

# DOUBLE CONVERGENCE OF A FAMILY OF DISCRETE DISTRIBUTED MIXED ELLIPTIC OPTIMAL CONTROL PROBLEMS WITH A PARAMETER

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**Abstract.** The convergence of a family of continuous distributed mixed elliptic optimal control problems  $(P_\alpha)$ , governed by elliptic variational equalities, when the parameter  $\alpha \rightarrow \infty$  was studied in Gariboldi - Tarzia, Appl. Math. Optim., 47 (2003), 213-230 and it has been proved that it is convergent to a distributed mixed elliptic optimal control problem  $(P)$ . We consider the discrete approximations  $(P_{h\alpha})$  and  $(P_h)$  of the optimal control problems  $(P_\alpha)$  and  $(P)$  respectively, for each  $h > 0$  and  $\alpha > 0$ . We study the convergence of the discrete distributed optimal control problems  $(P_{h\alpha})$  and  $(P_h)$  when  $h \rightarrow 0$ ,  $\alpha \rightarrow \infty$  and  $(h, \alpha) \rightarrow (0, +\infty)$  obtaining a complete commutative diagram, including the diagonal convergence, which relates the continuous and discrete distributed mixed elliptic optimal control problems  $(P_{h\alpha}), (P_\alpha), (P_h)$  and  $(P)$  by taking the corresponding limits. The convergent corresponds to the optimal control, and the system and adjoint system states in adequate functional spaces.

**Keywords.** Double convergence, Distributed optimal control problems, Elliptic variational equalities, Mixed boundary conditions, Numerical analysis, Finite element method, Fixed points, Optimality conditions, Error estimations.

## 1 Introduction

The purpose of this paper is to do the numerical analysis, by using the finite element method, of the convergence of the continuous distributed mixed optimal control problems with respect to a parameter (the heat transfer coefficient) given in [10, 11] obtaining a double convergence when the parameter of the finite element method goes to zero and the heat transfer coefficient goes to infinity.

We consider a bounded domain  $\Omega \subset \mathbf{R}^n$  whose regular boundary  $\Gamma = \partial\Omega = \Gamma_1 \cup \Gamma_2$  consists of the union of two disjoint portions  $\Gamma_1$  and  $\Gamma_2$  with  $\text{meas}(\Gamma_1) > 0$ . We consider the following elliptic partial differential problems with mixed boundary conditions, given by:

$$-\Delta u = g \quad \text{in } \Omega; \quad u = b \quad \text{on } \Gamma_1; \quad -\frac{\partial u}{\partial n} = q \quad \text{on } \Gamma_2, \quad (1)$$

$$-\Delta u = g \quad \text{in } \Omega; \quad -\frac{\partial u}{\partial n} = \alpha(u - b) \quad \text{on } \Gamma_1; \quad -\frac{\partial u}{\partial n} = q \quad \text{on } \Gamma_2 \quad (2)$$

where  $g$  is the internal energy in  $\Omega$ ,  $b = \text{Const.} > 0$  is the temperature on  $\Gamma_1$  for the system (1) and the temperature of the external neighborhood on  $\Gamma_1$  for the system (2) respectively,  $q$  is the heat flux on  $\Gamma_2$  and  $\alpha > 0$  is the heat transfer coefficient on  $\Gamma_1$ . The systems (1) and (2) can represent the steady-state two-phase Stefan problem for adequate data [21, 22]. We consider the following continuous distributed optimal control problem ( $P$ ) and a family of continuous distributed optimal control problems ( $P_\alpha$ ) for each parameter  $\alpha > 0$ , defined in [10], where the control variable is the internal energy  $g$  in  $\Omega$ , that is: Find the continuous distributed optimal controls  $g_{op} \in H = L^2(\Omega)$  and  $g_{\alpha_{op}} \in H$  (for each  $\alpha > 0$ ) such that:

$$\text{Problem (P): } J(g_{op}) = \min_{g \in H} J(g), \quad \text{Problem (P}_\alpha\text{): } J_\alpha(g_{\alpha_{op}}) = \min_{g \in H} J_\alpha(g) \quad (3)$$

where the quadratic cost functional  $J, J_\alpha : H \rightarrow \mathbb{R}_0^+$  are defined by [2, 18, 26]:

$$a) J(g) = \frac{1}{2} \|u_g - z_d\|_H^2 + \frac{M}{2} \|g\|_H^2, \quad b) J_\alpha(g) = \frac{1}{2} \|u_{\alpha g} - z_d\|_H^2 + \frac{M}{2} \|g\|_H^2 \quad (4)$$

with  $M > 0$  and  $z_d \in H$  given,  $u_g \in K$  and  $u_{\alpha g} \in V$  are the state of the systems defined by the mixed elliptic differential problems (1) and (2) respectively whose elliptic variational equality are given by [16]:

