

A multivalued variational inequality with unilateral constraints

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Abstract. The present paper represents a continuation of [3]. There, we studied a new class of variational inequalities involving a pseudomonotone univalued operator and a multivalued operator, for which we obtained an existence result, among others. In the current paper we prove that this result remains valid under significantly weaker assumption on the multivalued operator. Then, we consider a new mathematical model which describes the equilibrium of an elastic body attached to a nonlinear spring on a part of its boundary. We use our abstract result to prove the weak solvability of this elastic model.

Keywords: multivalued operator, variational inequality, cut-off operator, elastic material unilateral constraint, weak solution.

1 Introduction

The theory of variational inequalities plays an important role in the study of nonlinear boundary value problems arising in Mechanics, Physics and Engineering Sciences. Based on arguments of monotonicity and convexity, it started in early sixties and has gone through substantial development since then, as illustrated in the books [1, 4, 9, 10], for instance. There, the inequalities have been formulated in terms of univalued operators and subgradient of a convex function.

Variational inequalities with multivalued operators represent a more recent and challenging topic of nonlinear functional analysis. In particular, they represent a powerful instrument which allows to obtain new and interesting results in the study of various classes of variational-hemivariational inequalities. Such kind of inequalities involve both convex and nonconvex functions, have been introduced in [8] and have been investigated in various other works, as explained in [6, 7] and the references therein.

Recently, in [3], we considered a class of stationary variational inequalities with multivalued operators, for which we proved an existence result, under

a smallness assumption on the data. The proof was based on standard arguments of variational inequalities followed by a version of the Kakutani-Fan-Glicksberg fixed point theorem. Then, we considered a dual variational formulation of the problem, studied the link between the primal and dual formulations and provided an equivalence result. Finally, we applied this abstract formalism to the study of a frictional elastic contact model with normal compliance, for which we obtained existence and equivalence results.

The present paper represents a continuation of [3]. Our aim is twofold. The first one is to prove that the existence result of [3] remains valid under significantly weaker assumption on the corresponding multivalued operators. The second aim is to illustrate the use of this result in the study of a new mathematical model which describes the equilibrium of an elastic body attached to a nonlinear spring.

The manuscript is organized as follows. In Section 2 we recall the statement of the problem together with the existence result obtained in [3]. In Section 3, we state and prove our main abstract result, Theorem 2. Next, in Section 4, we introduce the elastic problem, list the assumption on the data and state its variational formulation. Finally, in Section 5, we apply Theorem 2 to prove its weak solvability.

2 Problem statement

Everywhere in this paper, we assume that $(V, \|\cdot\|)$ is a reflexive Banach space, we denote by V^* its topological dual, and $\langle \cdot, \cdot \rangle$ will represent the duality pairing between V and V^* . We use 0_V for the zero element of the space V and $\|\cdot\|_*$ for the norm on V^* . Assume in addition that $(U, \|\cdot\|_U)$ is a reflexive Banach space of topological dual U^* . We denote by $\|\cdot\|_{U^*}$ the norm on U^* and by $\langle \cdot, \cdot \rangle_{U^* \times U}$ the duality pairing between U and U^* . The symbol $w-U$ will represent the space U endowed with the weak topology while $s-U^*$ will represent the space U^* endowed with the strong topology. For a set D in a Banach space E , we define $\|D\|_E = \sup\{\|u\|_E \mid u \in D\}$.

Consider a set $K \subset V$, a single-valued operator $A: V \rightarrow V^*$, a multivalued operator $B: U \rightarrow 2^{U^*}$, a linear, continuous and compact operator $\iota: V \rightarrow U$, and a functional $f \in V^*$. We denote by $\|\iota\|$ the norm of ι in the space of linear continuous operators from V to U . With these data we state the following inequality problem.

