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Discontinuous model recovery anti-windup solution for
image based visual servoing ” submitted to Automatica**

Laurent Burlion, Luca Zaccarian, Henry de Plinval, Sophie Tarbouriech

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Detailed proof of the Lemma 1 used in the manuscript “Discontinuous model recovery anti-windup solution for image based visual servoing” [1]

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As mentioned in [1], the proof of technical Lemma 1 arises from brute force calculations of all possible cases. The details were omitted due to page restrictions. The proof of this result is reported below:

Lemma 1 *For any selection of $(\tilde{x}, x) = (\tilde{p}, \tilde{v}, p, v) \in \mathbb{R}^4$, it holds that*

$$\max_{\delta_p \in \Upsilon(\tilde{x}, x)} v \delta_p \leq k_p^g \frac{\lambda_m}{\lambda_0} (2|\tilde{x}||v| + |\tilde{x}|^2) + 3\bar{u}_p |\tilde{x}|. \quad (1)$$

To better follow the proof of Lemma 1, it is important to keep in mind the definition of M_I (defined in eq. (11),[1]) plus the following basic properties of the saturation function used in [1]: given $\bar{u}_p > 0$,

- **(P1)**: $\text{sat}_{\bar{u}_p}$ is 1-Lipschitz
- **(P2)**: $\forall x, |\text{sat}_{\bar{u}_p}(x)| \leq \bar{u}_p$
- **(P3)**: $\forall k_p^g > 0, \zeta > 0, \forall p \neq \tilde{p}$,

$$\frac{\text{sat}_{\bar{u}_p}(k_p^g \zeta p) - \text{sat}_{\bar{u}_p}(k_p^g \zeta \tilde{p})}{p - \tilde{p}} \geq 0 \quad (2)$$

- **(P4)**: $\forall \chi > 0, \forall x, y$ then

$$xy \geq 0 \implies y \text{sat}_{\bar{u}_p}(\chi x) \geq 0 \quad (3)$$

$$xy \leq 0 \implies y \text{sat}_{\bar{u}_p}(\chi x) \leq 0 \quad (4)$$

Proof. Given any pair $(\tilde{x}, x) = (\tilde{p}, \tilde{v}, p, v)$, consider set $\Upsilon(\tilde{x}, x)$ defined in eq. (44),[1]. In particular, consider any selection of $\zeta \in M_I(pv)$, of $\zeta_{aw} \in M_I((p - \tilde{p})(v - \tilde{v}))$ and any $\mu \in [\lambda_0, \lambda_m]$. (From the definition of M_I and eq. (4),[1], it is clear that $\zeta, \zeta_{aw}, \mu > 0$).

Then denote:

$$\psi := v (\text{sat}_{\bar{u}_p}(k_p^g \zeta \mu p) - \text{sat}_{\bar{u}_p}(k_p^g \zeta_{aw} \mu (p - \tilde{p}))) \quad (5)$$

where $k_p^g > 0$.

The proof of the lemma amounts to showing that

$$\psi \leq k_p^g \frac{\lambda_m}{\lambda_0} (2|\tilde{x}||v| + |\tilde{x}|^2) + 3\bar{u}_p |\tilde{x}| \quad (6)$$

This is done by way of a lengthy but simple study of five possible cases:

1. Suppose

$$(p - \tilde{p})(v - \tilde{v}) < 0 \quad \& \quad pv \geq 0 \quad (7)$$

In this case, $\zeta_{aw} = \frac{1}{\lambda_m}$ and (5) develops as follows:

$$\psi = v \left[\text{sat}_{\bar{u}_p}(k_p^g \zeta \mu p) - \text{sat}_{\bar{u}_p} \left(k_p^g \frac{\mu}{\lambda_m} (p - \tilde{p}) \right) \right] \quad (8)$$

$$\leq v \left[\text{sat}_{\bar{u}_p} \left(k_p^g \frac{\mu}{\lambda_m} p \right) - \text{sat}_{\bar{u}_p} \left(k_p^g \frac{\mu}{\lambda_m} u(p - \tilde{p}) \right) \right] + v \text{sat}_{\bar{u}_p}(k_p^g \zeta \mu p) \quad (9)$$

$$\leq k_p^g \frac{\lambda_m}{\lambda_0} |\tilde{x}| |v| + v \text{sat}_{\bar{u}_p}(k_p^g \zeta \mu p) \quad (10)$$

where the penultimate inequality was obtained using **(P4)**, i.e $0 \leq v \text{sat}_{\bar{u}_p} \left(k_p^g \frac{\mu}{\lambda_m} p \right)$.

