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Algebraic estimation in partial derivatives systems: parameters and differentiation problems

Rosane Ushirobira, Anja Korporal, Wilfrid Perruquetti

Abstract Two goals are sought in this paper: namely, to provide a succinct overview on algebraic techniques for numerical differentiation and parameter estimation for linear systems and to present novel algebraic methods in the case of several variables. The state-of-art in the introduction is followed by a brief description of the methodology in the subsequent sections. Our new algebraic methods are illustrated by two examples in the multidimensional case. Some algebraic preliminaries are given in the appendix.

1 Introduction

Many challenging questions in signal processing and control involve the estimation of derivatives of measured time signals, usually in noisy environment. This important issue is known as a numerical differentiation. Several approaches were proposed on this subject, based on different frameworks in applied mathematics and engineering. In control theory, designing a differentiator is an important problem, with many applications [2, 6, 17]. Some classical solutions are based on the least-squares polynomial interpolation and provide good offline results for this matter, see for example [15]. On another groundwork, just to mention a few works, numerical differentiators defined on an observer design basis were proposed in [4, 6, 13, 14, 41]

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and digital filter processing techniques used in [1, 3, 32, 37]. A high-order sliding mode based differentiator is designed in [8] by developing the results from [16] and it provides very satisfactory estimations despite some chattering in the response. We may also remark that the homogeneous finite-time-differentiator defined in [29] presents no chattering, but it is more sensitive to the signal amplitude.

The problem of estimating parameters in linear systems appears in the mathematical modeling of a physical phenomena. Differential equations in the considered model may contain parameters that are simply difficult to determine through data collecting, perhaps due to noisy measurements. This essential problem has attracted the attention of researchers in many fields. For instance, parameter estimation is a central subject in statics inference and several procedures can be applied to this problem, such as the maximum likelihood. Also, parameter estimation problems are also often related to optimization techniques.

There are countless works on numerical differentiation and parameter estimation. Among the recent advances on these issues, a promising solution is provided by differential algebra and operational calculus tools. This algebraic branch was initiated in the works by M. Fliess and H. Sira-Ramírez [10]. A clear description of the procedure, containing many useful explanations, can be found in [9, 19]. In despite of the innovative character of this framework, this algebraic approach remains quite underused. Nevertheless, some works do apply these ideas, see for instance [22, 42, 43] and for more practical developments, see for example [24, 27, 50]. For more details, the reader may refer to a quite extensive survey recently published in [39].

Algebraic methods within the numerical differentiation context were first applied to the univariable numerical differentiation by Mboup et al. in [20] where the authors use Jacobi projections to construct estimators for the derivatives. As described by the authors, the key idea of the method in this latter is to consider the nth-derivative of a smooth signal at a point τ as a single parameter to be estimated from a noisy observation of the signal. From that, a pointwise derivative can be estimated by varying τ. A truncated Taylor series expansion of the signal is the starting point in this technique, and the computations are then made in the operational domain. A slight drawback in the approximation by a truncated Taylor polynomial model may be its ephemeral character. To reduce this fast transient behavior, an improvement of the technique was proposed in [33]. A through study of these algebraic estimators, with emphasis on the error analysis, can be found in [18]. In that work, estimators based on fractional derivatives were introduced. An interesting computer architecture to accelerate the computation of the aforesaid algebraic derivative estimator was implemented in [28] using reconfigurable logic and implemented in an FPGA (field-programmable gate array).

In the multidimensional case, the estimation of derivatives of a noisy signal concern also many problems in engineering. For instance, in economy issues, in addition to the fields of signal processing or control. To tackle this numerical differential problem, most likely more problematic than the unidimensional case, several techniques were developed. The most commonly used is the finite differences method. The instability of possible solutions to these problems arise from the presence of noise due to the differentiation.

The use of algebraic tools for multidimensional numerical differentiation was addressed by Riachy et al. in [34–36]. Their inspiration comes from the original ideas in [10] and from the solutions proposed by [20].

The algebraic method developed in this paper is motivated by the parameter estimation methods elaborated in [45–47]. In those works, Weyl algebra based tools grant the estimation of amplitudes, signal and frequencies of a sinusoidal signal, providing faster estimates than known methods [5, 12, 48] (simulated examples in [12] provide fast estimates, however in more than a fraction of the period). The main advantage of our method is to give closed formulas for derivatives and parameter estimates. Furthermore, algebraic estimation techniques strongly rely on differential elimination. So, a number of different estimators (*i.e.* appropriate differential operators providing estimates) can be devised for a given estimation problem (this is well illustrated through a change-point detection problem in [21]). Hence, it appears that the quality of an estimator varies markedly with the order of the selected differential operators used in the elimination. The Weyl Algebra point of view introduced here within the algebraic context allows to characterize and to select the minimal order operators associated to any given estimation problem. Finally, let us stress that all algebraic approaches mentioned above in this Introduction share a very useful characteristic: obtained estimates are integrals of the noisy measured signal, so these integrals act as time varying filters.

Section 2 starts with a general introduction of the procedure of algebraic estimation, followed by the presentation of two estimation problems: numerical differentiation and parameter identification. In Section 4, the algebraic estimation of derivatives is illustrated through a significant example. To expose our method on a multidimensional parameter estimation problem, a particular partial differential equation was examined. It is the example of the heat conduction on a thin rod that is discussed and treated by algebraic estimations (this type of equation was considered in [38], also based on algebraic techniques). Proposed solutions to this problem in the algebraic framework are given in Section 5. The Appendix contains generalities on algebraic structures, as well as useful properties for the algebraic methods.

