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SIAM J. FINANCIAL MATH.
Vol. 8, pp. 532–559

Game Options in an Imperfect Market with Default*

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Abstract. We study pricing and superhedging strategies for game options in an imperfect market with default. We extend the results obtained by Kifer in [*Game Options*, Finance. Stoch., 4 (2000), pp. 443–463] in the case of a perfect market model to the case of an imperfect market with default, when the imperfections are taken into account via the nonlinearity of the wealth dynamics. We introduce the *seller's price* of the game option as the infimum of the initial wealths which allow the seller to be superhedged. We prove that this price coincides with the value function of an associated *generalized* Dynkin game, recently introduced in [R. Dumitrescu, M.-C. Quenez, and A. Sulem, *Elect. J. Probab.*, 21 (2016), 64], expressed with a nonlinear expectation induced by a nonlinear backward SDE with default jump. We, moreover, study the existence of superhedging strategies. We then address the case of ambiguity on the model—for example ambiguity on the default probability—and characterize the robust seller's price of a game option as the value function of a *mixed generalized* Dynkin game. We study the existence of a cancellation time and a trading strategy which allow the seller to be superhedged, whatever the model is.

Key words. game options, imperfect markets, generalized Dynkin games, nonlinear expectations, backward stochastic differential equations, nonlinear pricing, superhedging price, doubly reflected backward stochastic differential equations

AMS subject classifications. 93E20, 60J60, 60G40, 91A15, 91A05, 91G20, 91G80

DOI. 10.1137/16M1109102

1. Introduction.

Game options, which have been introduced by Kifer [22], are derivative contracts that can be terminated by both counterparties at any time before a maturity date T. More precisely, a game option allows the seller to cancel it and the buyer to exercise it at any stopping time smaller than T. If the buyer exercises at time τ before the seller cancels, then the seller pays the buyer the amount ξ_τ, but if the seller cancels before the buyer exercises, then he pays the amount $\zeta_\sigma \geq \xi_\tau$ to the buyer at the cancellation time σ. The difference $\zeta_\sigma - \xi_\sigma$ is interpreted as a penalty that the seller pays to the buyer for the cancellation of the contract. In short, if the buyer selects an exercise time τ and the seller selects a cancellation time σ, then the latter pays to the former the payoff $\xi_\tau \mathbf{1}_{\tau\leq\sigma} + \zeta_\sigma \mathbf{1}_{\tau>\sigma}$ at time $\tau \wedge \sigma$.

In the case of classical perfect markets, Kifer introduces the "fair price" of the game option, defined as the minimum initial wealth needed for the seller to cover his liability to pay the payoff to the buyer until a cancellation time, whatever is the exercise time chosen by the

*Received by the editors December 22, 2016; accepted for publication May 8, 2017; published electronically July 26, 2017.
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buyer. He shows, both in the Cox–Ross–Rubinstein discrete-time model and in the Black and Scholes model, that this price is equal to the value function of the following Dynkin game:

$$\sup_{\tau}\inf_{\sigma}\mathbb{E}_Q\left[\tilde{\xi}_\tau\mathbf{1}_{\tau\leq\sigma}+\tilde{\zeta}_\sigma\mathbf{1}_{\tau>\sigma}\right]=\inf_{\sigma}\sup_{\tau}\mathbb{E}_Q\left[\tilde{\xi}_\tau\mathbf{1}_{\tau\leq\sigma}+\tilde{\zeta}_\sigma\mathbf{1}_{\tau>\sigma}\right], \tag{1.1}$$

where $\tilde{\xi}_t$ and $\tilde{\zeta}_t$ are the discounted values of ξ_t and ζ_t, equal to $e^{-rt}\xi_t$ and $e^{-rt}\zeta_t$, respectively, in the Black and Scholes model, where r is the instantaneous interest rate. Here, $\mathbb{E}_Q$ denotes the expectation under the unique martingale probability measure Q of the market model. Further research on the pricing of game options and on more sophisticated game-type financial contracts includes, in particular, papers by Dolinsky and Kifer [11] and Dolinsky, Iron, and Kifer [10] in the discrete time case, and by Hamadène [17] in a continuous time perfect market model with continuous payoffs ξ and ζ. We also mention the paper by Bielecki and al. [2] which studies the pricing of game options in a market model with default. Note that in [21], Kallsen and Kuhn study game options in an incomplete market. They consider another type of pricing called *neutral valuation* via utility maximization.

The aim of the present paper is to study pricing and hedging issues for game options in the case of imperfections in the market model taken into account via the nonlinearity of the wealth dynamics, modeled via a nonlinear driver g. We, moreover, include the possibility of a default. A large class of imperfect market models can fit in our framework, like different borrowing and lending interest rates, or taxes on the profits from risky investments. Our model also includes the case when the seller of the option is a "large trader" whose hedging strategy may affect the market prices and the default probability.

Here, we suppose that the payoffs ξ and ζ associated with the game option are right-continuous left-limited (RCLL) only and they satisfy Mokobodzki's condition. We call *seller's price* of the game option, the infimum (denoted by u_0) of the initial wealths such that there exists a cancellation time σ, and a portfolio strategy which allow the seller to pay ξ_τ (at time τ) to the buyer if the buyer exercises at any time $\tau\leq\sigma$, and ζ_σ (at time σ) if the buyer has still not exercised at time σ. Note that this infimum is not necessarily attained. We provide a characterization of the seller's price u_0 of the game option as the (common) value of a corresponding *generalized* Dynkin game (recently introduced in [13]). More precisely, we show that

$$u_0=\sup_{\tau}\inf_{\sigma}\mathcal{E}^g[\xi_\tau\mathbf{1}_{\tau\leq\sigma}+\zeta_\sigma\mathbf{1}_{\tau>\sigma}]=\inf_{\sigma}\sup_{\tau}\mathcal{E}^g[\xi_\tau\mathbf{1}_{\tau\leq\sigma}+\zeta_\sigma\mathbf{1}_{\tau>\sigma}], \tag{1.2}$$

where $\mathcal{E}^g$ is a nonlinear expectation/evaluation induced by a nonlinear backward SDE (BSDE) with default jump solved under the primitive probability measure P with driver g. Note that in the particular case of a perfect market, the driver g is linear and one can show, by using an actualization procedure and a change of probability measure, that (1.2) corresponds to (1.1).

We prove that, under an additional left-regularity assumption on ζ (but not on ξ), there exist a cancellation time and a trading strategy which allow the seller to be superhedged. In this case, the infimum in the definition of the seller's price u_0 is attained. When ζ is only RCLL, the infimum is not necessarily attained. However, we show that for each $\varepsilon>0$, the amount u_0 allows the seller to be superhedged up to ε until a well-chosen cancellation time.

The proofs of these results rely on the links between generalized Dynkin games and nonlinear doubly reflected BSDEs with default jump.

The second main question we study is the pricing and superhedging problem of game options in the case of uncertainty on the (imperfect) market model. To the best of our knowledge, this problem has not been studied in the literature except by Dolinsky in [9] in a discrete time framework. In particular, our model can take into account an *ambiguity on the default probability* as illustrated in section 4.3. We prove that the robust seller's price of the game option under uncertainty, defined as the infimum of the initial wealths which allow the seller to be superhedged whatever the model is, coincides with the value function of a mixed generalized Dynkin game. We also study the existence of robust superhedging strategies.

The paper is organized as follows: in section 2, we introduce our imperfect market model with default and nonlinear wealth dynamics. In section 3, we study pricing and superhedging of game options and their links with *generalized Dynkin games*. In section 4, we address the case of an imperfect market with model ambiguity. Section 5 provides some complementary results concerning the buyer's point of view and the case with dividends. Some results on doubly reflected BSDEs with default jumps and a useful lemma of analysis are given in the appendix.

2. Imperfect market model with default.

2.1. Market model with default. Let $(\Omega, \mathcal{G}, P)$ be a complete probability space equipped with two stochastic processes: a unidimensional standard Brownian motion W and a jump process N defined by $N_t = \mathbf{1}_{\vartheta \le t}$ for any $t \in [0,T]$, where ϑ is a random variable which models a default time. We assume that this default can appear at any time, that is, $P(\vartheta \ge t) > 0$ for any $t \ge 0$. We denote by $\mathbb{G} = \{\mathcal{G}_t, t \ge 0\}$ the *augmented filtration* that is generated by W and N (in the sense of [8, IV-48]). We suppose that W is a $\mathbb{G}$-Brownian motion. We denote by $\mathcal{P}$ the $\mathbb{G}$-predictable σ-algebra. Let (Λ_t) be the predictable compensator of the nondecreasing process (N_t). Note that $(\Lambda_{t\wedge\vartheta})$ is then the predictable compensator of $(N_{t\wedge\vartheta}) = (N_t)$. By uniqueness of the predictable compensator, $\Lambda_{t\wedge\vartheta} = \Lambda_t$, $t \ge 0$ a.s. We assume that Λ is absolutely continuous w.r.t. Lebesgue's measure, so that there exists a nonnegative process λ, called the intensity process, such that $\Lambda_t = \int_0^t \lambda_s ds$, $t \ge 0$. Since $\Lambda_{t\wedge\vartheta} = \Lambda_t$, λ vanishes after ϑ. We denote by M the compensated martingale which satisfies

$$M_t = N_t - \int_0^t \lambda_s ds\,.$$

Let $T > 0$ be the finite horizon. We introduce the following sets:

- S^2 is the set of $\mathbb{G}$-adapted RCLL processes φ such that $\mathbb{E}[\sup_{0\le t\le T} |\varphi_t|^2] < +\infty$.
- $\mathcal{A}^2$ is the set of real-valued nondecreasing RCLL predictable processes A with $A_0 = 0$ and $\mathbb{E}(A_T^2) < \infty$.
- $\mathbb{H}^2$ is the set of $\mathbb{G}$-predictable processes Z such that $\|Z\|^2 := \mathbb{E}[\int_0^T |Z_t|^2 dt] < \infty$.
- $\mathbb{H}^2_\lambda := L^2(\Omega\times[0,T], \mathcal{P}, \lambda_t dt)$, equipped with the scalar product $\langle U, V\rangle_\lambda := \mathbb{E}[\int_0^T U_t V_t \lambda_t dt]$ for all U, V in $\mathbb{H}^2_\lambda$. For each $U \in \mathbb{H}^2_\lambda$, we set $\|U\|^2_\lambda := \mathbb{E}[\int_0^T |U_t|^2 \lambda_t dt] < \infty$.

Note that for each $U \in \mathbb{H}^2_\lambda$, we have $\|U\|^2_\lambda = \mathbb{E}[\int_0^{T\wedge\vartheta} |U_t|^2 \lambda_t dt]$ because the $\mathbb{G}$-intensity λ vanishes after ϑ. Moreover, we can suppose that for each U in $\mathbb{H}^2_\lambda = L^2(\Omega \times [0,T], \mathcal{P}, \lambda_t dt)$, U (or its representant in $\mathcal{L}^2(\Omega \times [0,T], \mathcal{P}, \lambda_t dt)$ still denoted by U) vanishes after ϑ.

Moreover, $\mathcal{T}$ denotes the set of stopping times τ such that $\tau \in [0,T]$ a.s. and for each S in $\mathcal{T}$, $\mathcal{T}_S$ is the set of stopping times τ such that $S \leq \tau \leq T$ a.s.

We recall the martingale representation theorem (see, e.g., [18]).

Lemma 2.1. *Any $\mathbb{G}$-local martingale $m = (m_t)_{0\leq t\leq T}$ has the representation*

$$m_t = m_0 + \int_0^t z_s dW_s + \int_0^t l_s dM_s \quad \forall\, t \in [0,T] \quad a.s., \tag{2.1}$$

where $z = (z_t)_{0\leq t\leq T}$ and $l = (l_t)_{0\leq t\leq T}$ are predictable such that the two above stochastic integrals are well defined. If m is a square integrable martingale, then $z \in \mathbb{H}^2$ and $l \in \mathbb{H}^2_\lambda$.

We consider now a financial market with three assets with price process $S = (S^0, S^1, S^2)'$ governed by the equations

$$\begin{cases} dS_t^0 = S_t^0 r_t dt, \\ dS_t^1 = S_t^1[\mu_t^1 dt + \sigma_t^1 dW_t], \\ dS_t^2 = S_{t^-}^2[\mu_t^2 dt + \sigma_t^2 dW_t - dM_t]. \end{cases}$$

The process $S^0 = (S_t^0)_{0\leq t\leq T}$ corresponds to the price of a nonrisky asset with interest rate process $r = (r_t)_{0\leq t\leq T}$, $S^1 = (S_t^1)_{0\leq t\leq T}$ to a nondefaultable risky asset, and $S^2 = (S_t^2)_{0\leq t\leq T}$ to a defaultable asset with total default. The price process S^2 vanishes after ϑ.

All the processes σ^1, σ^2, r, μ^1, μ^2 are predictable (that is $\mathcal{P}$-measurable). We set $\sigma = (\sigma^1, \sigma^2)'$. We make the following assumptions.

Assumption 2.2. The coefficients $\sigma^1, \sigma^2 > 0$, and r, σ^1, σ^2, μ^1, μ^2, λ, λ^{-1},$(\sigma^1)^{-1}$, $(\sigma^2)^{-1}$ are bounded.

We consider an investor, endowed with an initial wealth equal to x, who can invest his wealth in the three assets of the market. At each time $t < \vartheta$, he chooses the amount φ_t^1 (resp., φ_t^2) of wealth invested in the first (resp., second) risky asset. However, after time ϑ, the investor cannot invest his wealth in the defaultable asset since its price is equal to 0, and he only chooses the amount φ_t^1 of wealth invested in the first risky asset. Note that the process φ^2 can be defined on the whole interval $[0,T]$ by setting $\varphi_t^2 = 0$ for each $t \geq \vartheta$. A process $\varphi_. = (\varphi_t^1, \varphi_t^2)'_{0\leq t\leq T}$ is called a *risky assets stategy* if it belongs to $\mathbb{H}^2 \times \mathbb{H}^2_\lambda$. We denote by $V_t^{x,\varphi}$ (or simply V_t) the *wealth* or, equivalently, the value of the portfolio, at time t. The amount invested in the nonrisky asset at time t is then given by $V_t - (\varphi_t^1 + \varphi_t^2)$.

The perfect market model. In the classical case of a perfect market model, the wealth process and the strategy satisfy the self-financing condition:

$$dV_t = \left(r_t V_t + \varphi_t^1\left(\mu_t^1 - r_t\right) + \varphi_t^2\left(\mu_t^2 - r_t\right)\right)dt + \left(\varphi_t^1\sigma_t^1 + \varphi_t^2\sigma_t^2\right)dW_t - \varphi_t^2 dM_t. \tag{2.2}$$

Setting $K_t := -\varphi_t^2$, and $Z_t := \varphi_t^1\sigma_t^1 + \varphi_t^2\sigma_t^2$, which implies that $\varphi_t^1 = (Z_t + \sigma_t^2 K_t)(\sigma_t^1)^{-1}$, we get

$$\begin{aligned} dV_t &= r_t V_t + \left(Z_t + \sigma_t^2 K_t\right)\left(\mu_t^1 - r_t\right)\left(\sigma_t^1\right)^{-1} - K_t\left(\mu_t^2 - r_t\right)dt + Z_t dW_t + K_t dM_t \\ &= \left(r_t V_t + Z_t\theta_t^1 + K_t\theta_t^2\lambda_t\right)dt + Z_t dW_t + K_t dM_t, \end{aligned}$$

where $\theta_t^1 := \frac{\mu_t^1 - r_t}{\sigma_t^1}$ and $\theta_t^2 := \frac{\sigma_t^2\theta_t^1 - \mu_t^2 + r_t}{\lambda_t}\mathbf{1}_{\{t\leq\vartheta\}}$.

