  the tangent space to 101 X at x, and by TX the tangent bundle. A parametrized vector field will be a  $C^{\infty}$ 102mapping defined from  $X \times U$  into TX such that, for every  $u \in U$ ,  $f(\cdot, u)$  is a vector 103 field defined on X. The set of parametrized vector fields defined on X will be denoted 104by  $\Gamma_U(X)$ . If f belongs to  $\Gamma_U(X)$ , we denote by  $\varphi_t^u$  the flow generated by the vector 105field  $f(\cdot, u)$  (the parameter u being fixed); so for every  $x \in X$ , every  $u \in U$ , and every 106  $t \geq 0$ , we have 107

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$$\varphi_0^u(x) = x$$
 and  $\frac{\mathrm{d}\varphi_t^u(x)}{\mathrm{d}t} = f(\varphi_t^u(x), u)$ 

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109 Let  $u_0, u_1, \ldots$  be a sequence of controls (*i.e.* a sequence of elements of U), for  $k \ge 1$ , 110 we denote by  $\underline{u_k}$  the finite sequence  $\underline{u_k} = (u_0, \ldots, u_{k-1})$ .

111 Let  $\psi: M \to N$  be a differentiable mapping between two manifolds M and N, 112 the notation  $d\psi(x)$  will stand for the differential of  $\psi$  at x; let  $\xi \in T_x M$  be a tangent 113 vector,  $d\psi(x) \cdot \xi$  will denote the image of  $\xi$  under  $d\psi(x)$ .

Hereafter, together with a parametrized vector field, we consider a  $C^{\infty}$  mapping h from X to  $\mathbf{R}^{d_y}$  and, given a sampling time T > 0, we consider the mapping  $\Theta_T^{f,h}$ defined as

117 (3) 
$$\Theta_T^{f,h}: \begin{array}{ccc} X \times U^{2n} & \longrightarrow & \mathbf{R}^{(2n+1)d_y} \times U^{2n} \\ (x,\underline{u_{2n}}) & \longmapsto & \left(h(x_0),h(x_1),\dots,h(x_{2n}),\underline{u_{2n}}\right) \end{array}$$

118 where the sequence  $(x_0, x_1, \ldots, x_{2n})$  is defined recursively by  $x_0 = x$  and  $x_{k+1} =$ 119  $\varphi_T^{u_k}(x_k)$ . Also, we denote by  $y_i$  and  $\bar{y}_i$ , the values at  $x_i$  and  $\bar{x}_i$  under h:  $y_i = h(x_i)$ 120 and  $\bar{y}_i = h(\bar{x}_i)$ .

121 DEFINITION 1. We shall say that the sampled data system (2) is strongly observ-122 able if the mapping  $\Theta_T^{f,h}$  defined above is one-to-one.

We shall show that, generically, system (2) is strongly observable, to be more precise, we endow  $\Gamma_U(X) \times C^{\infty}(X, \mathbf{R}^{d_y})$  with the Whitney topology and we shall prove that the set of pairs (f, h) such that the mapping  $\Theta_T^{f,h}$  is injective is a residual subset of  $\Gamma_U(X) \times C^{\infty}(X, \mathbf{R}^{d_y})$  provided that  $d_y \ge d_u + 2$  (case  $d_u > 0$ ) or  $d_y \ge 1$ (case  $d_u = 0$ ). The tools used in this paper come from the transversality theory, hereafter, we recall the notion of transversality as well as the Abraham's theorem of density [1] which will be intensively in the proof of our main result. 130 DEFINITION 2 (Transversality). Let f be a smooth mapping between two smooth 131 manifolds X and Y, W a submanifold of Y and x a point in X. We shall say that f132 is transversal to W at x if either

133 •  $f(x) \notin W$ , or

134 •  $f(x) \in W$  and  $T_{f(x)}Y = T_{f(x)}W + df_x(T_xX)$ .

We shall say that f is transversal to W if it is transversal to W at every  $x \in X$ . We shall use the symbol  $\pitchfork$  to denote the transversality.

137 Concerning this definition, some elementary conditions show that the second 138 equality cannot be satisfied if  $\operatorname{codim} W > \dim X$ . Therefore if  $\operatorname{codim} W > \dim X$ , 139 transversality means non membership: in this case saying that f is transverse to W140 amounts to saying that  $f(x) \notin W$  for every  $x \in X$ . This trick will be used later in 141 the proofs of propositions 9 and 10.

We recall also the notion of representation: let  $\mathscr{A}, X$  and Y be  $C^r$  manifolds and  $\rho$  a map from  $\mathscr{A}$  to  $C^r(X, Y)$ . For  $a \in \mathscr{A}, \rho_a : X \to Y$  is the map defined as  $\rho_a(x) = \rho(a)(x)$ . We say that  $\rho$  is a  $C^r$  representation if the evaluation map:

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$$\begin{array}{cccc} \operatorname{ev}_{\rho} : \mathscr{A} \times X & \longrightarrow & Y \\ (a,x) & \longmapsto & \rho_{a}(x) = \rho(a)(x) \end{array}$$

146 is a  $C^r$  map from  $\mathscr{A} \times X$  to Y.

147 THEOREM 3 (Transversal density theorem). Let  $\mathscr{A}, X, Y$  be  $C^r$  manifolds,  $\rho$ : 148  $\mathscr{A} \to C^r(X, Y)$  a  $C^r$  representation,  $W \subset Y$  a submanifold (not necessarily closed), 149 and  $ev_{\rho} : \mathscr{A} \times X \to Y$  the evaluation map. Define  $\mathscr{A}_W \subset \mathscr{A}$  by:

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$$\mathscr{A}_W = \{ a \in \mathscr{A} \mid \rho_a \pitchfork W \}$$

151 Assume that:

152 1. X has a finite dimension n and W has a finite codimension q in Y; 153 2.  $\mathscr{A}$  and X are second countable; 154 3.  $r > \max(0, n - q);$ 155 4.  $\exp_{\rho} \pitchfork W.$ 

156 Then  $\mathscr{A}_W$  is residual in  $\mathscr{A}$ .

Notice that manifold *A* is not necessarily finite dimensional; it may be a Banach
space or an open subset of a Banach space.

#### 159 **2. Main result.**

160 THEOREM 4. Assume that  $d_y \ge d_u + 2$  or that  $d_u = 0$  and  $d_y \ge 1$ , and let T > 0161 a given sampling time. Then the set of pairs (f, h) such that system (2) is strongly 162 observable is a residual subset of  $\Gamma_U(X) \times C^{\infty}(X, \mathbf{R}^{d_y})$ .

In order to prove this theorem, we need some preliminary results, namely the propositions 5, 9 and 10 stated in sections 3 and 4.3–4.4. In these propositions, different possible configurations, denoted hereafter by  $\mathbf{C_0}$  through  $\mathbf{C_4}$ , of the lists  $(x_0, x_1, \ldots, x_{2n})$  and  $(\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{2n})$  are considered (the  $\bar{x}'_i s$  are defined as the  $x_i$ 's, cf. (3)). For each of this configuration  $\mathbf{C_k}$   $(k = 0, \ldots, 4)$ , we prove that, generically  $\Theta_T^{f,h}(x, \underline{u_{2n}}) \neq \Theta_T^{f,h}(\bar{x}, \underline{u_{2n}})$  if  $(x, \underline{u_{2n}}), (\bar{x}, \underline{u_{2n}})$  is under  $\mathbf{C_k}$  configuration.

To be more precise, given two different initial conditions x and  $\bar{x}$  and an integer  $s \leq 2n$ , we shall consider the two lists

where the  $\bar{x}_i$ 's are defined as the  $x_i$ 's. Concerning the two lists  $L_{2n}$  and  $L_{2n}$ , we 173174shall examine all the possible situations: the elements of these lists can be pairwise distinct, some equalities can occur among the elements of the first list while the ones 175of the second are pairwise distinct, etc. In the following section, we define five possible 176configurations and we prove that, necessarily the above-mentioned lists belong to one 177 of five configurations. 178

2.1. The different configurations. Hereafter we shall give an exhaustive list 179of all the possibilities concerning the equalities between the elements of the lists  $L_{2n}$ 180and  $\bar{L}_{2n}$ ; in the sequel, we shall say that the equalities  $x_i = \bar{x}_i$  and  $x_{i'} = \bar{x}_{j'}$  between 181elements of  $L_{2n}$  and elements of  $L_{2n}$  are in the same direction if the differences i - j182and i' - j' have the same sign. Take  $x \neq \bar{x} \in X$ , even if we have to invert the roles of 183 $L_{2n}$  and  $L_{2n}$ , the possible configurations for these lists are: 184

 $C_0$  The elements of  $L_{2n}$  are pairwise different; moreover the only possible equalities 185between elements of  $L_{2n}$  and elements of  $\overline{L}_{2n}$  are all in the "same direction". 186 187That is to say, let

$$I = \{ 0 \le i \le 2n \mid \exists j \in \{0, \dots, 2n\}, x_i = \bar{x}_j \},\$$

for  $i \in I$ , let  $E_i = \{ 0 \le j \le 2n \mid x_i = \bar{x}_j \}$ , then either for every  $i \in I$ , for 190every  $j \in E_i$ , j < i or for every  $i \in I$ , for every  $j \in E_i$ , j > i. 191  $C_1$  There exist some subscripts  $0 \le i and <math>0 \le j, m \le 2n$  such that 192

•  $x_p = x_i$  and  $\bar{x}_m = x_j$ ;

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- there is no equality between the elements of  $L_{p-1}$ ;
- letting  $q = \max(j, p, m)$ , there is no equality between the elements of  $L_{q-1};$
- the equalities between elements of  $L_{q-1}$  and  $\overline{L}_{q-1}$  have the same direction.

**C**<sub>2</sub> There exist some subscripts  $0 \le i , and <math>0 \le j < m \le 2n$ , with  $m \ge p$ 199200 and such that 201

•  $x_p = \bar{x}_m$  and  $x_i = \bar{x}_j$  with  $p - i \neq m - j$ ;

• there is no equalities between the elements of  $L_p$  nor between the elements of  $L_m$ ;

• the only possible equalities between elements of  $L_{m-1}$  and  $L_{m-1}$  are all in the same direction; moreover if these equalities write  $x_{i_1} = \bar{x}_{j_1}, \ldots$ ,  $x_{i_r} = \bar{x}_{j_r}$  the differences  $i_1 - j_1, \ldots, i_r - j_r$  are equal;

• if  $x_{i'} = \bar{x}_{j'}$  with  $i', j' \le m$ , then  $i' \ge i$  and  $j' \ge j$ .

**C**<sub>3</sub> There exist some subscripts  $0 \le i , and <math>0 \le m < j \le 2n$  with  $p \le j$  and 208209 such that

- $x_p = \bar{x}_m$  and  $x_i = \bar{x}_j$ ;
  - there is no equalities between the elements of  $L_p$  nor between the elements of  $L_i$ ;
  - the only possible equalities between elements of  $L_{j-1}$  and  $L_{j-1}$  are all in the same direction; moreover, if these equalities write  $x_{i_1}$  =  $\bar{x}_{j_1}, \ldots, x_{i_r} = \bar{x}_{j_r}$  the differences  $i_1 - j_1, \ldots, i_r - j_r$  are equal;
- if  $x_{i'} = \bar{x}_{j'}$  with  $i', j' \leq j$ , then  $i' \geq p$  and  $j' \geq m$ .

217  $C_4$  There exist some subscripts  $0 \le i , and <math>0 \le j < m \le 2n$  with  $m \ge p$ and such that 218

- $x_p = x_i$  and  $\bar{x}_m = \bar{x}_j$ ;
- there is no equality between one element of  $L_{m-1}$  and one element of 220 $L_{m-1}$ . 221

Denote by  $e_s$  the number of equalities between the elements of  $L_s \cup L_s$ , if  $e_{2n} = 0$ 2.2.2 223or 1, then we are under the  $C_0$  configuration.

Now, assume that  $e_{2n} \geq 2$ ; if we go from  $L_s \cup \overline{L}_s$  to  $L_{s-1} \cup \overline{L}_{s-1}$  we lose 0, 1 or 224 2 equalities. Thus if  $e_{2n} \geq 2$ , there exists a subscript  $m \leq 2n$  such that there exist 225exactly two or exactly three equalities between the elements of  $L_m \cup L_m$ ; we denote by 226 s the minimal subscript with this property. Notice that, if  $e_s$  is exactly three,  $x_s$  and 227  $\bar{x}_s$  must be equal to some elements of  $L_{s-1} \cup \bar{L}_{s-1}$  (because we always have  $x_s \neq \bar{x}_s$ ). 228 Assume first that  $e_s = 3$  and denote by  $x_s = z_1$ ,  $\bar{x}_s = z_2$  and  $z_3 = z_4$  the three 229equalities, then 230• if  $z_1 \in L_s$ , and  $z_3, z_4 \in L_s$ , then  $L_{s-1}$  and  $\overline{L}_s$  are under  $\mathbf{C_1}$  (if  $z_2 \in L_s$ ) or 231 $C_4$  configuration (if  $z_2 \in \overline{L}_s$ ); 232 • if  $z_1 \in L_s$ ,  $z_3 \in L_s$ , and  $z_4 \in \overline{L}_s$ , then  $L_s$  and  $\overline{L}_{s-1}$  are under  $C_1$  configura-233 tion 234• if  $z_1 \in L_s, z_3 \in \overline{L}_s$ , and  $z_4 \in \overline{L}_s$ , then  $L_s$  and  $\overline{L}_{s-1}$  are under  $C_4$  configura-235236tion: • if  $z_1 \notin L_s$ , then  $L_s$  and  $L_{s-1}$  are under  $C_1$ ,  $C_2$  or  $C_3$  configuration (we could 237238 have to invert the roles of  $L_s$  and  $L_s$ ). 239 If  $e_s = 2$  and if the lists  $L_s$  and  $L_s$  are not under  $C_1 - C_4$  configurations, we have two cases to consider. 240 In the first case, there exists some subscripts  $0 \le t < s$  and  $0 \le t' < s'$  such 241 that  $x_t = x_s$  and  $x_{t'} = x_{s'}$ ; if s' < s we set i = t' and p = s', if s' = s, we set 242 $i = \min(t, t')$  and  $p = \max(t, t')$ . As noticed above, when we go from lists  $L_s$  and 243 244  $L_s$  to the lists  $L_{s+1}$  and  $L_{s+1}$  we gain 0, 1 or two equalities. Denote by  $\sigma$  the least subscript greater than s such that there exist 3 or 4 equalities among the elements 245of  $L_{\sigma}$  and  $L_{\sigma}$ ; if such a subscript fails to exist, the lists  $L_{2n}$  and  $L_{2n}$  are under  $C_0$ 246configuration. Otherwise, assume that there exists exactly three equalities between 247the elements of  $L_{\sigma} \cup L_{\sigma}$ , this additional equality can be one of the followings 248•  $\bar{x}_{\sigma} = x_j$  (with  $j < \sigma$ ), the lists  $L_{2n}$  and  $\bar{L}_{2n}$  are then under  $\mathbf{C}_1$  configuration; 249250•  $\bar{x}_{\sigma} = \bar{x}_j$  (with  $j < \sigma$ ), the lists  $L_p$  and  $L_{\sigma}$  are under  $C_4$  configuration; •  $x_{\sigma} = \bar{x}_m$  (with  $m < \sigma$ ), the lists  $L_{2n}$  and  $L_{2n}$  are then under  $\mathbf{C}_1$  configuration; 251•  $x_{\sigma} = x_j$  in this case we seek for the least subscript  $\sigma' > \sigma$  (if any) such that 252one get 1 or 2 additional equalities by going from  $L_{\sigma} \cup L_{\sigma}$  to  $L_{\sigma'} \cup L_{\sigma'}$ . 253If we are in the case where there exist exactly 4 equalities between the elements of 254 $L_{\sigma} \cup L_{\sigma}$ , these equalities can be 255•  $x_{\sigma} = x_{j_1}$  and  $\bar{x}_{\sigma} = x_{j_2}$  (with  $j_1, j_2 < \sigma$ ), in this case the lists  $L_{2n}$  and  $\bar{L}_{2n}$  are 256then under  $C_1$  configuration; 257258

•  $x_{\sigma} = x_{j_1}$  and  $\bar{x}_{\sigma} = \bar{x}_{j_2}$ , the lists  $L_p$  and  $\bar{L}_{\sigma}$  are then under  $C_4$  configuration;

•  $x_{\sigma} = \bar{x}_{j_1}$  and  $\bar{x}_{\sigma} = x_{j_2}$ , the lists  $L_p$  and  $L_{j_1}$  are then under  $C_1$  configuration; 259•  $x_{\sigma} = \bar{x}_{j_1}$  and  $\bar{x}_{\sigma} = \bar{x}_{j_2}$ , the lists  $L_p$  and  $\bar{L}_{\sigma}$  are then under  $C_4$  configuration. 260In the second case there exist some subscripts  $0 \le t < s$  and  $0 \le t' < s' < s$  such that 261 s-t=s'-t',  $x_s=\bar{x}_t$  and  $x_{s'}=\bar{x}_{t'}$  (without loss of generality, we can exchange the 262 roles of x and  $\bar{x}$ ). Proceeding as in the first case, we prove that either the lists  $L_{2n}$ 263 and  $L_{2n}$  are under  $\mathbf{C}_0$  configuration or there exists a subscript  $s < \sigma \leq 2n$  such that 264265 $L_{\sigma}$  and  $L_{\sigma}$  are under one of the configurations  $C_1$ ,  $C_2$  or  $C_3$ .

Let the pair  $(f,h) \in \Gamma_U(X) \times C^\infty(X, \mathbf{R}^p)$  be fixed; hereafter, we shall say that 266the configuration of the triplet  $(x, \bar{x}, \underline{u}_{2n})$   $(x \neq \bar{x})$  is  $\mathbf{C}_i$  if the configuration of the 267lists  $L_{2n}$  and  $\bar{L}_{2n}$  related to x and  $\bar{x}$  is  $C_i$ . In the sequel, we shall assume that all the 268function spaces (such that  $C^{\infty}(X, \mathbf{R}^p), \ldots$ ) as well as the spaces  $\Gamma(X), \Gamma_U(X)$  and 269 $\mathscr{G}_2^U(a)$  are endowed with the  $C^r$  topology, where  $r \in \mathbf{N}^*$ . 270

271 **2.2.** Outline of the proof of Theorem 4. Without going into the tech-272 nique details, we shall explain our strategy for the proof. Take  $(f,h) \in \Gamma_U(X) \times$ 273  $C^{\infty}(X, \mathbf{R}^{d_y})$ , let  $x_0 \neq \bar{x}_0$  be two different initial conditions and  $\underline{u_{2n+1}}$  a sequence 274 of controls. The equality  $\Theta_T^{f,h}(x_0, \underline{u_{2n+1}}) = \Theta_T^{f,h}(\bar{x}_0, \underline{u_{2n+1}})$  can be formulated in 275 geometric terms. To show where are the difficulties, we make a first attempt by con-276 sidering the simplest way to make this formulation. Consider the mapping, denoted 277 by  $r_{f,h}$ , related to the pair (f, h) and defined as