Problem 1. Find $u \in V$ and $\xi \in U^$ such that*

$$u \in K, \quad \langle Au - f, v - u \rangle + \langle \xi, \iota(v - u) \rangle_{U^* \times U} \geq 0 \quad \text{for all } v \in K,$$

$$\xi \in B(\iota u).$$

Note that, since the operator B is multivalued, we refer to Problem 1 as a multivalued variational inequality. Let $\iota^*: U^* \rightarrow V^*$ be the adjoint operator to ι . Then, using the definition of the subgradient of the indicator function of

the set K , denoted $\partial_c J_K$, it is easy to see that Problem 1 is equivalent with the following subdifferential inclusion: find $u \in V$ such that

$$Au + \iota^* B(\iota u) + \partial_c J_K(u) \ni f \text{ in } V^*.$$

In the study of Problem 1 we need the following assumptions.

$H(A)$: $A: V \rightarrow V^*$ is an operator such that

- (1) A is coercive, i.e., $\langle Au, u \rangle \geq \alpha \|u\|^2 - \beta$ for all $u \in V$ with $\alpha, \beta > 0$.
- (2) $\|Au\|_{V^*} \leq a_1 + a_2 \|u\|$ for all $u \in V$ with $a_1, a_2 > 0$.
- (3) A is pseudomonotone, i.e. it is bounded and $u_n \rightarrow u$ weakly in V and $\limsup \langle Au_n, u_n - u \rangle \leq 0$ imply $\langle Au, u - v \rangle \leq \liminf \langle Au_n, u_n - v \rangle$ for all $v \in V$.
- (4) A is strictly monotone on K , i.e., for all $u, v \in K$ such that $u \neq v$, we have $\langle Au - Av, u - v \rangle > 0$.

$H(B)$: $B: U \rightarrow 2^{U^*}$ is an operator such that

- (1) B has nonempty and convex values.
- (2) the graph of B is sequentially closed in $(s-U) \times (w-U^*)$ topology.
- (3) $\|Bw\|_{U^*} \leq b_1 + b_2 \|w\|_U$ for all $w \in U$ with $b_1, b_2 > 0$.
- (4) the smallness condition $b_2 \|\iota\|^2 < \alpha$ holds.

$H(K)$: K is a nonempty, convex and closed subset of V .

The following result, obtained in [3], provides the solvability of Problem 1.

Theorem 1. *Under hypotheses $H(A)$, $H(B)$ and $H(K)$ Problem 1 has at least one solution.*

The proof of Theorem 1 is based on arguments of elliptic variational inequalities, various estimates and a version of the Kakutani-Fan-Glicksberg fixed point theorem.

3 An abstract existence result

We now consider the following assumptions on the data of the Problem 1.

$H(B)$: $B: U \rightarrow 2^{U^*}$ is an operator such that

- (5) the operator $B: U \rightarrow 2^{U^*}$ is bounded, i.e., it maps bounded sets in U into bounded sets in U^* .
- (6) there exist constants $b_3, b_4 \geq 0$ with $b_4 \|\iota\|^2 < \alpha$ such that for all $u \in U$ and all $\xi \in Bu$ we have

$$\langle \xi, u \rangle_{U^* \times U} \geq -b_3 - b_4 \|u\|_U^2.$$

Our main abstract existence result in the study of Problem 1 is the following.

Theorem 2. *Assume hypotheses $H(A)$, $H(B)(1)$, (2), (5) and (6), $H(K)$ and $0_V \in K$. Then Problem 1 has a solution.*

Proof. Fix $N \geq 1$ and define a cut-off operator $B^N : U \rightarrow 2^{U^*}$ by

$$B^N(v) = \begin{cases} Bv & \text{if } \|v\|_U \leq N \\ B\left(\frac{Nv}{\|v\|_U}\right) & \text{otherwise} \end{cases}$$

for $v \in U$. Since B satisfies assumptions $H(B)(1)$ and (2), it is straightforward to verify that the operator B^N satisfies these two assumptions as well. Moreover, since B satisfies $H(B)(5)$, it follows that B^N satisfies $H(B)(3)$ with a constant $b_1 > 0$ which depends on N and $b_2 > 0$ arbitrary small and independent of N . It is also clear that B^N satisfies $H(B)(4)$. We are now in a position to apply Theorem 1. Thus, we deduce that there exists $(u^N, \xi^N) \in V \times U^*$ such that

$$\begin{aligned} u^N \in K, \quad \langle Au^N - f, v - u^N \rangle + \langle \xi^N, \iota(v - u^N) \rangle_{U^* \times U} &\geq 0 \text{ for all } v \in K, \quad (1) \\ \xi^N \in B^N(\iota u^N). \end{aligned}$$