where the last inequality was obtained using **(P1)** and $0 < \frac{\mu}{\lambda_m} \leq 1 < \frac{\lambda_m}{\lambda_0}$. Moreover, it is readily seen that $(p - \tilde{p})(v - \tilde{v}) < 0$ and **(P3)** imply:

$$\left(\text{sat}_{\bar{u}_p}(k_p^g \zeta \mu p) - \text{sat}_{\bar{u}_p}(k_p^g \zeta \mu \tilde{p}) \right) (v - \tilde{v}) \leq 0 \quad (11)$$

Thus,

$$0 \leq v \text{sat}_{\bar{u}_p}(k_p^g \zeta \mu p) \leq \text{sat}_{\bar{u}_p}(k_p^g \zeta \mu \tilde{p}) (v - \tilde{v}) + \text{sat}_{\bar{u}_p}(k_p^g \zeta \mu p) \tilde{v} \quad (12)$$

Rewriting (7) as follows

$$0 \leq pv < \tilde{p}(v - \tilde{v}) + p\tilde{v} \quad (13)$$

and combining (10),(12),(13) one successively gets (using **(P1)** and **(P2)**):

$$\psi \leq k_p^g \frac{\lambda_m}{\lambda_0} |\tilde{x}| |v| + k_p^g \frac{\lambda_m}{\lambda_0} |\tilde{p}(v - \tilde{v})| + \bar{u}_p |\tilde{v}| \quad (14)$$

$$\leq k_p^g \frac{\lambda_m}{\lambda_0} |\tilde{x}| |v| + k_p^g \frac{\lambda_m}{\lambda_0} |\tilde{p}v| + k_p^g \frac{\lambda_m}{\lambda_0} |\tilde{p}\tilde{v}| + \bar{u}_p |\tilde{v}| \quad (15)$$

$$\leq 2k_p^g \frac{\lambda_m}{\lambda_0} |\tilde{x}| |v| + k_p^g \frac{\lambda_m}{2\lambda_0} |\tilde{x}|^2 + \bar{u}_p |\tilde{x}| \quad (16)$$

which means that (6) is satisfied in case (7).

2. Suppose

$$(p - \tilde{p})(v - \tilde{v}) < 0 \quad \& \quad pv < 0 \quad (17)$$

In this case, $\zeta = \frac{1}{\lambda_m}$ and (5) develops as follows:

$$\psi \leq v \left(\text{sat}_{\bar{u}_p} \left(k_p^g \frac{\mu}{\lambda_m} p \right) - \text{sat}_{\bar{u}_p} \left(k_p^g \frac{\mu}{\lambda_m} (p - \tilde{p}) \right) \right) \quad (18)$$

$$\leq k_p^g \frac{\mu}{\lambda_m} |v| |\tilde{p}| \leq k_p^g |\tilde{x}| |v| \leq k_p^g \frac{\lambda_m}{\lambda_0} |\tilde{x}| |v| \quad (19)$$

Where the second inequality was obtained using property **(P1)**. This means that (6) is satisfied in case (17).