2 Problem formulation

As mentioned in the Introduction, numerous engineering problems concern the estimation of state variables or parameters. In this section, we describe briefly how algebraic methods proceed, in general, to this estimation.

Most of the time, the mathematical modeling of physical phenomena provide a description of the aforementioned practical problems through a differential equations framework. States or parameters to be identified appear in the terms of these differential equations.

Roughly speaking, for such a given differential equations system, algebraic methods observe typically the following sequence of steps:

1. Passage to the operational domain through the Laplace transform or by using Mikusinski operational calculus [25,26,49]: thanks to this step, differential equations are converted into algebraic ones, consequently allowing algebraic concepts to be applied. The resulting algebraic expressions depend on the Laplace variable s.
2. Computations with the algebraic equations using structural properties: in this part, the aim is to apply algebraic tools on the equations in order to find expressions and closed formulas for the parameters or derivatives estimations.
3. Return to the time domain and identification: the algebraic expressions in the Laplace variable s found in the previous step are converted into the time domain through the inverse Laplace transform. Possibly a time dependent system on the parameters must be solved.

It is notably in Step 2. that the advantages among different algebraic approaches can be seen. Indeed, in the differential elimination necessary in this process, structural properties of differential algebra are useful. Most of these procedures result in estimates given by integrals (rather than derivatives) of the noisy measured signal and these integrals will then provide noise attenuation. Thanks to special forms for the *annihilators* (differential operators involved in the differential elimination) developed in the appendix, the identification process presented in this paper will result in faster and less noise sensitive estimates. Therefore, these particular annihilators allow a better choice of suitable differential operators allowing the elimination of problematic parameters, or yield a more convenient matrix representation that will ease the solution of a system.

2.1 Derivative estimation problem

Throughout this paper, $\mathbb{K}$ denotes a field of characteristic zero (usually $\mathbb{R}$ or $\mathbb{C}$ in many applications). Let $\mathbf{x} = (x_1, \ldots, x_m)$ be an element in $\mathbb{K}^m$ ($m \in \mathbb{N}$). An m-tuple $N \in \mathbb{N}^m$ will be written as $N = (N_1, \ldots, N_m)$. We consider the partial order $\preceq$ on $\mathbb{N}^m$ defined by $N \preceq M$ if $N_i \leq M_i$, for all $1 \leq i \leq m$.

Let $f : U \subset \mathbb{R}^m \to \mathbb{R}$ be a multivariate signal where U is some neighborhood of 0. For a given $I = (i_1, \ldots, i_m) \in \mathbb{N}^m$, we denote $|I| := i_1 + \cdots + i_m$, $I! := i_1! \ldots i_m!$, $\mathbf{x}^I = x_1^{i_1} \ldots x_m^{i_m}$ and $\frac{\partial^I}{\partial \mathbf{x}^I} = \frac{\partial^{i_1}}{\partial x_1^{i_1}} \cdots \frac{\partial^{i_m}}{\partial x_m^{i_m}}$.

In practical problems, the available signal f is usually corrupted by a noise. Denote by f_{ϖ} the noisy multivariate signal

$$f_{\varpi}(\mathbf{x}) = f(\mathbf{x}) + \varpi(\mathbf{x}),$$

where $\varpi(\mathbf{x})$ is an additive noise. Assume that f admits a Taylor series expansion at 0 and write:

$$f(\mathbf{x}) = \sum_{I \in \mathbb{N}^m} \frac{a_I}{I!} \mathbf{x}^I, \text{ where } a_I = \frac{\partial^I f}{\partial \mathbf{x}^I}(0).$$

For $N=(N_1,\ldots,N_m)\in\mathbb{N}^m$, the truncated Taylor series f_N at order N is given by:

$$f_N(\mathbf{x})=\sum_{I\preceq N}\frac{a_I}{I!}\mathbf{x}^I. \tag{1}$$

The multivariate Laplace transform of a function $g:\mathbb{R}^m\to\mathbb{R}$ is given by:

$$G(\mathbf{s}):=\mathcal{L}(g)(\mathbf{s})=\int_{\mathbb{R}_+^m}g(\mathbf{t})\,\mathrm{e}^{-\mathbf{s}\,^T\mathbf{t}}d\mathbf{t}, \tag{2}$$

where $\mathbf{s}=(s_1,\ldots,s_m)$ is the Laplace (multi)variable and $^T\mathbf{t}$ denotes the transpose of $\mathbf{t}\in\mathbb{R}^m$. That implies, for instance:

$$\mathcal{L}\left(\frac{\mathbf{x}^I}{I!}\right)=\frac{1}{\mathbf{s}^{I+\mathbf{1}}}$$

where $\mathbf{s}^I=s_1^{i_1}\ldots s_m^{i_m}$. To realize $f_N(\mathbf{x})$ in the operational domain, we apply the Laplace transform (2) on (1). It results:

$$F_N(\mathbf{s})=\sum_{I\preceq N}\frac{a_I}{\mathbf{s}^{I+1}}. \tag{3}$$

For $\mathbf{x},\mathbf{t}\in\mathbb{K}^m$, we use the notation:

$$\int_{\mathbf{0}}^{\mathbf{x}}g(\mathbf{t})d\mathbf{t}=\int_0^{x_1}\ldots\int_0^{x_m}g\left(t_1,\ldots,t_m\right)dt_1\ldots dt_m.$$