278

$$\begin{array}{cccc} r_{f,h}: & X^{(2)} \times U^{2n} & \longrightarrow & \mathbf{R}^{(2n+1)d_y} \\ & & (x,\bar{x},\underline{u_{2n+1}}) & \longmapsto & (y_0 - \bar{y}_0, \dots, y_{2n} - \bar{y}_{2n}) \end{array}$$

where  $y_i = h(x_i)$ ,  $\bar{y}_i = f(\bar{x}_i)$  and the  $x_i$ 's and the  $\bar{x}_i$ 's defined as in (3). The 279equality  $\Theta_T^{f,h}(x_0,\underline{u_{2n+1}}) = \Theta_T^{f,h}(\bar{x}_0,\underline{u_{2n+1}})$  means that  $r_{f,h}(x_0,\bar{x}_0,\underline{u_{2n}})$  belongs to 280the submanifold  $\overline{W} = \{0\} \subset \mathbf{R}^{(2n+1)\overline{d_y})}$ . Notice that the codimension of W is equal 281 to  $(2n+1)d_y$  and is greater than  $2n(d_u+1)$  the dimension of the domain of  $r_{f,h}$ . If 282  $r_{f,h}$  is transverse to W, this inequality on codim W implies that  $r_{f,h}(x_0, \bar{x}_0, \underline{u}_{2n})$  does 283not belong to W and therefore that  $\Theta_T^{f,h}(x_0, \underline{u_{2n+1}}) \neq \Theta_T^{f,h}(\bar{x}_0, \underline{u_{2n+1}})$ . Assume now that we are able to prove that, generically,  $\overline{r_{f,h}}$  is transversal to W, that is to say 284285assume that there exists a residual set  $\mathscr{R}$  such that  $r_{f,h}$  is transversal to W whenever 286(f,h) belongs to  $\mathscr{R},$  we then have proved that, generically,  $\Theta_T^{f,h}$  is one-to-one. In order 287to prove that, generically,  $r_{f,h}$  is transversal to W, we could try to apply Theorem 3 288 by proving that the evaluation map related to the representation r from  $\Gamma_U(X)$  × 289 $C^{\infty}(X, \mathbf{R}^{d_y})$  to  $C^r(X^{(2)} \times U^{2n}, \mathbf{R}^{(2n+1)d_y)}$  is transversal to W. This evaluation map 290 $ev_r$  is defined as  $ev_r(f, h, x, \bar{x}, u_{2n}) = r_{f,h}(x, \bar{x}, u_{2n})$ , as it is linear with respect to 291h, its differential with respect to h is given by  $\eta \mapsto (\eta(x_0) - \eta(\bar{x}_0), \dots, \eta(x_{2n}) - \eta(\bar{x}_0))$ 292 $\eta(\bar{x}_{2n})$ , if we were able to show that there exists  $\eta$  in  $C^{\infty}(X, \mathbf{R}^{d_y})$  such that  $\eta(x_i) - \eta(x_i)$ 293 $\eta(\bar{x}_i) = \mathfrak{Y}_i$ , for arbitrary vectors  $\mathfrak{Y}_0, \ldots, \mathfrak{Y}_{2n}$  of  $\mathbf{R}^{d_y}$ , then we would be done. The 294existence of such an  $\eta$  is generally not ensured but is certainly true if the elements 295 $x_0, \ldots, x_{2n}, \overline{x}_0, \ldots, \overline{x}_{2n}$  are all different or, more generally, if the two lists  $L_{2n}$  and 296 $L_{2n}$  are under  $\mathbf{C}_0$  configuration. In this case, modifying  $r_{f,h}$  and the definition of W 297as explained in the next section, we can prove that, generically  $r_{f,h}$  is transversal to 298W. Now the two lists are not always under such a configuration, the points  $x_0$  and  $\bar{x}_0$ 299300 could be located on a periodic trajectory of f or could be singular points. The case of singular points shows that we cannot disregard the cases where the two lists  $L_{2n}$  and 301  $\overline{L}_{2n}$  are not under configuration  $\mathbf{C}_0$ . Assume that  $x_0$  and  $\overline{x}_0$  are singular points for the 302 vector field  $f(\cdot, u_0)$  and take a sequence of identical controls:  $u_0 = u_1 = \ldots = u_{2n}$ 303 then we cannot argue as in the case of  $C_0$  configuration: a mapping  $\eta$  as above fails 304 to exist because  $x_0 = \cdots = x_{2n}$  and  $\bar{x}_0 = \ldots \bar{x}_{2n}$ . Notice that this situation is 305 unavoidable: on some manifolds, every vector field has at least one singular point; 306 this means that the particular configurations  $C_1$ - $C_4$  cannot be eliminated by using 307 an argument of density. 308

The outline of the rest of this section will be the following: for each configuration 309  $\mathbf{C}_0 - \mathbf{C}_4$ , we shall prove that there exists a residual subset of  $\Gamma_U(X) \times C^r(X, \mathbf{R}^{d_y})$ 310 (endowed with the  $C^r$  topology), denoted by  $\mathscr{C}_k^r$   $(k = 0, \ldots, 4)$ , such that if  $(f, h) \in$ 311  $\mathscr{C}_k^r$  and if  $(x_0, \bar{x}_0, \underline{u_{2n}})$  is in configuration  $\mathbf{C}_k$ , then  $\Theta_T^{f,h}(x_0, \underline{u_{2n}}) \neq \Theta_T^{f,h}(\bar{x}_0, \underline{u_{2n}})$ . 312Consider the intersection  $\mathscr{C}^r \triangleq \mathscr{C}_0^r \cap \cdots \cap \mathscr{C}_4^r$ , which also is a residual subset, and 313 take a pair (f, h) in  $\mathscr{C}^r$ ; let  $x_0 \neq \bar{x}_0$  be two different initial conditions and  $u_{2n}$  a finite 314 sequence of controls, as  $(x_0, \bar{x}_0, \underline{u}_{2n})$  must be in one of the  $\mathbf{C_0}$ - $\mathbf{C_4}$  configurations, we 315 have  $\Theta_T^{f,h}(x_0, u_{2n}) \neq \Theta_T^{f,h}(\bar{x}_0, u_{2n})$ . Now taking the intersections of the sets  $\mathscr{C}^r$  for 316

r = 1, 2..., we obtain a residual  $\mathscr{R}$  of  $\Gamma_U(X) \times C^r(X, \mathbf{R}^{d_y})$  endowed with the  $C^{\infty}$ 317 topology; this residual is such that if (f,h) belongs to  $\mathscr{R}$ , then  $\Theta_T^{f,h}$  is one-to-one. 318

In the following section, the existence of the residual sets  $\mathscr{C}_0^r - \mathscr{C}_4^r$  is stated in 319

propositions 5, 9 and 10. For the proofs of these propositions, our strategy will be the 320 321 following: we shall consider some submanifold W together with some representation  $\rho$ ; the choice of W and  $\rho$  being related to the considered configuration of the lists  $L_{2n}$ 322 and  $L_{2n}$ . Concerning W and  $\rho$ , we shall prove the following results: 323

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• by applying the *Transversal density theorem* [1], we shall see that the set of 324 pairs  $(f,h) \in \Gamma_U(X) \times C^{\infty}(X, \mathbf{R}^{d_y})$  which are transversal to W is dense; 325

- we shall prove also that the codimension of W is greater than the dimension 326 327 of the domain of  $\rho_{f,h}$ , which implies that the range of  $\rho_{f,h}$  does not intersect W: 328
  - due to our choice of W saying that  $\rho_{f,h}(x, \bar{x}, u_{2n}) \notin W$  will imply that  $\Theta_T^{f,h}(x, u_{2n}) \neq \Theta_T^{f,h}(\bar{x}, u_{2n}).$

We shall provide a detailed proof for the  $C_0$  and  $C_1$  configurations, the proofs 331 for the  $C_2$  and  $C_3$  configurations will be omitted because they are very similar to the previous ones. Concerning the configurations  $C_0-C_3$ , we only need the assumption 333  $d_y \geq d_u + 1$  to prove the existence of the residual subsets  $\mathscr{C}_0 - \mathscr{C}_3$ . We have to consider 334 apart the case of  $C_4$  configuration because to prove the existence of the residual subset 335 $\mathscr{C}_4$  we need the assumption  $d_y \ge d_u + 2$ . 336

3. The triplet  $(x, \bar{x}, u_{2n})$  is under configuration C<sub>0</sub>. In this section, we 337 338 deal first with the simplest case: the  $C_0$  configuration.

**PROPOSITION 5.** Assume that  $d_y > d_u$  (the number of observations is greater than 339 the number of controls). Denote by  $\overset{\circ}{\mathscr{C}}_{0}^{r}$  the set of pairs  $(f,h) \in \Gamma_{U}(X) \times \overset{\circ}{C}^{\infty}(X, \mathbf{R}^{d_{y}})$ 340 such that  $\Theta_T^{f,h}(x, u_{2n+1}) \neq \Theta_T^{f,h}(\bar{x}, u_{2n+1})$  whenever the triplet  $(x, \bar{x}, \underline{u_{2n}})$  (with  $x \neq \bar{x}$ ) 341 is in configuration  $\overline{\mathbf{C}_{\mathbf{0}}}$ . Then  $\mathscr{C}_{\mathbf{0}}^r$  contains a residual for the  $C^r$  topology. 342

343 *Proof.* We consider the representation

where  $\rho_{f,h}$  is the mapping 345

and where  $y_i = h(x_i)$  and  $\bar{y}_i = h(\bar{x}_i)$ , the  $x_i$ 's and the  $\bar{x}_i$ 's being defined above. We 347 consider the submanifold  $W_0$  included in  $(X^{2n+1})^2 \times \mathbf{R}^{(2n+1)d_y}$  defined as follows. 348Submanifold  $W_0$  is the set of those elements  $(a_0, \ldots, a_{2n}, \bar{a}_0, \ldots, \bar{a}_{2n}, 0, \ldots, 0)$  such 349 350 that 351

• the elements  $\bar{a}_0, \ldots, \bar{a}_{2n}$  are pairwise distinct;

• we have  $a_k \neq \bar{a}_l$  if k > l.

Notice that the codimension of  $W_0$  is equal to  $(2n+1)d_y$ , as  $d_y > d_u$ , we have 353  $(2n+1)d_y \ge (2n+1)d_u + 2n + 1$ , so the codimension of  $W_0$  is greater than the dimension of  $X^{(2)} \times U^{2n+1}$ , the domain of  $\rho_{f,h}$ . 354 355

Recall that the evaluation map  $ev_{\rho}$  is defined as: 356

357 
$$\operatorname{ev}_{\rho}(f,h,x,\bar{x},\underline{u_{2n}}) = \rho_{f,h}(x,\bar{x},\underline{u_{2n}})$$
8

We shall see that  $ev_{\rho}$  is transversal to  $W_0$  at every given point

359 
$$\mathscr{X} \triangleq (f, h, x, \bar{x}, u_{2n}) \in \Gamma_U(X) \times C^r(X, \mathbf{R}^{d_y}) \times X^{(2)} \times U^{2n}$$

Consider a point  $\mathscr{X}$  such that  $\operatorname{ev}_{\rho}(\mathscr{X}) \in W_0$  and a vector  $(\mathfrak{X}, \mathfrak{X}, \mathfrak{Y})$  that is tangent to the codomain of  $\operatorname{ev}_{\rho}$ , with  $\mathfrak{X}_i \in \operatorname{T}_{x_i}X$ ,  $\overline{\mathfrak{X}}_i \in \operatorname{T}_{\overline{x}_i}X$ , and  $\mathfrak{Y}_i \in \mathbf{R}^{d_y}$   $(i = 0, \ldots, 2n)$ ; we have to prove that there exist  $\phi \in \Gamma_U(X)$ ,  $\eta \in C^{\infty}(X, \mathbf{R}^{d_y})$ ,  $\xi \in \operatorname{T}_x X$ ,  $\overline{\xi} \in \operatorname{T}_{\overline{x}} X$ ,  $\nu_i \in \operatorname{T}_{u_i} U$  (for  $i = 0, \ldots, 2n$ ) and a vector  $\zeta$  in the tangent space to  $W_0$  at  $\operatorname{ev}_{\rho}(\mathscr{X})$ such that

365 (5) 
$$(\mathfrak{X}, \mathfrak{X}, \mathfrak{Y}) = d(ev_{\rho})(\mathscr{X}) \cdot (\phi, \eta, \xi, \overline{\xi}, \nu) + \zeta$$

We shall prove this relation with  $\phi = 0, \xi = 0, \bar{\xi} = 0$  and  $\nu = 0$ . We denote by

$$\alpha_0,\ldots,\alpha_{2n},\bar{\alpha}_0,\ldots,\bar{\alpha}_{2n},0,\ldots,0$$

the components of  $\zeta$ . In the right-hand member of (5), the 2(2n+1) first components are equal to  $\alpha_0, \ldots, \alpha_{2n}, \bar{\alpha}_0, \ldots, \bar{\alpha}_{2n}$ , and can be chosen such that  $\alpha_i = \mathfrak{X}_i$  and  $\bar{\alpha}_i = \bar{\mathfrak{X}}_i$ (*i* = 0,..., 2*n*). The 2*n* + 1 last equations in (5) are

$$\eta_0 - \bar{\eta}_0 = \mathfrak{Y}_0, \qquad \dots, \qquad \eta_{2n} - \bar{\eta}_{2n} = \mathfrak{Y}_{2n}$$

where we let  $\eta_i \triangleq \eta(x_i)$  and  $\bar{\eta}_i \triangleq \eta(\bar{x}_i)$ . We consider this system as a linear system whose unknown are  $\eta_0, \ldots, \eta_{2n}, \bar{\eta}_0, \ldots, \bar{\eta}_{2n}$ . If  $\operatorname{ev}_{\rho}(\mathscr{X})$  belongs to the submanifold  $W_0$ , the points  $\bar{x}_0, \ldots, \bar{x}_{2n}$  are pairwise distinct, so the unknown  $\bar{\eta}_0, \ldots, \bar{\eta}_{2n}$  can be arbitrarily and independently chosen. There could be some equalities between the elements of the list  $L_{2n}$  and between an element of  $L_{2n}$  and an element of  $\bar{L}_{2n}$ . If an equality such that  $x_k = \bar{x}_l$  exists, then, as  $\operatorname{ev}_{\rho}(\mathscr{X}) \in W_0$ , we necessarily have k < land  $\eta_k = \bar{\eta}_l$ . The matrix of system (6) is then

380 
$$M = (A \mid -I_{(2n+1)d_n} + B)$$

Where  $I_{(2n+1)d_y}$  is the  $(2n+1)d_y$  dimensional identity matrix and matrix B is a block 381 matrix, whose blocks are 0 or  $d_u$  dimensional identity matrices. Moreover, matrix B 382 is upper triangular: if we have an equality like  $x_k = \bar{x}_l$ , then we find in B the block 383  $I_{d_{y}}$  (identity matrix) at position (k, l) with l > k. From the form of matrix B, we can 384 conclude that the rank of M is equal to  $(2n+1)d_y$  which implies that we can find 385  $\bar{\eta}_0, \ldots, \bar{\eta}_{2n}, \eta_0, \ldots, \eta_{2n}$  such that system (6) has a solution. Denote by  $x_{k_1}, \ldots, x_{k_p}$ 386 the list of the elements of  $L_{2n}$  which are not equal to an element of  $L_{2n}$ . It is possible 387 to find a solution of (6) such that  $\eta_{k_1} = \cdots = \eta_{k_p} = 0$ , for such a solution, it is then 388 possible to find a mapping  $\eta$  such that  $\eta(x_i) = \eta_i$  and  $\eta(\bar{x}_i) = \bar{\eta}_i$   $(i = 0, \dots, 2n)$ . 389

We have shown that  $ev_{\rho}$  is transversal to  $W_0$ . The conclusion of the proposition 390 now follows from the application of the Transversal density theorem [1]: the set  $\mathscr{O}_{0}^{r}$ 391 of pairs  $(f,h) \in \Gamma_U(X) \times C^{\infty}(X, \mathbf{R}^{d_y})$  such that  $\rho_{f,h}$  is transversal to  $W_0$  is open 392 and dense in the  $C^r$  topology (for every r > 0). Take now a pair (f, h) in this set 393  $\mathscr{O}_0^r$  and take two initial conditions  $x \neq \bar{x}$ , and a finite sequence of controls  $u_{2n}$  such 394 that the triplet  $(x, \bar{x}, \underline{u}_{2n})$  is in configuration  $\mathbf{C}_0$ , the mapping  $\rho_{f,h}$  is transversal to 395  $W_0$  at  $(x, \bar{x}, u_{2n})$  but as  $\operatorname{codim} W_0 > \dim(X^{(2)} \times U^{2n})$ , transversality means that 396  $\rho_{f,h}(x, \bar{x}, \underline{u_{2n}}) \notin W_0$ , which implies that at least one of the equalities  $y_j = \bar{y}_j$  is not 397 satisfied and so  $\Theta_T^{f,h}(x, u_{2n}) \neq \Theta_T^{f,h}(\bar{x}, u_{2n}).$ 398 

4. The case of the  $C_1-C_4$  configurations. In the proofs of the next propo-399 400sitions 9 and 10, we shall have to consider the derivative of  $\varphi_T(x)$  with respect to the vector field f (we are no more able to take  $\phi = 0$  as in the proof of Proposition 5). 401 This is why we ned to state a technical lemma, which bears on the computation of 402 these derivatives. To prove this lemma, we have to take into account the periodic 403 trajectories of a vector field; these trajectories have some generic properties—which 404 we intend to use in the proof of Lemma 8-, which are stated in the Kupka-Smale 405 theorem. Now the Kupka-Smale theorem has not been stated for parametrized vector 406 fields, so we will show that it can be generalized to those vector fields. 407

408 **4.1.** Periodic trajectories and the Kupka-Smale theorem. Take a parametrized vector field  $f \in \Gamma_U(X)$ ,  $u \in U$ ,  $x \in X$ , and assume that x belongs to a periodic 409 trajectory of the vector field  $f(\cdot, u)$ . Then there exists  $\pi_0 > 0$  such that  $\varphi_{\pi_0}^u(x) = x$ , 410 this implies that  $d\varphi_{\pi_0}^u(x) \cdot f(x,u) = f(x,u)$ , so 1 is an eigenvalue of  $A \triangleq d\varphi_{\pi_0}^u(x)$ . In 411 the sequel we shall have to consider expressions like  $\mathrm{Id} + A + \cdots + A^k$  and we shall 412 need that this sum of linear mappings be invertible; this is certainly true if, apart 413 from 1, the other eigenvalues have modulus different from 1. The theorem of Kupka-414 Smale [15, 17] asserts that this is generically the case for a vector field. Let a > 0, 415 hereafter, we denote by  $\mathscr{G}_2(a)$  the subset of  $\Gamma(X)$  of those vector fields f such that 416• if x is a singular point of f (*i.e* f(x) = 0), then for every  $t \neq 0$ ,  $d\varphi_t(x)$ : 417

 $T_x X \to T_x X$  has no complex eigenvalue of modulus 1;

418

419 • if x belongs to a periodic trajectory of f with period  $0 < \pi_0 \leq a$ , then, 420 denoting by  $1, \lambda_2, \ldots, \lambda_n$  the eigenvalues of  $d\varphi_{\pi_0}(x)$ , we have  $|\lambda_i| \neq 1$  for 421  $i = 2, \ldots, n$ .

422 Hereafter, recall that the manifolds X and U are assumed to be compact. We 423 have

424 THEOREM 6 (Kupka-Smale). Let a > 0, the set  $\mathscr{G}_2(a)$  is residual; moreover for 425 the  $C^r$  topology  $(r < +\infty)$ ,  $\mathscr{G}_2(a)$  is open and dense.

426 This theorem can be generalized to parametrized vector fields, namely we have.

THEOREM 7. Let a > 0, the set  $\mathscr{G}_2^U(a)$  of parametrized vector fields such that  $f(\cdot, u) \in \mathscr{G}_2(a)$  for every  $u \in U$  is a residual; moreover  $\mathscr{G}_2^U(a)$  is open and dense for the  $C^r$  topology.

This theorem can be proved by adapting the steps of the proof of the Kupka-Smale's theorem which can be found in [1]. Owing to lack of space, we do not write here the proof of Theorem 7 but we give a sketch of this proof in Appendix A; moreover, the reader is referred to [19] where this result is proved in the case where the dimension of U is 1.