Consider a European contingent claim with maturity $T > 0$ and payoff ξ which is $\mathcal{G}_T$ measurable, belonging to L^2. The problem is to price and hedge this claim by constructing a replicating portfolio. From [14, Proposition 2.6], there exists a unique process $(X, Z, K) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ solution of the following BSDE with default jump:

$$-dX_t = -\left(r_t X_t + Z_t \theta^1_t + K_t \theta^2_t \lambda_t\right) dt - Z_t dW_t - K_t dM_t, \quad X_T = \xi. \tag{2.3}$$

The solution (X, Z, K) provides the replicating portfolio. More precisely, the process X corresponds to its value, and the hedging risky assets stategy $\varphi \in \mathbb{H}^2_\lambda$ is given by $\varphi = \Phi(Z, K)$, where Φ is the one to one map defined on $\mathbb{H}^2 \times \mathbb{H}^2_\lambda$ by the following.

Definition 2.3. *Let Φ be the functional defined by*

$$\Phi : \mathbb{H}^2 \times \mathbb{H}^2_\lambda \to \mathbb{H}^2 \times \mathbb{H}^2_\lambda; (Z, K) \mapsto \Phi(Z, K) := \varphi,$$

where $\varphi = (\varphi^1, \varphi^2)$ is given by

$$\varphi^2_t = -K_t, \quad \varphi^1_t = \frac{Z_t + \sigma^2_t K_t}{\sigma^1_t},$$

which is equivalent to $K_t = -\varphi^2_t$, $Z_t = \varphi^1_t \sigma^1_t + \varphi^2_t \sigma^2_t = \varphi_t{}' \sigma_t$.

Note that the processes φ^2 and K, which belong to $\mathbb{H}^2_\lambda$, both vanish after time ϑ.

The process X coincides with $V^{X_0,\varphi}$, the value of the portfolio associated with initial wealth $x = X_0$ and portfolio strategy φ. From the seller's point of view, this portfolio is a hedging portfolio. Indeed, by investing the initial amount X_0 in the reference assets along the strategy φ, the seller can pay the amount ξ to the buyer at time T (and similarly at each initial time t). We derive that X_t is the price at time t of the option, called the *hedging price*, and denoted by $X_t(\xi)$. By the representation property of the solution of a λ-linear BSDE with default jump (see [14, Theorem 2.13]), we have that the solution X of BSDE (2.3) can be written as follows:

$$X_t(\xi) = \mathbb{E}\left[e^{-\int_t^T r_s ds} \zeta_{t,T} \xi \,|\, \mathcal{G}_t\right], \tag{2.4}$$

where $\zeta_{t,\cdot}$ satisfies

$$d\zeta_{t,s} = \zeta_{t,s^-}\left[-\theta^1_s dW_s - \theta^2_s dM_s\right], \quad \zeta_{t,t} = 1. \tag{2.5}$$

This defines a *linear* price system X: $\xi \mapsto X(\xi)$. Suppose now that

$$\theta^2_t < 1, \ 0 \le t \le \vartheta \ dt \otimes dP\text{-a.s.} \tag{2.6}$$

Then $\zeta_{t,\cdot} > 0$. Let Q be the probability measure which admits $\zeta_{0,T}$ as density on $\mathcal{G}_T$. Using Girsanov's theorem, it can be shown that Q is the unique martingale probability measure. In this case, the price system X is increasing and corresponds to the classical arbitrage free price system (see [18, 3, 2]).

Remark 2.4. We have presented above the case of a defaultable asset with total default. A different model for the asset price S^2 (see, e.g., [18, Chapter 7, section 9.3]) could be considered:

$$dS_t^2 = S_{t-}^2 \left[\mu_t^2 dt + \sigma_t^2 dW_t + \beta_t dM_t\right],$$

where $\beta_t \neq 0$ and $\beta_t > -1$ with β_t, β_t^{-1} bounded. In this case, the price does not vanish after the default time ϑ. We suppose that

$$\frac{\mu_t^1 - r_t}{\sigma_t^1}\mathbf{1}_{\{t>\vartheta\}} = \frac{\mu_t^2 - r_t}{\sigma_t^2}\;\mathbf{1}_{\{t>\vartheta\}} \quad dt \otimes dP\text{-a.s.} \tag{2.7}$$

Let $\zeta_{0,\cdot}$ be defined by (2.5) with $\theta_t^1 = \frac{\mu_t^1 - r_t}{\sigma_t^1}, \theta_t^2 = \frac{\mu_t^2 - \sigma_t^2\theta_t^1 - r_t}{\beta_t\lambda_t}\mathbf{1}_{\{t\leq\vartheta\}}$. Assume that $\theta_t^2 < 1$, $0 \leq t \leq \vartheta$ $dt \otimes dP$-a.s. The assumption (2.7) ensures that the probability measure Q with $\zeta_{0,T}$ as density on $\mathcal{G}_T$ is the unique martingale probability measure. The arbitrage free price of the contingent claim ξ is given by (2.4) and satisfies BSDE (2.3); moreover, the hedging strategy $\varphi = (\varphi^1, \varphi^2)$ is given by $\varphi_t^2 = \frac{K_t}{\beta_t}$ and $\varphi_t^1 = \frac{Z_t - \varphi_t^2\sigma_t^2}{\sigma_t^1}$.

The imperfect market model $\mathcal{M}^g$. From now on, we assume that there are imperfections in the market which are taken into account via the *nonlinearity* of the dynamics of the wealth. More precisely, the dynamics of the wealth V associated with strategy $\varphi = (\varphi^1, \varphi^2)$ can be written via a *nonlinear* driver, defined as follows:

Definition 2.5 (driver, *λ-admissible* driver). *A function g is said to be a* driver *if g : $[0, T] \times \Omega \times \mathbb{R}^3 \to \mathbb{R}$, $(\omega, t, y, z, k) \mapsto g(\omega, t, y, z, k)$ which is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^3)$-measurable, and such that $g(., 0, 0, 0) \in \mathbb{H}^2$.*

A driver g is called a λ-admissible driver *if, moreover, there exists a constant $C \geq 0$ such that $dP \otimes dt$-a.s. for each (y_1, z_1, k_1), (y_2, z_2, k_2),*

$$|g(\omega, t, y_1, z_1, k_1) - g(\omega, t, y_2, z_2, k_2)| \leq C\left(|y_1 - y_2| + |z_1 - z_2| + \sqrt{\lambda_t}|k_1 - k_2|\right). \tag{2.8}$$

The positive real C is called the λ-constant *associated with driver g.*

Note that condition (2.8) implies that for each $t > \vartheta$, since $\lambda_t = 0$, g does not depend on k. In other terms, for each (y, z, k), we have: $g(t, y, z, k) = g(t, y, z, 0)$, $t > \vartheta$ $dP \otimes dt$-a.s. Let $x \in \mathbb{R}$ be the initial wealth and let $\varphi = (\varphi^1, \varphi^2)$ in $\mathbb{H}^2 \times \mathbb{H}^2_\lambda$ be a portfolio strategy.

We suppose that the associated *wealth* process $V_t^{x,\varphi}$ (or simply V_t) satisfies the following dynamics:

$$-dV_t = g\left(t, V_t, \varphi_t{}'\sigma_t, -\varphi_t^2\right)dt - \varphi_t{}'\sigma_t dW_t + \varphi_t^2 dM_t \tag{2.9}$$

with $V_0 = x$. Since g is Lipschitz with respect to y, this formulation makes sense. Indeed, setting $f_t^1 := \int_0^t \varphi_t{}'\sigma_t dW_s + \varphi_t^2 dM_s$ for each ω, the deterministic function $(V_t^{Y_0,\varphi}(\omega))$ is defined as the unique solution of the following deterministic differential equation:

$$V_t^{x,\varphi}(\omega) = x - \int_0^t g\left(\omega, s, V_s^{x,\varphi}(\omega), \varphi_s{}'\sigma_s(\omega), -\varphi_s^2(\omega)\right)ds + f_t^1(\omega),\ 0 \leq t \leq T. \tag{2.10}$$

Note that, equivalently, setting $Z_t = \varphi_t{}'\sigma_t$ and $K_t = -\varphi_t^2$, the dynamics (2.9) of the wealth process V_t can be written as follows:

$$-dV_t = g(t, V_t, Z_t, K_t)dt - Z_t dW_t - K_t dM_t. \tag{2.11}$$

In the following, our imperfect market model is denoted by $\mathcal{M}^g$.

Note that in the case of a perfect market (see (2.3)), we have

$$g(t, y, z, k) = -r_t y - \theta_t^1 z - \theta_t^2 k\lambda_t, \tag{2.12}$$

which is a λ-admissible driver by Assumption 2.2.

2.2. A nonlinear pricing system. Pricing and hedging European options in the imperfect market $\mathcal{M}^g$ leads to BSDEs with nonlinear driver g and a default jump. By [14, Proposition 2.6], we have the following.

Proposition 2.6. *Let g be a λ-admissible driver, let $\xi \in L^2(\mathcal{G}_T)$. There exists a unique solution $(X(T,\xi), Z(T,\xi), K(T,\xi))$ (denoted simply by (X, Z, K)) in $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}_\lambda^2$ of the following BSDE:*

$$-dX_t = g(t, X_t, Z_t, K_t)dt - Z_t dW_t - K_t dM_t, \quad X_T = \xi. \tag{2.13}$$

Let us consider a European option with maturity T and terminal payoff $\xi \in L^2(\mathcal{G}_T)$ in this market model. Let (X, Z, K) be the solution of BSDE (2.13). The process X is equal to the wealth process associated with initial value $x = X_0$, strategy $\varphi = \Phi(Z, K)$ (where Φ is defined in Definition 2.3) that is $X = V^{X_0,\varphi}$. Its initial value $X_0 = X_0(T,\xi)$ is thus a sensible price (at time 0) of the claim ξ for the seller since this amount allows him/her to construct a trading strategy $\varphi \in \mathbb{H}^2 \times \mathbb{H}_\lambda^2$, called the *hedging strategy* (for the seller), such that the value of the associated portfolio is equal to ξ at time T. Moreover, by the uniqueness of the solution of BSDE (2.13), it is the unique price (at time 0) which satisfies this hedging property. Similarly, $X_t = X_t(T,\xi)$ satisfies an analogous property at time t, and is called the hedging price at time t. This leads to a *nonlinear pricing* system, first introduced by El Karoui and Quenez [16] in a Brownian framework (later called g-*evaluation* in [23]) and denoted by $\mathcal{E}^g$. For each $S \in [0,T]$, for each $\xi \in L^2(\mathcal{G}_S)$ the associated g-evaluation is defined by $\mathcal{E}^g_{t,S}(\xi) := X_t(S,\xi)$ for each $t \in [0,S]$.

In order to ensure the (strict) monotonicity and the no arbitrage property of the nonlinear pricing system $\mathcal{E}^g$, we make the following assumption (see [14, section 3.3]).

Assumption 2.7. Assume that there exists a bounded map

$$\gamma : [0,T] \times \Omega \times \mathbb{R}^4 \to \mathbb{R},\ (\omega, t, y, z, k_1, k_2) \mapsto \gamma_t^{y,z,k_1,k_2}(\omega),$$

$\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^4)$-measurable and satisfying $dP \otimes dt$-a.s. for each $(y, z, k_1, k_2) \in \mathbb{R}^4$,

$$g(t, y, z, k_1) - g(t, y, z, k_2) \geq \gamma_t^{y,z,k_1,k_2}(k_1 - k_2)\lambda_t, \tag{2.14}$$

and P-a.s., for each $(y, z, k_1, k_2) \in \mathbb{R}^4$, $\gamma_t^{y,z,k_1,k_2} > -1$.

This assumption is satisfied, e.g., when $g(t,\cdot)$ is nondecreasing with respect to k or if g is $\mathcal{C}^1$ in k with $\partial_k g(t,\cdot) > -\lambda_t$ on $\{t \leq \vartheta\}$. In the special case of a perfect market, g is given by (2.12), which implies that $\partial_k g(t,\cdot) = -\theta_t^2 \lambda_t$. In this case, Assumption 2.7 is thus equivalent to $\theta_t^2 < 1$, which corresponds to the usual assumption (2.6) made in the literature on default risk.

Remark 2.8. Suppose that $g(t,0,0,0) = 0$ $dP \otimes dt$-a.s. Then the price of an option with a null payoff is equal to 0, that is, for each $S \in [0,T]$, $\mathcal{E}^g_{\cdot,S}(0) = 0$ a.s. Moreover, by the comparison theorem for BSDEs with default jump (see [14, Theorem 2.17]), it follows that the nonlinear pricing system $\mathcal{E}^g$ is nonnegative, that is, for each $S \in [0,T]$, for all $\xi \in L^2(\mathcal{G}_S)$, if $\xi \geq 0$ a.s., then $\mathcal{E}^g_{\cdot,S}(\xi) \geq 0$ a.s.

Definition 2.9. *Let $Y \in S^2$. The process (Y_t) is said to be a strong $\mathcal{E}$-supermartingale (resp., martingale) if $\mathcal{E}_{\sigma,\tau}(Y_\tau) \leq Y_\sigma$ (resp., $= Y_\sigma$) a.s. on $\sigma \leq \tau$ for all $\sigma, \tau \in \mathcal{T}_0$.*

Proposition 2.10. *For each $S \in [0,T]$ and for each $\xi \in L^2(\mathcal{G}_S)$, the associated price (or g-evaluation) $\mathcal{E}^g_{t,S}(\xi)$ is an $\mathcal{E}^g$-martingale. Moreover, for each $x \in \mathbb{R}$ and each portfolio strategy $\varphi \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$, the associated wealth process $V^{x,\varphi}$ is an $\mathcal{E}^g$-martingale.*

Proof. By the flow property of BSDEs, the solution of a BSDE with driver g is an $\mathcal{E}^g$-martingale. The first assertion follows. The second one is obtained by noting that $V^{x,\varphi}$ is the solution of the BSDE with driver g, terminal time T, and terminal condition $V^{x,\varphi}_T$. ∎

Example 2.11 (examples of market imperfections).