$$u_g \in K: \quad a(u_g, v) = (g, v) - \int_{\Gamma_2} q v d\gamma, \quad \forall v \in V_0 \quad (5)$$

$$u_{\alpha g} \in V: \quad a_\alpha(u_{\alpha g}, v) = (g, v) - \int_{\Gamma_2} q v d\gamma + \alpha \int_{\Gamma_1} b v d\gamma, \quad \forall v \in V \quad (6)$$

and their adjoint system states  $p_g \in V$  and  $p_{\alpha g} \in V$  are defined by the following elliptic variational equalities:

$$a) p_g \in V_0: a(p_g, v) = (u_g - z_d, v), \quad \forall v \in V_0; \quad b) p_{\alpha g} \in V: a_\alpha(p_{\alpha g}, v) = (u_{\alpha g} - z_d, v), \quad \forall v \in V \quad (7)$$

with the spaces and bilinear forms defined by:

$$V = H^1(\Omega), V_0 = \{v \in V, v|_{\Gamma_1} = 0\}, K = b + V_0, H = L^2(\Omega), Q = L^2(\Gamma_2) \quad (8)$$

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v dx, \quad a_{\alpha}(u, v) = a(u, v) + \alpha \int_{\Gamma_1} uv d\gamma, \quad (u, v) = \int_{\Omega} uv dx \quad (9)$$

where the bilinear, continuous and symmetric forms  $a$  and  $a_{\alpha}$  are coercive on  $V_0$  and  $V$  respectively, that is [16]:

$$\exists \lambda > 0 \text{ such that } \lambda \|v\|_V^2 \leq a(v, v), \quad \forall v \in V_0 \quad (10)$$

$$\exists \lambda_{\alpha} = \lambda_1 \min(1, \alpha) > 0 \text{ such that } \lambda_{\alpha} \|v\|_V^2 \leq a_{\alpha}(v, v), \quad \forall v \in V \quad (11)$$

and  $\lambda_1 > 0$  is the coercive constant for the bilinear form  $a_1$  [16, 21].

The unique continuous distributed optimal energies  $g_{op}$  and  $g_{\alpha, op}$  have been characterized in [10] as a fixed point on  $H$  for a suitable operators  $W$  and  $W_{\alpha}$  over their optimal adjoint system states  $p_{g_{op}} \in V_0$  and  $p_{\alpha g_{\alpha, op}} \in V$  defined by:

$$W, W_{\alpha} : H \rightarrow H \quad \text{such that} \quad a) W(g) = -\frac{1}{M} p_g, \quad b) W_{\alpha}(g) = -\frac{1}{M} p_{\alpha g}. \quad (12)$$

The limit of the optimal control problem ( $P_{\alpha}$ ) when  $\alpha \rightarrow \infty$  was studied in [10] and it was proven that:

$$\lim_{\alpha \rightarrow \infty} \|u_{\alpha g_{\alpha, op}} - u_{g_{op}}\|_V = 0, \quad \lim_{\alpha \rightarrow \infty} \|p_{\alpha g_{\alpha, op}} - p_{g_{op}}\|_V = 0, \quad \lim_{\alpha \rightarrow \infty} \|g_{\alpha, op} - g_{op}\|_H = 0 \quad (13)$$

for a large constant  $M > 0$  by using the characterization of the optimal controls as fixed points through operators (12a) and (12b); this restrictive hypothesis on data was eliminated in [11] by using the variational formulations. We can summarize the conditions (13) saying that the distributed optimal control problems ( $P_{\alpha}$ ) converges to the distributed optimal control problem (P) when  $\alpha \rightarrow +\infty$ .

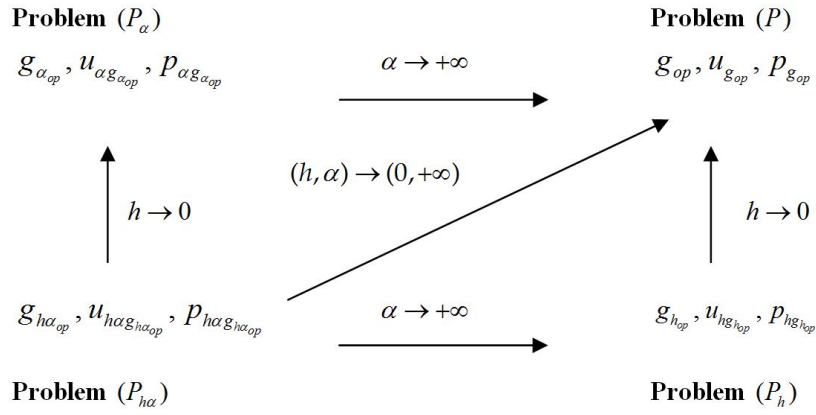
Now, we consider the finite element method and a polygonal domain  $\Omega \subset \mathbf{R}^n$  with a regular triangulation with Lagrange triangles of type 1, constituted by affine-equivalent finite element of class  $C^0$  being  $h$  the parameter of the finite element approximation which goes to zero [3,7]. Then, we discretize the elliptic variational equalities for the system states (6) and (5), the adjoint system states (7a) and (7b), and the cost functional (4a,b) respectively. In general, the solution of a mixed elliptic boundary problem belongs to  $H^r(\Omega)$  with  $1 < r \leq \frac{3}{2} - \varepsilon$  ( $\varepsilon > 0$ ) but there exist some examples which solutions belong to  $H^r(\Omega)$  with  $2 \leq r$  [1, 17, 20]. Note that mixed boundary conditions play an important role in various applications, e.g. heat conduction and electric potential problems [12].