435 **4.2.** Technical lemma. The proofs of propositions 9 and 10 below will follow the same scheme as the proof of Prop. 5. Nevertheless, in the following propositions, 436in order to prove that the mapping  $ev_{\rho}$  is transversal to some submanifold W, we 437 shall have to consider the derivative of the flow with respect to a vector field. Thus, 438before going further, we recall a result which will be used in the proof of Lemma 8 439and propositions 9 and 10. Take two vector fields f and  $\phi$  defined on X and denote by 440  $\varphi_t$  and  $\varphi_t^{\lambda}$  ( $\lambda \in \mathbf{R}$ ) the flows related to f and  $f + \lambda \phi$  respectively. In [1, Perturbation 441 theorem], the following formula is proved: for every  $x \in X$ , we have 442

443 (7) 
$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\varphi_t^{\lambda}(x)\Big|_{\lambda=0} = \int_0^t \mathrm{d}\varphi_{\sigma} \circ \phi \circ \varphi_{t-\sigma}(x)\mathrm{d}\sigma.$$

444 Obviously, this formula can be extended to the case of parametrized vector fields. 445 Consider f and  $\phi$  in  $\Gamma_U(X)$  and denote by  $\varphi_t^{u,\lambda}$  the flow generated by the vector field 446  $f(\cdot, u) + \lambda \phi(\cdot, u)$  (with u fixed). Starting from an initial condition  $x_0$ , consider now 447 the sequence  $x_0^{\lambda}, x_1^{\lambda}, \ldots$  defined recursively as  $x_0^{\lambda} = x_0$  and  $x_{i+1}^{\lambda} = \varphi_T^{u_i,\lambda}(x_i^{\lambda})$ , then 448 applying formula (7), we deduce easily that

449 
$$\frac{\mathrm{d}}{\mathrm{d}\lambda} x_{i+1}^{\lambda} \bigg|_{\lambda=0} = J_i + \delta_i(J_{i-1}) + \dots + \delta_1(J_0)$$

450 where

451

$$J_k = \int_0^T \mathrm{d}\varphi_{\sigma}^{u_k}(\varphi_{T-\sigma}^{u_k}(x_k)) \cdot \phi(\varphi_{T-\sigma}^{u_k}(x_k), u_k) \mathrm{d}\sigma;$$

the integral  $J_k$  belongs to the tangent space of X at  $\varphi_T^{u_k}(x_k) = x_{k+1}$ . Moreover the  $\delta_k$ 's are the mappings defined as

454 
$$\delta_k = \mathrm{d}(\varphi_T^{u_i} \circ \cdots \circ \varphi_T^{u_k})(x_k) \,.$$

455 We state now a preliminary result which will be used in the proofs of propositions 9 456 and 10. We shall say that these lists are under  $C'_3$  (resp.  $C'_4$ ) configuration if

• they are under  $C_3$  (resp.  $C_4$ ) configuration and if

 $\begin{array}{ll} \text{ + 58} & \bullet \text{ there exists a subscript } k \in \{i, \dots, p-1\} \cup \{m, \dots, j-1\} \text{ (resp. } k \in \{i, \dots, p-1\} \text{ (resp. } k \in \{i, \dots, p-1\} \text{ (sec)} \\ 1\} \cup \{j, \dots, m-1\} \text{ ) such that } u_k \neq u_{p-1} \text{ .} \end{array}$ 

460 The proof of the following lemma is postponed in Appendix B.

461 LEMMA 8. Let  $\mathfrak{X}_p \in T_{x_p}X$  be an arbitrary tangent vector to X at  $x_p$ . Assume 462 that the lists  $L_{2n}$  and  $\overline{L}_{2n}$  are under  $\mathbf{C}_1$  configuration, then one can find a vector field 463  $\phi \in \Gamma_U(X)$  such that we have

464 
$$\frac{\mathrm{d}x_p^{\lambda}}{\mathrm{d}\lambda}\bigg|_{\lambda=0} = \mathfrak{X}_p, \qquad \qquad \frac{\mathrm{d}x_i^{\lambda}}{\mathrm{d}\lambda}\bigg|_{\lambda=0} = 0.$$

466 If these lists are under  $C_2$ ,  $C'_3$  or  $C'_4$  configuration, then one can find a vector field 467  $\phi \in \Gamma_U(X)$  such that we have

$$468 \qquad \frac{\mathrm{d}x_p^{\lambda}}{\mathrm{d}\lambda}\bigg|_{\lambda=0} = \mathfrak{X}_p, \qquad \frac{\mathrm{d}x_i^{\lambda}}{\mathrm{d}\lambda}\bigg|_{\lambda=0} = 0, \qquad \frac{\mathrm{d}\bar{x}_m^{\lambda}}{\mathrm{d}\lambda}\bigg|_{\lambda=0} = 0, \qquad \frac{\mathrm{d}\bar{x}_j^{\lambda}}{\mathrm{d}\lambda}\bigg|_{\lambda=0} = 0.$$

### 470 **4.3.** The triplet $(x, \overline{x}, \underline{u_{2n}})$ is under one of the configurations $C_1, C_2$ or 471 $C_3$ .

472 PROPOSITION 9. Assume that  $d_y > d_u$  (the number of observations is greater 473 than the number of controls). For k = 1, 2, 3, denote by  $\mathscr{C}_k^r$  the subset of pairs  $(f, h) \in$ 474  $\Gamma_U(X) \times C^{\infty}(X, \mathbb{R}^p)$  such that  $\Theta_T^{f,h}(x, \underline{u_{2n+1}}) \neq \Theta_T^{f,h}(\bar{x}, \underline{u_{2n+1}})$  whenever the triplet 475  $(x, \bar{x}, \underline{u_{2n}})$  (with  $x \neq \bar{x}$ ) is in configuration  $\mathbf{C}_k$ . Then each subset  $\mathscr{C}_k^r$  contains a 476 residual for the  $C^r$  topology.

Hereafter, we write the proof of this proposition only in the case of  $C_1$  configuration, the case proofs for the other configurations being very similar.

479 **4.3.1.** Proof of the proposition in the case of  $C_1$  configuration. If the 480 triplet  $(x, \bar{x}, \underline{u_{2n}})$  is in the  $C_1$  configuration, there exist subscripts i, j, p, m such that 481  $x_p = x_i$  and  $\overline{x_m} = x_j$ . 482 We choose four subscripts  $0 \le i, j, p, m \le 2n$  such that i < p; letting q =483  $\max(i, j, p, m)$ , we consider the representation  $\rho$ 

484 
$$\rho: \mathscr{G}_2^U(a) \times C^r(X, \mathbf{R}^{d_y}) \longrightarrow C^r(X^{(2)} \times U^q, X^{q+1} \times X^{q+1} \times \mathbf{R}^{q d_y})$$
$$(f, h) \longmapsto \rho_{f, h}$$

485 defined through the mapping  $\rho_{f,h}$ 

486 
$$\rho_{f,h}: \begin{array}{ccc} X^{(2)} \times U^{q} & \longrightarrow & X^{q+1} \times X^{q+1} \times \mathbf{R}^{q \, d_{y}} \\ (x, \bar{x}, \underline{u_{q}}) & \longmapsto & (x_{0}, \dots, x_{q}, \bar{x}_{0}, \dots, \bar{x}_{q}, y_{0} - \bar{y}_{0}, \dots, y_{q-1} - \bar{y}_{q-1}) \,. \end{array}$$

487 We fix two lists of subscripts (possibly empty)  $i_1 < \cdots < i_r < q$  and  $j_1 < \cdots < j_r < q$ 488 such that the signs of  $i_1 - j_1, \ldots, i_r - j_r$  are the same ; if j, m < q, one has  $i_k = j$ 489 and  $j_k = m$  for some subscript k. We consider also the submanifold

490 
$$V_m^{j,i,p} \subset X^{q+1} \times X^{q+1} \times \mathbf{R}^{(2n+1)d_y}$$

491 defined as follows:  $V_m^{j,i,p}$  is the set of those elements

492 
$$(a_0, \ldots, a_q, \bar{a}_0, \ldots, \bar{a}_q, 0, \ldots, 0)$$

493 such that

• we have the equalities  $\bar{a}_m = a_j$  and  $a_i = a_p$ ;

- the elements of  $\{\bar{a}_0, \ldots, \bar{a}_{q-1}\}$  are pairwise different;
- 496  $a_{i'} \neq \bar{a}_{j'}$  if  $(i', j') \neq (i_k, j_k)$   $(i', j' < q, k = 1, \dots, r)$ .

Notice that the number of submanifold having the above properties is finite, moreover the codimension of  $V_m^{j,i,p}$  is equal to  $2n+q d_y$  and is greater or equal to  $2n+q(d_u+1) > 2n+q d_u$ ; so the codimension of  $V_m^{j,i,p}$  is greater than the dimension of the domain of  $\rho_{f,h}$ .

We shall show that  $\operatorname{ev}_{\rho}$  is transversal to  $V_m^{j,i,p}$ . Let  $\mathscr{X} \triangleq (f,h,x,\bar{x},\underline{u}_q)$  be a point such that  $\operatorname{ev}_{\rho}(\mathscr{X}) \in V_m^{j,i,p}$  and take a vector  $(\mathfrak{X}, \overline{\mathfrak{X}}, \mathfrak{Y})$  which is tangent to the codomain of  $\operatorname{ev}_{\rho}$  at  $\operatorname{ev}_{\rho}(\mathscr{X})$ , with  $\mathfrak{X}_k \in T_{x_k}X$ ,  $\overline{\mathfrak{X}}_k \in T_{\bar{x}_k}X$   $(k = 0, \ldots, q)$  and  $\mathfrak{Y}_k \in \mathbf{R}^{d_y}$  $(k = 0, \ldots, q - 1)$ . We have to prove that there exist  $\phi \in \Gamma_U(X)$ ,  $\eta \in C^r(X, \mathbf{R}^{d_y})$ ,  $\xi \in T_x X$ ,  $\overline{\xi} \in T_{\bar{x}}X$ ,  $\nu_l \in T_{u_l}U$  (for  $l = 0, \ldots, m - 1$ ) and a vector  $\zeta$  in the tangent space to  $V_m^{j,i,p}$  at  $\operatorname{ev}_{\rho}(\mathscr{X})$  such that

507 (8) 
$$(\mathfrak{X}, \overline{\mathfrak{X}}, \mathfrak{Y}) = d(ev_{\rho})(\mathscr{X}) \cdot (\phi, \eta, \xi, \overline{\xi}, \nu) + \zeta.$$

508 We shall prove this relation with  $\xi = 0$  and  $\nu = 0$ . We denote by

509 
$$\alpha_0, \ldots, \alpha_q, \bar{\alpha}_0, \ldots, \bar{\alpha}_q, \beta_0, \ldots, 0, \ldots, 0$$

510 the components of  $\zeta$ .

Among the 2q + 2 first equations in (8), the ones corresponding to subscripts different from j, i and p (first q + 1 equations) or m (last q + 1 equations) are trivial because the corresponding tangent vectors  $\alpha_k$  ( $k \neq j, i, p$ ) and  $\bar{\alpha}_k$  ( $k \neq m$ ) can be arbitrarily chosen. Thus we focus on the following four equations

$$\underline{\xi}_{1}\underline{\xi} \quad (9) \qquad \alpha_j + A_j = \mathfrak{X}_j, \qquad \alpha_i + A_i = \mathfrak{X}_i, \qquad \alpha_p + A_p = \mathfrak{X}_p, \qquad \bar{\alpha}_m + \bar{A}_m + \bar{\xi}_m = \mathfrak{X}_m.$$

517 Here  $A_l$  (resp.  $\bar{A}_l$ ), l = 0, ..., q, denotes the derivative of  $x_l^{\lambda}$  (resp.  $\bar{x}_l$ ) with respect 518 to  $\lambda$  evaluated at  $\lambda = 0$  while  $\bar{\xi}_i$  is defined recursively as

<sup>510</sup><sub>520</sub> (10) 
$$\bar{\xi}_0 = \bar{\xi}, \qquad \bar{\xi}_{k+1} = \mathrm{d}\varphi_T^{u_k}(\bar{x}_k) \cdot \bar{\xi}_k, \qquad k = 0, \dots, 2n-1.$$

Notice also that, as  $\zeta$  is a tangent vector to  $W_r^{j,i,p}$ , necessarily,  $\alpha_i = \alpha_p$  and  $\alpha_j = \bar{\alpha}_m$ . As we are under  $\mathbf{C_1}$  configuration, we can apply the preliminary lemma 8, thus we can find a vector field  $\phi$  such that  $A_i = 0$  while  $A_p$  is equal to an arbitrary tangent vector in  $\mathbf{T}_{x_p}X$  (notice that we cannot guarantee that  $A_j = 0$  because we do not know the position of j with respect to i and p). Taking into account that  $\alpha_p = \alpha_i$ and  $\bar{\alpha}_m = \alpha_j$ , the equations (9) then rewrite

$$\underbrace{525}_{525} \quad (11) \qquad \alpha_j + A_j = \mathfrak{X}_j, \qquad \alpha_i = \mathfrak{X}_i, \qquad \alpha_i + A_p = \mathfrak{X}_p, \qquad \alpha_j + \bar{A}_m + \bar{\xi}_m = \mathfrak{X}_m,$$

529 a solution to system (11) is then

$$\sum_{j=1}^{530} \alpha_i = \mathfrak{X}_i, \qquad A_p = \mathfrak{X}_p - \mathfrak{X}_i, \qquad \alpha_j = \mathfrak{X}_j - A_j, \qquad \bar{\xi}_m = \mathfrak{X}_m - \bar{A}_m - \mathfrak{X}_j + A_j$$

Notice that, once  $A_p$  has been chosen,  $A_j$  and  $A_m$  are fixed (they depend on  $\mathfrak{X}_p - \mathfrak{X}_i$ ), moreover  $\overline{\xi}$  can be chosen in such a way that  $\overline{\xi}_m$  is equal to an arbitrary tangent vector.

535 As for the last 2q equations, they can be written

$$\begin{array}{cccc} \underline{536} \\ \underline{537} \end{array} (12) \qquad \eta_0 - \bar{\eta}_0 - \bar{\chi}_0 = \mathfrak{Y}_0, \qquad \dots, \qquad \eta_{q-1} - \bar{\eta}_{q-1} - \bar{\chi}_{q-1} = \mathfrak{Y}_{q-1} \end{array}$$

where  $\bar{\chi}_k = dh(x_k) \cdot \bar{\xi}_k$ ,  $\eta_k = \eta(x_k)$ , and  $\bar{\eta}_k = \eta(\bar{x}_k)$   $(k = 0, \dots, q-1)$ . We regard this system as a linear system. As  $\bar{x}_m = x_j$  is the only equality between the  $x_k$ 's and the  $\bar{x}_k$ 's, we can consider that the unknowns for this system are  $\eta_0, \dots, \eta_{q-1}$  and  $\bar{\eta}_0, \dots, \bar{\eta}_{q-1}$ ; the matrix of this system then writes

542 
$$(A \mid -\mathbf{I}_{q\,d_u} + B)$$

here  $I_{q d_y}$  denotes the  $q d_y$  dimensional identity matrix while B is a  $d_y \times d_y$  block matrix which is a strictly upper or strictly lower triangular matrix. Thus  $I_{q d_y} - B$  is an upper or lower-triangular block matrix, the elements of the diagonal being equal to the  $d_y \times d_y$  identity matrix, hence  $-I_{q d_y} + B$  is non singular, which proves that system (12) admits a solution.

This proves that  $ev_{\rho}$  is transversal to  $V_m^{j,i,p}$ ; we achieve the proof of Proposition 9 as the one of Proposition 5.

550 **4.4. The triplet**  $(x, \bar{x}, \underline{u_{2n}})$  is under one configuration  $C_4$ . We examine 551 now the case of  $C_4$  configuration, in this case, the assumption  $d_y \ge d_u + 1$  is no more 552 sufficient.

PROPOSITION 10. Assume that  $d_u = 0$  and  $d_y \ge 1$ , or  $d_u > 0$  and  $d_y \ge d_u + 2$ (the number of observations is greater than the number of controls plus one). Denote by  $\mathscr{C}_{4}^{r}$  the set of pairs  $(f,h) \in \Gamma_{U}(X) \times C^{\infty}(X, \mathbb{R}^{p})$  such that  $\Theta_{T}^{f,h}(x, \underline{u_{2n+1}}) \neq$  $\Theta_{T}^{f,h}(\bar{x}, \underline{u_{2n+1}})$  whenever the triplet  $(x, \bar{x}, \underline{u_{2n}})$  (with  $x \neq \bar{x}$ ) is in configuration  $\mathbb{C}_{4}$ . Then  $\mathscr{C}_{4}^{r}$  contains a residual for the  $C^{r}$  topology.

558 Proof of Proposition 10. In the sequel, we shall say that the lists  $L_{2n}$  and  $\bar{L}_{2n}$ 559 are under  $\mathbf{C}''_{\mathbf{4}}$  configuration if they are under  $\mathbf{C}_{\mathbf{4}}$  configuration but not under  $\mathbf{C}'_{\mathbf{4}}$ 560 configuration. We shall consider these two subcases separately.

561 Configuration  $\mathbf{C}'_{\mathbf{4}}$ . In this case there exist subscripts  $0 \leq i and <math>0 \leq j < m \leq 2n$  such that  $x_i = x_p$  and  $\bar{x}_j = \bar{x}_m$ ; moreover there exists a subscripts 563  $k \in \{i, \ldots, p-1\} \cup \{j, \ldots, m-1\}$  such that  $u_k \neq u_{p-1}$ . Without loss of generality, 564 we can assume that  $m \geq p$ . 565 We choose four subscripts  $0 \le i and <math>0 \le j < m \le 2n$   $(m \ge p)$ , as well 566 as a subscript  $k_0 \in \{i, \ldots, p-1\} \cup \{j, \ldots, m-1\}$ . We consider the representation  $\rho$ 567 defined on  $\mathscr{G}_2^U(a) \times C^r(X, \mathbf{R}^{d_y})$  through the mapping  $\rho_{f,h}$  as

$$\rho_{f,h}: X^{(2)} \times U^m_{(k_0,p-1)} \longrightarrow X^{m+1} \times X^{m+1} \times \mathbf{R}^{m\,d_y}$$

$$(x, \bar{x}, \underline{u_m}) \longmapsto (x_0, \dots, x_m, \bar{x}_0, \dots, \bar{x}_m, y_0 - \bar{y}_0, \dots, y_{m-1} - \bar{y}_{m-1}).$$

569 Together with  $\rho$ , we consider the submanifold

570 
$$Z_{i,m}^{i,p} \subset X^{m+1} \times X^{m+1} \times \mathbf{R}^{m \, d_y}$$

571 defined as the set of those elements

$$(a_0,\ldots,a_m,\bar{a}_0,\ldots,\bar{a}_m,0,\ldots,0)$$

573 such that

572

• we have the equalities  $a_i = a_p$  and  $\bar{a}_j = \bar{a}_m$ ;

• the above equalities are the only ones between the elements of  $L_p \cup \overline{L}_m$ .

Notice first that the codimension of  $Z_{j,m}^{i,p}$  is equal to  $2n + m d_y$  and is greater than 2 $n + m d_u$  which is greater than the dimension of the domain of  $\rho_{f,h}$ .

578 We shall show that  $ev_{\rho}$  is transversal to  $Z_{j,m}^{i,p}$ . Let  $\mathscr{X} \triangleq (f,h,x,\bar{x},\underline{u_m})$  be a 579 point such that  $ev_{\rho}(\mathscr{X}) \in Z_{j,m}^{i,p}$  and take a tangent vector  $(\mathfrak{X},\bar{\mathfrak{X}},\mathfrak{Y})$  with  $\mathfrak{X}_k \in T_{x_k}X$ , 580  $\bar{\mathfrak{X}}_k \in T_{\bar{x}_k}X$  (k = 0, ..., m), and  $\mathfrak{Y}_k \in \mathbf{R}^{d_y}$  (k = 0, ..., m - 1). We have to prove 581 that there exist  $\phi \in \Gamma_U(X)$ ,  $\eta \in C^r(X, \mathbf{R}^{d_y})$ ,  $\xi \in T_x X$ ,  $\bar{\xi} \in T_{\bar{x}}X$ ,  $\nu_k \in T_{u_k}U$  (for 582 j = 0, ..., m - 1) and a vector  $\zeta$  in the tangent space to  $Z_{j,m}^{i,p}$  at  $ev_{\rho}(\mathscr{X})$  such that

583 (13) 
$$(\mathfrak{X}, \overline{\mathfrak{X}}, \mathfrak{Y}) = d(ev_{\rho})(f, h, x, \overline{x}, u_m) \cdot (\phi, \eta, \xi, \overline{\xi}, \nu) + \zeta.$$

584 We shall prove this relation with  $\xi = 0$ ,  $\overline{\xi} = 0$  and  $\nu = 0$ . We denote by

585 
$$\alpha_0, \ldots, \alpha_m, \bar{\alpha}_0, \ldots, \bar{\alpha}_m, 0, \ldots, 0$$

the components of  $\zeta$ ; notice that, as  $\zeta$  is a tangent vector to  $Z_{j,m}^{i,p}$ , we have  $\alpha_i = \alpha_p$ and  $\bar{\alpha}_j = \bar{\alpha}_m$ .