The hypothesis $0_V \in K$ allows to test (1) with $v = 0_V$. As a result, we obtain

$$\langle Au^N, u^N \rangle + \langle \xi^N, \iota u^N \rangle_{U^* \times U} \leq \|f\|_{V^*} \|u^N\|. \quad (2)$$

We estimate from below the expression $\langle \xi^N, \iota u^N \rangle_{U^* \times U}$. If $\|\iota u^N\|_U \leq N$, then $\xi^N \in Bu^N$ and

$$\langle \xi^N, \iota u^N \rangle_{U^* \times U} \geq -b_3 - b_4 \|\iota\|^2 \|u^N\|^2.$$

If, in contrast, $\|\iota u^N\|_U > N$, then $\xi^N \in B\left(\frac{N \iota u^N}{\|\iota u^N\|_U}\right)$, and assumption $H(B)(6)$ yields

$$\begin{aligned} \langle \xi^N, \iota u^N \rangle_{U^* \times U} &= \left\langle \xi^N, \frac{N \iota u^N}{\|\iota u^N\|_U} \right\rangle_{U^* \times U} \frac{\|\iota u^N\|_U}{N} \geq (-b_3 - b_4 N^2) \frac{\|\iota u^N\|_U}{N} \\ &\geq -b_3 \|\iota\| \|u^N\| - b_4 \|\iota\|^2 \|u^N\|^2. \end{aligned}$$

In either case, we have

$$\langle \xi^N, \iota u^N \rangle_{U^* \times U} \geq -b_3 - b_3 \|\iota\| \|u^N\| - b_4 \|\iota\|^2 \|u^N\|^2. \quad (3)$$

Combining inequality (3) with $H(A)(1)$ and (2) implies

$$(\alpha - b_4 \|\iota\|^2) \|u^N\|^2 \leq b_3 + (\|f\|_{V^*} + b_3 \|\iota\|) \|u^N\|. \quad (4)$$

Next, using (4) and $H(B)(6)$, we deduce that $\|u^N\|$ is bounded by a constant independent of N . Therefore, since $\|\iota u^N\|_U \leq \|\iota\| \|u^N\|$, it follows that $\|\iota u^N\|_U$ is also bounded by a constant independent of N . We now take N large enough so that the truncation in the definition of the operator B^N is inactive. It follows that u^N also solves Problem 1, which completes the proof. \square

We end this section with some comments on the assumptions on the multi-valued operator B . First, we note that, clearly, condition $H(B)(5)$ is significantly weaker than $H(B)(3)$. Moreover, we stress that assumption $H(B)(6)$ is weaker than $H(B)(3)$ and $H(B)(4)$. In addition, condition $H(B)(6)$ has various physical interpretations, when dealing with examples arising in mechanics. We also mention that if $\langle \xi, u \rangle_{U^* \times U} \geq 0$ for all $u \in U$ and $\xi \in Bu$, then this condition is satisfied.

4 Contact problem in elasticity

In this section we consider a boundary value problem which models a contact problem for elastic material and for which the abstract result of Section 3 can be applied.

The physical setting is the following. An elastic body occupies an open, bounded and connected set in $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) with a Lipschitz boundary $\partial\Omega = \Gamma$. The concept of measurability, used below, is considered with respect to the $d-1$ dimensional Hausdorff measure, denoted by m . The set Γ is partitioned into three disjoint and measurable parts Γ_1, Γ_2 and Γ_3 such that $m(\Gamma_1) > 0$. The body is clamped on Γ_1 , is submitted to surface tractions on Γ_2 and is attached to a nonlinear spring on Γ_3 .