The goal of this paper is to study the numerical analysis, by using the finite element method, of the convergence results (13) corresponding to the continuous distrib-

uted elliptic optimal control problems  $(P_\alpha)$  and  $(P)$  when  $\alpha \rightarrow +\infty$ . The main result of this paper can be characterized by the following result:

**Theorem 1**

We have the following complete commutative diagram which relates the continuous distributed mixed optimal control problems  $(P_\alpha)$  and  $(P)$ , with the discrete distributed mixed optimal control problems  $(P_{h\alpha})$  and  $(P_h)$  and it is obtained by taking the limits  $h \rightarrow 0, \alpha \rightarrow +\infty$  and  $(h, \alpha) \rightarrow (0, +\infty)$ , as in Figure 1, where  $g_{h\alpha_{op}}, u_{h\alpha g_{h\alpha_{op}}}$  and  $p_{h\alpha g_{h\alpha_{op}}}$  are respectively the optimal control, the system and the adjoint system states of the discrete distributed mixed optimal control problem  $(P_{h\alpha})$  for each  $h > 0$  and  $\alpha > 0$ , and the double convergence is the diagonal one.



**Fig. 1.** Relationship among optimal control problems  $(P_{h\alpha}), (P_\alpha), (P_h)$  and  $(P)$  by taking the limits  $h \rightarrow 0, \alpha \rightarrow +\infty$  and  $(h, \alpha) \rightarrow (0, +\infty)$ .

The study of the limit  $h \rightarrow 0$  of the discrete solutions of optimal control problems can be considered as a classical limit, see [4-6, 8, 9, 13-15, 19, 23, 24, 27, 28] but the limit  $\alpha \rightarrow +\infty$ , for each  $h > 0$ , and the double limit  $(h, \alpha) \rightarrow (0, +\infty)$  can be considered as a new ones.

The paper is organized as follows. In Section II we define the discrete elliptic variational equalities for the state systems  $u_{hg}$  and  $u_{h\alpha g}$ , we define the discrete distributed cost functional  $J_h$  and  $J_{h\alpha}$ , we define the discrete distributed optimal control problems  $(P_h)$  and  $(P_{h\alpha})$ , and the discrete elliptic variational equalities for the adjoint state systems  $p_{hg}$  and  $p_{h\alpha g}$  for each  $h > 0$  and  $\alpha > 0$ , and we obtain properties for the discrete optimal control problems  $(P_h)$  and  $(P_{h\alpha})$ . In Section III we study the classical convergences of the discrete distributed optimal control problems  $(P_h)$  to  $(P)$ , and  $(P_{h\alpha})$  to  $(P_\alpha)$  when  $h \rightarrow 0$  (for each  $\alpha > 0$ ) and the estimations for the

discrete cost functional  $J_h$  and  $J_{h\alpha}$ . In Section IV we study the new convergence of the discrete distributed optimal control problems  $(P_{h\alpha})$  to  $(P_h)$  when  $\alpha \rightarrow +\infty$  for each  $h > 0$  and we obtain a commutative diagram which relates the continuous and discrete distributed mixed optimal control problems  $(P_{h\alpha}), (P_\alpha), (P_h)$  and  $(P)$  by taking the limits  $h \rightarrow 0$  and  $\alpha \rightarrow +\infty$ . In Section V we study the new double convergence of the discrete distributed optimal control problems  $(P_{h\alpha})$  to  $(P)$  when  $(h, \alpha) \rightarrow (0, +\infty)$  and we obtain the diagonal convergence in the previous commutative diagram.

## 2 Discretization by Finite Element Method and Properties

We consider the finite element method and a polygonal domain  $\Omega \subset \mathbf{R}^n$  with a regular triangulation with Lagrange triangles of type 1, constituted by affine-equivalent finite element of class  $C^0$  being  $h$  the parameter of the finite element approximation which goes to zero [3, 7]. We can take  $h$  equal to the longest side of the triangles  $T \in \mathcal{T}_h$  and we can approximate the sets  $V, V_0$  and  $K$  by:

$$V_h = \left\{ v_h \in C^0(\bar{\Omega}) / v_h|_T \in P_1(T), \forall T \in \mathcal{T}_h \right\}, V_{0h} = \{ v_h \in V_h / v_h|_{\Gamma_1} = 0 \}; K_h = b + V_{0h} \quad (14)$$

where  $P_1$  is the set of the polynomials of degree less than or equal to 1. Let  $\pi_h : C^0(\bar{\Omega}) \rightarrow V_h$  be the corresponding linear interpolation operator. Then there exists a constant  $c_0 > 0$  (independent of the parameter  $h$ ) such that [3]:

$$a) \|v - \pi_h(v)\|_H \leq c_0 h^r \|v\|_r; b) \|v - \pi_h(v)\|_r \leq c_0 h^{r-1} \|v\|_r; \forall v \in H^r(\Omega), 1 < r \leq 2. \quad (15)$$

