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Maria Virginia Catalisano, Luca Chiantini, Anthony Vito Geramita, Alessandro Oneto

WARING-LIKE DECOMPOSITIONS OF POLYNOMIALS, 1

MARIA V. CATALISANO, LUCA CHIANTINI, ANTHONY V. GERAMITA*, AND ALESSANDRO ONETO

In memory of Tony Geramita (1942–2016)

Abstract. Let f be a homogeneous form of degree d in n variables. A Waring decomposition of f is a way to express f as a sum of d^{th} powers of linear forms. In this paper we consider the decompositions of a form as a sum of expressions, each of which is a fixed monomial evaluated at linear forms.

1. Introduction

Let $f \in k[x_1, \dots, x_n] = R = \oplus_{i \geq 0} R_i$, ($k = \bar{k}$ and $n \geq 2$). The necessities of a given problem involving f often make it useful to have different ways to *decompose* f. E.g. in many computations it is useful to express a polynomial as a sum of monomials ordered in a specific way. Different applications call for different kinds of decompositions. The following papers give some interesting uses of some non-standard decompositions (see, e.g. [22], [19], [18], [11]).

One particular kind of decomposition that has received a great deal of attention is the *Waring decomposition.* This decomposition asks us to write $f \in R_d$ in an efficient way as

$$f = \sum_{i=1}^{s} L_i^d$$

where each L_i is a linear form. There is an extensive literature on this decomposition with many interesting applications (see [18],[4],[23],[24], [10]).

In this paper we want to propose an extension of the notion of Waring decomposition. To explain the idea we introduce the following notation.

Fix an integer d. Let $\mathcal{P}_r = \mathbb{Z}[Z_1, \dots, Z_r]$ and let $\mathcal{M}_{r,d}$ be the subset of all monomials such that

$$M = Z_1^{d_1} \cdots Z_r^{d_r}$$

and

i) $d_i > 0$, for all i,
ii) $d_1 + \cdots + d_r = d$.

Of course, *i)* and *ii)* imply $r \leq d$.

Definition 1.1. Let M be as above and let $f \in R_d$.

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During the submission of this paper, Tony Geramita passed away. In memory of a friend, besides a colleague and a mentor, the three remaining authors dedicate this paper to him.

i) An *M-decomposition of f having length s* is an expression of the form

$$f = \sum_{j=1}^{s} L_{1,j}^{d_1} \cdot L_{2,j}^{d_2} \cdots L_{r,j}^{d_r},$$

where the $L_{i,j}$ are linear forms.

ii) The *M-rank* of f is the least integer s such that f has an M-decomposition of length s.

Remark 1.2. *i*) If $M = Z_1^d$ then the M-rank of f is known as the *Waring rank* of f.

ii) Every $f \in R_d$ has an $M = Z_1^{d_1} \cdots Z_r^{d_r}$-decomposition of finite length for any choice of M. This is immediate from the fact that every $f \in R_d$ has finite Waring rank and

$$L^d = L^{d_1} \cdots L^{d_r}.$$

iii) Two other special cases have received a great deal of attention recently: when $M = Z_1 \cdots Z_r$ ($r \geq 2$) then the forms in R_r which have M-rank equal to 1 are called *split forms* or *completely decomposable forms*. M-decompositions for these M are considered in [1], [2], [12], [27] and [30].

When $M = Z_1^{d-1} Z_2$, M-decompositions were first considered in [13] and then in [6] and [5]. They arose naturally from a consideration of the secant varieties of the tangential varieties to the Veronese varieties. The recent work [3] is a major contribution to this decomposition problem.

iv) For the case of binary forms (i.e. $n = 2$) this problem has its roots in the very foundations of modern algebra. Since the M-rank of a binary form is invariant under the usual $SL_2(k)$ action on $k[x_1, x_2]$, we see in the works of Cayley, Salmon, Sylvester ([8], [26], [28]) the search for the invariants which characterize forms of M-rank 1 for special choices of M.