To prove that Equation (13) admits a solution, the reasoning is analogous to the one made in the proofs of the previous propositions but here, we have to apply twice Lemma 8. Hereafter, given a vector field  $\phi$  we denote by  $x_i^{\lambda,\phi}$ , the sequence defined recursively as follows

$$\frac{593}{593} \qquad \qquad x_0^{\lambda,\phi} = x_0, \qquad \qquad x_{i+1}^{\lambda,\phi} = \varphi_T^{u_i,\lambda}(x_i^{\lambda,\phi})$$

where  $\varphi_t^{u_i,\lambda}$  denotes the flow related to the vector field  $f(\cdot, u_i) + \lambda \phi$ . According to 594Lemma 8, there exist a vector field  $\phi_0$  such that the derivatives of  $x_i^{\lambda,\phi_0}$ ,  $\bar{x}_i^{\lambda,\phi_0}$  and 595  $\bar{x}_m^{\lambda,\phi_0}$  with respect to  $\lambda$  are all zero while the derivative of  $x_p^{\lambda,\phi_0}$  can be arbitrarily 596 chosen. As there exists  $k_0$  such that  $u_{k_0} \neq u_{p-1}$ , we can also apply this lemma by 597 replacing  $u_{p-1}$  by  $u_{k_0}$  in the lemma. For example, if  $j \leq k_0 < m$ , and, assuming 598without loss of generality, that  $k_0$  is the greatest subscript less than m such that 599 $u_{k_0} \neq u_{p-1}$ , we deduce from Lemma 8 that there exists a vector field  $\phi_1$  such that the 600 derivatives of  $x_i^{\lambda,\phi_1}$ ,  $x_p^{\lambda,\phi_1}$  and  $\bar{x}_j^{\lambda,\phi_1}$  with respect to  $\lambda$  are all zero while the derivative 601

of  $\bar{x}_{k_0+1}^{\lambda,\phi_1}$  can be arbitrarily chosen. Noticing that 602

603 
$$\bar{x}_{m}^{\lambda,\phi_{1}} = \begin{cases} \bar{x}_{k_{0}+1}^{\lambda,\phi_{1}} & \text{if } k_{0} = m-1, \\ \varphi_{(m-k_{0})T}^{u_{p-1}}(\bar{x}_{k_{0}}^{\lambda,\phi_{1}}) & \text{if } k_{0} < m-1; \end{cases}$$

we see that the derivative of  $\bar{x}_m^{\lambda,\phi_1}$  with respect to  $\lambda$  can be arbitrarily chosen. 604

We chose now the vector field  $\phi \in \Gamma_U(X)$  as follows 605

•  $\phi(\cdot, u_{p-1}) = \phi_0$  and  $\phi(\cdot, u_{k_0}) = \phi_1$ ; 606

607

•  $\phi(\cdot, u_k) \equiv 0$  if  $u_k \notin \{u_{p-1}, u_{k_0}\}$ . Clearly the derivatives of  $x_i^{\lambda, \phi}$  and  $\bar{x}_j^{\lambda, \phi}$  with respect to  $\lambda$  are zero while the derivatives 608 of  $x_n^{\lambda,\phi}$  and  $\bar{x}_m^{\lambda,\phi}$  can be arbitrarily chosen. 609

As in the previous configurations, as regard the first 2m + 2 equations in (13), we 610 have to consider only the four following ones 611

$$\begin{array}{ll} \underline{\widehat{\alpha}}_{13} & (14) & \alpha_i + A_i = \mathfrak{X}_i, & \alpha_p + A_p = \mathfrak{X}_p, & \bar{\alpha}_j + \bar{A}_j = \bar{\mathfrak{X}}_j, & \bar{\alpha}_m + \bar{A}_m = \bar{\mathfrak{X}}_m \end{array}$$

where the  $A_k$ 's (resp. the  $\bar{A}_k$ 's) denote the derivatives of the  $x_k^{\lambda,\phi}$  (resp. of the  $\bar{x}_k^{\lambda,\phi}$ ) 614 with respect to  $\lambda$ ; so we have  $A_i = 0$  and  $\bar{A}_j = 0$ . Notice also that, from the definition 615of  $Z_{j,m}^{i,p}$ , it follows that  $\alpha_i = \alpha_p$  and  $\bar{\alpha}_j = \bar{\alpha}_m$ . Taking into account these equalities, 616the solution to the equations (14) is then 617

$$\begin{array}{ll} \underline{\widehat{\alpha}}_{1} \\ \underline{\widehat{\alpha}}_{i} = \mathfrak{X}_{i}, \\ \alpha_{i} = \mathfrak{X}_{i}, \\ A_{p} = \mathfrak{X}_{p} - \mathfrak{X}_{j}, \\ \bar{\alpha}_{j} = \bar{\mathfrak{X}}_{j}, \\ \bar{\alpha}_{j} = \bar{\mathfrak{X}}_{j}, \\ \overline{A}_{m} = \bar{\mathfrak{X}}_{m} - \bar{\mathfrak{X}}_{j}. \end{array}$$

As regard the last m equalities in (13), the proof is the same as the one of 620 Proposition 9: the two lists  $L_{m-1}$  and  $\bar{L}_{m-1}$  are disjoint and the elements of  $\bar{L}_{m-1}$ 621 are pairwise distinct, so one can find a function  $\eta \in C^{\infty}(X, \mathbf{R}^{d_y})$  such that  $\eta(x_k) = 0$ 622 for  $k = 0, \ldots, m-1$  while the values of  $\eta$  at  $\bar{x}_k$   $(k = 0, \ldots, m-1)$  can be chosen 623 arbitrarily. 624

625 Configuration  $C''_4$ . In this case, we have  $u_i = \ldots u_{p-1} = u_j = \ldots u_{m-1}$ ; the equalities  $x_i = x_p$  and  $\bar{x}_j = \bar{x}_m$  then imply that the trajectories of the vector field 626 $f(\cdot, u_{p-1})$  are periodic. 627

We choose some subscripts  $0 \le i and <math>0 \le j < m \le 2n$ , without loss 628 of generality, we assume that  $m \ge p$ . We consider first t representation  $\rho$  defined on 629  $\mathscr{G}_2^U(a) \times C^r(X, \mathbf{R})$  through the mapping  $\rho_{f,h}$  defined as 630

$$\rho_{f,h}: \quad X^{(2)} \times U^{m}_{(i,p,j,m)} \times \mathbf{R}^{*}_{+} \longrightarrow X^{m+1} \times X^{m+1} \times \mathbf{R}^{m \, d_{y}}$$

$$(x, \bar{x}, \underline{u_{m}}, t) \longmapsto (x_{0}, \dots, x_{p-1}, \varphi^{u_{p_{1}}}_{t}(x_{i}), x_{p+1}, \dots, x_{m}, x_{p-1}, \overline{x_{0}}, \dots, \overline{x_{m}}, y_{0} - \overline{y_{0}}, \dots, y_{m-1} - \overline{y_{m-1}}).$$

where  $U_{(i,p,j,m)}^m$  is the submanifold of  $U^m$  defined as the set of those  $\underline{u_m}$  such that  $u_i = \cdots = u_{p-1} = u_j = \cdots = u_{m-1}$ . 632 633

Together with  $\rho$ , we consider the submanifold  $Z_{j,m}^{i,p}$  defined as in the previous 634 case. We shall prove that  $ev_{\rho}$  is transversal to  $Z_{j,m}^{i,p}$ . Let  $\mathscr{X} \triangleq (f, h, x, \bar{x}, \underline{u_m}, t)$  be a 635 point such that  $ev_{\rho}(\mathscr{X}) \in Z_{j,m}^{i,p}$  and take a tangent vector  $(\mathfrak{X}, \overline{\mathfrak{X}}, \mathfrak{Y})$  with  $\overline{\mathfrak{X}_k} \in T_{x_k}X$ , 636  $k \in \{0, \ldots, m\} \setminus \{p\}, \ \mathfrak{X}_p \in \mathrm{T}_{x_i}X, \ \bar{\mathfrak{X}}_k, \in \mathrm{T}_{\bar{x}_k}X, \ k = 0, \ldots, m \text{ and } \mathfrak{Y}_k \in \mathrm{\mathbf{R}}^{d_y}, \ k = 0, \ldots, m - 1.$  Notice that  $\mathrm{ev}_p(\mathscr{X}) \in Z_{j,m}^{i,p}$  implies that  $x_i$  and  $\bar{x}_j$  belong to periodic 637 638trajectories of the vector field  $f(\cdot, u_{p-1})$ . 639

We have to prove that there exist  $\phi \in \Gamma_U(X)$ ,  $\eta \in C^r(X, \mathbf{R}^{d_y})$ ,  $\xi \in \mathbf{T}_x X$ ,  $\bar{\xi} \in \mathbf{T}_{\bar{x}} X$ ,  $\nu_k \in \mathbf{T}_{u_k} U$ ,  $\tau \in \mathbf{R}$ , and a vector  $\zeta$  in the tangent space to  $W_{j,m}^{i,p}$  at  $\mathrm{ev}_{\rho}(\mathscr{X})$ 640641

such that 642

665

643 (15) 
$$(\mathfrak{X}, \overline{\mathfrak{X}}, \mathfrak{Y}) = d(ev_{\rho})(\mathscr{X}) \cdot (\phi, \eta, \xi, \overline{\xi}, \nu, \tau) + \zeta.$$

We shall prove this relation with  $\nu = 0$  and  $\bar{\xi} = 0$ . We denote by 644

$$\alpha_0,\ldots,\alpha_m,\bar{\alpha}_0,\ldots,\bar{\alpha}_m,0,\ldots,0$$

the components of  $\zeta$ ; notice that, as  $\zeta$  is a tangent vector to  $Z_{j,m}^{i,p}$ , we have  $\alpha_i = \alpha_p$ 646 and  $\bar{\alpha}_j = \bar{\alpha}_m$ . 647

As in the previous cases, in order to prove that the first 2m + 2 equations in (15) 648 can be satisfied, it is sufficient to focus our attention to the four following equations 649

650 
$$\alpha_i + \xi_i + A_i = \mathfrak{X}_i, \qquad \alpha_p + \xi_p + A_p + \tau f(x_i, u_{p-1}) = \mathfrak{X}_p,$$
  
654 
$$\bar{\alpha}_i + \bar{A}_i = \bar{\mathfrak{X}}_i, \qquad \bar{\alpha}_m + \bar{A}_m = \bar{\mathfrak{X}}_m,$$

$$\bar{\mathfrak{g}}_{52} \qquad \bar{\alpha}_j + A_j = \mathfrak{X}_j, \qquad \bar{\alpha}_m + A_m = \mathfrak{X}_j$$

the notations are the same as in the previous cases except for  $A_p$  and  $\xi_p$ : 653

654 
$$A_p = \left. \frac{\mathrm{d}\varphi_t^{u_{p-1},\lambda}(x_i)}{\mathrm{d}\lambda} \right|_{\lambda=0} , \qquad \xi_p = \mathrm{d}\varphi_t^{u_{p-1}}(x_i) \cdot \xi_i$$

We can apply Lemma 8: there exists  $\phi$  such that  $\overline{A}_j = 0$  while  $A_m$  can be arbitrarily 656 chosen. Here we cannot ensure that we also have  $A_i = A_p = 0$  because  $x_i$  could 657 belong to the periodic trajectory of  $f(\cdot, u_{p-1})$  passing through  $\bar{x}_j$ , so we first choose  $\bar{\alpha}_j = \bar{\mathfrak{X}}_j$  and  $\bar{A}_m = \bar{\mathfrak{X}}_m - \bar{\mathfrak{X}}_j$ . Now as  $\varphi_t^{u_{p-1}}(x_i) = x_i$ , we have  $t = q \pi_0$  where  $\pi_0$ 658 659denotes the prime period of the periodic trajectory of  $f(\cdot, u_{p-1})$  passing through  $x_i$ ; 660 thus  $d\varphi_t^{u_{p-1}}(x_i) = (d\varphi_{\pi_0}^{u_{p-1}}(x_i))^q$ . As before, due to the fact that f belongs to  $\mathscr{G}_2^U(a)$ , 661 the linear mapping 662

663 
$$(\xi_i, \tau) \mapsto \left( (\mathrm{d}\varphi^v_{\pi_0}(x_i))^q - \mathrm{Id} \right) \cdot \xi_i + \tau f(x_i, v)$$

is onto. Thus we can find  $\xi_i$  and  $\tau$  such that 664

$$\left( (\mathrm{d}\varphi_{\pi_0}^v(x_i))^q - \mathrm{Id} \right) \cdot \xi_i + \tau f(x_i, v) = \mathfrak{X}_p - \mathfrak{X}_i + A_p - \mathcal{X}_i$$

we take also  $\alpha_i = \mathfrak{X}_i - A_i$ , with these choices of  $\alpha_i$ ,  $\xi_i$  and  $\tau$ , we see that the two first 666 equations are also satisfied. 667

 $A_i$ .

As regards the last m equations in (15), we argue as in the previous case. 668

At this point, the application of the Transversal Density Theorem, shows that  $\mathscr{R}_4$ , 669 the set of pairs (f,h) in  $\mathscr{G}_2^U(a) \times C^r(X, \mathbf{R}^{d_y})$  such that  $\rho_{f,h}$  is transversal to the finite 670 set of submanifolds  $Z_{j,m}^{i,p}$ , is a residual. Now, we have to compute the codimension of 671  $Z_{j,m}^{i,p}$ , it is equal to  $2n+m d_y$  and is greater or equal to  $2n+m d_u+m$ ; this codimension 672 is greater than the dimension of the domain of  $\rho_{f,h}$  if  $m \geq 2$  or  $d_y \geq d_u + 2$ . In this 673 case, to be transversal to  $Z_{j,m}^{i,p}$  means non membership and we can conclude the proof 674of Proposition 10 as for the previous propositions. If m = 1,  $d_u = 0$  and  $d_y \ge 1$ ,  $ev_{\rho}$  is 675still transversal to submanifold  $Z_{j,m}^{i,p}$  but  $\operatorname{codim}(Z_{0,1}^{0,1})$  is then equal to the dimension of the domain of  $\rho_{f,h}$ , so we need an additional argument to conclude in this case. 676 677 Hereafter, we shall see that, in this particular case, although the codimension of  $Z_{0,1}^{0,1}$  is 678 equal to the dimension of the domain of  $\rho_{f,h}$ , transversality implies non membership. 679

Case where m = 1,  $d_u = 0$  and  $d_y \ge 1$ . Take  $(f, h) \in \mathscr{R}_4^r$ , then  $\rho_{f,h}$  is transversal 680 to  $W_{0,1}^{0,1}$ . Assume that  $x_0 \neq \bar{x}_0$  are two points of X such that 681

$$\begin{array}{ll}
\text{682} & \bullet \text{ there exists some } t > 0 \text{ such that } \varphi_t(x_0) = x_0, \ \varphi_T^u(\bar{x}_0) = \bar{x}_0; \\
\text{683} & \bullet h(x_0) = h(\bar{x}_0); \\
\end{array}$$

then there exist  $\xi_0 \in T_{x_0}X$ ,  $\bar{\xi}_0 \in T_{\bar{x}_0}X$ ,  $\tau \in \mathbf{R}$ ,  $\alpha_0$  and  $\bar{\alpha}_0$  in the tangent spaces to 684 685 X at  $x_0$  and  $\bar{x}_0$  such that the following equations are satisfied

686 (16) 
$$\begin{cases} \alpha_0 + \xi_0 = \mathfrak{X}_0, & \alpha_0 + \mathrm{d}\varphi_t(x_0) \cdot \xi_0 + \tau f(x_0, u) = \mathfrak{X}_1, \\ \bar{\alpha}_0 + \bar{\xi}_0 = \bar{\mathfrak{X}}_0, & \bar{\alpha}_0 + \mathrm{d}\varphi_T(\bar{x}_0) \cdot \bar{\xi}_0 = \bar{\mathfrak{X}}_1 \\ \mathrm{d}h(x_0) \cdot \xi_0 - \mathrm{d}h(\bar{x}_0) \cdot \bar{\xi}_0 = \mathfrak{Y} \end{cases}$$

whatever  $\mathfrak{X}_0, \mathfrak{X}_1, \overline{\mathfrak{X}}_0, \overline{\mathfrak{X}}_1$  and  $\mathfrak{Y}$  tangent vectors to the appropriate spaces. Clearly 687 the four first equations in this system are equivalent to the two following ones 688

689 
$$(\mathrm{d}\varphi_t(x_0) - \mathrm{Id}) \cdot \xi_0 + \tau f(x_0, u) = \mathfrak{X}_1 - \mathfrak{X}_0$$

$$(\mathrm{d}\varphi_T(\bar{x}_0) - \mathrm{Id}) \cdot \xi_0 = \mathfrak{X}_1 - \mathfrak{X}_1$$

As 1 is an eigenvalue of the linear mapping  $d\varphi_T(\bar{x}_0)$ , clearly the second equation of 692 this system cannot be satisfied. This implies that if  $\varphi_T(x_0) = x_0$  and  $\varphi_T(\bar{x}_0) = \bar{x}_0$ , 693 the point  $\rho_{f,h}(x_0, \bar{x}_0)$  cannot belong to  $W_{0,1}^{0,1}$ , which means that we must have  $h(x_0) \neq 0$ 694  $h(\bar{x}_0)$ . This achieve the proof of Proposition 10. 695

5. The case  $d_u \ge 1$  and  $d_y = d_u + 1$ . Counterexample. We shall exhibit 696 here a simple counterexample which shows that if  $d_y = 2$  and  $d_u = 1$ , then the 697 conclusion of our main result (Theorem 4) is no more true. That it to say, we exhibit 698 a pair  $(f_0, h_0)$  such that for every (f, h) in some neighborhood of  $(f_0, h_0)$ , the related 699 mapping  $\Theta_T^{f,h}$  is not injective. We recall hereafter, the isotopy theorem [1] which will 700 be used to prove some optimality of our main result. 701

THEOREM 11 (Transversal isotopy theorem). Let  $\mathscr{A}$ , Z and Y be  $C^{r+1}$  manifolds 702  $(r \geq 1), \ \rho : \mathscr{A} \to C^{r+1}(Z,Y) \ a \ C^{r+1}$  representation,  $W \subset Y$  a submanifold and 703  $a_0 \in \mathscr{A}$  a point. For  $a \in \mathscr{A}$  let  $W_a = \rho_a^{-1}(W)$ . Assume that 704

1. W is closed in Y; 705

- 2. Z is compact and  $C^{r+3}$ ; 706
- 3.  $\rho_{a_0}$  is transversal to W. 707

Then there is an open neighborhood N of  $a_0$  in  $\mathscr{A}$  such that, for  $a \in N$ , there is a  $C^r$ 708 diffeomorphism  $F_a: Z \to Z$  such that  $F_a(W_{a_0}) = W_a$  and  $F_a$  is  $C^r$  isotopic to the 709 710 *identity*.

We shall apply the transversal isotopy theorem 11 to the following situation: we 711 712 take

•  $\mathscr{A} = \Gamma_U(X) \times C^{\infty}(X, \mathbf{R}^{d_y})$ , where X and U are compact manifolds, dim U =713  $\begin{aligned} &d_u > 0 \text{ with } d_y = d_u + 1 \text{ ;} \\ \bullet \ & Z = X^2 \times U \times S^1 \text{ ;} \\ \bullet \ & Y = X^3 \times \mathbf{R}^{d_y} \text{ .} \end{aligned}$ 714

- 715
- 716

We consider also a representation  $\rho$  which is slightly different from the one which has 717718been used in the proof of Proposition 10; we define this representation through  $\rho_{f,h}$ 719as

720

$$\rho_{f,h}: \begin{array}{ccc} X^2 \times U \times S^1 & \longrightarrow & X^3 \times \mathbf{R}^{d_y} \\ (x, \bar{x}, u, s) & \longmapsto & (x, \varphi^u_T(x), \varphi^u_{\gamma(s)}(\bar{x}), h(x) - h(\bar{x})) \end{array}$$

$$17$$

721 here  $\gamma \in C^{\infty}(S^1, \mathbf{R})$ .

The submanifold  $W \subset X^3 \times \mathbf{R}^{d_y}$  is then defined as

723 
$$W = \{ (z_1, z_2, z_3, 0) \in X^3 \times \mathbf{R}^{d_y} \mid z_1 = z_2 = z_3 \}.$$

Consider a pair  $(f_0, h_0)$  such that  $\rho_{f_0, h_0}$  is transversal to W and such that  $W_{f_0, h_0}$ 724is nonempty. Applying the Transversal isotopy theorem 11, we deduce that there 725 exists a neighborhood N of  $(f_0, h_0)$  such that if  $(f, h) \in N$ ,  $W_{f,h} = F(W_{f_0,h_0})$  with F 726 a diffeomorphism from Z to Z. Thus  $W_{f,h}$  is nonempty; notice that we can assume 727 that  $\rho_{f,h}$  is transversal to W for every pair  $(f,h) \in N$ , this is a direct consequence of 728 729 the Openness of Transversal Intersection Theorem [1] applied to this situation with K = Z. Let  $(x, \bar{x}, u, s)$  be an element of  $W_{f,h}$ , we shall show that  $x \neq \bar{x}$ : arguing by 730 contradiction, we shall see that if we have the equality  $x = \bar{x}$ , then  $\rho_{f,h}$  cannot be 731 transversal to W at  $(x, \bar{x}, u, s)$ . We introduce some notations 732

733 
$$A = \mathrm{d}\varphi^u_T(x), \qquad \bar{A} = \mathrm{d}\varphi^u_{\gamma(s)}(\bar{x}),$$

734 
$$B = \left. \frac{\partial \varphi_T^v(x)}{\partial v} \right|_{v=u}, \qquad \qquad \bar{B} = \left. \frac{\partial \varphi_{\gamma(s)}^v(\bar{x})}{\partial v} \right|_{v=u}$$

$$\overline{C} = \mathrm{d}h(x), \qquad \qquad \overline{C} = \mathrm{d}h(\overline{x}).$$

737 Consider now the following "matrix" M

738 
$$M = \begin{pmatrix} A - \mathrm{Id} & 0 & B & 0 \\ - \mathrm{Id} & \bar{A} & \bar{B} & f(x, u) \\ C & -\bar{C} & 0 & 0 \end{pmatrix}.$$

Arguing as in the proof of Proposition 10, we see that the transversality of  $\rho_{f,h}$  to W at  $(x, \bar{x}, u, s)$  is equivalent to the invertibility of the square matrix M. Since we assume that  $x = \bar{x}$ , we have  $C = \bar{C}$ , which implies that the determinant of M is equal to the one of the following matrix M':

743 
$$M' = \begin{pmatrix} A - \mathrm{Id} & 0 & B & 0\\ \bar{A} - \mathrm{Id} & \bar{A} & \bar{B} & f(x, u)\\ 0 & -\bar{C} & 0 & 0 \end{pmatrix}.$$

Now as f(x, u) belongs to the kernels of A – Id and  $\bar{A}$  – Id (because  $\varphi_{\gamma(t)}^{u}(\bar{x}) = x$ ), the vector  $(f(x, u), 0, 0, 0)^{\mathrm{T}}$  belongs to the kernel of M' which implies that the determinant of M' is zero : we have reached a contradiction. As a consequence, if  $(f, h) \in N$ , there exists  $x \neq \bar{x}$ , u and s such that  $h(x) = h(\bar{x})$ ; moreover, the trajectory of  $f(\cdot, u)$  passing through x is periodic and  $\bar{x}$  belongs to this trajectory, so we have  $\varphi_{kT}^{u}(x) = x$  and  $\varphi_{kT}^{u}(\bar{x}) = \bar{x}$ .