We define the discrete cost functional  $J_h, J_{h\alpha} : H \rightarrow \mathbf{R}_0^+$  by the following expressions:

$$a) J_h(g) = \frac{1}{2} \|u_{hg} - z_d\|_H^2 + \frac{M}{2} \|g\|_H^2, \quad b) J_{h\alpha}(g) = \frac{1}{2} \|u_{h\alpha g} - z_d\|_H^2 + \frac{M}{2} \|g\|_H^2 \quad (16)$$

where  $u_{hg}$  and  $u_{h\alpha g}$  are the discrete system states defined as the solution of the following discrete elliptic variational equalities [16, 24]:

$$u_{hg} \in K_h : a(u_{hg}, v_h) = (g, v_h) - \int_{\Gamma_2} q v_h d\gamma, \quad \forall v_h \in V_{0h}, \quad (17)$$

$$u_{h\alpha g} \in V_h : a_\alpha(u_{h\alpha g}, v_h) = (g, v_h) - \int_{\Gamma_2} q v_h d\gamma + \alpha \int_{\Gamma_1} b v_h d\gamma, \quad \forall v_h \in V_h. \quad (18)$$

The corresponding discrete distributed optimal control problems consists in finding  $g_{h_{op}}, g_{h_{\alpha op}} \in H$  such that:

$$a) \text{ Problem } (P_h): J_h(g_{h_{op}}) = \underset{g \in H}{\text{Min}} J_h(g), \quad b) \text{ Problem } (P_{h\alpha}): J_{h\alpha}(g_{h_{\alpha op}}) = \underset{g \in H}{\text{Min}} J_{h\alpha}(g) \quad (19)$$

and their corresponding discrete adjoint states  $p_{hg}$  and  $p_{h\alpha g}$  are defined respectively as the solution of the following discrete elliptic variational equalities:

$$p_{hg} \in V_{0h} : a(p_{hg}, v_h) = (u_{hg} - z_d, v_h), \quad \forall v_h \in V_{0h} \quad (20)$$

$$p_{h\alpha g} \in V_h : a_\alpha(p_{h\alpha g}, v_h) = (u_{h\alpha g} - z_d, v_h), \quad \forall v_h \in V_h \quad (21)$$

**Remark 1.** We note that the discrete (in the n-dimensional space) distributed optimal control problem  $(P_h)$  and  $(P_{h\alpha})$  are still infinite dimensional optimal control problems since the control space is not discretized.

**Lemma 2**

(i) There exist unique solutions  $u_{hg} \in K_h$  and  $p_{hg} \in V_{0h}$ , and  $u_{h\alpha g} \in V_h$  and  $p_{h\alpha g} \in V_h$  of the elliptic variational equalities (17) and (20), (18), and (21) respectively  $\forall g \in H, \forall q \in Q, b > 0$  on  $\Gamma_1$ .

(ii) The operators  $g \in H \rightarrow u_{hg} \in V$ , and  $g \in H \rightarrow u_{h\alpha g} \in V$  are Lipschitzians. The operators  $g \in H \rightarrow p_{hg} \in V_{0g}$ , and  $g \in H \rightarrow p_{h\alpha g} \in V_h$  are Lipschitzians and strictly monotone operators.

**Proof.** We use the Lax-Milgram Theorem, the variational equalities (17), (18), (20) and (21), the coerciveness (10) and (11) and following [10, 18, 25]. W

**Theorem 3**

(i) The discrete cost functional  $J_h$  and  $J_{h\alpha}$  are  $H$ -elliptic and strictly convex applications, that is  $(\forall g_1, g_2 \in H, \forall t \in [0, 1])$ :

$$(1-t)J_h(g_2) + tJ_h(g_1) - J_h(tg_1 + (1-t)g_2) \geq M \frac{t(1-t)}{2} \|g_2 - g_1\|_H^2 \quad (22)$$

$$(1-t)J_{h\alpha}(g_2) + tJ_{h\alpha}(g_1) - J_{h\alpha}(tg_1 + (1-t)g_2) \geq M \frac{t(1-t)}{2} \|g_2 - g_1\|_H^2 \quad (23)$$

(ii) There exist a unique optimal controls  $g_{h_{op}} \in H$  and  $g_{h_{\alpha op}} \in H$  that satisfy the optimization problems (19a) and (19b) respectively.