- Different borrowing and lending interest rates R_t and r_t, with $R_t \geq r_t$: the driver g is then of the form

$$g\left(t, V_t, \varphi_t'\sigma_t, -\varphi_t^2\right) = -r_t V_t - \varphi_t^1\left(\mu_t^1 - r_t\right) - \varphi_t^2\left(\mu_t^2 - r_t\right) + (R_t - r_t)\left(V_t - \varphi_t^1 - \varphi_t^2\right)^-,$$

  where φ_t^2 vanishes after ϑ (see, e.g., [6]).
- Large investor seller: Suppose that the seller of the option is a large trader whose hedging strategy φ and its associated cost V may influence the market prices (see, e.g., [5, 1]). Taking into account the possible feedback effects in the market model, the large trader-seller may suppose that the coefficients are of the form $\sigma_t(\omega) = \bar{\sigma}(\omega, t, V_t, \varphi_t)$, where $\bar{\sigma} : \Omega \times [0,T] \times \mathbb{R}^3 ; (\omega, t, x, z, k) \mapsto \bar{\sigma}(\omega, t, x, z, k)$ is a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^3)$-measurable map, and similarly for the other coefficients r, μ^1, μ^2. The driver is thus of the form

$$\begin{aligned} &g\left(t, V_t, \varphi_t'\bar{\sigma}_t(t, V_t, \varphi_t), -\varphi_t^2\right) \\ &\qquad = -\bar{r}(t, V_t, \varphi_t)V_t - \varphi_t^1\left(\bar{\mu}_t^1 - \bar{r}_t\right)(t, V_t, \varphi_t) - \varphi_t^2\left(\bar{\mu}_t^2 - \bar{r}_t\right)(t, V_t, \varphi_t). \end{aligned}$$

  Here, the map $\Psi : (\omega, t, y, \varphi) \mapsto (z, k)$ with $z = \varphi'\bar{\sigma}_t(\omega, t, y, \varphi)$ and $k = -\varphi^2$ is assumed to be one to one with respect to φ, and such that its inverse Ψ_φ^{-1} is $\mathcal{P} \otimes \mathcal{B}(\mathbf{R}^3)$-measurable.
- Taxes on risky investments profits: Let $\rho \in]0,1[$ represent an instantaneous tax coefficient (see, e.g., [15]). The driver is then given by

$$g\left(t, V_t, \varphi_t'\sigma_t, \varphi_t^2\beta_t\right) = -r_t V_t - \varphi_t^1\left(\mu_t^1 - r_t\right) - \varphi_t^2\left(\mu_t^2 - r_t\right) + \rho\left(\varphi_t^1 + \varphi_t^2\right)^+.$$

3. Pricing and hedging of game options in the imperfect market $\mathcal{M}^g$. Let $T > 0$ be the terminal time. Let ξ and ζ be adapted RCLL processes in S^2 with $\zeta_T = \xi_T$ a.s. and $\xi_t \leq \zeta_t$, $0 \leq t \leq T$ a.s. We suppose that Mokobodzki's condition is satisfied, that is, there exist two nonnegative RCLL supermartingales H and H' in $\mathcal{S}^2$ such that

$$\xi_t \leq H_t - H'_t \leq \zeta_t, \quad 0 \leq t \leq T \quad \text{a.s.}$$

The game option consists for the seller to select a cancellation time $\sigma \in \mathcal{T}$ and for the buyer to choose an exercise time $\tau \in \mathcal{T}$, so that the seller pays to the buyer at time $\tau \wedge \sigma$ the amount

$$I(\tau, \sigma) := \xi_\tau \mathbf{1}_{\tau \leq \sigma} + \zeta_\sigma \mathbf{1}_{\sigma < \tau}.$$

We now introduce the *seller's price* of the game option, denoted by u_0, defined as the infimum of the initial wealths which enable the seller to choose a cancellation time σ and to construct a portfolio which will cover his liability to pay the payoff to the buyer up to σ no matter the exercise time chosen by the buyer.

Definition 3.1. *For each initial wealth x, a* superhedge *against the game option is a pair (σ, φ) of a stopping time $\sigma \in \mathcal{T}$ and a portfolio strategy $\varphi \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ such that*[1]

$$V_t^{x,\varphi} \geq \xi_t, \ 0 \leq t \leq \sigma \ \ \text{a.s. and } V_\sigma^{x,\varphi} \geq \zeta_\sigma \ \text{a.s.} \tag{3.1}$$

We denote by $\mathcal{S}(x) = \mathcal{S}_{\xi,\zeta}(x)$ the set of all superhedges associated with initial wealth x.

We define the seller's price *as*

$$u_0 := \inf\{x \in \mathbb{R},\ \exists(\sigma, \varphi) \in \mathcal{S}(x)\}. \tag{3.2}$$

When the infimum in (3.2) is attained, the amount u_0 allows the seller to be superhedged, and is called the superhedging price.

Remark 3.2. We have $(0, 0) \in \mathcal{S}(\zeta_0)$ since $V_0^{\zeta_0,0} = \zeta_0$ and $\zeta_0 \geq \xi_0$. By (3.2), we thus get $u_0 \leq \zeta_0$.

Moreover, when $g(t, 0, 0, 0) = 0$ $dP \otimes dt$-a.s. and $\zeta \geq 0$, then we can restrict ourselves to nonnegative initial wealths, that is $u_0 = \inf\{x \geq 0,\ \exists(\sigma, \varphi) \in \mathcal{S}(x)\}$. Indeed, let $x \in \mathbb{R}$ be such that there exists $(\sigma, \varphi) \in \mathcal{S}(x)$. Then, $V_\sigma^{x,\varphi} \geq \zeta_\sigma \geq 0$ a.s. Now, by Proposition 2.10 the wealth process $V^{x,\varphi}$ is an $\mathcal{E}^g$-martingale. We thus have $x = \mathcal{E}^g_{0,\sigma}(V_\sigma^{x,\varphi})$. Since the pricing system $\mathcal{E}^g$ is nonnegative (see Remark 2.8), it follows that $x = \mathcal{E}^g_{0,\sigma}(V_\sigma^{x,\varphi}) \geq 0$.

We now provide a dual formulation of the seller's price, expressed in terms of the nonlinear pricing system $\mathcal{E}^g$.We introduce the following definition.

Definition 3.3. *We define the g-*value *of the game option as*

$$\inf_{\sigma \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \mathcal{E}^g_{0,\tau\wedge\sigma}[I(\tau, \sigma)]. \tag{3.3}$$

Our aim is to show that the seller's price u_0 of the game option is equal to its g-value. To this purpose, we first give the following characterization of the g-value.

[1]Note that condition (3.1) is equivalent to $V_{t\wedge\sigma}^{x,\varphi} \geq I(t, \sigma)$, $0 \leq t \leq T$ a.s.

Proposition 3.4 (characterization of the *g-value* of the game option). *Suppose that the payoffs ξ and ζ are (only) RCLL. The g-*value *of the game option satisfies*

$$\inf_{\sigma\in\mathcal{T}}\sup_{\tau\in\mathcal{T}}\mathcal{E}^g_{0,\tau\wedge\sigma}[I(\tau,\sigma)] = \sup_{\tau\in\mathcal{T}}\inf_{\sigma\in\mathcal{T}}\mathcal{E}^g_{0,\tau\wedge\sigma}[I(\tau,\sigma)] = Y_0, \tag{3.4}$$

where (Y,Z,K,A,A') is the unique solution in $S^2\times\mathbb{H}^2\times\mathbb{H}^2_\lambda\times\mathcal{A}^2\times\mathcal{A}^2$ of the doubly reflected BSDE (DRBSDE) associated with driver g and barriers ξ,ζ, that is,

(3.5)

$$-dY_t = g(t,Y_t,Z_t,K_t)dt + dA_t - dA'_t - Z_t dW_t - K_t dM_t;\ Y_T = \xi_T$$

with

(i) $\xi_t \le Y_t \le \zeta_t,\ 0\le t\le T$ *a.s.,*

(ii) $dA_t \perp dA'_t$ *(i.e., the measures dA_t and dA'_t are mutually singular),*

(iii) $\int_0^T (Y_t-\xi_t)dA^c_t = 0$ *a.s. and* $\int_0^T (\zeta_t - Y_t)dA'^c_t = 0$ *a.s.,*

$\Delta A^d_\tau = \Delta A^d_\tau \mathbf{1}_{\{Y_{\tau^-}=\xi_{\tau^-}\}}$ *and* $\Delta A'^d_\tau = \Delta A'^d_\tau \mathbf{1}_{\{Y_{\tau^-}=\zeta_{\tau^-}\}}$ *a.s.* $\forall \tau\in\mathcal{T}$ *predictable.*

Using the terminology introduced in [13], the first equality in (3.4) means that the generalized Dynkin game associated with the criterium $\mathcal{E}^g_{0,\tau\wedge\sigma}[I(\tau,\sigma)]$ is *fair.*

When g is linear and when there is no default, this corresponds to a well-known result on classical Dynkin games and linear DRBSDEs (see, e.g., [7, 17]).

Proof. The proof of existence and uniqueness of a solution (Y,Z,K,A,A') of the DRBSDE (3.5) is given in the appendix. Proceeding as in the proof of [13, Theorem 4.9] which was given in the framework of a random Poisson measure, we can prove that for each $S\in\mathcal{T}$, $Y_S = \operatorname{ess\,inf}_{\sigma\in\mathcal{T}_S}\operatorname{ess\,sup}_{\tau\in\mathcal{T}_S}\mathcal{E}^g_{S,\tau\wedge\sigma}[I(\tau,\sigma)] = \operatorname{ess\,sup}_{\tau\in\mathcal{T}_S}\operatorname{ess\,inf}_{\sigma\in\mathcal{T}_S}\mathcal{E}^g_{S,\tau\wedge\sigma}[I(\tau,\sigma)]$ a.s. The results of the proposition then follow by taking $S=0$. ■

Proposition 3.5. *Let (Y,Z,K,A,A') be the unique solution of the DRBSDE (3.5). When ξ (resp., $-\zeta$) is left-u.s.c. along stopping times, then A (resp., A') is continuous.*

Proof. Note first that for each predictable stopping time τ, by (3.5), we have $(\Delta Y_\tau)^+ = \Delta A'_\tau$ a.s. and $(\Delta Y_\tau)^- = \Delta A_\tau$ a.s. Suppose that now $-\zeta$ is left-u.s.c. along the stopping time. Let τ be a predictable stopping time. Using the equality $\Delta A'_\tau = (\Delta Y_\tau)^+$ together with the Skorokhod conditions satisfied by A', we get

$$\Delta A'_\tau = \mathbf{1}_{\{Y_{\tau^-}=\zeta_{\tau^-}\}}(Y_\tau - Y_{\tau^-})^+ = \mathbf{1}_{\{Y_{\tau^-}=\zeta_{\tau^-}\}}(Y_\tau - \zeta_{\tau^-})^+. \tag{3.6}$$

Now, since $-\zeta$ is left-u.s.c. along stopping times, we have $Y_\tau - \zeta_{\tau^-} \le Y_\tau - \zeta_\tau \le 0$ a.s., where the last equality follows from the inequality $Y\le\zeta$. Using (3.6), we derive that $\Delta A'_\tau = 0$ a.s. It follows that A' is continuous. By similar arguments, one can show that if ξ is left-u.s.c. along stopping times, then A is continuous. ■

Using the above propositions, we can now show the dual formulation for the seller's price. We first consider the simpler case when ζ is left lower-semicontinuous (or, equivalently, $-\zeta$ is left-u.s.c.) along stopping times. In this case, we prove below that the seller's price is equal to the g-value and that the infimum in (3.2) is attained. This implies that the seller's price is the *superhedging price*. Moreover, a superhedge strategy is provided via the solution of the associated DRBSDE.

Theorem 3.6 (seller's/superhedging price and superhedge of the game option). *Suppose that ζ is left lower-semicontinuous along stopping times (and ξ is only RCLL). The seller's price (3.2) of the game option coincides with the g-value of the game option, that is,*

$$u_0 = \inf_{\sigma \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \mathcal{E}^g_{0,\tau\wedge\sigma}[I(\tau,\sigma)] = \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathcal{E}^g_{0,\tau\wedge\sigma}[I(\tau,\sigma)]. \tag{3.7}$$

Let (Y, Z, K, A, A') be the solution of the DRBSDE associated with driver g and barriers ξ, ζ. The seller's price is equal to Y_0, that is,

$$u_0 = Y_0.$$

Moreover, the infimum in (3.2) is attained. The seller's price is thus the superhedging price and there exists a superhedge strategy (σ^*, φ^*) associated with the initial amount u_0, given by

$$\sigma^* := \inf\{t \geq 0,\ Y_t = \zeta_t\} \quad \text{and} \quad \varphi^* := \Phi(Z, K), \tag{3.8}$$

where Φ is defined in Definition 2.3.

Remark 3.7. In the special case of a perfect market model, our result gives that u_0 is characterized as the value function of a classical Dynkin game problem, which is shown in the literature (see, e.g., [22, 17]) under an additional regularity assumption on ξ, by using an actualization procedure, a change of probability measure, and some results on classical Dynkin games. Moreover, in this particular case, the characterization of u_0 and of the superhedge via the solution of a *linear* doubly reflected BSDE are shown in [17] by using the links between linear DRBSDEs and classical Dynkin games (first provided in [7]). To solve the problem in the case of an imperfect market model, when g is nonlinear, we need to use other arguments, in particular, some properties of the nonlinear g-evaluation $\mathcal{E}^g$, comparison theorems for BSDEs and for forward differential equations, and the links between *nonlinear* DRBSDEs and generalized Dynkin games (first provided in [13]).

Proof. By Proposition 3.4, the g-*value* of the game option is equal Y_0. Note that $u_0 = \inf \mathcal{H}$, where $\mathcal{H}$ is the set of initial capitals which allow the seller to be superhedged, that is,

$$\mathcal{H} = \{x \in \mathbb{R} : \exists (\sigma, \varphi) \in \mathcal{S}(x)\}.$$

Let us show that $Y_0 \geq u_0$. It is sufficient to prove that there exists $(\sigma^*, \varphi^*) \in \mathcal{S}(Y_0)$. By Proposition 3.5, since $-\zeta$ is left-u.s.c. along stopping times, the process A' is continuous. Let σ^* be defined as in (3.8). We have a.s. that $Y_t < \zeta_t$ for each $t \in [0, \sigma^*[$. Since Y is a solution of the DRBSDE (3.5), the process A' is thus constant on $[0, \sigma^*[$ a.s. and even on $[0, \sigma^*]$ by continuity. Hence, $A'_{\sigma^*} = A'_0 = 0$ a.s. For almost every ω, we thus have

$$Y_t(\omega) = Y_0 - \int_0^t g(s, \omega, Y_s(\omega), Z_s(\omega), K_s(\omega))ds + f_t(\omega) - A_t(\omega),\ 0 \leq t \leq \sigma^*(\omega), \tag{3.9}$$

where $f_t := \int_0^t Z_s dW_s + \int_0^t K_s dM_s$. Now, the wealth $V^{Y_0,\varphi^*}_{\cdot}$, associated with the initial capital Y_0 and the financial strategy $\varphi^* := \Phi(Z, K)$, satisfies for almost every ω the forward deterministic differential equation:

$$V_t^{Y_0,\varphi^*}(\omega) = Y_0 - \int_0^t g\left(s, V_s^{Y_0,\varphi^*}(\omega), Z_s(\omega), K_s(\omega)\right) ds + f_t(\omega),\ 0 \le t \le T. \tag{3.10}$$

Since A is nondecreasing, by applying the classical comparison result on $[0, \sigma^*(\omega)]$ (see, e.g., Lemma 6.2) for the two forward differential equations (3.9) and (3.10), with the same coefficient $(s,x) \mapsto -g(s,\omega,x,Z_s(\omega),K_s(\omega))$, we get

$$V_t^{Y_0,\varphi^*} \ge Y_t \ge \xi_t,\ 0 \le t \le \sigma^* \quad \text{a.s.},$$

where the last inequality follows from the inequality $Y \ge \xi$. We also have

$$V_{\sigma^*}^{Y_0,\varphi^*} \ge Y_{\sigma^*} = \zeta_{\sigma^*} \quad \text{a.s.},$$

where the last equality follows from the definition of the stopping time σ^* and the right-continuity of Y and ζ. Hence,

$$(\sigma^*, \varphi^*) \in \mathcal{S}(Y_0), \tag{3.11}$$

which implies that $Y_0 \in \mathcal{H}$. We thus get the inequality $Y_0 \ge u_0$.