Recall that for a multivariate function g and its Laplace transform G, the inverse Laplace transform satisfies

$$\mathcal{L}^{-1}\left(\frac{1}{\mathbf{s}^I}\frac{\partial^J G}{\partial\mathbf{s}^J}\right)=\frac{1}{(I-\mathbf{1})!}\int_{\mathbf{0}}^{\mathbf{x}}(\mathbf{x}-\tau)^{I-\mathbf{1}}(-\tau)^J g(\tau)d\tau \tag{4}$$

where $\tau=(\tau_1,\ldots,\tau_m)$ and $\mathbf{1}=(1,\ldots,1)\in\mathbb{K}^m$. For the sake of simplicity, we set:

$$v_{I,J}=v_{I,J}(\tau)=(\mathbf{x}-\tau)^I(-\tau)^J, \tag{5}$$

and a shorter notation can be used:

$$\mathcal{L}^{-1}\left(\frac{1}{\mathbf{s}^I}\frac{\partial^J G}{\partial\mathbf{s}^J}\right)=\frac{1}{(I-\mathbf{1})!}\int_{\mathbf{0}}^{\mathbf{x}}v_{I-1,J}(\tau)g(\tau)d\tau. \tag{6}$$

As we have seen in the introduction, a remarkable work on numerical differentiation by algebraic methods was written by Mboup *et al.* [20]. To illustrate their approach, we consider an example in the one-dimensional case. Consider the approximating polynomial function of degree N of a real-valued signal $f(t)$, analytic on some time interval:

$$f(t) = \sum_{i=0}^{N} \frac{a_i}{i!} t^i, \tag{7}$$

originated from its Taylor series expansion, hence

$$a_i = f^{(i)}(0),\ \forall 0 \leq i \leq N.$$

The goal is to estimate the derivatives of the signal $f(t)$, that means, the coefficients a_i in (7). Often, $f(t)$ will be assumed to be the measured signal from a signal $x(t)$ with some negligible noise, so we may consider only $f(t)$. For example, the estimation of the first derivative of $f(t)$ can be obtained in the following way: from the degree one polynomial

$$f(t) = a_0 + a_1 t,$$

we obtain the operational domain expression given by the action of the Laplace transform. That yields:

$$Y(s) = \frac{a_0}{s} + \frac{a_1}{s^2}.$$

In [20], a *minimal* annihilator Π is proposed to eliminate the term a_0. It consists of a suitable differential operator, in this case:

$$\Pi = \frac{1}{s^2} \frac{d}{ds}\, s.$$

(meaning that the expression is multiplied by s, then taking the derivative with respect to s and finally multiplying by $\frac{1}{s^2}$). The term a_0 is eliminated after the action of Π. The time domain representation obtained thanks to the inverse Laplace transform provides an estimate $\widehat{\dot{f}}(t)$ of the first derivative $a_1 = \dot{f}(0)$:

$$\widehat{\dot{f}}(t) = \frac{6}{T^3} \int_0^T (T - 2\tau)\, f(t-\tau) d\tau.$$

(in practice, f is replaced by its measure). The idea presented here is to individually estimate each derivative a_J for $J \preceq N$. To formalize our procedure, we consider the following sets:

$$\Theta = \{a_I \mid I \preceq N\},\ \Theta_{\text{est}} = \{a_J\} \text{ and } \Theta_{\overline{\text{est}}} = \Theta \setminus \Theta_{\text{est}}.$$

The definition of Θ, Θ_{est} and $\Theta_{\overline{\text{est}}}$ is clear: Θ contains all the parameters, Θ_{est} contains the parameters to be estimated and $\Theta_{\overline{\text{est}}}$ the remaining ones. The relation ($\mathcal{R}$) below follows from (3):

$$\mathcal{R} : P(\mathbf{s}) F_N(\mathbf{s}) + Q(\mathbf{s}) + \overline{Q}(\mathbf{s}) = 0 \tag{8}$$

where

$$P(\mathbf{s}) = \mathbf{s}^N,\ Q(\mathbf{s}) = -a_J \mathbf{s}^{N-J-1} \in \mathbb{K}_{\Theta_{\text{est}}}\left[\mathbf{s}, \frac{1}{\mathbf{s}}\right] \text{ and}$$

$$\overline{Q}(\mathbf{s}) = -\sum_{I \preceq N, I \neq J} a_I \mathbf{s}^{N-I-1} \in \mathbb{K}_{\Theta_{\overline{\text{est}}}}\left[\mathbf{s}, \frac{1}{\mathbf{s}}\right] \tag{9}$$

By $\mathbb{K}_{\Theta_{\text{est}}}$ and $\mathbb{K}_{\Theta_{\overline{\text{est}}}}$, we denote respectively the algebraic extensions $\mathbb{K}_{\Theta_{\text{est}}} = \mathbb{K}(\Theta_{\text{est}})$ and $\mathbb{K}_{\Theta_{\overline{\text{est}}}} = \mathbb{K}(\Theta_{\overline{\text{est}}})$.

Based on the relation $\mathcal{R}$ (8), three polynomials P, Q and $\overline{Q}$ are defined taking into account the coefficients to be identified: P is the polynomial multiplying $F_N(\mathbf{s})$, Q contains the coefficient to be estimated, while $\overline{Q}$ is formed by all the remaining terms. To obtain an equation containing only known terms and a_J, the polynomial $\overline{Q}$ must be eliminated. That will provide a formula for the estimate of a_J.

To annihilate $\overline{Q}$, some particular differential operators must be chosen to act on ($\mathcal{R}$). These operators are called *annihilators*. Such algebraic estimators for a_J will be constructed by using structural properties of the Weyl algebra (see the Appendix).

Let us stress that if Π is an annihilator estimating a_J, the partial derivative of f at any other point $\mathbf{p} \in \mathbb{K}^m$ can be obtained by computing $\Pi\left(\mathcal{L}\left(f(\mathbf{x}+\mathbf{p})\right)\right)$.