(iii)  $J_h$  and  $J_{h\alpha}$  are Gâteaux differentiable applications and their derivatives are given by the following expressions:

$$a) J'_h(g) = Mg + p_{hg}, \quad b) J'_{h\alpha}(g) = Mg + p_{h\alpha g}, \quad \forall g \in H, \quad \forall h > 0 \quad (24)$$

(iv) The optimality condition for the optimization problems (19a) and (19b) are given by:

$$a) J'_h(g_{h_{op}}) = 0 \Leftrightarrow g_{h_{op}} = -\frac{1}{M} p_{hg_{h_{op}}}; \quad b) J'_{h\alpha}(g_{h\alpha_{op}}) = 0 \Leftrightarrow g_{h\alpha_{op}} = -\frac{1}{M} p_{h\alpha g_{h\alpha_{op}}} \quad (25)$$

(v)  $J'_h$  and  $J'_{h\alpha}$  are Lipschitzians and strictly monotone operators.

**Proof.** We use the definitions (16a,b), the elliptic variational equalities (17) and (18) and the coerciveness (10) and (11), following [10,18,25].  $\square$

We define the operators:

$$W_h, W_{h\alpha} : H \rightarrow H \text{ such that } a) W_h(g) = -\frac{1}{M} p_{hg}, \quad b) W_{h\alpha}(g) = -\frac{1}{M} p_{h\alpha g}. \quad (26)$$

#### Theorem 4

We have that:

(i)  $W_h$  and  $W_{h\alpha}$  are Lipschitzian operators, and  $W_h$  ( $W_{h\alpha}$ ) is a contraction operator if and only if  $M$  is large, that is:

$$a) M > \frac{1}{\lambda^2}, \quad b) M > \frac{1}{\lambda_\alpha^2}. \quad (27)$$

(ii) If  $M$  verifies the inequalities (27a,b) then the discrete distributional optimal control  $g_{h_{op}} \in H$  ( $g_{h\alpha_{op}} \in H$ ) is obtained as the unique fixed point of  $W_h$  ( $W_{h\alpha}$ ), i.e.:

$$g_{h_{op}} = -\frac{1}{M} p_{hg_{h_{op}}} \Leftrightarrow W_h(g_{h_{op}}) = g_{h_{op}}, \quad g_{h\alpha_{op}} = -\frac{1}{M} p_{h\alpha g_{h\alpha_{op}}} \Leftrightarrow W_{h\alpha}(g_{h\alpha_{op}}) = g_{h\alpha_{op}}. \quad (28)$$

**Proof.** We use the definitions (26a,b), and the properties (25a,b) and Lemma 2.  $\square$

### 3 Convergence of the Discrete Distributed Optimal Control Problems $(P_h)$ to $(P)$ and $(P_{h\alpha})$ to $(P_\alpha)$ when $h \rightarrow 0$

We obtain the following error estimations between the continuous and discrete solutions:

#### Theorem 6

We suppose the continuous system states and adjoint system states have the regularities  $u_g, u_{\alpha g_{\alpha_{op}}} \in H^r(\Omega)$  and  $p_g, p_{\alpha g_{\alpha_{op}}} \in H^r(\Omega)$  ( $1 < r \leq 2$ ). If  $M$  verifies the inequalities (27a,b) then we have the following error bonds:

$$\left\| \mathbf{g}_{h_{op}} - \mathbf{g}_{op} \right\|_H \leq ch^{r-1}, \left\| \mathbf{u}_{h\mathbf{g}_{h_{op}}} - \mathbf{u}_{\mathbf{g}_{op}} \right\|_V \leq ch^{r-1}, \left\| p_{h\mathbf{g}_{h_{op}}} - p_{\mathbf{g}_{op}} \right\|_V \leq ch^{r-1} \quad (29)$$

$$\left\| \mathbf{g}_{h\alpha_{op}} - \mathbf{g}_{\alpha_{op}} \right\|_H \leq ch^{r-1}, \left\| \mathbf{u}_{h\alpha\mathbf{g}_{h\alpha_{op}}} - \mathbf{u}_{\alpha\mathbf{g}_{\alpha_{op}}} \right\|_V \leq ch^{r-1}, \left\| p_{h\alpha\mathbf{g}_{h\alpha_{op}}} - p_{\alpha\mathbf{g}_{\alpha_{op}}} \right\|_V \leq ch^{r-1} \quad (30)$$

where  $c$ 's are constants independents of  $h$ .