It remains to show that $u_0 \ge Y_0$. Since $Y_0 = \inf_{\sigma\in\mathcal{T}} \sup_{\tau\in\mathcal{T}} \mathcal{E}^g_{0,T}[I(\tau,\sigma)]$ (by Proposition 3.4), it is sufficient to show that

$$u_0 \ge \inf_{\sigma\in\mathcal{T}} \sup_{\tau\in\mathcal{T}} \mathcal{E}^g_{0,T}[I(\tau,\sigma)]. \tag{3.12}$$

Let $x \in \mathcal{H}$. There exists $(\sigma,\varphi) \in \mathcal{S}(x)$, that is, a pair (σ,φ) of a stopping time $\sigma \in \mathcal{T}$ and a portfolio strategy $\varphi \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ such that $V_t^{x,\varphi} \ge \xi_t$, $0 \le t \le \sigma$ a.s., and $V_\sigma^{x,\varphi} \ge \zeta_\sigma$ a.s., which implies that for all $\tau \in \mathcal{T}$ we have

$$V_{\tau\wedge\sigma}^{x,\varphi} \ge I(\tau,\sigma) \quad \text{a.s.}$$

By taking the $\mathcal{E}^g$-evaluation in the above inequality and then the supremum on $\tau \in \mathcal{T}$, using the monotonicity of the $\mathcal{E}^g$-evaluation and the $\mathcal{E}^g$-martingale property of the wealth process $V^{x,\varphi}$ (see Proposition 2.10), we obtain $x = \mathcal{E}^g_{0,\tau\wedge\sigma}[V^{x,\varphi}_{\tau\wedge\sigma}] \ge \mathcal{E}^g_{0,\tau\wedge\sigma}[I(\tau,\sigma)]$, for each $\tau \in \mathcal{T}$. By taking the supremum over $\tau \in \mathcal{T}$, and then the infimum over $\sigma \in \mathcal{T}$, we get

$$x \ge \inf_{\sigma\in\mathcal{T}} \sup_{\tau\in\mathcal{T}} \mathcal{E}^g_{0,\tau\wedge\sigma}[I(\tau,\sigma)].$$

This inequality holds for any $x \in \mathcal{H}$. By taking the infimum over $x \in \mathcal{H}$, we obtain the inequality (3.12), which yields that $u_0 \ge Y_0$. Since $Y_0 \ge u_0$, we get $Y_0 = u_0$. Moreover, this equality together with (3.11) implies that $(\sigma^*, \varphi^*) \in \mathcal{S}(u_0)$. The proof is thus complete. ■

Remark 3.8. Let $\hat\sigma$ be a stopping time such that $A'_{\hat\sigma} = 0$ a.s. and $Y_{\hat\sigma} = \zeta_{\hat\sigma}$ a.s. By the above proof, the pair $(\hat\sigma, \varphi^*)$ is a superhedge for the initial amount u_0, that is, $(\hat\sigma,\varphi^*) \in \mathcal{S}(u_0)$. For example, under the assumption of Theorem 3.6 (that is, the left-u.s.c. property along stopping times of $-\zeta$), the stopping time $\bar{\sigma} := \inf\{t \geq 0 : \ A'_t > 0\}$ satisfies these two equalities. Note that $\bar{\sigma} \geq \sigma^*$. In general, the equality does not hold.

Remark 3.9. Note that under the assumption of Theorem 3.6, there does not necessarily exist a saddle point for the generalized Dynkin game (3.4). However, if we suppose additionally that ξ is left-u.s.c. along the stopping time, there exists a saddle point. More precisely, in this case, by [13, Theorem 4.7], the pair (τ^*, σ^*), with σ^* defined in (3.8) and $\tau^* := \inf\{t \geq 0 : Y_t = \xi_t\}$, is a saddle point for the generalized Dynkin game (3.4), that is, for all $(\tau, \sigma) \in \mathcal{T}^2$ we have

$$\mathcal{E}^g_{0,\tau\wedge\sigma^*}[I(\tau, \sigma^*)] \leq Y_0 = \mathcal{E}^g_{0,\tau^*\wedge\sigma^*}[I(\tau^*, \sigma^*)] \leq \mathcal{E}^g_{0,\tau^*\wedge\sigma}[I(\tau^*, \sigma)],$$

which implies that τ^* is optimal for the optimal stopping problem $\sup_{\tau\in\mathcal{T}} \mathcal{E}^g[I(\tau, \sigma^*)]$.

The same properties also hold for the pair $(\bar{\tau}, \bar{\sigma})$, where $\bar{\tau} := \inf\{t \geq 0 : \ A_t > 0\}$.

We consider now the general case when ζ is only RCLL (as ξ). In this case, the seller's price u_0 is still equal to the g-value but it does not necessarily allow the seller to build a superhedge against the option. We introduce the definition of ε-*superhedges*:

Definition 3.10. *For each initial wealth x and for each $\varepsilon > 0$, an ε-superhedge against the game option is a pair (σ, φ) of a stopping time $\sigma \in \mathcal{T}$ and a risky-assets strategy $\varphi \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ such that*

$$V^{x,\varphi}_t \geq \xi_t, \ 0 \leq t \leq \sigma \ \text{ a.s.,} \quad \text{and} \quad V^{x,\varphi}_\sigma \geq \zeta_\sigma\text{-}\varepsilon \ \text{ a.s.}$$

In other terms, by investing the initial capital amount x in the market following the risky-assets strategy φ, the seller is completely hedged before σ, and at the cancellation time σ, he is hedged up to an amount ε.

We prove below that when ζ and ξ are only RCLL, the seller's price u_0 is equal to the g-value and that there exits an ε-superhedge for the game option.

Theorem 3.11 (*seller's price* and ε-*superhedge* of the game option). *Suppose that the processes ζ and ξ are only RCLL. The seller's price (3.2) of the game option coincides with the g-value of the game option, that is,*

$$u_0 = \inf_{\sigma\in\mathcal{T}} \sup_{\tau\in\mathcal{T}} \mathcal{E}^g_{0,\tau\wedge\sigma}[I(\tau, \sigma)] = \sup_{\tau\in\mathcal{T}} \inf_{\sigma\in\mathcal{T}} \mathcal{E}^g_{0,\tau\wedge\sigma}[I(\tau, \sigma)].$$

Let (Y, Z, K, A, A') be the solution of the DRBSDE associated with driver g and barriers ξ, ζ. The seller's price is equal to Y_0, that is,

$$u_0 = Y_0. \tag{3.13}$$

The infimum in (3.2) is not nessarily attained. Let $\varphi^* := \Phi(Z, K)$ and for each $\varepsilon > 0$, let

$$\sigma_\varepsilon := \inf\{t \geq 0 : \ Y_t \geq \zeta_t - \varepsilon\}. \tag{3.14}$$

The pair $(\sigma_\varepsilon, \varphi^*)$ is an ε-superhedge for the initial capital u_0.

Proof. By Proposition 3.4, the g-value is equal to Y_0. Let $\varepsilon > 0$. We have $Y_. \leq \zeta_. - \varepsilon$ on $[0, \sigma_\varepsilon[$. Since A' satisfies the Skorohod condition (iii), it follows that a.s., A' is constant on $[0, \sigma_\varepsilon[$. Also, $Y_{(\sigma^\varepsilon)^-} \leq \zeta_{(\tau^\varepsilon)^-} - \varepsilon$ a.s., which implies that $\Delta A'_{\sigma_\varepsilon} = 0$ a.s. Hence, $A'_{\sigma^\varepsilon} = 0$ a.s. It follows that for almost every ω, the deterministic function $Y_.(\omega)$ is the solution of the *forward* deterministic differential equation (3.9) on $[0, \sigma_\varepsilon(\omega)]$. Now, for almost every ω, the wealth $V^{Y_0,\varphi^*}_\cdot(\omega)$ is the solution of the deterministic differential equation (3.10). By applying the classical comparison result on differential equations (Lemma 6.2), we derive that $V_t^{Y_0,\varphi^*} \geq Y_t \geq \xi_t$, $0 \leq t \leq \sigma_\varepsilon$ a.s. Moreover, we have $V_{\sigma_\varepsilon}^{Y_0,\varphi^*} \geq Y_{\sigma_\varepsilon} \geq \zeta_{\sigma_\varepsilon} - \varepsilon$, where the last inequality follows from the definition of the stopping time σ_ε and the right continuity of Y and ζ. Hence, $(\sigma_\varepsilon, \varphi^*)$ is an ε-superhedge for the initial capital amount Y_0.

It remains to show that $Y_0 = u_0$. The proof of the inequality $u_0 \geq Y_0$, which uses Proposition 3.4, has been done in the second part of the proof of Theorem 3.6 and does not require the continuity of A'. Let us show the converse inequality. Let $\varepsilon > 0$. Let (Y', Z', K') be the solution of the BSDE associated with terminal time σ_ε and terminal condition $\zeta_{\sigma_\varepsilon} \vee V_{\sigma_\varepsilon}^{Y_0,\varphi^*}$. Now $(V^{Y_0,\varphi^*}, Z, K)$ is the solution of the BSDE associated with terminal time σ_ε and terminal condition $V_{\sigma_\varepsilon}^{Y_0,\varphi^*}$. By an a priori estimate on BSDEs with default jump (see [14, Proposition 2.4]), since $V_{\sigma_\varepsilon}^{Y_0,\varphi^*} \geq \zeta_{\sigma_\varepsilon} \vee V_{\sigma_\varepsilon}^{Y_0,\varphi^*} - \varepsilon$ a.s., we derive that $V_0^{Y_0,\varphi^*} = Y_0 \geq Y'_0 - K\varepsilon$ a.s., where K is a constant which only depends on T and the λ-constant C. By the comparison theorem for BSDEs, $Y'_t \geq V_t^{Y_0,\varphi^*} \geq \xi_t$. We derive that the amount Y'_0 $(\leq Y_0 + K\varepsilon)$ allows the seller to be superhedged, and the associated superhedge is given by σ_ε and $\varphi' := \Phi(Z', K')$. By the definition of u_0, we derive that $u_0 \leq Y'_0 \leq Y_0 + K\varepsilon$ for each $\varepsilon > 0$. Hence, $u_0 \leq Y_0$. Since $u_0 \geq Y_0$, we get $u_0 = Y_0$. ■

4. Pricing and hedging of game options with model uncertainty. We now study game options with uncertainty on the model, which includes, in particular, the case of uncertainty on the default probability (see Example 4.3 below).

4.1. Market model with ambiguity. In this section, we need to use a measurable selection theorem, which requires us to work on an appropriate probability space. We consider a Cox process model, which is a typical example of a default model. We work on the canonical space constructed as follows: let Ω_W be the Wiener space defined by $\Omega_W := \mathcal{C}(\mathbb{R}^+)$, that is, the set of continuous functions ω from $\mathbb{R}^+$ into $\mathbb{R}$ such that $\omega(0) = 0$. Recall that Ω_W is a Polish space for the norm $\|\cdot\|_\infty$. The space Ω_W is equipped with the σ-algebra $\mathcal{F}_W$ generated by the coordinate process $(W_t)_{t\geq 0}$ (which is equal to its Borelian σ-algebra). Let P_W be the probability under which $(W_t)_{t\geq 0}$ is a standard Brownian motion. Let $\Omega_\Theta := \mathbb{R}$, equipped with its Borelian σ-algebra $\mathcal{F}_\Theta = \mathcal{B}(\mathbb{R})$, and the probability P_Θ such that the identity map Θ admits an exponential law with parameter 1. We consider the product space $\Omega := \Omega_W \times \Omega_\Theta$, which is a Polish space. It is equipped with the σ-algebra $\mathcal{F}_W \otimes \mathcal{F}_\Theta$ and the probability $P := P_W \otimes P_\Theta$. Let $\mathcal{G}$ be the σ-algebra $\mathcal{F}_W \otimes \mathcal{F}_\Theta$ completed with respect to P. Let $\mathbb{F} = (\mathcal{F}_t, t \geq 0)$ be the filtration $\mathcal{F}_W$ *completed* with respect to $\mathcal{G}$ and P (in the sense of [19, p. 3] or [8, IV]). Let $(\bar{\lambda}_t)_{t\geq 0}$ be a bounded positive $\mathbb{F}$-predictable process. We introduce the following random variable, which represents the *default time*:

$$\vartheta := \inf\left\{t \geq 0,\ \int_0^t \bar{\lambda}_s ds \geq \Theta\right\}.$$

We have $P(\vartheta > t \,|\, \mathcal{F}_\infty) = P(\vartheta > t \,|\, \mathcal{F}_t) = \exp(-\int_0^t \bar{\lambda}_s ds)$, which corresponds to the so-called condition (H) (see, e.g., [18]). We now define the *default process*

$$N_t := \mathbf{1}_{\{\vartheta \leq t\}}, \quad t \geq 0.$$

We denote by $\mathbb{G} = (\mathcal{G}_t, t \geq 0)$ the filtration generated by W and N *augmented* with respect to $\mathcal{G}$ and P (in the sense of [8, IV-48]). By classical results, since condition (H) holds, we derive that W is a $\mathbb{G}$-Brownian motion. Moreover, the process M defined by

$$M_t := N_t - \int_0^{t\wedge\vartheta} \bar{\lambda}_s ds, \quad t \geq 0 \quad \text{a.s.},$$

is a $\mathbb{G}$-martingale. For each $t \geq 0$, let $\lambda_t := \bar{\lambda}_t \mathbf{1}_{\{t \leq \vartheta\}}$. The process λ, usually called the $\mathbb{G}$-*intensity* of ϑ, thus vanishes after ϑ. Let T be a given terminal time. The sets $\mathcal{P}$, S^2, $\mathbb{H}^2$, $\mathbb{H}^2_\lambda$, and $\mathcal{A}^2$ are defined as before.