We use the symbol \mathbb{S}^d for the space of second order symmetric $d \times d$ matrices. The canonical inner product on \mathbb{R}^d and \mathbb{S}^d will be denoted by “ \cdot ” and the Euclidean norm on the space \mathbb{S}^d will be denote by $\|\cdot\|_{\mathbb{S}^d}$. We also use the notation ν for the outward unit normal at Γ and, for a vector field \mathbf{v} , v_ν and \mathbf{v}_τ will represent the normal and tangential components of \mathbf{v} on Γ given by $v_\nu = \mathbf{v} \cdot \nu$ and $\mathbf{v}_\tau = \mathbf{v} - v_\nu \nu$, respectively. The normal and tangential components of the stress field σ on the boundary are defined by $\sigma_\nu = (\sigma \nu) \cdot \nu$ and $\sigma_\tau = \sigma \nu - \sigma_\nu \nu$. The mathematical model which describe the equilibrium of the elastic body in the physical setting above is the following.

Problem 2. Find a displacement field $\mathbf{u}: \Omega \rightarrow \mathbb{R}^d$, a stress field $\sigma: \Omega \rightarrow \mathbb{S}^d$ and the reactive interface force $\xi: \Gamma_3 \rightarrow \mathbb{R}$ such that

$$\sigma = \mathcal{F}\varepsilon(\mathbf{u}) \quad \text{in } \Omega, \tag{5}$$

$$\text{Div } \sigma + \mathbf{f}_0 = \mathbf{0} \quad \text{in } \Omega, \tag{6}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1, \tag{7}$$

$$\sigma \nu = \mathbf{f}_2 \quad \text{on } \Gamma_2, \tag{8}$$

$$\left. \begin{aligned} -g_1 \leq u_\nu \leq g_2, \quad \xi \in h(u_\nu) \\ u_\nu = -g_1 \implies -\sigma_\nu \leq \xi, \\ u_\nu = g_2 \implies -\sigma_\nu \geq \xi, \\ g_1 < u_\nu < g_2 \implies -\sigma_\nu = \xi \end{aligned} \right\} \quad \text{on } \Gamma_3, \tag{9}$$

$$\sigma_\tau = \mathbf{0} \quad \text{on } \Gamma_3. \tag{10}$$

Equation (5) is the constitutive law for elastic materials in which \mathcal{F} represents the elasticity operator and $\varepsilon(\mathbf{u})$ denotes the linearized strain tensor. Equation (6) is the equilibrium equation in which \mathbf{f}_0 represents the density of body forces. Conditions (7) and (8) are the displacement and traction conditions, respectively, in which \mathbf{f}_2 denotes the density of surface tractions. Condition (9) represents the interface law in which h is a multivalued function which will be described below and, finally, condition (10) shows that the shear on the surface Γ_3 vanishes.

We now provide additional comments on the conditions (9) which represent the novelty of our elastic model. First, this condition shows that spring prevents the normal displacement of the body in such a way that $-g_1 \leq u_\nu \leq g_2$. When $-g_1 < u_\nu < g_2$ the spring is active and exerts a normal reaction on the body, denoted ξ , which depends on the the normal displacement, i.e., $-\sigma_\nu = \xi \in h(u_\nu)$. Note that, for physical reason, h must be negative for a positive argument (since, in this case the spring is in compression and, therefore, its reaction is towards the body) and positive for a negative argument (since, in this case the spring is in extension and, therefore, it pulls the body). A typical example is the function $h(r) = kr$ which models the behavior of a linear spring of stiffness $k > 0$. Nevertheless, from mathematical point of view, we do not need this restriction, as shown in assumption $H(h)$ below. When $u_\nu = g_2$ the spring is completely compressed, thus, besides its reaction, an additional force oriented towards the body becomes active and blocks the normal displacements u_ν . Therefore, in this case we have $-\sigma_\nu \geq \xi$. When $u_\nu = -g_1$ the spring is completely extended and, besides its reaction, an extra force pulling body becomes active, which implies that $-\sigma_\nu \leq \xi$.