We give now an explicit example of a pair  $(f_0, h_0)$  such that  $\rho_{f_0, h_0}$  is transversal to W. In what follows, for the sake of simplicity and without loss of generality, we assume that  $T = 2\pi$ . The manifold X will be equal to the circle  $S^1$  and the set of controls U will also be equal to the circle  $S^1$ . We denote by  $u_1$  and  $u_2$  (resp.  $s_1$  and  $s_2$ ) the components of u (resp. of s) and we consider the following vector field:

755 
$$f_0(x,u) = R_1 \cdot x + u_1 R_2 \cdot x$$
18

with  $R_1$  and  $R_2$  the following skew-symmetric matrices

757  
758 
$$R_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad R_2 = \begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix},$$

759 with  $\beta \in (0, 1)$ .

761

760 Mapping  $h_0$  is defined by:

$$\begin{array}{rccc} h_0: & X & \longrightarrow & \mathbf{R}^2 \\ & x & \longmapsto & (x_1, x_1 x_2) \end{array}$$

762 while function  $\gamma$  is defined by:

$$\gamma(s) = \left(-\frac{1}{2} + \frac{1}{4}s_1\right)2\pi$$

The set  $W_{f_0,h_0}$ . Let  $x = (x_1, x_2)$  and  $\bar{x} = (\bar{x}_1, \bar{x}_2)$  be two points in  $S^1$ , and  $u = (u_1, u_2) \in S^1$  a control and  $s \in S^1$ ; assume that  $\rho_{f_0,h_0}(x, \bar{x}, u, s) \in W$ . Letting  $z = x_1 + ix_2 \in \mathbf{C}$ , the equality  $\varphi_T^u(x) = x$  is equivalent to  $e^{i(1+u_1\beta)2\pi}z = z$  which is equivalent to  $1 + u_1\beta = k$  with  $k \in \mathbf{Z}$ . Now, as  $\beta \in (0, 1)$ , we have  $|u_1\beta| < 1$  and so the equality  $u_1\beta = k - 1 \in \mathbf{Z}$  is possible if and only if  $u_1 = 0$ . From the definition of h, we can easily see that the equality  $h(x) = h(\bar{x})$  amounts to  $x = \bar{x}$  or  $x_1 = \bar{x}_1 = 0$ . If  $x = \bar{x}$ , as  $u_1 = 0$ , the equality  $\varphi_{\gamma(s)}^u(\bar{x}) = x$  is possible only if  $\gamma(s) \in \mathbf{Z}$ , but from the definition of  $\gamma$ , we have  $\gamma(s) \in [-3/4, -1/4]$ . Thus  $x \neq \bar{x}$  and  $x_1 = \bar{x}_1 = 0$ , so, taking into account that  $u_1 = 0$ , the equality  $\varphi_{\gamma(s)}^u(\bar{x}) = x$  is equivalent to the following ones

$$-\sin(\gamma(s))\bar{x}_2 = 0,$$
  $\cos(\gamma(s))\bar{x}_2 = x_2 = -\bar{x}_2.$ 

These two equalities are true iff  $\gamma(s) = (2k+1)\pi$  with  $k \in \mathbb{Z}$ , which is equivalent to  $s_1 = 4k + 4$ , as  $|s_1| \leq 1$ , this is possible only if  $s_1 = 0$ .

In conclusion,  $W_{f_0,h_0}$  is the set consisting of the following eight elements

778 
$$W_{f_0,h_0} = \{ ((0,\varepsilon_0), (0,-\varepsilon_0), (0,\varepsilon_1), (0,\varepsilon_2)) \}$$

779 where  $\varepsilon_0, \varepsilon_1, \varepsilon_2 \in \{-1, 1\}$ .

780 The transversality of  $\rho_{f_0,h_0}$ . Take  $(x, \bar{x}, u, s) \in X \times X \times U \times S^1$  an element such 781 that  $\rho_{f_0,h_0}(x, \bar{x}, u, s)$  belongs to W. Thus we have

$$\begin{array}{l} 782\\ 783\\ 783 \end{array} (17) \qquad x = (0, \varepsilon_0), \qquad \bar{x} = (0, -\varepsilon_0), \qquad u = (0, \varepsilon_1), \qquad s = (0, \varepsilon_2) \end{array}$$

with  $\varepsilon_i \in \{-1, 1\}$  (i = 0, 1, 2). Let  $(\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3, \mathfrak{Y})$  be a tangent vector to  $X^3 \times \mathbb{R}^2$  at  $\rho_{f_0,h_0}(x, \bar{x}, u, s)$ . Thus,  $\mathfrak{X}_1, \mathfrak{X}_2$  and  $\mathfrak{X}_3$  are tangent to  $S^1$  at  $(0, \varepsilon_0)$  while  $\mathfrak{Y}$  is a vector in  $\mathbb{R}^2$ . We write a tangent vector to submanifold W as  $\zeta = (\alpha, \alpha, \alpha, 0)$  where  $\alpha$  is a tangent vector to  $S^1$  at  $(0, \varepsilon_0)$ . We have to prove that there exists  $\xi$  (resp.  $\bar{\xi}$ ) in the tangent space to  $S^1$  at  $(0, \varepsilon_0)$  (resp.  $(0, -\varepsilon_0)$ ) as well as  $\nu$  a vector in the tangent space to  $S^1$  at  $(0, \varepsilon_1)$  and  $\sigma$  a vector tangent to  $S^1$  at s such that the following equalities are satisfied

791 (18) 
$$\mathfrak{X}_1 = \xi + \alpha,$$

792 (19) 
$$\mathfrak{X}_2 = \mathrm{d}\varphi_T^u(x) \cdot \xi + \frac{\partial \varphi_T^u(x)}{\partial u} \cdot \nu + \alpha \,,$$

793 (20) 
$$\mathfrak{X}_{3} = \mathrm{d}\varphi_{\gamma(s)}^{u}(\bar{x}) \cdot \bar{\xi} + \frac{\partial \varphi_{\gamma(s)}^{u}(\bar{x})}{\partial u} \cdot \nu + \frac{\partial \varphi_{\gamma(s)}}{\partial s} \cdot \sigma + \alpha$$

<sup>794</sup>/<sub>795</sub> (21) 
$$\mathfrak{Y} = \mathrm{d}h(x) \cdot \xi - \mathrm{d}h(\bar{x}) \cdot \bar{\xi}.$$

We examine first the fourth equation (21), taking into account that  $\xi = (\xi_1, 0)^T$ and  $\bar{\xi} = (\bar{\xi}_1, 0)^T$ , it writes

798 
$$\begin{pmatrix} 1 & 0 \\ \varepsilon_0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ -\varepsilon_0 & 0 \end{pmatrix} \begin{pmatrix} \bar{\xi}_1 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathfrak{Y}_1 \\ \mathfrak{Y}_2 \end{pmatrix}.$$

799 Clearly, the solution of this equation is

$$\xi_1 = \frac{\varepsilon_0 \mathfrak{Y}_2 + \mathfrak{Y}_1}{2}, \qquad \bar{\xi}_1 = \frac{\varepsilon_0 \mathfrak{Y}_2 - \mathfrak{Y}_1}{2}.$$

Taking into account that  $\mathfrak{X}_i = (\mathfrak{X}_i^1, 0)^{\mathrm{T}}$  and  $\alpha = (\alpha_1, 0)^{\mathrm{T}}$ , the solution of the first equation (18) is then given by (22) and by the equality

804 (23) 
$$\alpha_1 = \mathfrak{X}_1^1 - \frac{\varepsilon_0 \mathfrak{Y}_2 + \mathfrak{Y}_1}{2}.$$

805 Concerning the second equation (19), we notice that  $d\varphi_T^u(x) = \text{Id}$  and that

806 
$$\frac{\partial \varphi_T^u(x)}{\partial u} = \begin{pmatrix} -2\pi \,\beta \,\varepsilon_0 & 0\\ 0 & 0 \end{pmatrix}$$

so the solution of equation (19) is given by (22), (23) and by

808 (24) 
$$\nu_1 = -\frac{\mathfrak{X}_2^1 - \mathfrak{X}_1^1}{2\pi\,\beta\,\varepsilon_0}\,.$$

So far, the values of  $\xi$ ,  $\overline{\xi}$  and  $\nu$  are fixed, therefore showing that equation (20) comes down showing that the third term in the right-hand member of (20) can take arbitrary value. This third term writes

812 
$$\frac{\partial \varphi_{\gamma(s)}}{\partial s} \cdot \sigma = \frac{\pi}{2} \sigma_1 f(\varphi_{\gamma(s)}^u(x)) = \frac{\pi}{2} \sigma_1 \begin{pmatrix} \varepsilon_0 \\ 0 \end{pmatrix},$$

obviously, thanks to a suitable choice of  $\sigma_1$ , this expression can be made equal to an arbitrary tangent vector  $\tilde{\mathfrak{X}}_3$  of  $S^1$  at x.

815 This achieves the proof of the transversality of  $\rho_{f_0,h_0}$  to W.

Appendix A. Proof of Theorem 7. The proof of the Kupka-Smale theorem can obviously be found in the original papers [15, 16] and [17] but the reader can also find a very detailed proof of this result in [1]. For the proof of the generalization of this theorem, we shall follow closely the arguments given in this book.

#### A.1. The different sets in the theorem of Kupka-Smale.

**G1** Set  $\mathscr{G}_1$  is the set of vector fields whose critical points are elementary. A point x is **critical** for the vector field f if f(x) = 0, it is elementary if the differential

823 
$$d\varphi_t(x): T_x X \longrightarrow T_x X$$

has no complex eigenvalue of modulus 1 for every  $t \neq 0$ .

- **G** $\Delta$  Let *a* be a positive number, set  $\mathscr{G}\Delta(a)$  is the set of vector fields *f* such that if  $\mathcal{O}$ is a closed orbit of *f* with period  $0 < \tau \leq a$ , then this period is **transversal**, that is to say, the eigenvalue 1 of the differential
- 828  $d\varphi_{\tau}(x): T_x X \longrightarrow T_x X$
- has an algebraic multiplicity equal to 1 (notice that, in this case, f(x) is an eigenvector of  $d\varphi_{\tau}(x)$ ).

- 831 **G3/2** Set  $\mathscr{G}_{3/2}(a)$  is defined as  $\mathscr{G}_{3/2}(a) \triangleq \mathscr{G}_1 \cap \mathscr{G}\Delta(a)$ .
- **G2** Set  $\mathscr{G}_2(a)$  is the set of vector fields included in  $\mathscr{G}_1$  and whose periods  $0 < \tau \leq a$ of closed orbits are elementary. A period is **elementary** if the modulus of eigenvalues of  $d\varphi_{\tau}(x)$  that are distinct from 1, are different from 1.
- A consequence of the Kupka-Smale theorem is that all of these sets are open and dense for the  $C^r$  topology in  $\Gamma(X)$ .

We recall that  $\Gamma_U(X)$  denotes the set of parametrized vector fields over X; we define the sets  $\mathscr{G}_1^U, \mathscr{G}_\Delta^U, \ldots$  as the sets of vector fields belonging to one of the above categories for every  $u \in U$ . For instance we define  $\mathscr{G}_1^U$  as the set of parametrized vector fields f such that  $f(\cdot, u) \in \mathscr{G}_1$  for every  $u \in U$ . Notice that every vector field  $f \in \Gamma(X)$ can be regarded as a parametrized vector field, so we can write  $\Gamma(X) \subset \Gamma_U(X)$ ,  $\mathscr{G}_1 \subset \mathscr{G}_1^U$ , etc.

The tangent bundle  $T(X \times U)$  is diffeomorphic to the cartesian product  $TX \times TU$ , so a vector field f defined on  $X \times U$  can be regarded as a pair  $(f_1, f_2)$  of two vector fields defined respectively on X and U. We have then the following result.

846 LEMMA 12. The mapping  $\Pi$  defined as

847  $\Pi: \ \Gamma(X \times U) \longrightarrow \ \Gamma_U(X)$  $(f_1, f_2) \longmapsto \ f_1$ 

848 is open for the  $C^r$  topology (for any r > 0).

Proof. The set  $C^{\infty}(X \times U, TX \times TU)$  is diffeomorphic to  $C^{\infty}(X \times U, TX) \times C^{\infty}(X \times U, TU)$  (see [12, Prop. 3.6]); we deduce easily that  $\Gamma(X \times U)$  is diffeomorphic to  $\Gamma_U(X) \times \Gamma_X(U)$ . On the other hand the first projection map from  $\Gamma_U(X) \times \Gamma_X(U)$ to  $\Gamma_U(X)$  is an open and continuous mapping, it follows that  $\Pi$  is continuous and open as a composite mapping of an open and continuous mapping with a diffeomorphism.

As a consequence of this lemma, the direct image of an open and dense subset of  $\Gamma(X \times U)$  is an open and dense subset of  $\Gamma_U(X)$ .

We recall also that a mapping  $f : E \to F$  between topological spaces is called quasi-open if, for every open subset  $\mathcal{O} \subset E$ ,  $f(\mathcal{O})$  has a non empty interior. We shall need also, the following lemma.

EEMMA 13. The mapping  $\Phi$  defined as

860

$$\begin{array}{cccc} & & & & \\ &$$

is continuous. Here  $f_u$  denotes the vector field defined on X by  $f_u(x) = f(x, u)$ .

A.2. The set  $\mathscr{G}_1^U$  is open and dense in  $\Gamma_U(X)$ . The proof of the Kupka-862 Smale theorem 6 as made in [1] needs seven steps. Beforehand, one proves that  $\mathscr{G}_1^U$  is 863 open and dense for the  $C^r$  topology. The proof follows closely the one provided in [1, 864 page 98ff, in this book, the authors introduce a submanifold W of the 1-jet bundles 865  $J^1(X, TX)$  as follows. Consider a chart  $(P, \varphi)$  of X, to this chart we relate, as usual, 866 the chart  $(Q, \psi)$  of TX, where  $Q = \pi^{-1}(P)$  (here  $\pi : TX \to X$  denotes the canonical 867 projection) and  $\psi(v) = (\pi(v), \bar{v})$  with  $\bar{v}$  the local expression of  $v \in TX$  in the chart 868  $(P,\varphi)$ . Define then the chart  $\tau_{P,Q}$  as 869

870  
$$\tau_{P,Q}: \quad J^1(P,Q) \quad \longrightarrow \quad P' \times Q' \times \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n) \\ \sigma \qquad \longmapsto \quad (x, \psi \circ f \circ \varphi^{-1}(x), A)$$

đ

where  $P' = \varphi(P)$ ,  $Q' = \psi(Q)$  and x is the source of the 1-jet  $\sigma$ . The submanifold W is defined as the set of 1-jets such that  $\psi \circ f \circ \varphi^{-1}(x) = 0$  and A has at least

one eigenvalue with real part zero (clearly, this definition does not depend on the 873 particular choice of charts). To W, one relates the subset M of  $\mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$  of linear 874 maps having an eigenvalue with real part zero. In [1], the proof of the density of 875  $\mathscr{G}^1(X)$  relies on the fact that M is closed and is a finite union of submanifolds of 876 codimension  $\geq 1$ , which implies that codim  $W \geq n+1$ . In our case, we consider the 877 same 1-jet bundle but we replace X by  $X \times U$ , so we can see A as a linear map from 878  $\mathbf{R}^{n+d_u}$  to  $\mathbf{R}^{n+d_u}$ , we can identify A with its matrix written in the canonical basis of  $R^{n+d_u}$  and we write A as a block matrix  $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$  with  $A_{11} \in \mathbf{R}^{n \times n}$  and 879 880  $A_{22} \in \mathbf{R}^{d_y} \times \mathbf{R}^{d_y}$ . We modify the definition of W as follows: W is the set of 1-jets 881 such that  $A_{11}$  has at least one eigenvalue with real part zero. In a similar way as 882 above, we can prove that W has a codimension  $\geq n + d_u + 1$ . Consider now the set 883  $\mathscr{G}'_1(X \times U)$  of vector fields f defined on  $X \times U$  such that the 1-jet of f does not belong 884 to W; arguing as in [1], the fact that  $\operatorname{codim} W \ge n + d_u + 1$ , proves that the set is open 885 and dense. As  $\Pi$  is an open mapping (cf Lemma 12), and as  $\Pi(\mathscr{G}'_1(X \times U)) = \mathscr{G}^U_1$ , 886 we have proved that  $\mathscr{G}_1^U$  is also open and dense. 887

#### A.3. The seven steps of the proof. 888

Step 1: for every a > 0, the set  $\mathscr{G}_2^U(a)$  is open . First notice that  $\mathscr{G}_2^U(a) \neq \varnothing$ 889 because  $\mathscr{G}_2(a) \subset \mathscr{G}_2^U(a)$ ; moreover, we shall use the fact that  $\mathscr{G}_2(a)$  is open (proven 890 in [1]). 891

Let  $f \in \mathscr{G}_2^U(a)$  and  $u \in U$  be fixed, denote by  $f_u$  the vector fields in  $\Gamma(X)$  defined 892 by  $f_u(x) = f(x, u)$ ;  $f_u$  belongs to  $\mathscr{G}_2(a)$ , hence there exists  $\mathscr{N}_u$  a neighborhood of  $f_u$  in 893  $\Gamma(X)$  with  $\mathcal{N}_u \subset \mathscr{G}_2(a)$ . As the mapping  $\Phi$  is continuous (*cf* Lemma 13), there exists 894  $\mathscr{N}_{u}^{U}$  a neighborhood of f in  $\Gamma_{U}(X)$  and  $\mathscr{V}_{u}$  neighborhood of u such that  $\Phi(g, v) \in \mathscr{N}_{u}$ 895 for every  $(g, v) \in \mathscr{N}_u^U \times \mathscr{V}_u$ . The neighborhoods  $\mathscr{V}_u$  cover U, as U is compact, there exists a finite subcover:  $U \subset \bigcup_{i=1}^N \mathscr{V}_{u_i}$ , let  $\mathscr{N}^U = \bigcap_{i=1}^N \mathscr{N}_{u_i}^U$ ,  $\mathscr{N}^U$  is a neighborhood of f in  $\Gamma_U(X)$ . Let  $(g, v) \in \mathscr{N}^U \times U$ , there exists  $1 \leq i \leq N$  such that  $v \in \mathscr{V}_{u_i}$ , on the other hand  $g \in \mathscr{N}^U \subset \mathscr{N}_{u_i}^U$ , therefore  $\Phi(g, v) = g_v$  belongs to  $\mathscr{G}_2(a)$ . In conclusion, every element g of  $\mathscr{N}^U$  is such that  $g_v \in \mathscr{G}_2(a)$  for every v in U, 896 897 898 899

900 therefore  $f \in \mathscr{N}^U \subset \mathscr{G}_2^U(a)$ , we have showed that  $\mathscr{G}_2^U(a)$  is open. Step 2: for every a > 0, the set  $\mathscr{G}_{3/2}^U$  is open. The proof is very similar to the 901

902 previous one. 903

Step 3: if  $f \in \mathscr{G}_1^U$ , there exists a neighborhood  $\mathscr{N}$  of f with  $\mathscr{N} \subset \mathscr{G}_1^U$  and there 904exists  $a_0 > 0$  such that  $\mathscr{G}_2^U(a_0) \cap \mathscr{N} = \mathscr{N}$ . Let  $f \in \mathscr{G}_1^U$ , for every  $u \in U$ ,  $f_u \in \mathscr{G}_1$ , 905 so there exists a neighborhood  $\mathcal{N}_u \subset \mathcal{G}_1$  of  $f_u$  in  $\mathcal{G}_2(a)$   $(f_u \in \mathcal{N}_u \subset \mathcal{G}_1)$  and there 906 exists  $a_u > 0$  such that  $\mathcal{N}_u \subset \mathcal{G}_2(a_u)$ . As in step 1, as  $\Phi$  is continuous, we construct a family of neighborhoods  $\mathcal{N}_u^U$  of f in  $\Gamma_U(X)$  and a family  $\mathcal{V}_u$  of neighborhoods of u such that for every  $(g, v) \in \mathcal{N}_u^U \times \mathcal{V}_u$ , the vector field  $g_v$  belongs to  $\mathcal{N}_u \subset \mathcal{G}_1$  and  $\mathcal{N}_u \subset \mathcal{G}_2(a_u)$ . Notice that, since  $\mathcal{G}_1^U$  is open, we can assume that  $\mathcal{N}_u^U \subset \mathcal{G}_1^U$ . 907 908 909 910