Let U be a nonempty closed subset of $\mathbb{R}$. Let $g : [0,T]\times\Omega\times\mathbb{R}^3\times U \to \mathbb{R}$; $(t,\omega,y,z,k,\alpha) \mapsto g(t,\omega,y,z,k,\alpha)$, be a given $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^3)\otimes\mathcal{B}(U)$-measurable function. Suppose $g(\cdot,\alpha)$ is *uniformly λ-admissible* with respect to (y,z,k), that is, it satisfies the inequalities (2.8) with a constant C which does not depend on α. We also assume that $g(\cdot,\alpha)$ is continuous with respect to α, and such that $\sup_{\alpha\in U} |g(t,.,0,0,0,\alpha)| \in \mathbb{H}_2$. Suppose also that

$$g(t,y,z,k_1,\alpha) - g(t,y,z,k_2,\alpha) \geq \theta_t^{y,z,k_1,k_2}(k_1-k_2)\lambda_t, \tag{4.1}$$

where θ_t^{y,z,k_1,k_2} satisfies the conditions of Assumption 2.7, in particular, the inequality $\theta_t^{y,z,k_1,k_2} > -1$. We also suppose that $g(t,0,0,0,\alpha) \geq 0$ for all $\alpha \in U$.

Let $\mathcal{U}$ be the set of U-valued predictable processes. For each $\alpha \in \mathcal{U}$, to simplify notation, we introduce the map g^α defined by

$$g^\alpha(t,\omega,y,z,k) := g(t,\omega,y,z,k,\alpha_t(\omega)). \tag{4.2}$$

Note that these maps g^α, $\alpha \in \mathcal{U}$, are all λ-admissible drivers with the same λ-constant C. The control α represents the ambiguity parameter of the model. To each ambiguity parameter α, corresponds a market model $\mathcal{M}_\alpha$, where the wealth process $V^{\alpha,x,\varphi}$ associated with an initial wealth x and a risky assets stategy $\varphi \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ satisfies

$$-dV_t^{\alpha,x,\varphi} = g\left(t, V_t^{\alpha,x,\varphi}, \varphi_t\sigma_t, -\varphi_t^2, \alpha_t\right) dt - \varphi_t\sigma_t dW_t + \varphi_t^2 dM_t, \;\; V_0^{\alpha,x,\varphi} = x. \tag{4.3}$$

In the market model $\mathcal{M}_\alpha$, the nonlinear pricing system is given by

$$\mathcal{E}^{g^\alpha} := \{\mathcal{E}^{g^\alpha}_{t,S},\; S \in [0,T], t \in [0,S]\},$$

also called g^α-evaluation.

4.2. Robust superhedging of game options. In our framework with ambiguity, the *seller's robust price* of the game option denoted by $\mathbf{u_0}$ is defined as the infimum of the initial wealths which enable the seller to be superhedged for any ambiguity parameter $\alpha \in \mathcal{U}$.

Definition 4.1. *For an initial wealth $x \in \mathbb{R}$, a robust superhedge against the game option is a pair (σ,φ) of a stopping time $\sigma \in \mathcal{T}$ and a portfolio strategy $\varphi \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ such that*[2]

$$V_t^{\alpha,x,\varphi} \geq \xi_t,\; 0 \leq t \leq \sigma \;\; a.s. \text{ and } V_\sigma^{\alpha,x,\varphi} \geq \zeta_\sigma \;\; a.s. \quad \forall \alpha \in \mathcal{U}. \tag{4.4}$$

We denote by $\mathcal{S}^r(x)$ the set of all robust superhedges associated with initial wealth x.

[2]Condition (4.4) is equivalent to $V_{t\wedge\sigma}^{\alpha,x,\varphi} \geq I(t,\sigma)$, $0 \leq t \leq T$ a.s. for all $\alpha \in \mathcal{U}$.

The seller's robust price *is defined as*[3]

$$\mathbf{u_0} := \inf\{x \in \mathbb{R},\ \exists(\sigma, \varphi) \in \mathcal{S}^r(x)\}. \tag{4.5}$$

When the infimum is reached, $\mathbf{u_0}$ *is called the robust superhedging price.*

Let $\alpha \in \mathcal{U}$. By Theorem 3.11, the seller's price of the game option in the market $\mathcal{M}_\alpha$ is characterized as its g^α-*value*. Moreover, it is equal to Y_0^α, where $(Y^\alpha, Z^\alpha, K^\alpha, A^\alpha, A'^\alpha)$ is the unique solution in $S^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\lambda \times \mathcal{A}^2 \times \mathcal{A}^2$ of the DRBSDE associated with driver g^α and barriers ξ and ζ. We now introduce an associated dual problem.

Definition 4.2. *The* dual problem *associated with the seller's superhedging problem is*

$$\mathbf{v_0} := \sup_{\alpha \in \mathcal{U}} Y_0^\alpha. \tag{4.6}$$

By Theorem 3.11, the seller's price Y_0^α of the game option in the market $\mathcal{M}_\alpha$ is equal to the common value function of the generalized Dynkin game associated with driver g^α, that is,

$$Y_0^\alpha = \inf_{\sigma \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \mathcal{E}^{g^\alpha}_{0,\tau\wedge\sigma}[I(\tau,\sigma)] = \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathcal{E}^{g^\alpha}_{0,\tau\wedge\sigma}[I(\tau,\sigma)].$$

Hence, the value function $\mathbf{v_0}$ of the dual problem is equal to the value function of a *mixed generalized Dynkin game*, that is,

$$\mathbf{v_0} = \sup_{\alpha \in \mathcal{U}} Y_0^\alpha = \sup_{\alpha \in \mathcal{U}} \inf_{\sigma \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \mathcal{E}^{g^\alpha}_{0,\tau\wedge\sigma}[I(\tau,\sigma)] = \sup_{\alpha \in \mathcal{U}} \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathcal{E}^{g^\alpha}_{0,\tau\wedge\sigma}[I(\tau,\sigma)]. \tag{4.7}$$

Remark 4.3. We shall see below (see Proposition 4.8) that $\mathbf{v_0}$ is also equal to

$$\inf_{\sigma \in \mathcal{T}} \sup_{\alpha \in \mathcal{U}} \sup_{\tau \in \mathcal{T}} \mathcal{E}^{g^\alpha}_{0,\tau\wedge\sigma}[I(\tau,\sigma)].$$

In order to show that $\mathbf{u_0} = \mathbf{v_0}$, we will first prove that $\mathbf{v_0}$ can be characterized as the solution of a DRBSDE.

Now, by definition, we have $\mathbf{v_0} = \sup_\alpha Y_0^\alpha$, where Y^α is the solution of the DRBSDE associated with barriers ξ and ζ, and with driver $g(\cdot, \alpha_t)$. We will show that $\mathbf{v_0}$ coincides with the solution of the DRBSDE associated with the same barriers ξ and ζ, and with the driver $\sup_\alpha g(\cdot, \alpha)$.

More precisely, let $\mathbf{G}$ be the map defined for each (t, ω, z, k) by

$$\mathbf{G}(t, \omega, y, z, k) := \sup_{\alpha \in U} g(t, \omega, y, z, k, \alpha). \tag{4.8}$$

Lemma 4.4. *The map* $\mathbf{G}$ *is a* λ-admissible *driver and satisfies Assumption* 2.7.

Proof. Since U is a closed subset of a Polish space, there exists a numerable subset D of U, dense in U. Since g is continuous with respect to u, the supremum in (4.8) can be taken

[3] Remark 3.2 also holds for the seller's robust price, that is, $\mathbf{u_0} \le \zeta_0$. Moreover, when $g(t, 0, 0, 0) = 0$ and $\zeta \ge 0$, then $\mathbf{u_0} = \inf\{x \ge 0,\ \exists(\sigma, \varphi) \in \mathcal{S}^r(x)\}$.

in D. It follows that $\mathbf{G}$ is $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^3)$-measurable. Let us show that $\mathbf{G}$ satisfies Assumption 2.7. By the definition of $\mathbf{G}(t,y,z,k_1)$ and by (4.1), we have, for all $\alpha\in\mathcal{U}$,

$$\mathbf{G}(t,y,z,k_1)-g(t,y,z,k_2,\alpha)\geq g(t,y,z,k_1,\alpha)-g(t,y,z,k_2,u)\geq\theta_t^{y,z,k_1,k_2}(k_1-k_2)\lambda_t.$$

Taking the infimum on $\alpha\in\mathcal{U}$ in this inequality, and using the definition of $\mathbf{G}(t,y,z,k_2)$, we derive that $\mathbf{G}(t,y,z,k_1)-\mathbf{G}(t,y,z,k_2)\geq\theta_t^{y,z,k_1,k_2}(k_1-k_2)\lambda_t$, which gives the desired result. The proof of conditions (2.8) relies on similar arguments and is left to the reader. Hence, $\mathbf{G}$ is a λ-admissible driver. ∎

We now prove that the dual function $\mathbf{v_0}$ is characterized as the solution of the DRBSDE associated with driver $\mathbf{G}$ and barriers ξ and ζ.

Theorem 4.5 (characterization of the dual value function $\mathbf{v_0}$). *Let $\mathbf{v_0}$ be defined by (4.6). We have $\mathbf{v_0}=Y_0$, where (Y,Z,K,A,A') is the solution of the DRBSDE associated with driver $\mathbf{G}$ and barriers ξ and ζ. If U is compact, there exists $\bar{\alpha}\in\mathcal{U}$ such that $\mathbf{v_0}=Y_0^{\bar{\alpha}}$, which means that the dual value function $\mathbf{v_0}$ is equal to the $g^{\bar{\alpha}}$*-value *of the game option in the market model $\mathcal{M}_{\bar{\alpha}}$.*

Proof. By the definition of $\mathbf{G}$ (see (4.8)), for each $(t,\omega,y,z,k)\in[0,T]\times\Omega\times\mathbb{R}^3\times U$, we have

$$\mathbf{G}(t,\omega,y,z,k)\geq g(t,\omega,y,z,k,\alpha_t(\omega)).$$

By the comparison theorem for DRBSDEs (see Theorem 5.1 in [13]), we thus have $Y\geq Y^\alpha$ a.s. for each $\alpha\in\mathcal{U}$. It follows that $Y_0\geq\sup_\alpha Y_0^\alpha$.

Let $\varepsilon>0$. By the definition of $\mathbf{G}$ as a supremum, for each $(t,\omega,y,z,l)\in\Omega\times[0,T]\times\mathbb{R}^2\times\mathbb{R}$, there exists $\alpha^\varepsilon\in U$ such that $\mathbf{G}(t,\omega,y,z,k)-\varepsilon\leq g(t,\omega,y,z,k,\alpha^\varepsilon)$. Now, the set

$$\{(t,\omega,\alpha)\in[0,T]\times\Omega\times U:\ \mathbf{G}(t,\omega,Y_{t^-}(\omega),Z_t(\omega),K_t(\omega))-\varepsilon\leq g(t,\omega,Y_{t^-}(\omega),Z_t(\omega),K_t(\omega),\alpha)\}$$

belongs to $\mathcal{P}\otimes\mathcal{B}(U)$. Hence, since the canonical space Ω is a Polish space, by applying a measurable selection theorem (see, e.g., [8, section 81, Appendix of Chap. III]) and [4, Lemma 1.2] (or [12, Lemma 26]), there exists a U-valued predictable process (α_t^ε) such that

$$\mathbf{G}(t,Y_t,Z_t,K_t)-\varepsilon\leq g(t,\omega,Y_t,Z_t,K_t,\alpha_t^\varepsilon),\quad 0\leq t\leq T,\quad dt\otimes dP\text{-a.s.}$$

By using the estimate (6.1) on DRBSDEs with default jump, with $\eta=\frac{1}{C^2}$ and $\beta=3C^2+2C$, we derive that there exists a constant $K\geq 0$, which depends only on C and T, such that, for each $\varepsilon>0$,

$$Y_0-K\,\varepsilon\leq Y_0^{\alpha^\varepsilon}.$$

Since $Y_0\geq\sup_\alpha Y_0^\alpha$, we thus get $Y_0=\sup_\alpha Y_0^\alpha=\mathbf{v_0}$.

Let us show the second assertion. If U is compact, for each $(t,\omega,y,z,l)\in[0,T]\times\Omega\times\mathbb{R}^2\times L_\lambda^2$, there exists $\bar{\alpha}\in U$ such that the supremum in (4.8) is attained at $\bar{\alpha}$. By the measurable selection theorem of [8] and [4, Lemma 1.2], there exists a U-valued predictable process $(\bar{\alpha}_t)$ such that

$$\mathbf{G}(t,Y_t,Z_t,K_t)=g(t,Y_t,Z_t,K_t,\bar{\alpha}_t),\quad 0\leq t\leq T,\quad dt\otimes dP\text{-a.s.}$$

It follows that Y and $Y^{\bar{\alpha}}$ are both solutions of the DRBSDE associated with driver $g^{\bar{\alpha}}$. Hence, by the uniqueness of the solution of a DRBSDE, $Y = Y^{\bar{\alpha}}$. ■

Using this result, we now provide the following theorem.

Theorem 4.6 (seller's robust price and superhedge). *Suppose that ζ is left-lower semicontinuous along stopping times (and ξ is only RCLL). The* seller's robust price *of the game option defined by (4.5) is equal to the dual value function $\mathbf{v_0}$ defined by (4.6), that is,*

$$\mathbf{u_0} = \mathbf{v_0}.$$

Let (Y, Z, K, A, A') be the solution of the DRBSDE associated with driver $\mathbf{G}$ defined by (4.8) and barriers ξ and ζ. The seller's robust price is equal to Y_0, that is,

$$\mathbf{u_0} = Y_0.$$

Moreover, the infimum in (4.6) is attained. The robust seller's price is thus the robust superhedging price of the game option. Let $\sigma^* := \inf\{t \geq 0,\ Y_t = \zeta_t\}$ and $\varphi^* := \Phi(Z, K)$. The pair (σ^*, φ^*) is a robust superhedge for the initial capital $\mathbf{u_0}$.

If U is compact, there exists $\bar{\alpha} \in \mathcal{U}$ such that the robust superhedging price of the game option is equal to the superhedging price in the market model $\mathcal{M}_{\bar{\alpha}}$, that is, $\mathbf{u_0} = Y_0^{\bar{\alpha}}$. The ambiguity parameter $\bar{\alpha}$ corresponds to a worst case scenario *among all the possible ambiguity parameters $\alpha \in \mathcal{U}$.*

Proof. By Theorem 4.5, $\mathbf{v_0} = Y_0$. Let $\mathcal{H}^r$ be the set of initial capitals which allow the seller to be superhedged, that is $\mathcal{H}^r = \{x \in \mathbb{R} : \exists(\sigma, \varphi) \in \mathcal{S}^r(x)\}$. Note that $\mathbf{u_0} = \inf \mathcal{H}^r$.