In the study of Problem 2, we use standard notation for Lebesgue and Sobolev spaces. For the displacement we use the space

$$V = \{ \mathbf{u} \in H^1(\Omega; \mathbb{R}^d) \mid \mathbf{u} = \mathbf{0} \text{ on } \Gamma_1 \}.$$

It is well known that the trace operator $\gamma: V \rightarrow L^2(\partial\Omega; \mathbb{R}^d)$ is compact. Moreover, we put $U = L^2(\Gamma_3)$ and define the operator $\iota: V \rightarrow U$ by $\iota(\mathbf{u}) = u_\nu|_{\Gamma_3}$ for all $\mathbf{u} \in V$. For the spaces V and U , we use the notation already used for the corresponding abstract spaces in Sections 2 and 3. In particular, $\|\cdot\|$ and $\|\cdot\|_*$ will represent the norm on V and V^* , respectively, $\langle \cdot, \cdot \rangle$ denotes the duality pairing between V and V^* , $\|\cdot\|_U$ and $\|\cdot\|_{U^*}$ are the norms on U and U^* .

We also need the following assumptions on the problem data.

$H(g)$: $g_1, g_2 \geq 0$ are two real constants.

$H(\mathcal{F})$: $\mathcal{F}: \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d$ is such that

- (1) there exists $L_{\mathcal{F}} > 0$ such that $\|\mathcal{F}(\mathbf{x}, \varepsilon_1) - \mathcal{F}(\mathbf{x}, \varepsilon_2)\|_{\mathbb{S}^d} \leq L_{\mathcal{F}} \|\varepsilon_1 - \varepsilon_2\|_{\mathbb{S}^d}$ for all $\varepsilon_1, \varepsilon_2 \in \mathbb{S}^d$, a.e. $\mathbf{x} \in \Omega$.
- (2) there exists $m_{\mathcal{F}} > 0$ such that $(\mathcal{F}(\mathbf{x}, \varepsilon_1) - \mathcal{F}(\mathbf{x}, \varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq m_{\mathcal{F}} \|\varepsilon_1 - \varepsilon_2\|_{\mathbb{S}^d}^2$ for all $\varepsilon_1, \varepsilon_2 \in \mathbb{S}^d$, a.e. $\mathbf{x} \in \Omega$.
- (3) $\mathcal{F}(\mathbf{x}, \varepsilon)$ is measurable on Ω for all $\varepsilon \in \mathbb{S}^d$.
- (4) $\mathcal{F}(\mathbf{x}, \mathbf{0}_{\mathbb{S}^d}) = \mathbf{0}_{\mathbb{S}^d}$ a.e. $\mathbf{x} \in \Omega$.

- $\underline{H}(h)$: $h: \Gamma_3 \times [-g_1, g_2] \rightarrow 2^{\mathbb{R}}$ is a multifunction such that
- (1) the sets $h(\mathbf{x}, r)$ are nonempty and convex for all $r \in [-g_1, g_2]$, a.e. $\mathbf{x} \in \Gamma_3$.
 - (2) $h(\cdot, r)$ has a measurable selection for all $r \in [-g_1, g_2]$.
 - (3) the graph of multifunction $h(\mathbf{x}, \cdot)$ is closed in \mathbb{R}^2 for a.e. $\mathbf{x} \in \Gamma_3$.
 - (4) $|h(\mathbf{x}, r)| \leq \bar{h}$ for all $r \in [-g_1, g_2]$ and a.e. $\mathbf{x} \in \Gamma_3$ with $\bar{h} \geq 0$.

- $\underline{H}(f)$: the densities of body forces and surface tractions are such that
- (1) if $d = 2$, then $\mathbf{f}_0 \in L^{p'}(\Omega; \mathbb{R}^2)$, $\mathbf{f}_2 \in L^{p'}(\Gamma_2; \mathbb{R}^2)$ for some $p' \in (1, \infty)$.
 - (2) if $d = 3$, then $\mathbf{f}_0 \in L^{\frac{6}{5}}(\Omega; \mathbb{R}^3)$, $\mathbf{f}_2 \in L^{\frac{4}{3}}(\Gamma_2; \mathbb{R}^3)$.