Due to the compactness of U, we have  $U \subset \bigcup_{i=1}^{N} \mathscr{V}_{u_i}$ ; let 911

912  
913 
$$\mathcal{N} = \bigcap_{i=1}^{N} \mathcal{N}_{u_i}^U, \qquad a_0 = \min_{1 \le i \le N} a_{u_i}.$$

Let  $(g, v) \in \mathcal{N}^U \times U$ , there exists  $1 \leq i \leq N$  such that  $v \in \mathscr{V}_{u_i}$  and we have  $g \in \mathcal{N} \subset \mathcal{V}$ 914  $\mathcal{N}_{u_i}^U$ , therefore  $\Phi(g, v) = g_v \in \mathcal{N}_{u_i}$  and  $\mathcal{N}_{u_i} \subset \mathscr{G}_2(a_{u_i})$ , but, clearly,  $\mathscr{G}_2(a_{u_i}) \subset \mathscr{G}_2(a_0)$ 915 since  $a_0 \leq a_{u_i}$ . Thus, we have showed that 916

917 
$$f \in \mathcal{N} \subset \mathcal{G}_1^U \cap \mathcal{G}_2^U(a_0).$$

which achieves the proof. 918

22

919 Step 4: if  $\mathcal{N} \subset \mathscr{G}_1^U$  is open and  $\mathscr{G}_2^U(a) \cap \mathcal{N}$  is dense in  $\mathcal{N}$  then  $\mathscr{G}_{3/2}^U(\frac{3}{2}a) \cap \mathcal{N}$  is 920 dense in  $\mathcal{N}$ .. This step is based on lemmas 31.7 and 31.8 in [1, p. 102]. Lemma 30.7 921 asserts that if x is a point located on a closed orbit of a vector field  $f \in \Gamma(X)$ , given 922 a tangent vector  $v \in T_x X$ , there exists a vector field  $g \in \Gamma(X)$  such that, denoting 923 by  $\tau$  the prime period of the closed trajectory and by  $\varphi_t^{\lambda}$  the flow related to  $f + \lambda g$ , 924 we have

925  $\frac{\mathrm{d}\varphi_{\tau}^{\lambda}(x)}{\mathrm{d}\lambda}\Big|_{\lambda=0} = v \,.$ 

As pointed out in section 4.2, Lemma 31.7 is still true for parametrized vector fields. Moreover, Lemma 31.8 can be proved in exactly the same way for parametrized vector fields, the representation  $\rho$  being modified as follows (we define it through the evaluation map):

930 (25) 
$$\operatorname{ev}_{\rho}: \Gamma_{U}(X) \times U \times X \times \mathbf{R}^{*}_{+} \longrightarrow X \times \mathbf{R}^{*}_{+} \times X \\ (f, u, x, t) \longmapsto (x, t, \varphi^{u}_{t}(x))$$

Finally, we can prove that  $ev_{\rho}$  is transverse to  $\Delta$  on  $\mathscr{G}_{2}^{U}(a) \times X \times U \times [0, \frac{3}{2} + \varepsilon)$  in the same way as for the proof written in [1] because  $X \times U$  is compact; we can then conclude exactly as in [1].

conclude exactly as in [1]. Step 5: if  $\mathcal{N} \subset \mathscr{G}_1^U$  is open and if  $\mathscr{G}_{3/2}^U(a) \cap \mathcal{N}$  is dense in  $\mathcal{N}$  then  $\mathscr{G}_2^U(a) \cap \mathcal{N}$ 934 is dense in  $\mathcal{N}$ . In [1], the proof of this step is needs four lemmas. We shall briefly 935 indicate how they can be extended to the case of parametrized vector fields. In the 936 sequel, we shall identify the tangent bundle  $T(X \times U)$  with the product  $TX \times TU$ . 937 Take a parametrized vector field,  $f \in \Gamma_U(X)$  and consider the related vector field 938  $\tilde{f}$  defined on  $X \times U$  as  $\tilde{f}_1(x, u) = f(x, u)$  and  $\tilde{f}_2(x, u) = 0$ . Assume that  $x \in X$ 939 belongs to a periodic trajectory, denoted by  $\gamma$ , of  $f(\cdot, u)$ , then (x, u) belongs to the 940 periodic trajectory  $\tilde{\gamma} \triangleq \gamma \times \{u\}$  of  $\tilde{f}$ . We can apply the tangent perturbation lemma 941 (Lemma 32.4 in [1]), to  $\tilde{f}$  and  $\tilde{\gamma}$ : we then obtain a vector field  $\tilde{g}$  defined on  $X \times U$ 942 that is zero on  $\tilde{\gamma}$  and zero outside some neighborhood of  $\tilde{\gamma}$ . Moreover if we denote by 943  $\tilde{\varphi}^{\lambda}$  the flow of  $\tilde{f} + \lambda \tilde{g}$ , we have  $\left. \frac{\mathrm{d}}{\mathrm{d}\lambda} \{ d\varphi_{\tau}(x, u) \} \right|_{\lambda=0} = \tilde{A}$  where  $\tilde{A}$  is endomorphism of 944  $T_x X \times T_u U$  that vanishes at  $(f(x, u), 0)^T$ . In particular, if we choose  $\tilde{A}$  such that its 945 second component is zero, then looking at the proof of Lemma 32.4, we see that,  $\tilde{g}_2$ , 946the second component of  $\tilde{g}$ , is zero. This lemma is therefore still valid for parametrized 947 948 vector fields.

Consider now a parametrized vector field  $f \in \mathscr{G}_{3/2}^U(a)$  and assume that  $x_0$  be a 949 point on a periodic trajectory  $\gamma_0$  of  $f(\cdot, u_0)$ , if u is closed to  $u_0$ , then the vector field 950  $f(\cdot, u)$  is closed to  $f(\cdot, u_0)$ . Invoking Lemma 24.4 in [1], we know that there exists 951 952 some neighborhood of  $x_0$  such that the vector field  $f(\cdot, u)$  admits a unique periodic trajectory in this neighborhood if u is closed enough to  $u_0$ . We can be more specific 953 by using the implicit function theorem. Take a submanifold  $Y \subset X$  of codimension 954 1, passing through  $x_0$  and which intersects  $\gamma_0$  transversally at  $x_0$ . We consider the 955 Poincaré map  $\mathcal{P}$ , related to the vector field  $f(\cdot, u_0)$ , defined in a neighborhood of  $x_0$ 956 957 in Y. This map is also defined for u closed enough to  $u_0$ , that is to say, there exists neighborhoods  $N_1 \subset Y$  of  $x_0$  in Y and  $N_2$  of  $u_0$  such that if  $(x, u) \in N_1 \times N_2$ , there 958 exists a first time  $\tau_u > 0$  such that  $\varphi_{\tau_u}^u(x, u) \in N_1$ . Now,  $\mathcal{P}(x_0, u_0) = x_0$  and we can 959 invoke the implicit function theorem to prove the existence of  $x_u$  and  $\tau_u$  defined for 960 u in some neighborhood of  $u_0$  included in  $N_2$  and such that  $\mathcal{P}(x_u, u) = x_u$  (as well 961 as  $\varphi_{\tau_u}^u(x_u) = x_u$ ). To see why we can use this theorem, we use the assumption that 962 $f \in \mathscr{G}_{3/2}^{U}(a)$ . We know that 1 is an eigenvalue of  $d\varphi_{\tau_{u_0}}^{u_0}(x_0)$  with multiplicity one and 963

that the other eigenvalues of  $d\varphi_{\tau_{u_0}}^{u_0}(x_0)$  are different from 1. The vector space  $T_{x_0}X$ 964 can then be written as  $T_{x_0}X = \mathbf{R}f(x_0, u_0) \oplus E$  where E is the n-1 dimensional 965

subspace of  $T_{x_0}X$  generated by the eigenvectors of  $d\varphi^{u_0}_{\tau_{u_0}}(x_0)$  that are different from 966

1. If we choose the submanifold Y such as its tangent space at  $x_0$  is equal to E, we 967

968 then see that the eigenvalues of the differential of  $\mathcal{P}$  with respect to the first variable

x are the n-1 eigenvalues (counted with multiplicities) of  $d\varphi_{\tau_{u_0}}^{u_0}(x_0)$  that are different 969

from 1. In conclusion, there exists  $x_u \in X$ ,  $\tau_u > 0$  defined in a neighborhood of  $u_0$ , 970 which depend smoothly on u and which satisfy  $\varphi_{\tau_u}^u(x_u, u) = x_u$ . 971

Now the M-structure lemma (Lemma 31.10 in [1]) can be generalized as follows. 972

LEMMA 14. Let E be a finite dimensional Banach space. Consider  $u \mapsto L_u$  a 973 smooth mapping defined from a neighborhood N of  $u_0 \in U$  to  $\mathcal{L}(E, E)$ , and  $u \mapsto v_u$ , 974 a smooth mapping defined from N to  $E \setminus \{0\}$ . Assume that  $L_u$  is transversal for  $v_u$ 975 (i.e.  $L_u v_u = v_u$  and the algebraic multiplicity of the eigenvalue 1 is equal to 1). Then 976 for every  $u \in N$ , there exists  $A_u \in \mathcal{L}(E, E)$  such that 977

978 1. 
$$A_u v_u = 0$$

2. for every mapping  $\mathscr{L}: I \times U \to \mathcal{L}(E, E)$  (I denotes the interval (0, b), b > 0) 979 980 satisfying the three conditions

(a)  $\mathscr{L}(0,u) = L_u;$ 981

- 982
- 983

(b)  $\left. \frac{\partial}{\partial s} \mathscr{L}(s, u) \right|_{s=0} = A_u;$ (c) for every  $s \in I$ ,  $\mathscr{L}(s, u)$  is transversal for  $v_u$ there exists  $\varepsilon > 0$  and  $N' \subset N$  (with N' open neighborhood of  $u_0$ ) such that  $\mathscr{L}(s, u)$ 984 is elementary for  $v_u$  for every  $(s, u) \in (0, \varepsilon) \times N'$  (i.e.  $\mathscr{L}(s, u)$  is transversal for  $v_u$ 985 and has no complex eigenvalue of modulus 1, except 1). Moreover  $u \mapsto A_u$  is smooth. 986

We sketch a proof of this lemma. 987

988 *Proof.* We consider the set, denoted by W, defined as follows

989 
$$W = \{ (B, u) \in L(E, E) \times N \mid B v_u = v_u \}$$

It is not difficult to see that W is a submanifold of  $\mathcal{L}(E, E) \times N$  of codimension dim E 990 (the mapping  $(B, u) \mapsto Bv_u - v_u$  is a submersion). 991

Then, consider M the subset of  $\mathcal{L}(E, E)$  defined as in the proof of Lemma 31.10 992 in [1]. Recall that M is a finite union of submanifolds of  $\mathcal{L}(E, E)$   $(M = \bigcup_{i=1}^{k} M_i)$ , 993 hence,  $\mathcal{M} \triangleq M \times N$  is a finite union of submanifolds of  $\mathcal{L}(E, E) \times N$  ( $\mathcal{M} = \bigcup_{i=1}^{k} M_i \times$ 994N); moreover every  $M_i \times N$  has a codimension  $\geq 1$  in W. 995

Denoting by  $X_u$  the point  $(L_u, u)$  of W, there exists a tangent vector  $\xi_u \in T_{X_u} W$ 996 but  $\xi_u \notin T_{X_u} M_i \times N$  (i = 1, ..., k); moreover  $\xi_u$  depends smoothly on u. This is 997 possible because each  $M_i \times N$  has a codimension  $\geq 1$  in W. The tangent vector  $\xi_u$ 998 may be written  $\xi_u = (A_u, v)$ , where  $A_u$  can be regarded as an element of  $\mathcal{L}(E, E)$ 999 (and  $v \in T_u U$ ). Now, due to the property of  $A_u$ , we can conclude that there exist 1000  $\varepsilon > 0$  such that  $\mathscr{L}(s, u) \notin M \times N$  as soon as  $0 < s < \varepsilon$ . 1001 

Returning to the vector field  $f \in \mathscr{G}^U_{3/2}(a)$ , we argue now as in Lemma 31.12, in 1002 order to prove the existence of a neighborhood  $N_1$  of  $\gamma_{u_0}$  and a neighborhood  $N_2$  of  $u_0$ 10031004 such that, for every  $u \in N_2$ , the vector field  $f(\cdot, u)$  admits a unique periodic trajectory  $\gamma_u$  located in  $N_1$ . Moreover, there exists a parametrized vector field  $g \in \Gamma_U(X)$  such 1005that  $g(\cdot, u)|_{\gamma_u} = 0$ ,  $g(\cdot, u)|_{X \setminus N_1} = 0$ ,  $g(\cdot, u) \equiv 0$  for  $u \in U \setminus N_2$ , and  $\gamma_u$  is an elementary 1006 periodic orbit for  $f(\cdot, u) + \lambda g(\cdot, u)$  for sufficiently small  $\lambda \in \mathbf{R}^+_+$ . We can find such a 1007 vector field q for every periodic trajectory of  $f(\cdot, u_0)$  (the number of these trajectories 1008

is finite), and so, we can conclude as in lemma 31.13 in [1], that there exists a vector field g such that the critical elements of  $f(\cdot, u) + \lambda g(\cdot, u)$  of period  $\leq a$  are exactly the same as the critical elements of  $f(\cdot, u)$  of period  $\leq a$  for every  $u \in N_2$ . Moreover the periodic trajectories of  $f(\cdot, u) + \lambda g(\cdot, u)$  are elementary for  $u \in N_2$ .

1013 Now, the open neighborhoods  $N_2$  cover U, which is compact. So there exist a 1014 finite set of elements  $u_0, \ldots, u_s$  together with neighborhoods  $N_2^0, \ldots, N_2^s$  of the  $u_i$ 's, 1015 there exist vector fields  $g_0, \ldots, g_s$  as in the above lemma. Take  $\rho_0, \ldots, \rho_s$  a partition 1016 of unity of U subordinated to the covering  $N_2^0, \ldots, N_2^s$  and consider the vector field 1017  $g \triangleq \rho_0 g_0 + \cdots + \rho_r g_s$ , we claim that  $f + \lambda g$  belong to  $\mathscr{G}_2^U(a)$  for every positive  $\lambda$ 1018 sufficiently small.

1019 We conclude this step as in [1].

1020 Step 6: If  $\mathcal{N} \subset \mathscr{G}_1^U$  is an open set, and  $\mathscr{G}_2^U(a) \cap \mathcal{N}$  is dense, then  $\mathscr{G}_2^U(\frac{3}{2}a) \subset \mathcal{N}$ 1021 is dense.. This step results trivially from steps 4 and 5.

1012 Step 7: If  $f \in \mathscr{G}_1^U$ , there exists an open neighborhood  $\mathscr{N}$  of f in  $\mathscr{G}_1^U$  such that 1023  $\mathscr{G}_2^U(a) \cap \mathscr{N} \subset \mathscr{N}$  is dense.. From step 3, if  $f \in \mathscr{G}_1^U$ , there exists  $a_0 > 0$  and a 1024 neighborhood  $\mathscr{N}$  of f in  $\mathscr{G}_1^U$  such that  $\mathscr{G}_2^U(a_0)\mathscr{N} = \mathscr{N}$ . In particular, this implies 1025 that  $\mathscr{G}_2^U(a_0)\mathscr{N}$  is dense in  $\mathscr{N}$ . Hence iterating step 5 k times, one has that  $\mathscr{G}_2^U(\frac{3^k}{2^k}a_0) \cap$ 1026  $\mathscr{N}$  is dense in  $\mathscr{N}$ . As one can find k such that  $\frac{3^k}{2^k}a_0 > a$ , we are done.

#### 1027 Appendix B. Proof of Lemma 8.

1028 The derivative of  $x_p^{\lambda}$  with respect to  $\lambda$  can be written as

1029 (26) 
$$\frac{\mathrm{d}}{\mathrm{d}\lambda} x_p^{\lambda} \Big|_{\lambda=0} = A_p + B_p \,.$$

1031 where  $A_p$  is zero or a sum of terms of the form  $\delta_{j_k}^p(J_{j_k})$ , with  $J_{j_k}$  is an integral that 1032 can be written as

1033 (27) 
$$J_{j_k} = \int_0^T \mathrm{d}\varphi_{\sigma}^{u_{p-1}}(\varphi_{T-\sigma}^{u_{p-1}}(x_{j_k})) \cdot \phi(\varphi_{T-\sigma}^{u_{p-1}}(x_{j_k}), u_{p-1}) \mathrm{d}\sigma \,,$$

1034 where the subscripts  $j_k$  in  $A_p$  are less or equal to p-1 and are such that  $u_{j_k} = u_{p-1}$ ; 1035 moreover  $\delta_{j_k}^p$  is the mapping defined as

1036 
$$\delta_{j_k}^a = \begin{cases} \mathrm{Id} & \text{if } j_k = p - 1, \\ \mathrm{d}(\varphi_T^{u_{p-1}} \circ \dots \circ \varphi_T^{u_{j_k+1}})(x_{j_k+1}) & \text{if } j_k$$

1037 As for the term  $B_p$ , it is zero or the sum of terms  $\delta_{j_k}^p(J'_{j_k})$ , where the  $J'_{j_k}$  are 1038 integrals that we write

1039 
$$J'_{j_k} = \int_0^T \mathrm{d}\varphi_{\sigma}^{u_{j_k}}(\varphi_{T-\sigma}^{u_{j_k}}(x_{j_k})) \cdot \phi(\varphi_{T-\sigma}^{u_{j_k}}(x_{j_k}), u_{j_k}) \mathrm{d}\sigma,$$

1040 with  $u_{j_k} \neq u_{p-1}$ .

1045

1041 We write the derivative of  $x_i^{\lambda}$  and, if we are under  $\mathbf{C_2}$ ,  $\mathbf{C'_3}$  or  $\mathbf{C'_4}$  configurations, 1042 the ones of  $\bar{x}_j^{\lambda}$  and  $\bar{x}_m^{\lambda}$ , in the same way; we denote by  $\bar{J}_{j_k}$  and  $\bar{J}'_{j_k}$  the integrals 1043 appearing in the derivatives of  $\bar{x}_j^{\lambda}$  and  $\bar{x}_m^{\lambda}$ . We shall see that  $\phi$  can be chosen such 1044 that

- the terms  $B_p$ ,  $B_i, \overline{B}_j$  and  $\overline{B}_m$  are zero;
- all the integrals  $\bar{J}_{j_k}$  that occur in the terms  $\bar{A}_j$  and  $\bar{A}_m$  are zero;
- all the integrals  $J_{j_k}$  are zero but the one corresponding to the subscript  $j_k = p 1$ , which can be arbitrarily chosen.

1049 To this end, the perturbation  $\phi$  that we shall consider will be zero outside some 1050 neighborhood of  $u_{p-1}$ ; to be more precise, given  $\phi_0$  a vector field defined on X, it is 1051 possible to find  $\phi \in \Gamma_U(X)$  such that

1052 •  $\phi(\cdot, u_{p-1}) = \phi_0;$ 

1053 • for  $j = 0, \dots, 2n$ ,  $\phi(\cdot, u_j) \equiv 0$  as soon as  $u_j \neq u_{p-1}$ .

With this choice of  $\phi$ , we have  $B_p = 0$ , and  $B_i = 0$  as well as  $\bar{B}_j = 0$ , and  $\bar{B}_m = 0$  if we are under  $C_2$ ,  $C'_3$  or  $C'_4$  configuration.