Let us show that $Y_0 \geq \mathbf{u_0}$. It is sufficient to show that there exists $(\sigma^*, \varphi^*) \in \mathcal{S}^r(Y_0)$. By Proposition 3.5, since $-\zeta$ is left-u.s.c. along stopping times, the process A' is continuous. By definition of σ^*, the process A' is constant on $[0, \sigma^*[$ a.s. and even on $[0, \sigma^*]$ by continuity. Hence, $A'_{\sigma^*} = A'_0 = 0$ a.s. We thus have

$$Y_t = Y_0 - \int_0^t \mathbf{G}(s, Y_s, Z_s, K_s)ds + \int_0^t Z_s dW_s + \int_0^t K_s dM_s - A_t,\ 0 \leq t \leq \sigma^* \quad \text{a.s.}$$

Let $\alpha \in \mathcal{U}$. In the market model $\mathcal{M}_\alpha$, the wealth process $V^{\alpha, Y_0, \varphi^*}_{\cdot}$ associated with the initial capital Y_0 and the financial strategy $\varphi^* := \Phi(Z, K)$ satisfies

$$V_t^{\alpha, Y_0, \varphi^*} = Y_0 - \int_0^t g(s, V_s^{\alpha, Y_0, \varphi^*}, Z_s, K_s, \alpha_s)ds + \int_0^t Z_s dW_s + \int_0^t K_s dM_s.$$

By the definition of $\mathbf{G}$ (see (4.8)), we have $-g(t, \omega, y, z, k, \alpha_t(\omega)) \geq -\mathbf{G}(t, \omega, y, z, k)$. Hence, since A is a nondecreasing process, by the comparison property for deterministic differential equations (see Lemma 6.2) applied to the two above forward equations, we derive that

$$V_t^{\alpha, Y_0, \varphi^*} \geq Y_t \geq \xi_t,\ 0 \leq t \leq \sigma^* \quad \text{a.s.},$$

where the last inequality follows from the inequality $Y \geq \xi$.

Moreover, we have $V_{\sigma^*}^{\alpha, Y_0, \varphi^*} \geq Y_{\sigma^*} = \zeta_{\sigma^*}$ a.s., and this holds for any $\alpha \in \mathcal{U}$. Hence, $(\sigma^*, \varphi^*) \in \mathcal{S}^r(Y_0)$, which implies $Y_0 \in \mathcal{H}^r$. Thus, $Y_0 \geq \mathbf{u_0}$.

Let us now show that $\mathbf{u_0} \geq Y_0$. Let $x \in \mathcal{H}^r$. There exists $(\sigma, \varphi) \in \mathcal{S}^r(x)$, that is, a pair (σ, φ) of a stopping time $\sigma \in \mathcal{T}$ and a portfolio strategy $\varphi \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ such that for each $\alpha \in \mathcal{U}$, we have $V^{\alpha,x,\varphi}_{t\wedge\sigma} \geq I(t,\sigma)$, $0 \leq t \leq T$ a.s. By the same arguments as in the proof of Theorem 3.6, we derive that for each $\alpha \in \mathcal{U}$,

$$x \geq \inf_{\sigma\in\mathcal{T}} \sup_{\tau\in\mathcal{T}} \mathcal{E}^{g^\alpha}_{0,\tau\wedge\sigma}[I(\tau,\sigma)].$$

By taking the supremum over $\alpha \in \mathcal{U}$ in this inequality, we obtain

$$x \geq \sup_{\alpha\in\mathcal{U}} \inf_{\sigma\in\mathcal{T}} \sup_{\tau\in\mathcal{T}} \mathcal{E}^{g^\alpha}_{0,\tau\wedge\sigma}[I(\tau,\sigma)] = \mathbf{v_0},$$

where the last equality follows from the fact that $\mathbf{v_0}$ is equal to the value function of the mixed generalized Dynkin game (4.7). By taking the infimum over $x \in \mathcal{H}^r$, we obtain $\mathbf{u_0} \geq \mathbf{v_0} = Y_0$. Since $Y_0 \geq \mathbf{u_0}$, we thus get $Y_0 = \mathbf{u_0}$. Since $(\sigma^*, \varphi^*) \in \mathcal{S}^r(Y_0)$, we derive that $(\sigma^*, \varphi^*) \in \mathcal{S}^r(\mathbf{u_0})$. The last assertion of the theorem follows from Theorem 4.5. ∎

When ζ is only RCLL, by using similar arguments to those used in the above proof and in the proof of Theorem 3.11, one can show the following result.

Theorem 4.7 (seller's robust price and ε-superhedge). *Suppose that the process ζ and ξ are only RCLL. The* seller's robust price *of the game option is equal to the dual value function, that is, $\mathbf{u_0} = \mathbf{v_0}$. We also have $\mathbf{u_0} = Y_0$, where (Y, Z, K, A, A') is the solution of the DRBSDE associated with driver $\mathbf{G}$ defined by (4.8) and barriers ξ and ζ.*

*Moreover, the infimum in (4.6) is not necessarily attained. For each $\varepsilon > 0$, let $\sigma_\varepsilon := \inf\{t \geq 0 : \ Y_t \geq \zeta_t - \varepsilon\}$. The pair $(\sigma_\varepsilon, \varphi^*)$, where $\varphi^* := \Phi(Z, K)$, is an ε-*robust superhedge *for the seller, in the sense that*

$$V^{\alpha,u_0,\varphi^*}_t \geq \xi_t, \ 0 \leq t \leq \sigma_\varepsilon \ \text{a.s.} \quad \text{and} \quad V^{\alpha,u_0,\varphi^*}_{\sigma_\varepsilon} \geq \zeta_{\sigma_\varepsilon} - \varepsilon \ \text{a.s.} \quad \forall \alpha \in \mathcal{U}.$$

We will now show that the infimum over σ and the supremum over α can be interchanged in the expression of the dual value function $\mathbf{v_0}$ (see (4.7)), which, since $\mathbf{u_0} = \mathbf{v_0}$, can be written as follows.

Proposition 4.8. *The seller's robust price $\mathbf{u_0}$ of the game option satisfies*

$$\mathbf{u_0} = \sup_{\alpha\in\mathcal{U}} \inf_{\sigma\in\mathcal{T}} \sup_{\tau\in\mathcal{T}} \mathcal{E}^{g^\alpha}_{0,\tau\wedge\sigma}[I(\tau,\sigma)] = \inf_{\sigma\in\mathcal{T}} \sup_{\alpha\in\mathcal{U}} \sup_{\tau\in\mathcal{T}} \mathcal{E}^{g^\alpha}_{0,\tau\wedge\sigma}[I(\tau,\sigma)]. \tag{4.9}$$

Proof. The first equality in (4.9) holds by the above theorem. Let us prove the second one. By the above theorem, we have $\mathbf{u_0} = Y_0$, where (Y, Z, K, A, A') is the solution of the DRBSDE associated with driver $\mathbf{G}$ defined by (4.8) and barriers ξ and ζ. To obtain the desired result, it is thus sufficient to prove that

$$Y_0 = \inf_{\sigma\in\mathcal{T}} \sup_{\alpha\in\mathcal{U}} \sup_{\tau\in\mathcal{T}} \mathcal{E}^{g^\alpha}_{0,\tau\wedge\sigma}[I(\tau,\sigma)]. \tag{4.10}$$

Since by definition (4.8), $\mathbf{G} = \sup_{\alpha\in U} g(\cdot, \alpha)$, by using similar arguments to those used in the proof of Theorem 4.5 (in particular, a measurable selection theorem), one can show that the solution of the BSDE associated with driver $\mathbf{G}$ and terminal condition $I(\tau, \sigma)$ is equal to the supremum over α of the solutions of the BSDEs associated with drivers $g(\cdot, \alpha)$ and the same terminal condition, that is,

$$\mathcal{E}^{\mathbf{G}}_{0,\tau\wedge\sigma}[I(\tau,\sigma)] = \sup_{\alpha\in\mathcal{U}} \mathcal{E}^{g^\alpha}_{0,\tau\wedge\sigma}[I(\tau,\sigma)]. \tag{4.11}$$

On the other hand, applying Proposition 3.4 to the generalized Dynkin game associated with driver $\mathbf{G}$, we obtain the equality

$$Y_0 = \inf_{\sigma\in\mathcal{T}} \sup_{\tau\in\mathcal{T}} \mathcal{E}^{\mathbf{G}}_{0,\tau\wedge\sigma}[I(\tau,\sigma)]. \tag{4.12}$$

Combining (4.11) and (4.12), we obtain the desired equality (4.10). ■

4.3. Application to the case of ambiguity on the default probability. We consider a family of a priori probability measures parametrized by $\alpha \in \mathcal{U}$. More precisely, for each $\alpha \in \mathcal{U}$, let Q^α be the probability measure equivalent to P, which admits Z^α_T as density with respect to P, where (Z^α_t) is the solution of the following SDE:

$$dZ^\alpha_t = Z^\alpha_t \nu(t, \alpha_t) dM_t, \quad Z^\alpha_0 = 1,$$

where $\nu : (\omega, t, \alpha) \mapsto \nu(t, \omega, \alpha)$ is a bounded $\mathcal{P} \otimes \mathcal{B}(U)$-measurable function defined on $\Omega \times [0, T] \times U$ with $\nu(t, \alpha) > C_1 > -1$.

By Girsanov's theorem, we derive that under Q^α, W is a $\mathbb{G}$-Brownian motion and $M^\alpha_t := N_t - \int_0^t \lambda_s(1+\nu(s, \alpha_s))ds$ is a $\mathbb{G}$-martingale. Hence, under Q^α, the $\mathbb{G}$-default intensity is equal to $\lambda_t(1 + \nu(t, \alpha_t))$. The process $\nu(t, \alpha_t)$ represents the *uncertainty on the default intensity*.

To each $\alpha \in \mathcal{U}$ corresponds a market model $\mathcal{M}_\alpha$ associated with the a priori probability measure Q^α. In the market $\mathcal{M}_\alpha$, the dynamics of the wealth process $V^{\alpha,x,\varphi}$ associated with an initial wealth x and a risky assets stategy $\varphi \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ are supposed to satisfy

$$-dV^{\alpha,x,\varphi}_t = f(t, V^{\alpha,x,\varphi}_t, \varphi'_t\sigma_t, -\varphi^2_t, \alpha_t)dt - \varphi'_t\sigma_t dW_t + \varphi^2_t dM^\alpha_t, \;\; V^{\alpha,x,\varphi}_0 = x, \tag{4.13}$$

where $f : (t, \omega, y, z, k, \alpha) \mapsto f(t, \omega, y, z, k, \alpha)$ is a map supposed to be uniformly λ-admissible with respect to (y, z, k), satisfying (4.1) with $\theta^{t,y,z,k_1,k_2} > (-1 - C_1) \vee (-1)$ and $\sup_{\alpha\in U} |f(t, ., 0, 0, 0, \alpha)| \in \mathbb{H}^p$ for some $p > 2$. For example, f can be given as in (2.12) in the case of a perfect market, or as in Example 2.11 of market imperfections, with coefficients which may depend on α.

By [14, Proposition A.3], there is a martingale representation theorem for $\mathbb{G}$-martingales under Q^α with respect to W and M^α. Let $\xi \in L^p(\mathcal{G}_T)$, where $p > 2$. By [14, Proposition 2.11], the density Z^α_T of Q^α with respect to P belongs to L^q for all $q \geq 2$. Let $p' \in]2, p[$. Applying Hölder's inequality, we derive that $E_{Q^\alpha}(\xi^{p'}) < +\infty$. Similarly, since by assumption $f(t, 0, 0, 0, \alpha_t) \in \mathbb{H}^p$, we derive that $f(t, 0, 0, 0, \alpha_t) \in \mathbb{H}^{p'}_{Q^\alpha}$. By [14, Corollary A.4], there exists a unique solution $(X^\alpha, Z^\alpha, K^\alpha)$ in $\mathcal{S}^{p'}_{Q^\alpha} \times \mathbb{H}^{p'}_{Q^\alpha} \times \mathbb{H}^{p'}_{Q^\alpha,\lambda}$ of the following Q^α-BSDE:

$$-dX^\alpha_t = f(t, X^\alpha_t, Z^\alpha_t, K^\alpha_t, \alpha_t)dt - Z^\alpha_t dW_t - K^\alpha_t dM^\alpha_t, \quad X^\alpha_T = \xi. \tag{4.14}$$

As in the previous section, to simplify notation, for each $\alpha \in \mathcal{U}$, we denote by f^α the driver $f^\alpha(t,y,z,k) = f(t,y,z,k,\alpha_t)$. The nonlinear price system in the market model $\mathcal{M}_\alpha$, denoted by $\mathcal{E}^{f^\alpha}_{Q^\alpha}$, is thus the f^α-evaluation under the *a priori probability* measure Q^α, defined on $L^{p'}$. The robust superhedges are defined as in Definition 4.1 and the seller's robust price $\mathbf{u_0}$ is defined by (4.5).

Since $M^\alpha_t = M_t - \int_0^t \lambda_s \nu(s,\alpha_s)ds$, the dynamics (4.13) of the wealth process $V^{\alpha,x,\varphi}$ in the market model $\mathcal{M}_\alpha$ can be written as follows:

$$-dV^{\alpha,x,\varphi}_t = -\lambda_t \nu(t,\alpha_t)\varphi_t^2 dt + f(t, V^{\alpha,x,\varphi}_t, \varphi_t\sigma_t, -\varphi_t^2, \alpha_t)dt - \varphi_t\sigma_t dW_t + \varphi_t^2 dM_t.$$

This example thus corresponds to the model with ambiguity defined in section 4.1 with $g(\cdot,\alpha)$ defined by

$$g(t,\omega,y,z,k,\alpha) := \lambda_t(\omega)\nu(t,\omega,\alpha)k + f(t,\omega,y,z,k,\alpha).$$

By the assumptions on f, the map g satisfies the required conditions, in particular, inequality (4.1). Theorems 4.6 and 4.7 as well as Proposition 4.8 hold. In particular, the seller's robust price $\mathbf{u_0}$ of the game option admits the following dual representation:

$$\mathbf{u_0} = \sup_{\alpha\in\mathcal{U}} \inf_{\sigma\in\mathcal{T}} \sup_{\tau\in\mathcal{T}} \mathcal{E}^{g^\alpha}_{0,\tau\wedge\sigma}[I(\tau,\sigma)] = \inf_{\sigma\in\mathcal{T}} \sup_{\alpha\in\mathcal{U}} \sup_{\tau\in\mathcal{T}} \mathcal{E}^{g^\alpha}_{0,\tau\wedge\sigma}[I(\tau,\sigma)]. \tag{4.15}$$

We now show that for each $\alpha \in \mathcal{U}$, $\mathcal{E}^{g^\alpha}$ is equal to the nonlinear price system $\mathcal{E}^{f^\alpha}_{Q^\alpha}$ relative to the market model $\mathcal{M}_\alpha$. First, we have $(Z^\alpha_T)^{-1} \in L^q$ for all $q \geq 1$. Indeed, The process $(Z^\alpha_t)^{-1}$ satisfies the following Q^α-SDE: $d(Z^\alpha_t)^{-1} = -(Z^\alpha_{t^-})^{-1}\nu(t,\alpha_t)dM^\alpha_t$ with $(Z^\alpha_0)^{-1} = 1$. By [14, Proposition 2.11], $(Z^\alpha_T)^{-1}$ belongs to $L^{q'}_{Q^\alpha}$ for all $q' \geq 1$, which implies that $(Z^\alpha_T)^{-1} \in L^q$ for all $q \geq 1$. Since $p' > 2$, by Hölder's inequality, we derive that $(X^\alpha, Z^\alpha, K^\alpha)$ (solution of (4.14)) belongs to $S^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ and is, thus, the unique solution in $S^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ of the P-BSDE:

$$-dX^\alpha_t = g^\alpha(t, X^\alpha_t, Z^\alpha_t, K^\alpha_t)dt - Z^\alpha_t dW_t - K^\alpha_t dM_t, \quad X^\alpha_T = \xi.$$

Hence, for each maturity S and each payoff $\eta \in L^p(\mathcal{G}_S)$, we have

$$\mathcal{E}^{f^\alpha}_{Q^\alpha,\cdot,S}(\eta) = \mathcal{E}^{g^\alpha}_{\cdot,S}(\eta),$$

which gives that $\mathcal{E}^{g^\alpha}$ is equal to the nonlinear price system $\mathcal{E}^{f^\alpha}_{Q^\alpha}$ relative to the market model $\mathcal{M}_\alpha$. Using this property together with equalities (4.15) and Theorem 4.7, we derive the following result.