We now turn to the variational formulations of Problem 2. To this end, we introduce the set of admissible displacements fields

$$K = \{ \mathbf{u} \in V \mid u_\nu \in [-g_1, g_2] \text{ a.e. on } \Gamma_3 \}. \tag{11}$$

Also, we define the operator $A: V \rightarrow V^*$ by the formula

$$\langle A\mathbf{u}, \mathbf{v} \rangle = \int_{\Omega} \mathcal{F}\varepsilon(\mathbf{u}) \cdot \varepsilon(\mathbf{v}) \, dx \quad \text{for all } \mathbf{u}, \mathbf{v} \in V. \tag{12}$$

Note that, since $H(\mathcal{F})$ holds, the operator A is well defined. Next, we note that the hypothesis $H(f)$ implies that the mapping

$$V \ni \mathbf{v} \mapsto \int_{\Omega} \mathbf{f}_0(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) \, dx + \int_{\Gamma_2} \mathbf{f}_2(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) \, d\Gamma \in \mathbb{R},$$

is linear and continuous. Indeed, if $d = 2$, then the embedding $V \subset L^r(\Omega; \mathbb{R}^2)$ and the restriction of the trace operator $\gamma_{\Gamma_2}: V \rightarrow L^r(\Gamma_2; \mathbb{R}^2)$ is linear and continuous, for any $r \in (1, \infty)$. On the other hand, if $d = 3$, then the embedding $V \subset L^6(\Omega; \mathbb{R}^3)$ and the restriction of the trace operator $\gamma_{\Gamma_2}: V \rightarrow L^4(\Gamma_2; \mathbb{R}^3)$ is linear and continuous. Hence, we can define $\mathbf{f} \in V^*$ by the formula

$$\langle \mathbf{f}, \mathbf{v} \rangle = \int_{\Omega} \mathbf{f}_0(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) \, dx + \int_{\Gamma_2} \mathbf{f}_2(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) \, d\Gamma.$$

Next, with a slight abuse of notation, we extend the multifunction h to $\Gamma_3 \times \mathbb{R}$ by setting $h(\mathbf{x}, s) = h(\mathbf{x}, -g_1)$ and $h(\mathbf{x}, s) = h(\mathbf{x}, g_2)$ for $s < -g_1$ and $s > g_2$, respectively. We use the same symbol h to denote this extended multifunction and we introduce the multifunction $B: U \rightarrow 2^{U^*}$ by the formula

$$B(u) = \{ \xi \in U^* \mid \xi(\mathbf{x}) \in h(\mathbf{x}, u(\mathbf{x})) \text{ a.e. on } \Gamma_3 \} \quad \text{for all } u \in U. \tag{13}$$

The variational formulation for Problem 2, obtained by using standard arguments, reads as follows.

Problem 3. Find a displacement field $\mathbf{u} \in V$ and a contact interface force $\xi \in U$ such that

$$\begin{aligned} \mathbf{u} \in K, \quad \langle A\mathbf{u}, \mathbf{v} - \mathbf{u} \rangle + \langle \xi, \iota(\mathbf{v} - \mathbf{u}) \rangle_{U^* \times U} &\geq \langle \mathbf{f}, \mathbf{v} - \mathbf{u} \rangle \quad \text{for all } \mathbf{v} \in K, \\ \xi &\in B(\iota\mathbf{u}). \end{aligned}$$

Note that Problem 3 represents a multivalued variational inequality. Its solvability will be proved in the next section, based on the abstract existence result provided by Theorem 2.

5 Existence of the solution

The main result in this section is the following.

Theorem 3. *Under hypotheses $H(g)$, $H(\mathcal{F})$, $H(h)$ and $H(f)$, Problem 3 has at least one solution.*

The proof of the theorem is carried out in several steps, based on two lemmas that we state and prove in what follows. The first lemma, already proved in [3], is given here for the convenience of the reader.