2.2 Parameter estimation

An example of parametric identification was given in the seminal paper by M. Fliess and H. Sira-Ramírez [10] and it concerns a first order input-output system:

$$\dot{y}(t) = ay(t) + u(t) + \gamma_0$$

where a is a parameter to be identified and γ_0 is a constant perturbation. In the operational domain, thanks to the Laplace transform, the above equation becomes:

$$sY(s) - y(0) = aY(s) + U(s) + \frac{\gamma_0}{s}$$

where s is the Laplace variable, $Y(s)$ and $U(s)$ denote the Laplace transform of $y(t)$ and $u(t)$ respectively. The action of the differential operator $\frac{1}{s^2}\ \frac{d^2}{ds^2}\ s$ on this expression yields:

$$\left(\frac{1}{s}\frac{d}{ds}Y(s) + \frac{2}{s^2}\frac{d^2}{ds^2}Y(s)\right)\ a = \frac{d^2}{ds^2}Y(s) + \frac{4}{s}\frac{d}{ds}Y(s) + \frac{2}{s^2}Y(s) - \left(\frac{1}{s}\frac{d^2}{ds^2}U(s) + \frac{2}{s^2}\frac{d}{ds}U(s)\right).$$

Operational calculus rules yield the following estimation for a:

$$a = \frac{\int_0^1 \left(\left(2\nu^2 - 3\nu + 1\right) \nu\, t\, u(t\nu) + \left(-6\nu^2 + 6\nu - 1\right) y(t\nu)\right) d\nu}{t \int_0^1 \nu\, (2\nu^2 - 3\nu + 1)\, y(t\nu)\, d\nu}$$

The parameter identification for a partial differential equation can be thought in a similar way. To illustrate this, a simple example of the one-dimensional heat equation is studied in the sequel. A similar algebraic approach was studied, for instance, in [38] for this same problem and in [11] for the parameter identification of a linear model of the planar motion of a heavy rope. The unidimensional Laplace transform was used in both these examples providing operational functions a as the solutions of an initial value problem. In this work, the Laplace transform in two variables is used to convert the partial differential equation into the operational domain representation.

3 Annihilators via the Weyl algebra

In the previous section, we have indicated that our aim is to annihilate the polynomial $\overline{Q}$ in the relation $\mathcal{R}$ (8), containing undesired parameters, see (9). That will be done by the action of *annihilators*: these are differential operators (or polynomials in the variables $\frac{\partial^l}{\partial \mathbf{s}^l}$) with polynomial coefficients (or rational functions) in the Laplace variables $s_1, \ldots, s_m$. A practical realization of differential operators acting on polynomial variables is the Weyl algebra. So, this algebra appears naturally in this context and its structural properties will be quite useful in the choice of the annihilators.

This algebraic viewpoint is inspired by the work of M. Fliess *et al.* [9,10,19]. Details about the algebraic notions defined in the sequel can be found in the appendix and in [23,31] as well.

Next, we keep the notation defined in the appendix (see Section 7) to define the differential operators *annihilators*. They will help to construct algebraic estimators, either of derivatives or partial derivatives, or also of parameters.

Recall that A_m denotes the Weyl algebra $\mathsf{A}_m = \mathbb{K}[\mathbf{s}]\left[\frac{\partial}{\partial \mathbf{s}}\right]$ and $\mathsf{B}_m = \mathbb{K}(\mathbf{s})\left[\frac{\partial}{\partial \mathbf{s}}\right]$, respectively the polynomial rings in $\frac{\partial}{\partial \mathbf{s}}$ with coefficients in the polynomial ring $\mathbb{K}[\mathbf{s}]$ and in the fraction field $\mathbb{K}(\mathbf{s})$.

Definition 1. Let $R \in \mathbb{K}_{\Theta_{\text{est}}}\left[\mathbf{s}, \frac{1}{\mathbf{s}}\right]$. A *R-annihilator with respect to* B_m is an element of $\mathrm{Ann}_{\mathsf{B}_m}(R) = \{F \in \mathsf{B}_m \mid F(R) = 0\}$.

Consider $m \geq 2$. Let us remark that $\mathrm{Ann}_{\mathsf{B}_m}(R)$ is a left ideal of B_m. Therefore, by Stafford's theorem (Theorem 2, Appendix), $\mathrm{Ann}_{\mathsf{B}_m}(R)$ is generated by two generators Π_1 and $\Pi_2 \in \mathsf{B}_m$:

$$\mathrm{Ann}_{\mathsf{B}_m}(R) = \mathsf{B}_m \Pi_1 + \mathsf{B}_m \Pi_2.$$

We call the annihilators Π_1 and Π_2 *minimal R-annihilators with respect to* B_m. The attribute *minimal* comes from the order of the differential operators. Notice that

$\mathrm{Ann}_{\mathsf{B}_m}(R)$ contains annihilators in finite integral form, *i.e.* operators with coefficients in $\mathbb{K}\left[\frac{1}{\mathbf{s}}\right]$.

Let us stress that thanks to the above Stafford's theorem, only two generators for the ideal are needed for a given $m \geq 2$.

Lemma 1. *Consider* $R(\mathbf{s}) = \alpha \mathbf{s}^N = \alpha s_1^{n_1} \dots s_m^{n_m}$, $N = (n_1, \dots, n_m) \in \mathbb{Z}^m$ *with* $\alpha \in \mathbb{K}_{\Theta_{\overline{est}}}$. *A minimal R-annihilator is given by*

$$s_i \frac{\partial}{\partial s_i} - n_i, \ \forall 1 \leq i \leq m.$$

Recall that the *degree* of a monomial $\mathbf{s}^I \in \mathbb{K}\left[\mathbf{s}, \frac{1}{\mathbf{s}}\right]$ is $|I|$. The *total degree* of a polynomial in $\mathbf{s}$ is the maximum degree of its monomials.