**Proof.** It is useful to use the restriction  $\alpha > 1$  by splitting  $a_\alpha$  by [21, 24,25]

$$a_\alpha(u, v) = a_1(u, v) + (\alpha - 1) \int_{\Gamma_1} uv d\gamma \quad (31)$$

but then it can be replaced by  $\alpha \geq \alpha_0$  for any  $\alpha_0 > 0$ . We follow a similar method to the one developed in [25] for Neumann boundary optimal control problems by using the elliptic variational equalities (17), (18), (20) and (21), the thesis holds.  $\square$

**Remark 2.** If  $M$  verifies the inequalities (27a,b) we can obtain the convergence in Theorem 6 by using the characterization of the fixed point (28a,b), and the uniqueness of the optimal controls  $\mathbf{g}_{op} \in H$  and  $\mathbf{g}_{\alpha_{op}} \in H$ .

Now, we give some estimations for the discrete cost functional  $J_{h\alpha}$  and  $J_h$ .

**Lemma 7** *If  $M$  verifies the inequality (27a,b) and the continuous system states and adjoint system states have the regularities  $\mathbf{u}_g, \mathbf{u}_{\alpha g} \in H^r(\Omega)$   $p_g, p_{\alpha g} \in H^r(\Omega)$  ( $1 < r \leq 2$ ) then we have the following error bonds:*

$$\frac{M}{2} \left\| \mathbf{g}_{h_{op}} - \mathbf{g}_{op} \right\|_H^2 \leq J(\mathbf{g}_{h_{op}}) - J(\mathbf{g}_{op}) \leq Ch^{2(r-1)}, \quad \frac{M}{2} \left\| \mathbf{g}_{h\alpha_{op}} - \mathbf{g}_{\alpha_{op}} \right\|_H^2 \leq J_\alpha(\mathbf{g}_{h\alpha_{op}}) - J_\alpha(\mathbf{g}_{\alpha_{op}}) \leq Ch^{2(r-1)} \quad (32)$$

$$\frac{M}{2} \left\| \mathbf{g}_{h_{op}} - \mathbf{g}_{op} \right\|_H^2 \leq J_h(\mathbf{g}_{op}) - J_h(\mathbf{g}_{h_{op}}) \leq Ch^{2(r-1)}, \quad \frac{M}{2} \left\| \mathbf{g}_{h\alpha_{op}} - \mathbf{g}_{\alpha_{op}} \right\|_H^2 \leq J_{h\alpha}(\mathbf{g}_{\alpha_{op}}) - J_{h\alpha}(\mathbf{g}_{h\alpha_{op}}) \leq Ch^{2(r-1)} \quad (33)$$

$$\left| J_h(\mathbf{g}_{op}) - J(\mathbf{g}_{op}) \right| \leq Ch^{r-1}, \quad \left| J_h(\mathbf{g}_{h_{op}}) - J(\mathbf{g}_{op}) \right| \leq Ch^{r-1} \quad (34)$$

$$\left| J_{h\alpha}(\mathbf{g}_{op}) - J_\alpha(\mathbf{g}_{op}) \right| \leq Ch^{r-1}, \quad \left| J_{h\alpha}(\mathbf{g}_{h\alpha_{op}}) - J_\alpha(\mathbf{g}_{\alpha_{op}}) \right| \leq Ch^{r-1} \quad (35)$$

where  $C$ 's are constants independents of  $h$  and  $\alpha$ .

**Proof.** Estimations (32) and (33) follow from the estimations (29), and the equalities (similar relationship for  $J$  and  $J_\alpha$ ):

$$J_\alpha(\mathbf{g}_{h\alpha_{op}}) - J_\alpha(\mathbf{g}_{\alpha_{op}}) = \frac{1}{2} \left\| \mathbf{u}_{h\alpha\mathbf{g}_{h\alpha_{op}}} - \mathbf{u}_{\alpha\mathbf{g}_{\alpha_{op}}} \right\|_H^2 + \frac{M}{2} \left\| \mathbf{g}_{h\alpha_{op}} - \mathbf{g}_{\alpha_{op}} \right\|_H^2 \quad (36)$$



$$J_{h\alpha}(\mathbf{g}_{\alpha_{op}}) - J_{h\alpha}(\mathbf{g}_{h\alpha_{op}}) = \frac{1}{2} \left\| u_{h\alpha g_{h_{op}}} - u_{h\alpha g_{h\alpha_{op}}} \right\|_H^2 + \frac{M}{2} \left\| \mathbf{g}_{h\alpha_{op}} - \mathbf{g}_{\alpha_{op}} \right\|_H^2 \quad (37)$$

$$|J_{h\alpha}(\mathbf{g}) - J_{\alpha}(\mathbf{g})| \leq \left( \frac{1}{2} \left\| u_{h\alpha g} - u_{\alpha g} \right\|_H + \left\| u_{\alpha g} - z_d \right\|_H \right) \left\| u_{h\alpha g} - u_{\alpha g} \right\|_H, \quad \forall \mathbf{g} \in H. \quad (38)$$