We split the points  $x_{j_k}$  (resp.  $\bar{x}_{j_k}$ ) that appear under the integrals  $J_{j_k}$  (resp.  $\bar{J}_{j_k}$ ) into two classes: the first class, denoted by  $\mathscr{P}_1$ , contains the points  $x_{j_k}$  and  $\bar{x}_{j_k}$ that belong to the trajectory of the vector field  $f(\cdot, u_{p-1})$  passing through  $x_{p-1}$  (and so  $\mathscr{P}_1$  contains the point  $x_{p-1}$  itself), the second class, denoted by  $\mathscr{P}_2$ , contains the points  $x_{j_k}$  and  $\bar{x}_{j_k}$  that do not belong to this trajectory. Denote by  $\mathscr{T}_1$  and  $\mathscr{T}_2$  the union of these trajectories restricted to the interval [0, T], namely

1062 
$$\mathscr{T}_i = \{ \varphi_t^{u_{p-1}}(z) \mid t \in [0,T], z \in \mathscr{P}_i \} \quad i = 1, 2$$

1063 The sets  $\mathscr{T}_1$  and  $\mathscr{T}_2$  being disjoint and compact, let  $\mathscr{U}_1$  and  $\mathscr{U}_2$  be two open sets 1064 of X such that  $\mathscr{T}_i \subset \mathscr{U}_i$  (i = 1, 2) and  $\mathscr{U}_1 \cap \mathscr{U}_2 = \varnothing$ . In the sequel, we shall 1065 assume that the vector field  $\phi_0$  is zero when restricted to  $\mathscr{U}_2$ ; this implies that the 1066 integrals  $J_{j_k}$  (resp.  $\bar{J}_{j_k}$ ) such that  $x_{j_k}$  (resp.  $\bar{x}_{j_k}$ ) belongs to  $\mathscr{P}_2$  are zero. Denote 1067 by  $j_1, \ldots, j_a$   $(0 \leq j_1 < \cdots < j_a = p - 1)$  the subscripts such that  $x_{j_k}$  belongs to 1068  $\mathscr{P}_1$  and let  $t_1, \ldots, t_a$  be such that  $x_{j_k} = \varphi_{t_k}^{u_{p-1}}(x_{p-1})$   $(k = 1, \ldots, a)$ . Denote also by 1069  $l_1 < \cdots < l_b < \max(m, j)$  the subscripts such that  $\bar{x}_{l_k}$  belongs to  $\mathscr{P}_1$  and let  $t'_1, \ldots, t'_b$ 1070 be such that  $\bar{x}_{l_k} = \varphi_{t'_k}^{u_{p-1}}(x_{p-1})$   $(k = 1, \ldots, b)$ .

Notice that, excepted when  $j_k = j_a$ , we cannot have  $t_k = 0$  since this would imply that  $x_{p-1} = x_{j_k}$  with  $j_k < p-1$ , which is not possible under C<sub>1</sub>-C<sub>4</sub> configuration. 1072 1073Also, all the  $t'_k$  (k = 1, ..., b) are non zero, because, if there existed a subscript k such that  $t'_k = 0$ , then, as  $u_{l_k} = u_{p-1}$ , we would have  $\bar{x}_{l_k+1} = x_p$ , which implies 1074 that we are under  $C_2$  or  $C'_3$  configuration and that  $l_k + 1 = m$ . Thus we have 1075  $\bar{x}_{m-1} = x_{p-1}$ , if we are under C'\_2 configuration this implies (m-1) - (p-1) = i - j1076 and so m - p = i - j, which contradicts the definition of the C<sub>2</sub> configuration. If we 1077are under  $\mathbf{C}'_{\mathbf{3}}$  configuration, we found a pair  $(i', j') \triangleq (p-1, m-1)$  such that  $x_{i'} = \bar{x}_{j'}$ 1078 with i' - j' = p - m and i' < p, which contradicts the definition of  $C_3$  configuration. 1079The trajectory related to the vector field  $f(\cdot, u_{p-1})$  passing through  $x_{p-1}$  may be 1080 periodic or aperiodic, we have to distinguish between these two cases. 1081

1082 The trajectory passing through  $x_{p-1}$  is not periodic. Assume that the trajectory 1083 of the vector field  $f(\cdot, u_{p-1})$  passing through  $x_{p-1}$  is not periodic. Taking into account 1084 that, with our notations,  $\phi(\cdot, u_{p-1}) = \phi_0$ , the terms  $J_{j_k}$  appearing in  $A_p$  and (possibly) 1085 in  $A_i$  can also be written as

1086 
$$J_{j_k} = \mathrm{d}\varphi_{t_k}^{u_{p-1}}(x_p) \cdot \int_{-t_k}^{T-t_k} \mathrm{d}\varphi_{\sigma}^{u_{p-1}} \circ \phi_0 \circ \varphi_{T-\sigma}^{u_{p-1}}(x_{p-1}) \mathrm{d}\sigma.$$

In the same way the integrals  $J_{l_k}$  appearing (possibly) in  $A_m$  and  $A_j$  write

1088

$$\bar{J}_{l_k} = \mathrm{d}\varphi_{t'_k}^{u_{p-1}}(x_p) \cdot \int_{-t'_k}^{T-t'_k} \mathrm{d}\varphi_{\sigma}^{u_{p-1}} \circ \phi_0 \circ \varphi_{T-\sigma}^{u_{p-1}}(x_{p-1}) \mathrm{d}\sigma.$$

$$26$$

1089 We set

1090  $T_{\min} = \min(\{t_k \mid k = 1, \dots, a\} \cup \{t'_k \mid k = 1, \dots, b\})$ 

$$1091 T_{\max} = \max\{\{T + t_k \mid k = 1, \dots, a\} \cup \{T + t'_k \mid k = 1, \dots, b\}\}$$

and we introduce the set  $\mathcal{T} = \{ \varphi_t^{u_{p-1}}(x_{p-1}) \mid T_{\min} \leq t \leq T_{\max} \}$ . For  $z = \varphi_t^{u_{p-1}}(x_{p-1})$ in  $\mathcal{T}$ , we define  $\phi_0(z)$  as

1095 
$$\phi_0(z) = \mu(t) \,\mathrm{d}\varphi_{t-T}^{u_{p-1}}(x_p) \cdot \mathfrak{X}_p$$

where  $\mu$  is a smooth function defined on  $[T_{\min}, T_{\max}]$  and  $\mathfrak{X}_p$  is an arbitrary vector tangent to X at  $x_p$ . As the trajectory passing through  $x_{p-1}$  is not periodic,  $\phi_0$  is unambiguously defined on  $\mathcal{T}$ , moreover  $\phi_0$  extends to a smooth vector field defined on the whole manifold X (and which is zero on  $\mathscr{U}_2$ ). With this choice of  $\phi_0$ , the integrals occurring in  $A_i$  and  $A_p$  write

1101 
$$J_{j_k} = \left(\int_{-t_k}^{T-t_k} \mu(T-\sigma) \,\mathrm{d}\sigma\right) \,\mathrm{d}\varphi_{t_k}^{u_{p-1}}(x_p) \cdot \mathfrak{X}_p = \left(\int_{t_k}^{T+t_k} \mu(\sigma) \,\mathrm{d}\sigma\right) \,\mathrm{d}\varphi_{t_k}^{u_{p-1}}(x_p) \cdot \mathfrak{X}_p;$$

1102 while the integrals occurring in the terms  $A_j$  and  $A_m$  write

1103 
$$\bar{J}_{l_k} = \left( \int_{-t'_k}^{T-t'_k} \mu(T-\sigma) \,\mathrm{d}\sigma \right) \mathrm{d}\varphi_{t'_k}^{u_{p-1}}(x_p) \cdot \mathfrak{X}_p = \left( \int_{t'_k}^{T+t'_k} \mu(\sigma) \,\mathrm{d}\sigma \right) \mathrm{d}\varphi_{t'_k}^{u_{p-1}}(x_p) \cdot \mathfrak{X}_p;$$

here all the  $t_k$ 's are non-zero but  $t_a$ , and all the  $t'_k$ 's are non zero. Choose now a smooth function M defined on  $[T_{\min}, T_{\max}]$  and such that

1106 • 
$$M(T) = M(2T) = \cdots = M(cT) = 1$$
; here c denotes the integer part of  $T_{\text{max}}/T$ ;

1108  
• 
$$M(T+t_k) = M(t_k) = M(T+t'_{k'}) = M(t'_{k'}) = 0, k = 1, ..., a-1, k' = 1, ..., b$$
  
1109  
where  $t_k, t'_k \neq \alpha T, \alpha = 1, ..., c;$ 

1110 • and M(0) = 0;

1111 We take now  $\mu(t) = \frac{\mathrm{d}M(t)}{\mathrm{d}t}$ , with this choice of  $\mu$ , all the integrals  $J_{j_k}$  and  $\bar{J}_{l_k}$  are 1112 zero except  $J_{p-1}$  which is equal to  $\mathfrak{X}_p$ .

1113 Point  $x_{p-1}$  is singular.. In this case, we have  $x_{p-1} = x_p$ , so necessarily we are 1114 under  $\mathbf{C_1}$  or  $\mathbf{C'_4}$  configuration. The vector field  $\phi_0$  is then chosen such that  $\phi_0$  is zero 1115 outside an open neighborhood  $\mathcal{N}$  of  $x_{p-1}$ , this neighborhood being chosen so small 1116 that the trajectories of  $f(\cdot, u_{p-1})$  passing through the points  $x_{j_k}$   $(j_k \neq p-1)$  and  $\bar{x}_{l_k}$ 1117 do not cross  $\mathcal{N}$ . With this choice of  $\phi_0$  all the integrals  $J_{j_k}$  and  $\bar{J}_{l_k}$  are zero but the 1118 one which correspond to  $j_k = p-1$  which is equal to

1119 
$$J_{p-1} = \int_0^T \mathrm{d}\varphi_s^{u_{p-1}}(x_{p-1}) \cdot \phi_0(x_{p-1}) \,\mathrm{d}s \,.$$

1120 Now  $d\varphi_s^{u_{p-1}}(x_{p-1}) = e^{sA}$  where A is the differential of f at  $x_{p-1}$ ; notice that, as 1121  $f \in \mathscr{G}_2^U(a)$ , A does not have any purely imaginary eigenvalue (and so is invertible). 1122 Hence we can compute explicitly integral  $J_{p-1}$ :

1123 
$$J_{p-1} = A^{-1}(e^{TA} - \mathrm{Id}) \cdot \phi_0(x_{p-1});$$

1124 as  $e^{TA}$  does not admit 1 as an eigenvalue,  $e^{TA}$  – Id is invertible, which proves that 1125  $J_{p-1}$  can be made equal to any tangent vector of  $T_{x_{p-1}}$  thanks to an appropriate 1126 choice of  $\phi_0(x_{p-1})$ .

The trajectory passing through  $x_{p-1}$  is periodic . We shall show now the same 1127result in the case when the trajectory passing through  $x_{p-1}$  is periodic; in other words, 1128we assume that the mapping  $t \mapsto \varphi_t^{u_{p-1}}(x_{p-1})$  is periodic and we denote by  $\pi_0$  its 1129prime period. In this case, the function  $\mu$  which appears in the definition of  $\phi_0$  must 1130 be periodic. Writing  $T = q\pi_0 + \tau$  with  $q \in \mathbf{N}$  and  $0 \leq \tau < \pi_0$ , we have 1131

1132 
$$J_{j_k} = \int_0^T \mathrm{d}\varphi_{\sigma}^{u_{p-1}} \circ \phi_0 \circ \varphi_{T-\sigma}^{u_{p-1}}(x_{j_k}) \mathrm{d}\sigma$$
$$\frac{q-1}{2} \int_0^{(l+1)\pi_0} d\varphi_{T-\sigma}^{u_{p-1}}(x_{j_k}) \mathrm{d}\sigma$$

133
$$= \sum_{l=0}^{q-1} \int_{l\pi_0}^{(l+1)\pi_0} \mathrm{d}\varphi_{\sigma}^{u_{p-1}} \circ \phi_0 \circ \varphi_{T-\sigma}^{u_{p-1}}(x_{j_k}) \mathrm{d}\sigma + \int_{q\pi_0}^{T} \mathrm{d}\varphi_{\sigma}^{u_{p-1}} \circ \phi_0 \circ \varphi_{T-\sigma}^{u_{p-1}}(x_{j_k}) \mathrm{d}\sigma$$
134
$$= \sum_{l=0}^{q-1} \mathrm{d}\varphi_{l\pi_0}^{u_{p-1}}(\varphi_T^{u_{p-1}}(x_{j_k})) \cdot \int_{0}^{\pi_0} \mathrm{d}\varphi_{\sigma}^{u_{p-1}} \circ \phi_0 \circ \varphi_{T-\sigma}^{u_{p-1}}(x_{j_k}) \mathrm{d}\sigma$$

$$= \sum_{l=0}^{1134} \operatorname{d} \varphi_{l \pi_{0}}^{\dagger} (\varphi_{T}^{\dagger} (x_{j_{k}})) \cdot \int_{0}^{\tau} \operatorname{d} \varphi_{\sigma}^{p-1} \circ \varphi_{0} \circ \varphi_{T-\sigma}^{\dagger} (x_{j_{k}}) \operatorname{d} \sigma$$

1135 (28) 
$$+ d\varphi_{q\pi_0}^{u_{p-1}}(\varphi_T^{u_{p-1}}(x_{j_k})) \int_0^{\cdot} d\varphi_{\sigma}^{u_{p-1}} \circ \phi_0 \circ \varphi_{T-\sigma}^{u_{p-1}}(x_{j_k}) d\sigma.$$

Now the  $x_{j_k}$ 's in the above integrals are such that  $x_{j_k} = \varphi_{t_k}^{u_{p-1}}(x_{p-1})$ . Notice that, due to the periodicity of the trajectory passing through  $x_{p-1}$ , we can assume 1137 1138that  $0 \le t_k < \pi_0$ . Writing the  $x_{j_k}$ 's in terms of  $x_{p-1}$ , the above integrals between 0 1139 and  $\pi_0$  can be re-written as. 1140

1141 
$$\int_{0}^{\pi_{0}} \mathrm{d}\varphi_{\sigma}^{u_{p-1}} \circ \phi_{0} \circ \varphi_{T-\sigma}^{u_{p-1}}(x_{j_{k}}) \mathrm{d}\sigma = \mathrm{d}\varphi_{t_{k}}^{u_{p-1}}(x_{p}) \cdot \int_{-t_{k}}^{\pi_{0}-t_{k}} \mathrm{d}\varphi_{\sigma}^{u_{p-1}} \circ \phi_{0} \circ \varphi_{T-\sigma}^{u_{p-1}}(x_{p-1}) \mathrm{d}\sigma.$$

In the same way, the integral between 0 and  $\tau$  in (28) can be written 1142

1143 
$$\int_{0}^{\tau} \mathrm{d}\varphi_{\sigma}^{u_{p-1}} \circ \phi_{0} \circ \varphi_{T-\sigma}^{u_{p-1}}(x_{j_{k}}) \mathrm{d}\sigma = \mathrm{d}\varphi_{t_{k}}^{u_{p-1}}(x_{p}) \int_{-t_{k}}^{\tau-t_{k}} \mathrm{d}\varphi_{\sigma}^{u_{p-1}} \circ \phi_{0} \circ \varphi_{T-\sigma}^{u_{p-1}}(x_{p-1}) \mathrm{d}\sigma$$

It follows from these considerations and from the equality  $\varphi_T^{u_{p-1}}(x_{p-1}) = x_p$ , that we 1144 can write  $J_{j_k}$  under the form 1145

1146 
$$J_{j_k} = \sum_{l=0}^{q-1} \mathrm{d}\varphi_{l\,\pi_0+t_k}^{u_{p-1}}(x_p) \cdot \int_{-t_k}^{\pi_0-t_k} \mathrm{d}\varphi_{\sigma}^{u_{p-1}} \circ \phi_0 \circ \varphi_{-\sigma}^{u_{p-1}}(x_p) \mathrm{d}\sigma$$

1147 
$$+ \mathrm{d}\varphi_{q\pi_0+t_k}^{u_{p-1}}(x_p) \cdot \int_{-t_k}^{\tau-t_k} \mathrm{d}\varphi_{\sigma}^{u_{p-1}} \circ \phi_0 \circ \varphi_{-\sigma}^{u_{p-1}}(x_p) \mathrm{d}\sigma$$

1148 
$$= \mathrm{d}\varphi_{t_k}^{u_{p-1}}(x_p) \cdot \left(Q \cdot \int_{-t_k}^{\pi_0 - t_k} \mathrm{d}\varphi_{\sigma}^{u_{p-1}} \circ \phi_0 \circ \varphi_{-\sigma}^{u_{p-1}}(x_p) \mathrm{d}\sigma\right)$$

1149 (29) 
$$+ \delta^q \cdot \int_{-t_k}^{\tau-t_k} \mathrm{d}\varphi_{\sigma}^{u_{p-1}} \circ \phi_0 \circ \varphi_{-\sigma}^{u_{p-1}}(x_p) \mathrm{d}\sigma \bigg);$$

where we let 1151

1152 (30) 
$$\delta = \mathrm{d}\varphi_{\pi_0}^{u_{p-1}}(x_p)$$
 and  $Q = \sum_{l=0}^{q-1} \delta^l$ .

We introduce also the following notation, let  $t \in [0, \pi_0]$  and denote by  $I_t$  the integral 1153

1154 
$$I_t = \int_0^t \mathrm{d}\varphi_{\sigma}^{u_{p-1}} \circ \phi_0 \circ \varphi_{-\sigma}^{u_{p-1}}(x_p) \mathrm{d}\sigma \,.$$
28

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1155 We now rewrite  $J_{j_k}$  in terms of integrals  $I_t$ , from (29), we get 1156 • if  $t_k < \tau$ ,

1157 (31) 
$$J_{j_k} = \mathrm{d}\varphi_{t_k}^{u_{p-1}}(x_p) \cdot \left( (Q \circ \delta^{-1} + \delta^{q-1}) \cdot I_{\pi_0} - \delta^{-1} \cdot I_{\pi_0 - t_k} + \delta^q \cdot I_{\tau - t_k} \right)$$

1158 • if 
$$\tau \leq t_k$$

1159 (32) 
$$J_{j_k} = \mathrm{d}\varphi_{t_k}^{u_{p-1}}(x_p) \cdot \left(Q \circ \delta^{-1} \cdot I_{\pi_0} - \delta^{-1} \cdot I_{\pi_0 - t_k} + \delta^{q-1} \cdot I_{\pi_0 + \tau - t_k}\right);$$

1160 notice that, in each case,  $\pi_0 - t_k$  and  $\pi_0 + \tau - t_k$  belong to  $[0, \pi_0]$ ; 1161 As regards the integrals  $\bar{J}_{l_k}$  (k = 1, ..., b), analogous computations lead to the same 1162 above formulas (31) and (32).

1163 We choose now  $\phi_0$  as follows, for  $z = \varphi_{-\sigma}^{u_{p-1}}(x_p)$  with  $\sigma \in [0, \pi_0]$ , we define  $\phi_0(z)$ 1164 as

1165 (33) 
$$\phi_0(z) = \mathrm{d}\varphi^{u_{p-1}}_{-\sigma}(x_p) \cdot \vartheta(\sigma)$$

1166 where  $\vartheta : \mathbf{R} \to T_{x_p} X$  is a  $\pi_0$ -periodic mapping to be determined; notice first that, as 1167 we want  $\phi_0$  to be  $C^r$ , we must have  $\vartheta(0) = \vartheta(\pi_0) = 0$  as well as  $\vartheta^{(l)}(0) = \vartheta^{(l)}(\pi_0) = 0$ 1168 for  $l = 1, \ldots, r$ .

We deal first with the special case  $\tau = 0$ , in this case  $T = q \pi_0$  and we have 1169  $x_{p-1} = x_p$ , so the lists  $L_p$  and  $L_m$  must be in the  $C_1$  or  $C'_4$  configuration. If we are 1170 under  $\mathbf{C}'_{\mathbf{4}}$  configuration and if there exists a subscript  $j_1 < p-1$  such that  $u_{j_1} = u_{p-1}$ 1171and  $x_{j_1} = \varphi_{t_1}^{u_{p-1}}(x_{p-1})$ , then  $x_{j_1} = x_{j_1+1}$  which is impossible from the definition of 1172 the  $\mathbf{C}'_{\mathbf{4}}$  configuration. If there exists a subscript  $l_1 < m$  such that  $u_{l_1} = u_{p-1}$  and 1173 $\bar{x}_{l_1} = \varphi_{t'_1}^{u_{p-1}}(\bar{x}_{p-1})$ , then we would have  $\bar{x}_{l_1} = \bar{x}_{l_1+1}$ ; as the lists  $L_p$  and  $\bar{L}_m$  are in the 1174 $C_4$  configuration, this implies that  $l_1 = m - 1$  and  $u_{m-1} = u_{p-1}$  which is incompatible 1175with the definition of the  $C'_4$  configuration. We conclude that the terms  $A_{p-1}$ ,  $A_i$ 1176and  $\bar{A}_m$  are zero in this case and that  $A_p$  is equal to  $Q \cdot I_{\pi_0}$ ; so, with our choice of 11771178 $\phi_0$ , we have

1179 
$$A_p = Q \cdot \int_0^{\pi_0} \vartheta(\sigma) \, \mathrm{d} \sigma$$

and it is obviously possible to find a periodic function  $\vartheta$  satisfying the above constraints and whose integral over the interval  $[0, \pi_0]$  is equal to  $Q^{-1} \cdot \mathfrak{X}_p$ ; clearly this choice of  $\vartheta$  is also possible if we are under  $\mathbf{C}_1$  configuration.

We assume now that  $\tau \neq 0$ ; we deal first with the case of  $C_1$  configuration. We introduce the following sets.