Lemma 1. *Assume $H(\mathcal{F})$. Then, the operator $A: V \rightarrow V^*$ defined by (12) satisfies conditions $H(A)(1)$ –(4) with $\alpha = m_{\mathcal{F}}$ in $H(A)(1)$.*

Proof. By conditions $H(\mathcal{F})(1)$ and (3), we have

$$\left| \int_{\Omega} \mathcal{F}\varepsilon(\mathbf{u}) \cdot \varepsilon(\mathbf{v}) \, dx \right| \leq L_{\mathcal{F}} \|\mathbf{u}\| \|\mathbf{v}\| \quad \text{for all } \mathbf{u}, \mathbf{v} \in V.$$

This implies that $\|A\mathbf{u}\|_* \leq L_{\mathcal{F}} \|\mathbf{u}\|$ for all $\mathbf{u} \in V$, which proves $H(A)(2)$. In addition, assumption $H(\mathcal{F})(2)$ yields $\langle A\mathbf{u} - A\mathbf{v}, \mathbf{u} - \mathbf{v} \rangle \geq m_{\mathcal{F}} \|\mathbf{u} - \mathbf{v}\|^2$ for all $\mathbf{u}, \mathbf{v} \in V$. This shows that condition $H(A)(4)$ is satisfied. Furthermore, for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, by $H(\mathcal{F})(1)$, we have $\langle A\mathbf{u} - A\mathbf{v}, \mathbf{w} \rangle \leq L_{\mathcal{F}} \|\mathbf{u} - \mathbf{v}\|_V \|\mathbf{w}\|$ for all $\mathbf{u}, \mathbf{v} \in V$. This proves that $\|A\mathbf{u} - A\mathbf{v}\|_* \leq L_{\mathcal{F}} \|\mathbf{u} - \mathbf{v}\|_V$ for all $\mathbf{u}, \mathbf{v} \in V$, which implies that A is Lipschitz continuous and hence hemicontinuous. Since we already know that A is bounded and monotone, by Proposition 27.6 in [11], it follows that A is pseudomonotone and $H(A)(3)$ holds. By $H(\mathcal{F})(4)$, we have $A\mathbf{0}_V = \mathbf{0}_V$. Thus, from (5), we get $\langle A\mathbf{u}, \mathbf{u} \rangle \geq m_{\mathcal{F}} \|\mathbf{u}\|^2$ for all $\mathbf{u} \in V$. Therefore, $H(A)(1)$ holds, which completes the proof. \square

Next, we proceed with the following result.

Lemma 2. *Assume $H(g)$ and $H(h)$. Then, the multivalued operator B defined by (13) satisfies $H(B)(1)$, (2), (5) and (6).*

Proof. First we prove $H(B)(1)$. Convexity of values of B is a simple consequence of the convexity in $H(h)(1)$. To prove nonemptiness, let $u \in U$ and $(u_n)_{n=1}^\infty$ be a sequence of simple (i.e. piecewise constant) functions converging to u for a.e. $\mathbf{x} \in \Gamma_3$. The hypothesis $H(h)(2)$ implies that the multifunction $\Gamma_3 \ni \mathbf{x} \rightarrow h(\mathbf{x}, u_n(\mathbf{x})) \subset \mathbb{R}$ has a measurable selection ξ_n for all $n \in \mathbb{N}$. From $H(h)(4)$, it follows that $\|\xi_n\|_{U^*}^2 = \int_{\Gamma_3} \xi_n(\mathbf{x})^2 d\Gamma \leq \bar{h}^2 m(\Gamma_3)$, so passing to a subsequence, if necessary, we may assume that $\xi_n \rightarrow \xi$ weakly in U^* with $\xi \in U^*$. As $\xi_n(\mathbf{x}) \in [-\bar{h}, \bar{h}]$ for a.e. $\mathbf{x} \in \Gamma_3$, we are in a position to apply Proposition 3.16 of [5] to obtain $\xi(\mathbf{x}) \in \overline{\text{conv}} \limsup_{n \rightarrow \infty} (\xi_n(\mathbf{x}))$ for a.e. $\mathbf{x} \in \Gamma_3$, where $\limsup_{n \rightarrow \infty}$ is the Kuratowski-Painlevé upper limit of sets defined by