#### 4 Convergence of the Discrete Optimal Control Problems $(P_{h\alpha})$ to $(P_h)$ when $\alpha \rightarrow +\infty$

##### Theorem 9

We have the following limits:

$$\lim_{\alpha \rightarrow +\infty} \left\| u_{h\alpha g_{h\alpha_{op}}} - u_{hg_{h_{op}}} \right\|_V = \lim_{\alpha \rightarrow +\infty} \left\| p_{h\alpha g_{h\alpha_{op}}} - p_{hg_{h_{op}}} \right\|_V = \lim_{\alpha \rightarrow +\infty} \left\| \mathbf{g}_{h\alpha_{op}} - \mathbf{g}_{h_{op}} \right\|_H = 0, \quad \forall h > 0. \quad (39)$$

**Proof.** We omit this proof because we prefer to prove the next one with more details.

#### 5 Double Convergence of the Discrete Distributed Optimal Control Problem $(P_{h\alpha})$ to $(P)$ when $(h, \alpha) \rightarrow (0, +\infty)$

For the discrete distributed optimal control problem  $(P_{h\alpha})$  we will now consider the double limit  $(h, \alpha) \rightarrow (0, +\infty)$ .

##### Theorem 10

We have the following limits:

$$\lim_{(h, \alpha) \rightarrow (0, +\infty)} \left\| u_{h\alpha g_{h\alpha_{op}}} - u_{g_{op}} \right\|_V = \lim_{(h, \alpha) \rightarrow (0, +\infty)} \left\| p_{h\alpha g_{h\alpha_{op}}} - p_{g_{op}} \right\|_V = \lim_{(h, \alpha) \rightarrow (0, +\infty)} \left\| \mathbf{g}_{h\alpha_{op}} - \mathbf{g}_{op} \right\|_H = 0 \quad (40)$$

**Proof.** From now on we consider that  $c$ 's represent positive constants independent simultaneously of  $h > 0$  and  $\alpha > 0$  (see (31)). We show a sketch of the proof by obtaining the following estimations (for  $\forall h > 0$  and  $\forall \alpha > 1$ ):

$$\left\| u_{h0} \right\|_V \leq c_1, \quad \left\| u_{h\alpha 0} \right\|_V \leq c_2, \quad (\alpha - 1) \int_{\Gamma_1} (u_{h\alpha 0} - b)^2 d\gamma \leq c_3 \quad (41)$$

$$\left\| \mathbf{g}_{h\alpha_{op}} \right\|_H \leq c_4, \quad \left\| u_{h\alpha g_{h\alpha_{op}}} \right\|_H \leq c_5, \quad \left\| \mathbf{g}_{h_{op}} \right\|_H \leq c_6 \quad (42)$$

$$\left\| u_{hg_{h_{op}}} \right\|_V \leq c_7, \quad \left\| u_{h\alpha g_{h\alpha_{op}}} \right\|_V \leq c_8, \quad (\alpha - 1) \int_{\Gamma_1} (u_{h\alpha g_{h\alpha_{op}}} - b)^2 d\gamma \leq c_9 \quad (43)$$

$$\left\| p_{hg_{hop}} \right\|_V \leq c_{10}, \quad \left\| p_{h\alpha g_{h\alpha op}} \right\|_V \leq c_{11}, \quad (\alpha - 1) \int_{\Gamma_1} p_{h\alpha g_{h\alpha op}}^2 d\gamma \leq c_{12}. \quad (44)$$

For example, the constant  $c_{11}$  is a positive constant independent simultaneously of  $h > 0$  and  $\alpha > 0$ , and it is given by the following expression:

$$\begin{aligned} c_{11} = & \|z_d\|_H \left[ \frac{1}{\lambda_1} \left( 1 + \frac{1}{\sqrt{M}} \left( \frac{1}{\lambda_1} + \frac{1}{\lambda} + \frac{1}{\lambda\lambda_1} \right) \right) + \frac{1}{\lambda} \left( 1 + \frac{1}{\lambda_1} \right) \left( 1 + \frac{1}{\lambda\sqrt{M}} \right) \right] \\ & + b \left[ \frac{1}{\lambda_1} \left( 1 + \frac{1}{\lambda_1} \right) \left( 1 + \frac{1}{\lambda\sqrt{M}} + \frac{1}{\lambda_1\sqrt{M}} \right) + \frac{1}{\lambda} \left( 1 + \frac{1}{\lambda_1} \right) \left( 1 + \frac{1}{\lambda\sqrt{M}} \right) \right] \\ & + \|q\|_Q \|v_0\| \left[ \frac{1}{\lambda_1} \left[ \frac{1}{\lambda_1} + \frac{1}{\lambda} \left( 1 + \frac{1}{\lambda_1} \right) + \frac{1}{\sqrt{M}} \left( \frac{1}{\lambda\lambda_1} + \frac{1}{\lambda^2} \left( 1 + \frac{1}{\lambda_1} \right) + \frac{1}{\lambda_1^2} \left( 1 + \frac{1}{\lambda} \right) \right) \right] + \frac{1}{\lambda^2} \left( 1 + \frac{1}{\lambda_1} \right) \left( 1 + \frac{1}{\lambda\sqrt{M}} \right) \right] \end{aligned} \quad (45)$$