Remark 1. Consider $R \in \mathbb{K}_{\Theta_{\overline{\text{est}}}}\left[\mathbf{s}, \frac{1}{\mathbf{s}}\right]$ with a monomial s^I of maximal degree. So R has total degree $|I|$. Let $i_k = \max\{i_j \mid j = 1, \dots, m\}$. If $|I| > 0$, then $\frac{\partial^{i_k+1}}{\partial s_k^{i_k+1}}$ is clearly an R-annihilator.

Now, recall that the polynomial to be annihilated in this differentiation problem is (see (9)):

$$\overline{Q}(\mathbf{s}) = -\sum_{I \preceq N, I \neq J} a_I \mathbf{s}^{N-I-1} \in \mathbb{K}_{\Theta_{\overline{\text{est}}}}\left[\mathbf{s}, \frac{1}{\mathbf{s}}\right].$$

By the previous remark, it results immediately:

Lemma 2. *The differential operators* $\frac{\partial^{n_k}}{\partial s_k^{n_k}}$ *are* $\overline{Q}$*-annihilators, for all* $1 \leq k \leq m$.

To construct an alternative annihilator, an algorithm is sketched below:

Algorithm 1

Input: *A polynomial* $R = \sum_{\substack{I \in \mathbb{Z}^m \\ finite}} b_I \mathbf{s}^I$ *in* $\mathbb{K}_{\Theta_{\overline{est}}}\left[\mathbf{s}, \frac{1}{\mathbf{s}}\right]$ *of total degree* $d \in \mathbb{N}$

Output: *An R-annihilator*

1. *Set* $\Pi = 1 \in D$.
2. *Choose a monomial of degree d in R, say* $b_J \mathbf{s}^J$ *with* $J = (j_1, \dots, j_m)$ *(so* $|J| = d$*).*
3. *Choose* $j_k = \min\{j_\ell > 0 \mid \ell = 1, \dots, m\}$.
4. *Apply* $\pi = s_k \frac{\partial}{\partial s_k} - j_k$ *(see Lemma 1) on R.*
5. *(a) If* $\pi(R) = 0$*, then return* π *and stop the algorithm.*
   (b) If $\pi(R) \neq 0$*, then set* $\Pi = \pi \circ \Pi$ *and return to step (2) with* $R \leftarrow \pi(R)$.

Example 1. Consider $m = 2$ and the polynomial $R(s_1, s_2) = a_{00} s_1^2 s_2 + a_{01} s_1^2 + a_{10} s_1 s_2 + a_{20} s_2 \in \mathbb{K}_{\Theta_{\overline{\text{est}}}}[s_1, s_2]$. A R-annihilator constructed with the above algorithm is $\left(s_1 \frac{\partial}{\partial s_1} - 2\right) \circ \left(s_2 \frac{\partial}{\partial s_2} - 1\right)$.

The concept of an estimator must be defined in order to take into account the remaining terms in the relation ($\mathcal{R}$), after the action of an annihilator. Notice that the parameters to be estimated might appear in the set of coefficients of both polynomials Q and P, but they might as well be present exclusively in one of the two. Therefore a $\overline{Q}$-annihilator must not eliminate all terms in Θ_{est}, as formalized in the definition below:

Definition 2. An *estimator* $\pi \in \mathsf{B}$ is a $\overline{Q}$-annihilator satisfying

$$\text{coeffs}\,(\pi(\mathcal{R})) \cap \mathbb{K}_{\Theta_{\text{est}}} \neq \emptyset,$$

where coeffs (R) denotes the set of coefficients of a polynomial $R \in \mathbb{K}_{\Theta}\left[\mathbf{s}, \frac{1}{\mathbf{s}}\right]$.

It is implied by this definition that the criterion on the coefficients must be considered in the choice of annihilators in the Algorithm 1.

It is important to stress that in some cases, it may be interesting to adopt another way of proceeding. For instance, several different annihilators can be constructed for each $\overline{Q}$. Then, Stafford's theorem can be applied to provide two minimal generators by using the package `Stafford` [30, 31]. The final step is to observe the criterion in Definition 2 in these generators to obtain an estimator.

4 Derivative estimation

To illustrate the algebraic method for numerical differentiation, we present here the estimation for a derivative in the two-dimensional case. Hence, the equation (1) is considered for $m = 2$. For this example, we assume that the parameter to be estimated is $a_{21} = \frac{\partial^3 f}{\partial x_1^2 \partial x_2}(0,0)$. Based on (1), a truncated Taylor series at $N = (2,1)$ will then be used:

$$f(x_1,x_2) = a_{00} + a_{10}x_1 + a_{01}x_2 + a_{20}x_1^2 + a_{11}x_1x_2 + a_{21}x_1^2x_2.$$

The coefficient to be estimated is a_{21} in the truncated Taylor series above, so we may distinguish the following polynomials P, Q and $\overline{Q} \in \mathbb{K}\left[\mathbf{s}, \frac{1}{\mathbf{s}}\right]$ in the relation ($\mathcal{R}$) (see (8)):

$$\begin{aligned}
P(s_1,s_2) &= s_1^2 s_2, \\
Q(s_1,s_2) &= -a_{21}s_1^{-1}s_2^{-1} \in \mathbb{K}_{\Theta_{\text{est}}}\left[\mathbf{s}, \frac{1}{\mathbf{s}}\right] \quad \text{and} \\
\overline{Q}(s_1,s_2) &= -\sum_{\substack{(i,j)\preceq N \\ (i,j)\neq(2,1)}} a_{ij}s_1^{1-i}s_2^{-j} \in \mathbb{K}_{\Theta_{\overline{\text{est}}}}\left[\mathbf{s}, \frac{1}{\mathbf{s}}\right].
\end{aligned}$$