Proposition 4.9 (seller's robust price). *The seller's robust price of the game option in this model admits the following dual representation:*

$$\mathbf{u_0} = \sup_{\alpha\in\mathcal{U}} \inf_{\sigma\in\mathcal{T}} \sup_{\tau\in\mathcal{T}} \mathcal{E}^{f^\alpha}_{Q^\alpha,0,\tau\wedge\sigma}[I(\tau,\sigma)] = \inf_{\sigma\in\mathcal{T}} \sup_{\alpha\in\mathcal{U}} \sup_{\tau\in\mathcal{T}} \mathcal{E}^{f^\alpha}_{Q^\alpha,0,\tau\wedge\sigma}[I(\tau,\sigma)]. \tag{4.16}$$

Let $\mathbf{G}$ *be the map defined for each* (t,ω,z,k) *by*

$$\mathbf{G}(t,\omega,y,z,k) := \sup_{\alpha\in U} \left(\lambda_t(\omega)\nu(t,\omega,\alpha)k + f(t,\omega,y,z,k,\alpha)\right). \tag{4.17}$$

We have $\mathbf{u_0} = Y_0$, *where* Y *is the solution of the* P*-DRBSDE associated with driver* $\mathbf{G}$ *and barriers* ξ *and* ζ.

5. Complementary results.

5.1. Pricing of European options from the buyer's point of view. Let us consider the pricing and hedging problem of a European option with maturity T and payoff $\xi \in L^2(\mathcal{G}_T)$ from the buyer's point of view. Supposing the initial price of the option is z, he starts with the amount $-z$ at time $t = 0$, and looks to find a risky-assets strategy $\tilde{\varphi}$ such that the payoff that he receives at time T allows him to recover the debt he incurred at time $t = 0$ by buying the option, that is, such that $V_T^{-z,\tilde{\varphi}} + \xi = 0$ a.s. or, equivalently, $V_T^{-z,\tilde{\varphi}} = -\xi$ a.s.

The buyer's price of the option is thus equal to the opposite of the seller's price of the option with payoff $-\xi$, that is, $-\mathcal{E}^g_{0,T}(-\xi) = -\tilde{X}_0$, where $(\tilde{X}, \tilde{Z}, \tilde{K})$ is the solution of the BSDE associated with driver g and terminal condition $-\xi$. Let us specify the hedging strategy for the buyer. Suppose that the initial price of the option is $z := -\tilde{X}_0$. The process $\tilde{X}$ is equal to the value of the portfolio associated with initial value $-z = \tilde{X}_0$ and strategy $\tilde{\varphi} := \Phi(\tilde{Z}, \tilde{K})$ (where Φ is defined in Definition 2.3), that is, $\tilde{X} = V^{\tilde{X}_0,\tilde{\varphi}} = V^{-z,\tilde{\varphi}}$. Hence, $V_T^{-z,\tilde{\varphi}} = \tilde{X}_T = -\xi$ a.s., which yields that $\tilde{\varphi}$ is the hedging risky-assets strategy for the buyer. Similarly, $-\mathcal{E}^g_{t,T}(-\xi) = -\tilde{X}_t$ satisfies an analogous property at time t, and is called the *hedging price for the buyer* at time t.

This leads to the *nonlinear pricing system* $\tilde{\mathcal{E}}^g$ *relative to the buyer* in the market $\mathcal{M}^g$ defined for each $(S, \xi) \in [0, T] \times L^2(\mathcal{G}_S)$ by

$$\tilde{\mathcal{E}}^g_{\cdot,S}(\xi) := -\mathcal{E}^g_{\cdot,S}(-\xi). \tag{5.1}$$

Remark 5.1. When $g(t, 0, 0, 0) = 0$, then $\tilde{\mathcal{E}}^g_{\cdot,S}(0) = 0$. Moreover, by the comparison theorem for BSDEs with default, if $\xi \geq 0$, then $\tilde{\mathcal{E}}^g_{\cdot,S}(\xi) \geq 0$.

Note that $\tilde{\mathcal{E}}^g_{\cdot,S}(\xi)$ is equal to the solution of the BSDE with driver $-g(t, -y, -z, -k)$ and terminal condition ξ. Hence, if we suppose that $-g(t, -y, -z, -k) \leq g(t, y, z, k)$ (which is satisfied if, for example, g is convex with respect to (y, z, k)), then, by the comparison theorem for BSDEs, we have $\tilde{\mathcal{E}}^g_{\cdot,S}(\xi) = -\mathcal{E}^g_{\cdot,S}(-\xi) \leq \mathcal{E}^g_{\cdot,S}(\xi)$ for each $(S, \xi) \in [0, T] \times L^2(\mathcal{G}_S)$.[4]

Moreover, when $-g(t, -y, -z, -k) = g(t, y, z, k)$ (which is satisfied if, for example, g is linear with respect to (y, z, k), as in the perfect market case), we have $\tilde{\mathcal{E}}^g = \mathcal{E}^g$.

5.2. Pricing of the game option from the buyer's point of view. In this section, we consider the point of view of the buyer of the game option. Supposing the initial price of the game option is z, he starts with the amount $-z$ at time $t = 0$, and looks to find a superhedge, that is, an exercise time τ and a risky-assets strategy φ, such that the payoff that he receives allows him to recover the debt he incurred at time $t = 0$ by buying the game option, no matter the cancellation time chosen by the seller. This notion of superhedge for the buyer can be defined more precisely as follows.

Definition 5.2. *A* buyer's superhedge *against the game option with initial price* $z \in \mathbb{R}$ *is a pair* (τ, φ) *of a stopping time* $\tau \in \mathcal{T}$ *and a risky-assets strategy* $\varphi \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ *such that*

$$V_t^{-z,\varphi} \geq -\zeta_t, \ 0 \leq t < \tau \ \text{ a.s. and } V_\tau^{-z,\varphi} \geq -\xi_\tau \ \text{ a.s.} \tag{5.2}$$

[4]Note that a price functional p generally satisfies $-p(-\xi) \leq p(\xi)$ (see, e.g., [20, section 2]).

We denote by $\mathcal{B}_{\xi,\zeta}(z)$ *the set of all buyer's superhedges against the game option with payoffs* (ξ,ζ) *associated with initial price* $z \in \mathbb{R}$.

The buyer's price of the game option in the market model $\mathcal{M}^g$, *denoted by* $\tilde{u}_0$, *is defined as the supremum of the initial prices which allow the buyer to be superhedged, that is,*[5]

$$\tilde{u}_0 := \sup\{z \in \mathbb{R},\ \exists(\tau,\varphi) \in \mathcal{B}_{\xi,\zeta}(z)\}. \tag{5.3}$$

The first inequality of (5.2) also holds at time $t = \tau$ because $\xi \leq \zeta$. It follows that $\mathcal{B}_{\xi,\zeta}(z) = \mathcal{S}_{-\zeta,-\xi}(-z)$, where $\mathcal{S}_{-\zeta,-\xi}(-z)$ is the set of seller's superhedges against the game option with payoffs $(-\zeta,-\xi)$ associated with initial capital $-z$.

Hence, $-\tilde{u}_0 = \inf\{x \in \mathbb{R},\ \exists(\tau,\varphi) \in \mathcal{S}_{-\zeta,-\xi}(x)\}$. We thus have the following.

Theorem 5.3. *The buyer's price of the game option with payoffs* (ξ,ζ) *is equal to the opposite of the* seller's price *of the game option with payoffs* $(-\zeta,-\xi)$.

The previous results (Theorems 3.6 and 3.11) can thus be applied. In particular, we have the following dual formulation of the buyer's price:

$$\tilde{u}_0 = \sup_{\tau\in\mathcal{T}} \inf_{\sigma\in\mathcal{T}} \tilde{\mathcal{E}}^g_{0,\tau\wedge\sigma}[I(\tau,\sigma)] = \inf_{\sigma\in\mathcal{T}} \sup_{\tau\in\mathcal{T}} \tilde{\mathcal{E}}^g_{0,\tau\wedge\sigma}[I(\tau,\sigma)], \tag{5.4}$$

where $\tilde{\mathcal{E}}^g_{0,\tau\wedge\sigma}[I(\tau,\sigma)] = -\mathcal{E}^g_{0,\tau\wedge\sigma}[-I(\tau,\sigma)]$. The quantity $\tilde{\mathcal{E}}^g_{0,\tau\wedge\sigma}[I(\tau,\sigma)]$ corresponds to the buyer's price of the European option with payoff $I(\tau,\sigma)$ and terminal time $\tau\wedge\sigma$ (see (5.1)).

Remark 5.4. In the special case of a perfect market, the dynamics of the wealth process X are linear with respect to (X,φ), which implies that the buyer's price $\tilde{u}_0$ is equal to the seller's price u_0 (and $\tilde{\mathcal{E}}^g = \mathcal{E}^g$, as seen in Remark 5.1).

Let $(\tilde{Y},\tilde{Z},\tilde{K},\tilde{A},\tilde{A}')$ be the solution of the DRBSDE associated with driver g and barriers $(-\zeta,-\xi)$. By Theorem 3.11, the buyer's price is equal to the opposite of the solution, that is, $\tilde{u}_0 = -\tilde{Y}_0$.

Moreover, by Theorem 3.6, when ξ is left-u.s.c. along stopping times (but not necessarily $-\zeta$), the pair $(\tilde{\tau},\tilde{\varphi})$, where $\tilde{\tau} := \inf\{t \geq 0 :\ -\tilde{Y}_t = \xi_t\}$ and $\tilde{\varphi} := \Phi(\tilde{Z},\tilde{K})$, is a buyer's superhedge.

Buyer's robust price of the game option in the case with ambiguity.. In this paragraph, we consider the market model with ambiguity described in section 4.1.

Definition 5.5. *A* buyer's robust superhedge *against the game option with initial price* $z \in \mathbb{R}$ *is a pair* (τ,φ) *of a stopping time* $\tau \in \mathcal{T}$ *and a strategy* $\varphi \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ *such that*

$$V_t^{\alpha,z,\varphi} \geq -\zeta_t,\ 0 \leq t < \tau \ \ a.s. \text{ and } V_\tau^{\alpha,z,\varphi} \geq -\xi_\tau \ a.s. \quad \forall\alpha \in \mathcal{U}. \tag{5.5}$$

We denote by $\mathcal{B}^r_{\xi,\zeta}(z)$ *the set of all buyer's robust superhedges against the game option with payoffs* (ξ,ζ) *associated with initial price* $z \in \mathbb{R}$.

[5]We have $(0,0) \in \mathcal{B}_{\xi,\zeta}(\xi_0)$. Hence, $\tilde{u}_0 \geq \xi_0$. Moreover, similarly to Remark 3.2, if $g(t,0,0,0) = 0$ and $\xi_0 \geq 0$, then $\tilde{u}_0 = \sup\{z \geq 0,\ \exists(\tau,\varphi) \in \mathcal{B}_{\xi,\zeta}(z)\}$.

The buyer's robust price of the game option is defined as the supremum of the initial prices which allow the buyer to construct a robust superhedge, that is,

$$\tilde{\mathbf{u}}_{\mathbf{0}} := \sup\{z \in \mathbb{R},\ \exists(\tau, \varphi) \in \mathcal{B}^r_{\xi,\zeta}(z)\}. \tag{5.6}$$

Since $\xi \leq \zeta$, condition (5.5) is equivalent to

$$V_t^{-z,\varphi} \geq -\zeta_t,\ 0 \leq t \leq \tau \ \text{ a.s. and } V_\tau^{-z,\varphi} \geq -\xi_\tau \text{ a.s.} \quad \forall \alpha \in \mathcal{U}.$$

It follows that $\mathcal{B}^r_{\xi,\zeta}(z) = \mathcal{S}^r_{-\zeta,-\xi}(-z)$, where $\mathcal{S}^r_{-\zeta,-\xi}(-z)$ is the set of seller's robust superhedges against the game option with payoffs $(-\zeta, -\xi)$ associated with initial capital $-z$. We thus have the following.

Theorem 5.6. *The buyer's robust price of the game option with payoffs (ξ, ζ) is equal to the opposite of the seller's robust price of the game option with payoffs $(-\zeta, -\xi)$.*

The previous results (Theorems 4.6 and 4.7) can thus be applied. In particular, we have the following dual formulation of the *buyer's robust price*:

$$\tilde{\mathbf{u}}_{\mathbf{0}} = \inf_{\alpha \in \mathcal{U}} \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \tilde{\mathcal{E}}^{g^\alpha}_{0,\tau\wedge\sigma}[I(\tau, \sigma)] = \inf_{\alpha \in \mathcal{U}} \inf_{\sigma \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \tilde{\mathcal{E}}^{g^\alpha}_{0,\tau\wedge\sigma}[I(\tau, \sigma)], \tag{5.7}$$

where $\tilde{\mathcal{E}}^{g^\alpha}_{0,\tau\wedge\sigma}[I(\tau, \sigma)] = -\mathcal{E}^{g^\alpha}_{0,\tau\wedge\sigma}[-I(\tau, \sigma)]$. Using (5.4), we derive that the buyer's robust price $\tilde{\mathbf{u}}_{\mathbf{0}}$ is equal to the infimum over $\alpha \in \mathcal{U}$ of the buyer's prices in $\mathcal{M}_\alpha$.

Remark 5.7. By Proposition 4.8, we derive that

$$\tilde{\mathbf{u}}_{\mathbf{0}} = \sup_{\tau \in \mathcal{T}} \inf_{\alpha \in \mathcal{U}} \inf_{\sigma \in \mathcal{T}} \tilde{\mathcal{E}}^{g^\alpha}_{0,\tau\wedge\sigma}[I(\tau, \sigma)].$$

Note that for each $\alpha \in \mathcal{U}$, the quantity $\tilde{\mathcal{E}}^{g^\alpha}_{0,\tau\wedge\sigma}[I(\tau, \sigma)]$ is the buyer's price in the market model $\mathcal{M}_\alpha$ of the European option with payoff $I(\tau, \sigma)$ and terminal time $\tau \wedge \sigma$ (see (5.1)).