1185 
$$\mathbf{T}_{\alpha} = \{ t_k, k = 1, \dots, a-1 \mid t_k \equiv -\alpha \tau \pmod{\pi_0} \}.$$

1186 If  $\mathbf{T}_1 = \emptyset$ , we set  $\alpha_0 = 0$ , if not, we denote by  $\alpha_0$  the largest integer  $\alpha$  such that 1187  $\mathbf{T}_1 \neq \emptyset, \mathbf{T}_2 \neq \emptyset, \dots, \mathbf{T}_{\alpha} \neq \emptyset$ . If  $\alpha_0 \neq 0$ , we introduce the integers  $\gamma_{\alpha} = [\alpha \tau / \pi_0] + 1$ 1188 (where [x] denotes the integer part of x); so if  $t_k \in \mathbf{T}_{\alpha}$ , we have  $t_k = \gamma_{\alpha} \pi_0 - \alpha \tau$ .

1189 It could happen that  $\pi_0$  is divisible by  $\tau$ ; hereafter, we distinguish two cases. 1190 First we assume that there does not exist any  $\alpha \leq \alpha_0$  such that  $(\alpha + 1)\tau \equiv 0$ 1191 (mod  $\pi_0$ ); this assumption implies that, for every pair  $0 \leq \alpha < \alpha' \leq \alpha_0$  we have 1192  $\alpha'\tau - \alpha\tau \not\equiv 0 \pmod{\pi_0}$ . Thus, there exists a mapping  $V : \mathbf{R} \to T_{x_p} X$  such that

• V is 
$$\pi_0$$
-periodic and  $V(0) = 0$ ,  $V^{(l)}(0) = 0$  for  $l = 1, ..., r + 1$ ;

1194 •  $V(\tau) = \delta^{-q} \cdot \mathfrak{X}_p$ , and

1195 
$$V(2\tau) = \begin{cases} \delta^{-q-1} \cdot V(\tau) & \text{if } \pi_0 < 2\tau \\ \delta^{-q} \cdot V(\tau) & \text{if } \pi_0 > 2\tau \end{cases},$$

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$$V((\alpha_0+1)\tau) = \begin{cases} \delta^{-q-1} \cdot V(\alpha_0\tau) & \text{if } \gamma_{\alpha_0}\pi_0 < (\alpha_0+1)\tau \\ \delta^{-q} \cdot V(\alpha_0\tau) & \text{if } \gamma_{\alpha_0}\pi_0 > (\alpha_0+1)\tau; \end{cases}$$

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1201

•  $V(\pi_0 - t_k) = 0$ ,  $V(\tau - t_k) = 0$  (case  $t_k < \tau$ ) or  $V(\pi_0 + \tau - t_k) = 0$  (case  $\tau \le t_k$ ) if  $t_k \notin \mathbb{T}_1 \cup \cdots \cup \mathbb{T}_{\alpha_0}$  (notice that, for such a  $t_k$ ,  $\pi_0 - t_k$  and  $\pi_0 + \tau - t_k$  do no more belong to  $\mathbb{T}_1 \cup \cdots \cup \mathbb{T}_{\alpha_0}$ ).

1202 We take then the mapping  $\vartheta$  equal to the derivative of V, with this choice, all the 1203 integrals  $J_{j_k}$  are zero but  $J_{p-1}$  which is equal to  $\mathfrak{X}_p$ ; this implies that  $A_p$  is equal to 1204  $\mathfrak{X}_p$  while we have  $A_i = 0$ .

Case where  $\tau$  divides  $\pi_0$ . Assume now that there exists  $\alpha \leq \alpha_0$  such that  $(\alpha + \alpha)$ 12051) $\tau \equiv 0 \pmod{\pi_0}$ , then necessarily  $\alpha = \alpha_0$  (if  $\alpha < \alpha_0$ , the time  $t_a = 0$  would belong 1206to  $\mathbf{T}_{\alpha+1}$ ). There exists a subscript  $j_k$  such that  $x_{j_k} = \varphi_{-\alpha_0\tau}^{u_{p-1}}(x_{p-1}) = \varphi_{\tau}^{u_{p-1}}(x_{p-1}) =$ 1207 $x_p$ , as  $j_k < p$ , from this equality and from the definition of configuration  $C_1$ , we 1208 deduce that  $j_k = i$  and that  $u_i = u_{p-1}$ . Arguing by induction, assume that, for some 1209  $0 \leq r , we have <math>u_i = \dots = u_{i+r} = u_{p-1}$  and  $x_i = \varphi_{-\alpha\sigma\tau}^{u_{p-1}}(x_{p-1}), \dots, x_{i+r} = \varphi_{-(\alpha_0 - r)\tau}^{u_{p-1}}(x_{p-1})$ . There exists a subscript  $j_k$  such that  $x_{j_k} = \varphi_{-(\alpha_0 - r-1)\tau}^{u_{p-1}}(x_{p-1}) = \varphi_{-(\alpha_0 - r-1)\tau}^{u_{p-1}}(x_{p-1})$ . 12101211 $\varphi_{\tau}^{u_{p-1}}(x_{i+r}) = x_{i+r+1}$ , as above, this equality implies that  $u_{i+r+1} = u_{p-1}$ . We have proved that  $u_i = u_{i+1} = \cdots = u_{p-1}$  and that  $\alpha_0 = p - i - 1$ . As  $x_p = \varphi_{(p-i)T}^{u_{p-1}}(x_i)$ , we 1212 1213 1214 re-write  $A_p$  as follows:

1215 
$$A_p = \int_0^{(p-i)T} \mathrm{d}\varphi_{\sigma}^{u_{p-1}} \circ \phi_0 \circ \varphi_{-\sigma}^{u_{p-1}}(x_p) \,\mathrm{d}\sigma + \mathrm{d}\varphi_{(p-i)T}^{u_{p-1}}(x_p) \cdot A_i \,.$$

1216 As  $(\alpha_0 + 1)\tau \equiv 0 \pmod{\pi_0}$ , there exists  $r \in \mathbf{N}$  such that  $(p - i)T = q'\pi_0$ , thus the 1217 first term in this new expression of  $A_p$  can be re-written as

1218 
$$\int_0^{(p-i)T} \mathrm{d}\varphi_{\sigma}^{u_{p-1}} \circ \phi_0 \circ \varphi_{-\sigma}^{u_{p-1}}(x_p) \,\mathrm{d}\sigma = Q' \cdot I_{\pi_0}$$

1219 where  $Q' = \operatorname{Id} + \delta + \dots \delta^{q'-1}$ . We shall see that  $\phi_0$  can be chosen such that  $A_i = 0$ 1220 and  $A_p$  equal to any tangent vector field. Hereafter, we call chain of length  $c \ (c \ge 2)$ 1221 a sequence of c pairs  $((x_{j_{k_1}}, t_{k_1}), \dots, (x_{j_{k_c}}, t_{k_c}))$  such that

• the points  $x_{j_{k_1}}, \ldots, x_{j_{k_c}}$  belong to  $\mathscr{P}_1$  and the subscripts  $j_{k_1}, \ldots, j_{k_c}$  are pairuse distinct;

• the times  $t_{k_1}, \ldots, t_{k_c}$  belong to  $\mathbf{T}_1 \cup \cdots \cup \mathbf{T}_{\alpha_0} \cup \{0\}$  and are such that

1225 
$$t_{k_2} \equiv t_{k_1} + \tau \pmod{\pi_0}, \dots, t_{k_c} \equiv t_{k_1} + (c-1)\tau \pmod{\pi_0}.$$

1226 Notice that two chains are either disjoint or equal. The chain

1227 
$$\mathbf{c}_0 \triangleq ((x_i, \tau), (x_{i+1}, 2\tau), \dots, (x_{p-1}, (\alpha_0 + 1)\tau))$$

has a length equal to  $\alpha_0 + 1$ ; the lengths of all the other chains are less than  $\alpha_0 + 1$ because otherwise, we could find at least two equalities between the elements of  $L_p$ .

We choose vector field  $\phi_0$  as in the case where  $(\alpha_0 + 1)\tau \not\equiv 0 \pmod{\pi_0}$ , but the 1230 mappings  $\vartheta$  and V are chosen as follows: 1231

1232 • 
$$\vartheta(t) = V'(t)$$
 and  $V(t) = V_0(t) + tv_0$  where  $V_0$  is  $\pi_0$ -periodic with  $V_0(\pi_0) = 1233$   
 $V_0(0) = 0$ ; notice that this choice of  $\vartheta$  implies  $I_t = V_0(t) + tv_0$ ;

1234 • 
$$V'_0(0) = V'_0(\pi_0) = -v_0$$
 and  $V^{(k)}_0 = V^{(k)}_0(\pi_0) = 0$  with  $k = 2, \dots, r+1$ ;

•  $v_0 = \frac{1}{\pi_0} \cdot (Q')^{-1} \mathfrak{X}_p;$ 1235

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1238

• if 
$$((x_{j_{k_1}}, t_{k_1}), \dots, (x_{j_{k_c}}, t_{k_c}))$$
 is a chain of maximal length distinct from the chain  $\mathfrak{c}_0$ , we must have

(34) 
$$\begin{cases} (Q \circ \delta^{-1} + \delta^{q-1}) \cdot I_{\pi_0} - \delta^{-1} \cdot I_{\pi_0 - t_{k_s}} + \delta^q \cdot I_{\tau - t_{k_s}} = 0, & \text{if } t_{k_s} < \tau, \\ Q \circ \delta^{-1} \cdot I_{\pi_0} - \delta^{-1} \cdot I_{\pi_0 - t_{k_s}} + \delta^{q-1} \cdot I_{\pi_0 + \tau - t_{k_s}} = 0, & \text{if } \tau \le t_{k_s}. \end{cases}$$

These equalities can also be written as 1239

1240

$$\begin{cases} (Q \circ \delta^{-1} + \delta^{q-1}) \cdot I_{\pi_0} - \delta^{-1} \cdot (V_0(-t_{k_s}) + (\pi_0 - t_{k_s})v_0) \\ & + \delta^q \cdot (V_0(\tau - t_{k_s}) + (\tau - t_{k_s})v_0) = 0, & \text{if } t_{k_s} < \tau, \\ Q \circ \delta^{-1} \cdot I_{\pi_0} - \delta^{-1} \cdot (V_0(-t_{k_s}) + (\pi_0 - t_{k_s})v_0) \\ & + \delta^{q-1} \cdot (V_0(\tau - t_{k_s}) + (\pi_0 + \tau - t_{k_s})v_0 = 0, & \text{if } \tau \le t_{k_s}. \end{cases}$$

 $< \tau$ ,

As  $c < \alpha_0 + 1$ , the numbers  $\tau - t_{k_1}, -t_{k_1}, \dots, -t_{k_1} - (c-1)\tau$  are pairwise dis-1241tinct modulo  $\pi_0$ , the values of  $V_0$  at these points can be chosen independently 1242from each other; this proves that the above equalities can be achieved. 1243

1244 • If  $t_k$  does not belong to a chain, nor is the case for  $\tau - t_k$ , so equality (34) can be satisfied thanks to an appropriate choice of  $V(-t_k)$  and  $V(\tau - t_k)$ . 1245

With this choice of  $\vartheta$ , taking into account the formulas (31) and (32), we see that we 1246have  $A_p = \mathfrak{X}_p$  and  $A_i = 0$ . 1247

If we are under the configurations  $C_2$ ,  $C'_3$  or  $C'_4$ , the sets  $T_{\alpha}$  are defined as 1248 1249follows. 1250

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1251 
$$\mathbf{T}_{\alpha} = \{ t_k, k = 1, \dots, a-1 \mid t_k \equiv -\alpha \tau \pmod{\pi_0} \}$$

$$\cup \{ t'_k, k = 1, \dots, b \mid t'_k \equiv -\alpha \tau \pmod{\pi_0} \},$$

the integer  $\alpha_0$  being defined as above. If there does not exist any  $\alpha \leq \alpha_0$  such that 12541255 $(\alpha + 1)\tau \equiv 0 \pmod{\pi_0}$ , the reasoning is exactly the same than in the case of  $\mathbf{C}_1$ configuration: choosing  $\phi_0$  as explained above, we get  $A_i = 0$ ,  $A_m = 0$  and  $A_j = 0$ 12561257while  $A_p$  can be arbitrarily chosen.

Assume now that there exists  $\alpha$  such that  $(\alpha + 1)\tau \equiv 0 \pmod{\pi_0}$ , then, as above, 1258necessarily  $\alpha = \alpha_0$ . As we are under  $C_2$ ,  $C'_3$  or  $C'_4$  configuration, the definition of 1259a chain takes into account the elements of  $\overline{L}_{2n}$ : we call chain of length c a sequence 12601261  $((z_1, \mathfrak{t}_{k_1}), \ldots, (z_c, \mathfrak{t}_{k_c}))$  such that

• the points 
$$z_1, \ldots, z_c$$
 belong to  $\mathscr{P}_1$ 

the points z<sub>1</sub>,..., z<sub>c</sub> belong to F<sub>1</sub>,
the times t<sub>k1</sub>,..., t<sub>kc</sub> belong to T<sub>1</sub> ∪ ··· ∪ T<sub>α0</sub> ∪ {0} and are such that

1264

1263

$$\mathfrak{t}_{k_2} \equiv \mathfrak{t}_{k_1} + \tau \pmod{\pi_0}, \ \dots, \ \mathfrak{t}_{k_c} \equiv \mathfrak{t}_{k_1} + (c-1)\tau \pmod{\pi_0};$$

• if 
$$\mathbf{t}_k = t_k$$
 (resp.  $\mathbf{t}_k = t'_k$ ), then  $z_k = x_{j_k}$  (resp.  $z_k = \bar{x}_{l_k}$ ); moreover the  
subscripts  $j_k$  (resp.  $l_k$ ) related to those  $x_{j_k}$  (resp.  $\bar{x}_{l_k}$ ) are pairwise distinct.

If we are under configuration  $C'_4$ , the only possible chain of length  $\alpha_0 + 1$  is either 1267

1268

$$\mathfrak{c}_0 = ((x_i, \tau), \dots, (x_{p-1}, (\alpha_0 + 1)\tau)),$$
31

1269 1270 or

$$\bar{\mathfrak{c}}_0 = \left( (\bar{x}_i, \tau), \dots, (\bar{x}_{m-1}, (\alpha_0 + 1)\tau) \right).$$

Notice that, from the definition of  $C_4'$  configuration, these two chains,  $\mathfrak{c}_0$  and  $\bar{\mathfrak{c}}_0,$ 1271cannot coexist. All the other chains have a length less than  $\alpha_0 + 1$ . To show this 1272fact, let  $((z_1, \mathfrak{t}_{k_1}), \ldots, (z_c, \mathfrak{t}_{k_c}))$  be a chain such that  $c = \alpha_0 + 1$ . All the elements 1273  $(z_1,\ldots,z_c)$  belong either to  $L_{p-1}$  or to  $L_{m-1}$  (if not, we could find an equality between 12741275an element of  $L_{p-1}$  and an element of  $\overline{L}_{m-1}$ ). If all the  $z_k$ 's belong to  $L_{p-1}$ , then we have  $z_{c+1} = z_1$ ; from the definition of  $\mathbf{C}'_4$  configuration, it follows that  $z_1 = x_i$ 1276and  $z_{c+1} = x_p$ . If all the  $z_k$ 's are in  $L_{m-1}$ , we obtain that  $z_1 = \bar{x}_j$ ,  $z_{c+1} = \bar{x}_m$  and 1277 $u_i = \cdots = u_{m-1} = u_{p-1}$ . As all the chain but  $\mathfrak{c}_0$  have a length less than  $\alpha_0 + 1$ , we 1278can conclude, as for  $\mathbf{C_1}$  configuration, that there exists a function  $\mathscr V$  defining a vector 1279field  $\phi_0$  which ensures that  $A_i = 0$ ,  $A_j = 0$ , and  $A_m = 0$  while  $A_p$  can be arbitrarily 1280chosen. 1281

We shall see that a chain of length  $\alpha_0 + 1$  is not possible under  $C_2$  or  $C'_3$  configuration. Assume that we are not under  $C_4$  configuration and denote by

1284 
$$\mathfrak{c}_0 = ((z_1, \mathfrak{t}_1), \dots, (z_{\alpha_0+1}, \mathfrak{t}_{\alpha_0+1}))$$

1285 a chain with length  $\alpha_0 + 1$ ; notice that, from the definition of a chain, we have

1286 (35) 
$$z_{k+1} = \varphi_{\tau}^{u_{p-1}}(z_k) = \varphi_{T}^{u_{p-1}}(z_k).$$

1287 There exists  $0 \le \alpha_1 \le \alpha_0$  such that  $\mathfrak{t}_{k_1} \equiv -\alpha_1 \tau \pmod{\pi_0}$  so, from the definition of a 1288 chain, we have

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1289 
$$\mathfrak{t}_{k_1} \equiv -\alpha_1 \tau \pmod{\pi_0}, \qquad \mathfrak{t}_{k_2} \equiv -(\alpha_1 - 1)\tau \pmod{\pi_0}, \dots$$
1290 
$$\mathfrak{t}_{k_1} \equiv -(\alpha_1 - \alpha_2)\tau \pmod{\pi_2}$$

$$\begin{array}{ccc} 1290\\ 1291 \end{array} \qquad \qquad \mathfrak{t}_{k_{\alpha_0+1}} \equiv -(\alpha_1 - \alpha_0)\tau \pmod{\pi}$$

1292 therefore  $\mathbf{t}_{k_{\alpha_1+1}} \equiv 0 \pmod{\pi_0}$  and  $z_{\alpha_1+1} = x_{p-1}$ . If  $\alpha_1 < \alpha_0$ , this equality implies 1293  $z_{\alpha_1+2} = x_p$ ; if  $\alpha_1 = \alpha_0$ , we have  $z_1 = \varphi_{-\alpha_0\tau}^{u_{p-1}}(x_{p-1}) = \varphi_{\tau}^{u_{p-1}}(x_{p-1}) = x_p$ . Thus, 1294 in chain  $\mathbf{c}_0$ , there exists an element  $z_i$  equal to  $x_p$ , this implies that we cannot be 1295 under  $\mathbf{C}_2$  configuration because, in this case we can neither have  $z_i = x_{j_k}$  because 1296  $j_k < p$  nor  $z_i = x_p = \bar{x}_m = \bar{x}_{l_k}$  because  $l_k < m$ . Thus we are under  $\mathbf{C}'_3$  configuration, 1297 reordering the elements of the chain, we can assume that

1298 
$$\mathfrak{t}_{k_1} \equiv -\alpha_0 \tau \pmod{\pi_0}, \mathfrak{t}_{k_2} \equiv -(\alpha_0 - 1)\tau \pmod{\pi_0}, \dots, \mathfrak{t}_{k_{\alpha_0+1}} \equiv 0 \pmod{\pi_0}.$$

We have  $z_1 = \varphi_{-\alpha_0\tau}^{u_{p-1}}(x_{p-1}) = \varphi_{\tau}^{u_{p-1}}(x_{p-1}) = x_p$ , so, as all the subscripts  $j_k$  are less 1299 than p, we have  $z_1 = \bar{x}_m \in L_{2n}$ . Let r be the greatest subscript such that  $z_1, \ldots, z_r \in L_{2n}$ 1300  $\overline{L}_{2n}$ ; from (35), we have  $z_r = \overline{x}_{m+r-1}$  and  $z_{r+1} = x_{j_k}$  for some subscript  $j_k < p$ , so 1301  $x_{j_k} = \bar{x}_{m+r}$ ; from the definition of configuration  $\mathbf{C}'_{\mathbf{3}}$  this implies that we cannot have 1302m+r < j, so m+r = j and  $z_{r+1} = x_i$ . Let s be the greatest subscript greatest 1303than or equal to r+1 such that  $z_{r+1}, \ldots, z_s \in L_{2n}$ . We have  $z_s = x_{s+i-r-1}$  and we 1304claim that  $z_s$  is the last element of  $\mathfrak{c}_0$  because if there exist  $z_{s+1} \in L_{2n}$ , then we have 1305 $z_{s+1} = \varphi_{\tau}^{u_{p-1}}(z_s) = x_{s+i-r}$ ; from the definition of configuration  $\mathbf{C}'_{\mathbf{3}}$  this implies that 1306s+i-r=p and  $z_{s+1}=\bar{x}_m$  which contradicts the definition of a chain. The element 1307  $z_s$  being the last element of the chain, we have  $z_s = \varphi_{\alpha_0\tau}^{u_{p-1}}(z_1) = \varphi_{-\tau}^{u_{p-1}}(\bar{x}_m) = x_{p-1}$ , 1308therefore chain  $\mathfrak{c}_0$  can be written 1309

1310 
$$\mathbf{c}_0 = ((\bar{x}_m, -\alpha_0\tau), \dots, (\bar{x}_{j-1}, -(\alpha_0 - j + m + 1)\tau), (x_i, -(\alpha_0 - j + m)\tau), \dots, (x_{p-1}, 0))$$

1311 from which we deduce that  $u_i = \cdots = u_{p-1} = u_m = \cdots = u_{j-1}$  which contradicts the 1312 definition of  $\mathbf{C}'_{\mathbf{3}}$ .

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