The first step of the estimation is to determine minimal $\overline{Q}$-annihilators. To begin, Lemma 1 helps to find two $\overline{Q}$-annihilators $\frac{\partial^2}{\partial s_1^2}\left(s_1\frac{\partial}{\partial s_1}+1\right)$ and $\frac{\partial}{\partial s_2}\left(s_2\frac{\partial}{\partial s_2}+1\right)$. However, they are not estimators since they clearly eliminate Q as well and Q is the only term in $\mathcal{R}$ with coefficients in $\mathbb{K}_{\Theta_{\overline{\text{est}}}}$ (see Definition 2).

We then follow the Algorithm 1 to determine an alternative $\overline{Q}$-annihilator that may also be a estimator. In the case of a_{21}, we obtain:

$$\Pi = \frac{1}{s_1^2 s_2}\frac{\partial^2}{\partial s_1 s_2}\left(s_1\frac{\partial}{\partial s_1}-1\right).$$

Let us remark that for other coefficients $a_{k\ell}$, some annihilators are proposed in [36] and in [44] as well. Using Remark 4 in the Appendix, it can be shown that the annihilator Π is a minimal annihilator.

The action of Π on the relation $\mathcal{R}$ with P, Q and $\overline{Q}$ defined above, provides the following expression:

$$2\frac{F(s_1,s_2)}{{s_1}^3{s_2}^2}+4\frac{\frac{\partial}{\partial s_1}F(s_1,s_2)}{{s_1}^2{s_2}^2}+2\frac{\frac{\partial}{\partial s_2}F(s_1,s_2)}{{s_1}^3 s_2}+4\frac{\frac{\partial^2}{\partial s_2\partial s_1}F(s_1,s_2)}{{s_1}^2 s_2}+\frac{\frac{\partial^2}{\partial {s_1}^2}F(s_1,s_2)}{s_1{s_2}^2}$$
$$+\frac{\frac{\partial^3}{\partial s_2\partial {s_1}^2}F(s_1,s_2)}{s_1 s_2}+2\frac{a_{2,1}}{{s_1}^6{s_2}^4}=0.$$

Isolating the term with a_{21} and applying the inverse Laplace transform (6) provides the consequent estimate:

$$a_{21}=-\frac{360}{x_1^5x_2^3}\int_{\mathbf{0}}^{(x_1,x_2)}\left(v_{2,1,0,0}+v_{0,1,2,0}-4v_{1,1,0,0}-4v_{1,0,0,1}+v_{2,0,0,1}+v_{0,0,2,1}\right)f(\tau)d\tau,$$

where we use the notation (5):

$$v_{I,J}=(x_1-\tau)^{i_1}(x_2-\eta)^{i_2}(-\tau)^{j_1}(-\eta)^{j_2},$$

for all $I=(i_1,i_2)$, $J=(j_1,j_2)\in\mathbb{N}^2$.

5 Parameter estimation

In the previous section, we examined the case of numerical differentiation where annihilators were used to eliminate the undesired terms of the truncated Taylor series seen in the operational domain. Moreover, as we have seen in the introduction, a similar procedure may provide estimates for parameters in an ordinary differential equation.

In this section, we present a parameter identification problem for a two-dimensional partial differential equation. The following classical example was studied in [38,44], for instance. Consider the problem of the heat conduction in a thin rod of length 1.

Let $w : (z,t) \mapsto w(z,t)$ be the function representing the temperature at position z at time t. The partial differential equation describing this problem is given by:

$$\frac{\partial^2}{\partial z^2} w(z,t) - \beta \frac{\partial}{\partial t} w(z,t) - \alpha w(z,t) = 0 \tag{10}$$

The rod is assumed to be perfectly isolated at $z = 0$, so $\frac{\partial}{\partial z} w(0,t) = 0$. The condition at $z = 1$ is not of interest to us and we assume that the initial temperature is 0. In addition, we suppose that the temperature $w(z,t)$ at any time t and position z at the rod can be measured and used in the parameter estimation. To simplify the notation, we write $q_0 : t \mapsto w(0,t)$ and $f : t \mapsto w(z,0)$.

The goal is to identify the parameters α and β. The algebraic method used in the previous subsection is applied here. Using the notation $r : t \mapsto \frac{\partial}{\partial z} w(0,t)$, the Laplace transform (2) is employed to realize the partial differential equation (10) as an algebraic equation in the Laplace variable $\mathbf{s} = (s_1, s_2)$ (s_1 corresponds to z and s_2 corresponds to t):

$$\left(s_1^2 - \beta s_2 - \alpha\right) W(s_1,s_2) + \beta F(s_1) - s_1 Q_0(s_2) - R(s_2) = 0, \tag{11}$$

where $W(\mathbf{s})$, $F(s_1)$, $Q_0(s_2)$ and $R(s_2)$ denote the Laplace transforms of $w(z,t)$ with respect to z and t, of $f(z)$ with respect to z, and of $q_0(t)$ and $r(t)$ with respect to t respectively. Since by hypothesis, $R \equiv 0$ and $F \equiv 0$, the equation (11) leads to:

$$\left(-\beta\, s_2 + {s_1}^2 - \alpha\right) W\left(s_1,s_2\right) - s_1 Q_0\left(s_2\right) = 0. \tag{12}$$