Therefore, from the above estimations we have that:

$$\exists f \in H / g_{h\alpha op} \longrightarrow f \text{ in } H \text{ weak as } (h, \alpha) \rightarrow (0, +\infty) \quad (46)$$

$$\exists \eta \in V / u_{h\alpha g_{h\alpha op}} \longrightarrow \eta \text{ in } V \text{ weak (} H \text{ strong) as } (h, \alpha) \rightarrow (0, +\infty) \text{ with } \eta / \Gamma_1 = b \quad (47)$$

$$\exists \xi \in V / p_{h\alpha g_{h\alpha op}} \longrightarrow \xi \text{ in } V \text{ weak (} H \text{ strong) as } (h, \alpha) \rightarrow (0, +\infty) \text{ with } \xi / \Gamma_1 = 0 \quad (48)$$

$$\exists f_h \in H / g_{h\alpha op} \longrightarrow f_h \text{ in } H \text{ weak as } \alpha \rightarrow +\infty \quad (49)$$

$$\exists \eta_h \in V / u_{h\alpha g_{h\alpha op}} \longrightarrow \eta_h \text{ in } V \text{ weak (in } H \text{ strong) as } \alpha \rightarrow +\infty \text{ with } \eta_h / \Gamma_1 = b \quad (50)$$

$$\exists \xi_h \in V / p_{h\alpha g_{h\alpha op}} \longrightarrow \xi_h \text{ in } V \text{ weak (in } H \text{ strong) as } \alpha \rightarrow +\infty \text{ with } \xi_h / \Gamma_1 = 0 \quad (51)$$

$$\exists f_\alpha \in H / g_{h\alpha op} \longrightarrow f_\alpha \text{ in } H \text{ weak as } h \rightarrow 0 \quad (52)$$

$$\exists \eta_\alpha \in V / u_{h\alpha g_{h\alpha op}} \longrightarrow \eta_\alpha \text{ in } V \text{ weak (in } H \text{ strong) as } h \rightarrow 0 \text{ with } \eta_\alpha / \Gamma_1 = b \quad (53)$$

$$\exists \xi_\alpha \in V / p_{h\alpha g_{h\alpha op}} \longrightarrow \xi_\alpha \text{ in } V \text{ weak (in } H \text{ strong) as } h \rightarrow 0 \text{ with } \xi_\alpha / \Gamma_1 = 0 \quad (54)$$

$$\exists f^* \in H / g_{hop} \longrightarrow f^* \text{ in } H \text{ weak as } h \rightarrow 0 \quad (55)$$

$$\exists \eta^* \in V / u_{hg_{hop}} \longrightarrow \eta^* \text{ in } V \text{ weak (} H \text{ strong) as } h \rightarrow 0 \text{ with } \eta^* / \Gamma_1 = b \quad (56)$$

$$\exists \xi^* \in V / p_{hg_{hop}} \longrightarrow \xi^* \text{ in } V \text{ weak (} H \text{ strong) as } h \rightarrow 0 \text{ with } \xi^* / \Gamma_1 = 0 \quad (57)$$

Taking into account the uniqueness of the distributed optimal control problems  $(P_{h\alpha})$ ,  $(P_\alpha)$ ,  $(P_h)$  and  $(P)$ , and the uniqueness of the elliptic variational equalities corresponding to their state systems we get

$$\eta_h = u_{hf_h} = u_{hg_{hop}}, \quad \xi_h = p_{hf_h} = p_{hg_{hop}}, \quad f_h = g_{hop} \quad (58)$$

$$\eta_\alpha = u_{\alpha f_\alpha} = u_{\alpha g_{\alpha op}}, \quad \xi_\alpha = p_{\alpha f_\alpha} = p_{\alpha g_{\alpha op}}, \quad f_\alpha = g_{\alpha op} \quad (59)$$

$$\eta = \eta^* = u_f = u_{g_{op}}, \quad \xi = \xi^* = p_f = p_{g_{op}}, \quad f = f^* = g_{op}. \quad (60)$$

Now, by using [11] we obtain

$$\lim_{\alpha \rightarrow +\infty} \|f_\alpha - g_{op}\|_H = 0, \quad \lim_{\alpha \rightarrow +\infty} \|\eta_\alpha - u_{g_{op}}\|_V = 0, \quad \lim_{\alpha \rightarrow +\infty} \|\xi_\alpha - p_{g_{op}}\|_V = 0 \quad (61)$$

and therefore the three double limits (40) hold when  $(h, \alpha) \rightarrow (0, +\infty)$ .

**Proof of Theorem 1.** It is a consequence of the properties (29), (30), (39), (40) and [10,11].

**Remark 3.** We note that this double convergence is a novelty with respect to the recent results obtained for a family of discrete Neumann boundary optimal control problems [25].

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