3. Suppose

$$(p - \tilde{p})(v - \tilde{v}) > 0 \quad \& \quad pv \leq 0 \quad (20)$$

In this case, $\zeta_{aw} = \frac{1}{\lambda_0}$ and (5) develops as follows

$$\psi = v \left[\text{sat}_{\bar{u}_p}(k_p^g \zeta \mu p) - \text{sat}_{\bar{u}_p} \left(k_p^g \frac{\mu}{\lambda_0} (p - \tilde{p}) \right) \right] \quad (21)$$

$$\leq 0 - \text{vsat}_{\bar{u}_p} \left(k_p^g \frac{\mu}{\lambda_0} (p - \tilde{p}) \right) \quad (22)$$

$$\leq -(v - \tilde{v}) \text{sat}_{\bar{u}_p} \left(k_p^g \frac{\mu}{\lambda_0} (p - \tilde{p}) \right) - \tilde{v} \text{sat}_{\bar{u}_p} \left(k_p^g \frac{\mu}{\lambda_0} (p - \tilde{p}) \right) \quad (23)$$

$$\leq 0 + \bar{u}_p |\tilde{v}| \leq \bar{u}_p |\tilde{x}| \quad (24)$$

where the first inequality is obtained using **(P4)**

where the penultimate inequality is obtained using **(P2)**

From the last inequality, (6) is thus satisfied in case (20).

4. Suppose

$$(p - \tilde{p})(v - \tilde{v}) > 0 \quad \& \quad pv > 0 \quad (25)$$

In this case, $\zeta = \zeta_{aw} = \frac{1}{\lambda_0}$ and (5) develops as follows

$$\psi = v \left(\text{sat}_{\bar{u}_p} \left(k_p^g \frac{\mu}{\lambda_0} p \right) - \text{sat}_{\bar{u}_p} \left(k_p^g \frac{\mu}{\lambda_0} (p - \tilde{p}) \right) \right) \quad (26)$$

$$\leq k_p^g \mu \frac{|v\tilde{p}|}{\lambda_0} \leq k_p^g \frac{\lambda_m}{\lambda_0} |\tilde{x}| |v| \quad (27)$$

which means that (6) is satisfied in case (25).

5. Suppose

$$(p - \tilde{p})(v - \tilde{v}) = 0 \quad (28)$$

In this case, (5) develops as follows:

$$\psi = v(\text{sat}_{\bar{u}_p}(k_p^g \zeta \mu p) - \text{sat}_{\bar{u}_p}(k_p^g \zeta_{aw} \mu (p - \tilde{p}))) \quad (29)$$

$$= \text{vsat}_{\bar{u}_p}(k_p^g \zeta \mu p) - \tilde{v} \text{sat}_{\bar{u}_p}(k_p^g \zeta_{aw} \mu (p - \tilde{p})) \quad (30)$$

$$\leq \text{vsat}_{\bar{u}_p}(k_p^g \zeta \mu p) + \bar{u}_p |\tilde{v}| \quad (31)$$

Moreover, it is readily seen that $(p - \tilde{p})(v - \tilde{v}) = 0$ implies that $p = \tilde{p}$ or $v = \tilde{v}$, which implies:

$$(\text{sat}_{\bar{u}_p}(k_p^g \zeta \mu p) - \text{sat}_{\bar{u}_p}(k_p^g \zeta \mu \tilde{p}))(v - \tilde{v}) = 0 \quad (32)$$

Combining (31) and (32), one successively gets:

$$\psi \leq \tilde{v} \text{sat}_{\bar{u}_p}(k_p^g \zeta \mu p) + \text{sat}_{\bar{u}_p}(k_p^g \zeta \mu \tilde{p})(v - \tilde{v}) + \bar{u}_p |\tilde{v}| \quad (33)$$

$$\leq \bar{u}_p |\tilde{x}| + \text{sat}_{\bar{u}_p}(k_p^g \zeta \mu \tilde{p})v + |\text{sat}_{\bar{u}_p}(k_p^g \zeta \mu \tilde{p})\tilde{v}| + \bar{u}_p |\tilde{v}| \quad (34)$$

$$\leq k_p^g \frac{\lambda_m}{\lambda_0} |\tilde{x}| |v| + 3\bar{u}_p |\tilde{x}| \quad (35)$$

where the last inequality is obtained using **(P1)** and **(P2)**. which means that (6) is satisfied in case (28). The proof is now completed.

References

- [1] L. Burlion, L. Zaccarian, H. de Plinval and S. Tarbouriech "*Discontinuous model recovery anti-windup solution for image based visual servoing*", submitted to Automatica, 2017.