Here the set of parameters Θ_{est} to be estimated is

$$\Theta_{\text{est}} = \{\alpha, \beta\},$$

while $\Theta_{\overline{\text{est}}} = \emptyset$. Following the procedure described at the beginning of Section 2, a system on the indeterminates α and β will be determined by acting suitable annihilators on (12). So a two-steps procedure will be applied. In the first step, we rewrite (12) in the form of a $\mathcal{R}$-relation (see (8)):

$$\mathcal{R} : P(\mathbf{s})W(\mathbf{s}) + Q(\mathbf{s}) + \overline{Q}(\mathbf{s}) = 0, \tag{13}$$

where

$$P(\mathbf{s}) = s_1^2 - \beta s_2 - \alpha,\ Q(\mathbf{s}) = -s_1 Q\left(s_2\right) \text{ and } \overline{Q}(\mathbf{s}) = 0. \tag{14}$$

As mentioned before, the problem of annihilating $\overline{Q}$ is tackled by finding suitable $\overline{Q}$-annihilators that will lead to a system in Θ_{est}. Notice in this example that the parameters to be estimated also appear in the coefficients of the polynomial P.

Here, the polynomial $\overline{Q}$ to be annihilated in (14) is 0, so we may consider the above equation (14). In order to apply the inverse Laplace transform (4), we divide this equation by $s_1^3 s_2^2$. Using the notation $v_{I,J} = (z-\tau)^{i_1}\,(t-\eta)^{i_2}\,(-\tau)^{j_1}(-\eta)^{j_2}$, for all $I = (i_1,i_2)$, $J = (j_1,j_2) \in \mathbb{N}^2$, we obtain in the spatial domain:

$$A_{11}\alpha + A_{12}\beta = B_1$$

where

$$\begin{cases} A_{11} = -\dfrac{1}{2}\displaystyle\int_0^{(z,t)} v_{2,1,0,0}\, w(\tau,\eta)\, \mathrm{d}\tau\, \mathrm{d}\eta, \\ \\ A_{12} = -\dfrac{1}{2}\displaystyle\int_0^{(z,t)} v_{2,0,0,0}\, w(\tau,\eta)\, \mathrm{d}\tau\, \mathrm{d}\eta, \\ \\ B_1 = -\displaystyle\int_0^{(z,t)} v_{0,1,0,0}\, (w(\tau,\eta) - q_0(\eta))\, \mathrm{d}\tau\ \mathrm{d}\eta. \end{cases}$$

In the second step, we will try to eliminate the term with the polynomial Q_0: in this case, the polynomial $\overline{Q}$ to be annihilated in (14) is $\overline{Q}(\mathbf{s}) = -s_1 Q_0(s_2)$ while $Q(\mathbf{s}) = \beta F(s_1)$. We propose the $\overline{Q}$-annihilator $\pi = \frac{\partial^2}{\partial s_1^2}$. Applying π on the relation (13) gives:

$$-\alpha \frac{\partial^2}{\partial {s_1}^2} W + \left(\frac{\mathrm{d}^2}{\mathrm{d}{s_1}^2} F(s_1) - s_2 \frac{\partial^2}{\partial {s_1}^2} W \right) \beta + {s_1}^2 \frac{\partial^2}{\partial {s_1}^2} W + 4 s_1 \frac{\partial}{\partial s_1} W + 2W = 0. \tag{15}$$

After dividing the above equation by a suitable monomial in $\mathbf{s}$, namely $s_1^2 s_2^2$, we obtain:

$$\begin{cases} A_{21} = -\dfrac{1}{2}\displaystyle\int_0^{(z,t)} v_{2,1,0,2}\, w(\tau,\eta)\, \mathrm{d}\tau\, \mathrm{d}\eta \\ \\ A_{22} = -\dfrac{1}{2}\displaystyle\int_0^{(z,t)} v_{2,0,0,2} w(\tau,\eta)\, \mathrm{d}\tau\, \mathrm{d}\eta, \\ \\ B_2 = \displaystyle\int_0^{(z,t)} (4v_{1,1,0,0} - v_{2,1,0,0} - v_{0,1,2,0})\, w(\tau,\eta)\, \mathrm{d}\tau\, \mathrm{d}\eta. \end{cases}$$

A system on Θ_{est} results from the actions of $\overline{Q}$-annihilators:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.$$

Solving the system provides the estimates of α and β.

Remark 2. In [44], other annihilators were proposed since the statement of the problem and its initial and boundary conditions were different. Again, recall that a very special property of the two-dimensional Weyl algebra is Stafford's theorem (see Theorem 2). This important result allows the computation of two minimal annihila-

tors. Moreover the package `Stafford` [30, 31] uses a highly efficient algorithm to calculate these differential operators.

6 Conclusion

In this paper, we provided a short preview on algebraic estimation for derivatives and for parameters in linear systems. Advantages and possible drawbacks of this algebraic framework were evoked in a brief state-of-art. More detailed problem statements were given in the subsequent sections, followed by proposed solutions within the algebraic context. The algebraic properties in the appendix, notably concerning the Weyl algebra, support these solutions. Furthermore, we illustrate our algebraic method with two typical examples: in the case of two-dimensional numerical differentiation, while in the case of parameter estimation for partial derivatives systems, the thin rod example is studied. An essential point deserves to be emphasized: the algebraically obtained estimated are based on integrals of measured signals. These particular integrals play the role of time-varying filters. Furthermore, closed formulas for derivatives and parameters estimates that obtained with our method, via the Weyl algebra tools, are presented in this paper.

7 Appendix

We recall below some basic definitions and properties of the Weyl algebra.