Let $(\tilde{Y}, \tilde{Z}, \tilde{K}, \tilde{A}, \tilde{A}')$ be the solution of the DRBSDE associated with driver $\mathbf{G}$ defined by (4.8) and barriers $(-\zeta, -\xi)$. By Theorem 4.7, the buyer's robust price of the game option is equal to $-\tilde{Y}_0$, that is, $\tilde{\mathbf{u}}_{\mathbf{0}} = -\tilde{Y}_0$. Moreover, by Theorem 4.6, when ξ is left-u.s.c. along stopping times (but not necessarily $-\zeta$), the pair $(\tilde{\tau}, \tilde{\varphi})$, where $\tilde{\tau} := \inf\{t \geq 0 : \ -\tilde{Y}_t = \xi_t\}$ and $\tilde{\varphi} := \Phi(\tilde{Z}, \tilde{K})$, is a buyer's robust superhedge of the game option.

5.3. Seller's price and buyer's price processes of the game option. We can define the seller's price of the game option at each stopping time $S \in \mathcal{T}$. More precisely, for each wealth $X \in L^2(\mathcal{F}_S)$ (at initial time S), an S-superhedge against the game option is a pair (σ, φ) of a stopping time $\sigma \in \mathcal{T}_S$ and a portfolio strategy $\varphi \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ such that $V_t^{S,X,\varphi} \geq \xi_t$, $S \leq t \leq \sigma$ a.s. and $V_\sigma^{S,X,\varphi} \geq \zeta_\sigma$ a.s., where $V^{S,X,\varphi}$ denotes the wealth process associated with initial time S and initial condition X. The seller's price at time S is defined by $u(S) :=$ $\operatorname{ess\,inf}\{X \in L^2(\mathcal{F}_S),\ \exists(\sigma, \varphi) \in \mathcal{S}_S(X)\}$, where $\mathcal{S}_S(X)$ is the set of all S-superhedges associated with initial wealth X. Using similar arguments to those used in the proof of Theorem 3.11, we obtain

$$u(S) = \text{ess}\inf_{\sigma\in\mathcal{T}_S} \text{ess}\sup_{\tau\in\mathcal{T}_S} \mathcal{E}^g_{S,\tau\wedge\sigma}(I(\tau,\sigma)) = \text{ess}\sup_{\tau\in\mathcal{T}_S} \text{ess}\inf_{\sigma\in\mathcal{T}_S} \mathcal{E}^g_{S,\tau\wedge\sigma}(I(\tau,\sigma)) = Y_S \quad \text{a.s.},$$

where (Y, Z, K, A, A') is the solution of DRBSDE (3.5).

Similarly, we can define the buyer's price at time S.

5.4. Game options with intermediate dividends. Suppose that a European option pays a terminal payoff ξ at terminal time S and an intermediate dividend, modeled by a nondecreasing RCLL adapted process (D_t) with $D_0 = 0$. There exists a unique solution (X, Z, K) in $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ of the following BSDE:

$$-\,dX_t = g(t, X_t, Z_t, K_t)dt + dD_t - Z_t dW_t - K_t dM_t, \quad X_S = \xi. \tag{5.8}$$

The process X is the wealth process associated with initial value $x = X_0$ and strategy $\varphi = \Phi(Z, K)$. Here, dD_t represents the amount withdrawn from the portfolio between t and $t + dt$ in order to pay the dividends to the buyer. Hence, the amount X_0 allows the seller to be perfectly hedged against the option, in the sense that it allows him/her to pay the intermediate dividends and the terminal payoff to the buyer, by investing the amount X_0 along the strategy φ in the market. The price for the seller (at time 0) of this option is thus given by X_0 and the associated hedging strategy is equal to φ. Note that the driver of BSDE (5.8) is given by the λ-admissible generalized driver $g(t, X_t, Z_t, K_t)dt + dD_t$. This leads to the following nonlinear pricing system: for each $S \in [0, T]$, for each $\xi \in L^2(\mathcal{G}_S)$, and for each $D \in \mathcal{A}^2$, the associated g-value is defined by $\mathcal{E}^{g,D}_{t,S}(\xi) := X^D_t(S, \xi)$ for each $t \in [0, S]$. Note that $\mathcal{E}^{g,D}_{t,S}(\xi)$ can be defined on the whole interval $[0, T]$ by setting $\mathcal{E}^{g,D}_{t,S}(\xi) := \mathcal{E}^{g^S,D^S}_{t,T}(\xi)$ for $t \geq S$, where $g^S(t, .) := g(t, .)\mathbf{1}_{t\leq S}$ and $D^S_t := D_{t\wedge S}$. Some properties of this nonlinear pricing system are provided in [14].

Concerning the pricing of the game option, the approach is the same, replacing the driver g by the "generalized" driver $g(\cdot)dt + dD_t$, and $\mathcal{E}^g$ by $\mathcal{E}^{g,D}$.

6. Appendix. We show the following estimates for DRBSDEs in our framework, with universal constants.

Proposition 6.1 (a priori estimate for DRBSDEs). *Let f^1 be a λ-admissible driver with λ-constant C and let f^2 be a driver. Let ξ and ζ be two adapted RCLL processes with $\zeta_T = \xi_T$ a.s., $\xi \in \mathcal{S}^2$, $\zeta \in \mathcal{S}^2$, $\xi_t \leq \zeta_t$, $0 \leq t \leq T$ a.s., and satisfying Mokobodzki's condition.*

For $i = 1, 2$, let $(Y^i, Z^i, K^i, A^i, A'^i)$ be a solution of the DRBSDE associated with terminal time T, driver f^i, and barriers ξ and ζ. Let $\eta, \beta > 0$ be such that $\beta \geq \frac{3}{\eta} + 2C$ and $\eta \leq \frac{1}{C^2}$. Let $\bar{f}(s) := f^1(s, Y^2_s, Z^2_s, K^2_s) - f^2(s, Y^2_s, Z^2_s, K^2_s)$. For each $t \in [0, T]$, we then have

$$e^{\beta t}(Y^1_s - Y^2_s)^2 \leq \eta\, \mathbb{E}\left[\int_t^T e^{\beta s} \bar{f}(s)^2 ds \mid \mathcal{G}_t\right] \quad \text{a.s.} \tag{6.1}$$

Moreover, $\|\bar{Y}\|^2_\beta \leq T\eta\|\bar{f}\|^2_\beta$, and if $\eta < \frac{1}{C^2}$, we then have $\|\bar{Z}\|^2_\beta + \|\bar{K}\|^2_{\lambda,\beta} \leq \frac{\eta}{1-\eta C^2}\|\bar{f}\|^2_\beta$.

Proof. For s in $[0,T]$, denote $\bar{Y}_s := Y_s^1 - Y_s^2$, $\bar{Z}_s := Z_s^1 - Z_s^2$, $\bar{K}_s := K_s^1 - K_s^2$. By Itô's formula applied to the semimartingale $e^{\beta s}\bar{Y}_s$ between t and T, we get

$$
\begin{aligned}
& e^{\beta t}\bar{Y}_t^2 + \beta \int_t^T e^{\beta s}\bar{Y}_s^2 ds + \int_t^T e^{\beta s}\bar{Z}_s^2 ds + \int_t^T e^{\beta s}\bar{K}_s^2 \lambda_s ds \\
&\quad + \sum_{0<s\leq T} e^{\beta s}\left(\Delta A_s^1 - \Delta A_s^2 - \Delta A_s^{\prime 1} + \Delta A_s^{\prime 2}\right)^2 \\
&\quad = 2\int_t^T e^{\beta s}\bar{Y}_s \left(f^1\left(s, Y_s^1, Z_s^1, K_s^1\right) - f^2\left(s, Y_s^2, Z_s^2, K_s^2\right)\right) ds \\
&\quad - 2\int_t^T e^{\beta s}\bar{Y}_s\bar{Z}_s dW_s - \int_t^T e^{\beta s}\left(2\bar{Y}_{s-}\bar{K}_s + \bar{K}_s^2\right) dM_t \\
&\quad + 2\int_t^T e^{\beta s}\overline{Y}_{s-} dA_s^1 - 2\int_t^T e^{\beta s}\overline{Y}_{s-} dA_s^2 - 2\int_0^T e^{\beta s}\overline{Y}_{s-}\, dA_s^{\prime 1} - \int_0^T e^{\beta s}\overline{Y}_{s-}\, dA_s^{\prime 2}.
\end{aligned} \tag{6.2}
$$

Now, we have $\overline{Y}_s dA_s^{1,c} = (Y_s^1 - \xi_s)dA_s^{1,c} - (Y_s^2 - \xi_s)dA_s^{1,c} = -(Y_s^2 - \xi_s)dA_s^{1,c} \leq 0$, and, by symmetry, $\overline{Y}_s dA_s^{2,c} \geq 0$. By similar arguments, we obtain $\overline{Y}_{s-}\Delta A_s^{1,d} \leq 0$, $\overline{Y}_{s-}\Delta A_s^{2,d} \geq 0$, $\overline{Y}_s dA_s^{\prime 1,c} \geq 0$, $\overline{Y}_{s-}\Delta A_s^{\prime 1,d} \geq 0$, and $\overline{Y}_{s-}\Delta {A'^{2,d}}_s \leq 0$. Hence, the four last terms of the right-hand side of (6.2) are nonpositive. Taking the conditional expectation given $\mathcal{G}_t$, we obtain

$$
\begin{aligned}
& e^{\beta t}\bar{Y}_t^2 + E\left[\beta\int_t^T e^{\beta s}\bar{Y}_s^2 ds + \int_t^T e^{\beta s}\left(\bar{Z}_s^2 + \bar{K}_s^2\lambda_s\right) ds \mid \mathcal{G}_t\right] \\
&\quad \leq 2E\left[\int_t^T e^{\beta s}\bar{Y}_s\left(f^1\left(s, Y_s^1, Z_s^1, K_s^1\right) - f^2\left(s, Y_s^2, Z_s^2, K_s^2\right)\right) ds \mid \mathcal{G}_t\right].
\end{aligned} \tag{6.3}
$$

Also, $|f^1(s, Y_s^1, Z_s^1, K_s^1) - f^2(s, Y_s^2, Z_s^2, K_s^2)| \leq |f^1(s, Y_s^1, Z_s^1, K_s^1) - f^1(s, Y_s^2, Z_s^2, K_s^2)| + |\bar{f}_s|$. Using the λ-admissibility property of f^1, we derive that

$$
\left|f^1\left(s, Y_s^1, Z_s^1, K_s^1\right) - f^2\left(s, Y_s^2, Z_s^2, K_s^2\right)\right| \leq C|\bar{Y}_s| + C|\bar{Z}_s| + C|\bar{K}_s|\sqrt{\lambda_s} + |\bar{f}_s|.
$$

Note now that, for all nonnegative numbers λ, y, z, k, f, and $\varepsilon > 0$, we have $2y(Cz + Ck\sqrt{\lambda} + f) \leq \frac{y^2}{\varepsilon^2} + \varepsilon^2(Cz + Ck\sqrt{\lambda} + f)^2 \leq \frac{y^2}{\varepsilon^2} + 3\varepsilon^2(C^2y^2 + C^2k^2\lambda + f^2)$. Hence,

$$
\begin{aligned}
& e^{\beta t}\bar{Y}_t^2 + \mathbb{E}\left[\beta\int_t^T e^{\beta s}\bar{Y}_s^2 ds + \int_t^T e^{\beta s}\left(\bar{Z}_s^2 + \bar{K}_s^2\lambda_s\right) ds \mid \mathcal{G}_t\right] \\
& \leq \mathbb{E}\left[\left(2C + \frac{1}{\varepsilon^2}\right)\int_t^T e^{\beta s}\bar{Y}_s^2 ds + 3C^2\varepsilon^2\int_t^T e^{\beta s}\left(\bar{Z}_s^2 + \bar{K}_s^2\lambda_s\right) ds + 3\varepsilon^2\int_t^T e^{\beta s}\bar{f}_s^2 ds \mid \mathcal{G}_t\right].
\end{aligned} \tag{6.4}
$$

Let us make the change of variable $\eta = 3\epsilon^2$. Then, for each $\beta, \eta > 0$ chosen as in the proposition, these inequalities lead to (6.1). By integrating (6.1), we obtain $\|\bar{Y}\|_\beta^2 \leq T\eta\|\bar{f}\|_\beta^2$. Using inequality (6.4), the last assertion of the proposition follows. ■

From these estimates, we derive an existence and uniqueness result for DRBSDEs.

Proof of the existence and the uniqueness of the solution of a DRBSDE (3.5). Let us first consider the case when the driver $g(t)$ does not depend on the solution. By using the representation property of $\mathbb{G}$-martingales (Lemma 2.1) and some results of Dynkin games theory, one can show, proceeding as in [13], that there exists a unique solution of the associated DRBSDE (3.5). The proof in the general case is the same as for nonreflected BSDEs with default jump (see the proof of [14, Proposition 2.6]). It is based on a fixed point argument, using the previous estimates. ∎

We state a comparison result of analysis for differential equations (deterministic). Let $L^2 = L^2([0,T], dt)$ be the space of square integrable Borelian real valued maps on $[0,T]$.

Lemma 6.2 (comparison for differential equations). *For $i = 1,2$, let $b_i : [0,T] \times \mathbb{R} \to \mathbb{R}; (t,y) \mapsto b_i(t,y)$ be a Borelian map with $b_i(.,0) \in L^2$, and supposed to be uniformly Lipschitz with respect to y. Let f^1, f^2 be right-continuous maps in L^2 and let $x_1, x_2 \in \mathbb{R}$. For $i = 1,2$, let y^i be the unique right-continuous map in L^2 satisfying the differential equation*

$$y_t^i := x_i + \int_0^t b_i(s, y_s^i)ds + f^i(t).$$

Suppose that $x_1 \geq x_2$ and $b_1(t, y_t^2) \geq b_2(t, y_t^2)$, $0 \leq t \leq T$ ds-a.e. Suppose also that $f^1 = f^2 + A$, where A is a nondecreasing right-continuous map on $[0,T]$ with $A_0 = 0$. We then have $y_t^1 \geq y_t^2$ for each $t \in [0,T]$. Moreover, if $x_1 > x_2$, then $y_t^1 > y_t^2$ for each $t \in [0,T]$.

Proof. The proof is classical. We have $d\bar{y}_t = \lambda_t \bar{y}_t dt + (b_1(t, y_t^2) - b_2(t, y_t^2))dt + dA_t$, where $\bar{y} := y^1 - y^2$ and $\lambda_t := (b_1(t, y_t^1) - b_1(t, y_t^2))(\bar{y}_t)^{-1}\mathbf{1}_{y_t^1 \neq y_t^2}$. Hence, $\bar{y}_t = x_1 - x_2 + \int_0^t e^{-\int_s^t \lambda_u du}(b_1(s, y_s^2) - b_2(s, y_s^2))ds + \int_0^t e^{-\int_s^t \lambda_u du} dA_s \geq 0$. Moreover, if $x_1 > x_2$, then the inequality is strict. ∎

Acknowledgment. The authors thank an anonymous referee for insightful comments which led to a significantly improved version of the paper. The authors also thank Marek Rutkowski for useful discussions.

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