$$\limsup_{n \rightarrow \infty} A_n = \{ s \in \mathbb{R} \mid s_{n_k} \rightarrow s, s_{n_k} \in A_{n_k} \}$$

for a sequence of sets $A_n \subset \mathbb{R}$. Hence $\xi(\mathbf{x}) \in \overline{\text{conv}} \limsup_{n \rightarrow \infty} h(\mathbf{x}, u_n(\mathbf{x}))$ for a.e. $\mathbf{x} \in \Gamma_3$. From $H(h)(1)$, (3) and the pointwise convergence $u_n(\mathbf{x}) \rightarrow u(\mathbf{x})$ a.e. on Γ_3 , it follows that

$$\xi(\mathbf{x}) \in \overline{\text{conv}} \limsup_{n \rightarrow \infty} h(\mathbf{x}, u_n(\mathbf{x})) \subset \overline{\text{conv}} h(\mathbf{x}, u(\mathbf{x})) = h(\mathbf{x}, u(\mathbf{x})),$$

for a.e. $\mathbf{x} \in \Gamma_3$ and ξ is the sought measurable selection.

We pass to the proof of $H(B)(2)$. Let $u_n \rightarrow u$ strongly in U and $\xi_n \rightarrow \xi$ weakly in U^* be such that $\xi_n \in B(u_n)$. We show that $\xi \in B(u)$. At least for a subsequence, not relabeled, we have $u_n(\mathbf{x}) \rightarrow u(\mathbf{x})$ for a.e. $\mathbf{x} \in \Gamma_3$. The proof that $\xi(\mathbf{x}) \in h(\mathbf{x}, u(\mathbf{x}))$ for a.e. $\mathbf{x} \in \Gamma_3$ follows the line of the corresponding proof of condition $H(B)(1)$.

By a straightforward calculation and $H(h)(4)$, we deduce that hypothesis $H(B)(5)$ holds. To prove $H(B)(6)$, let $u \in U$ and $\xi \in Bu$. From the following inequality

$$\begin{aligned} \langle \xi, u \rangle_{U^* \times U} &= \int_{\Gamma_3} \xi(\mathbf{x}) u(\mathbf{x}) d\Gamma \geq -\bar{h} \int_{\Gamma_3} |u(\mathbf{x})| d\Gamma \geq -\bar{h} m(\Gamma_3)^{\frac{1}{2}} \|u\|_U \\ &\geq -\frac{\alpha}{2\|\iota\|^2} \|u\|_U^2 - \frac{\|\iota\|^2 \bar{h}^2 m(\Gamma_3)}{2\alpha}, \end{aligned}$$

we deduce that $H(B)(6)$. The proof is complete. □

We are now in a position to provide the proof of Theorem 3.

Proof. Note that Problem 3 is a particular case of Problem 1. Indeed, both V and U are reflexive Banach spaces, and the normal trace operator ι is linear, continuous and compact. Moreover, the set K defined by (11) is nonempty, convex, closed in V , and $\mathbf{0}_V \in K$. In addition, from Lemma 1, it follows that $H(A)(1)$ –(4) hold, and Lemma 2 shows that the operator B satisfies $H(B)(1)$, (2), (5) and (6). Theorem 3 is now a direct consequence of Theorem 2. □

Theorem 3 provides the weak solvability of the contact Problem 2, since once the displacement field is obtained by solving Problem 3, then the stress field σ is determined by using the constitutive law (5). The question of the uniqueness of the solution is left open.

Remark 1. Note that because of $H(h)(4)$, the multivalued map given by (13) satisfies hypothesis $H(B)(4)$, and Theorem 1 is sufficient to obtain the existence result in Theorem 3. However, if in the place of (9), we consider the law $-\sigma_\nu = ku_\nu$ with $k > 0$, and set $K = V$, then to apply Theorem 1, we need a smallness assumption on the constant k . On the other hand, the above law satisfies $H(B)(5)$ –(6) without any limitations on the value of k .

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