Definition 3. Let $m \in \mathbb{N} \setminus \{0\}$. The *Weyl algebra* $\mathsf{A}_m(\mathbb{K})$ (or A_m) is the $\mathbb{K}$-algebra with generators $p_1, q_1, \ldots, p_m, q_m$ and relations

$$[p_i, q_j] = \delta_{ij},\ [p_i, p_j] = [q_i, q_j] = 0, \forall\ 1 \leq i, j \leq m$$

where $[\cdot,\cdot]$ is the commutator defined by $[u, v] := uv - vu$, for all u, $v \in \mathsf{A}_m(\mathbb{K})$.

The Weyl algebra A_m can be realized as the algebra of polynomial differential operators on the polynomial ring $\mathbb{K}[\mathbf{s}]$ by setting:

$$p_i = \frac{\partial}{\partial s_i} \text{ and } q_i = s_i \times \cdot\ , \forall\ 1 \leq i \leq m,$$

where $\times$ denotes the multiplication map. That implies that A_m can be written as $\mathsf{A}_m = \mathbb{K}[\mathbf{q}][\mathbf{p}] = \mathbb{K}[\mathbf{s}]\left[\frac{\partial}{\partial \mathbf{s}}\right]$. The algebra of differential operators $\mathsf{B}_m(\mathbb{K})$ (or B_m) on $\mathbb{K}[\mathbf{s}]$ with coefficients in the rational functions field $\mathbb{K}(\mathbf{s})$ is naturally related to $\mathsf{A}_m(\mathbb{K})$. In this case, we can write $\mathsf{B}_m := \mathbb{K}(\mathbf{q})[\mathbf{p}] = \mathbb{K}(\mathbf{s})\left[\dfrac{\partial}{\partial \mathbf{s}}\right]$.

A $\mathbb{K}$-basis for A_m is given by $\left\{\mathbf{q}^I\mathbf{p}^J \mid I,J \in \mathbb{N}^m\right\}$ where $\mathbf{q} = q_1^{i_1}\dots q_m^{i_m}$ and $\mathbf{p} = p_1^{i_1}\dots p_m^{i_m}$. An operator $F \in \mathsf{A}_m$ can be written in a *canonical form*,

$$F = \sum_{I,J} \lambda_{IJ}\mathbf{q}^I\mathbf{p}^J \text{ with } \lambda_{IJ} \in \mathbb{K}.$$

Similarly, an element $F \in \mathsf{B}_m$ can be written as

$$F = \sum_I g_I(\mathbf{s})\frac{\partial^I}{\partial s^I}, \text{ where } g_I(\mathbf{s}) \in \mathbb{K}(\mathbf{s}).$$

The *order* of F is defined as $\mathrm{ord}(F) = \max\{[I] \mid g_I(\mathbf{s}) \neq 0\}$. This definition holds for the Weyl algebra A_m as well, since $\mathsf{A}_m \subset \mathsf{B}_m$. Some useful properties of A_m and B_m are given by the following propositions (see for instance [7]):

Proposition 1. *The algebra A_m is a domain. Moreover, A_m is a simple algebra (*i.e. *it contains no nontrivial ideals) and also a left Noetherian ring (*i.e. *every left ideal is finitely generated).*

These properties are shared by B_m. In addition, A_m is neither a principal right domain, nor a principal left domain. Nevertheless this is true for B_1:

Proposition 2. *B_1 admits a left division algorithm, that is, if F, $G \in \mathsf{B}_1$, then there exists Q, $R \in \mathsf{B}_1$ such that $F = QG+R$ and $\mathrm{ord}(R) < \mathrm{ord}(G)$. Consequently, B_1 is a principal left domain.*

Alas, this proposition does not hold for B_m for $m \geq 2$. But an important theorem by T. Stafford (see [40]) provides an remarkable property on the number of generators of a left ideal in the Weyl algebra. Namely, Stafford proved that every left ideal of D ($D = \mathsf{A}_m$ or B_m) can be generated by two elements in D:

Theorem 2 (Stafford). *Let $\mathfrak{a}$ be a left ideal of D generated by three elements F_1, F_2 and $F_3 \in D$. Then, there exist G_1 and $G_2 \in D$ such that*

$$\mathfrak{a} = D(F_1 + G_1F_3) + D(F_2 + G_2F_3).$$

An effective implementation in `Maple`, named `Stafford`, of this important theorem can be found in the work of A. Quadrat and D. Robertz [30].

Remark 3. It is important to notice that the principality of B_1 was largely used in the initial works on algebraic methods applied to univariate numerical differentiation, such as [20] or parameter estimation in ordinary differential equations, see for instance [47]. In the multivariate case, the principality holds no longer, therefore the importance of Stafford's theorem.

To close this part, we remark a useful identity:

Remark 4. For arbitrary $N, M \in \mathbb{N}^r$, we have

$$\frac{\partial^N}{\partial \mathbf{s}^N}\frac{1}{\mathbf{s}^M} = \sum_{0\preceq J\preceq N}\binom{N}{J}(-1)^{|N-J|}\frac{M^{\overline{N-J}}}{\mathbf{s}^{M+N-J}}\frac{\partial^J}{\partial \mathbf{s}^J},$$

where $\binom{N}{J} = \binom{n_1}{j_1}\dots\binom{n_r}{j_r}$, $M^{\overline{N}} = m_1^{\overline{n_1}}\dots m_r^{\overline{n_r}}$ and $m_i^{\overline{n_i}}$ denotes the rising factorial ($m_i^{\overline{n_i}} = m_i(m_i+1)\dots(m_i+n_i-1)$).

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