A modern treatment of these classical investigations (as well as advances on them) can be found in the lovely papers of Chipalkatti (see [14], [15], [16], [17], [9]). The bibliographies in these papers give a quick entry into the classical literature on the subject.

v) The M-rank of a form obviously depends on M. E.g. recall that the Waring rank of xyz in $k[x, y, z]$ is 4 (see e.g. [10]) while if $M = Z_1^2 Z_2$, the M-rank of xyz is 2. To see this note that the M-rank is bigger than 1, but

$$4xyz = x((y + z)^2 - (y - z)^2).$$

There is a geometric way of considering the problem of finding the M-rank of a polynomial $f \in R_d$, $M \in \mathcal{M}_{r,d}$, $M = Z_1^{d_1} \cdots Z_r^{d_r}$.

Let $\mathbb{P}(R_1)$ be the projective space based on the k-vector space R_1. We define the morphism

$$\varphi_M : (\mathbb{P}(R_1))^r \longrightarrow \mathbb{P}(R_d) \simeq \mathbb{P}^{\binom{d+n-1}{n-1}-1},$$

where $(\mathbb{P}(R_1))^r$ denotes the cartesian product of r copies of $\mathbb{P}(R_1)$, by

$$\varphi_M([L_1], \ldots, [L_r]) = [L_1^{d_1} L_2^{d_2} \cdots L_r^{d_r}],$$

and denote the image of φ_M by $\mathbb{X}_M$.

Remark 1.3. *i*) Notice that when we have $M = Z_1^d$, then $\mathbb{X}_M$ is precisely the d^{th} Veronese embedding of $\mathbb{P}(R_1) = \mathbb{P}^{n-1}$ in $\mathbb{P}(R_d)$.

ii) In general, the variety $\mathbb{X}_M$ is a projection of an appropriate Segre-Veronese variety. However, we will not use that fact in this paper.

iii) The equations which define the Veronese variety are well-known (see e.g. [25]). It would be interesting to find equations for the variety $\mathbb{X}_M$ when $M \neq Z_1^d$. (See, however, [7] for the variety of split forms).

If $\mathbb{X} \subseteq \mathbb{P}^t$ is any projective variety, then

$$\sigma_s(\mathbb{X}) := \overline{\{P \in \mathbb{P}^t \mid P \in < P_1, \dots, P_s >, P_i \in \mathbb{X}, \}}$$

is the s^{th}-*secant variety* of $\mathbb{X}$.

$$\sigma_1(\mathbb{X}) = \mathbb{X},$$

$$\sigma_2(\mathbb{X}) = \text{ secant line variety to } \mathbb{X}, \quad \text{etc.}$$

Remark 1.4. When $\mathbb{X} = \mathbb{X}_M$ and $M = Z_1^d$, so $\mathbb{X}$ is the d^{th} Veronese embedding of $\mathbb{P}^n$ then, if $F \in R_d$ has Waring rank s we have $[F] \in \sigma_s(\mathbb{X})$. In particular, if s is the least integer for which $\sigma_s(\mathbb{X}) = \mathbb{P}(R_d)$ then the generic element $[F] \in \mathbb{P}(R_d)$ has Waring rank s.

This is the fundamental connection between the algebraic problem of finding the Waring rank of a generic form and the geometric problem of finding the dimensions of secant varieties to Veronese varieties.

One can ask about the dimensions of the secant varieties of any projective variety. More precisely: given $\mathbb{X} \subset \mathbb{P}^t$ what is $\dim \sigma_s(\mathbb{X})$ for $s \geq 2$?

There is a reasonable guess which gives an upper bound for $\dim \sigma_s(\mathbb{X})$. It is obtained by counting parameters and observing that $\sigma_s(\mathbb{X}) \subseteq \mathbb{P}^t$, namely

$$\dim \sigma_s(\mathbb{X}) \leq \min\{s \dim \mathbb{X} + s - 1, t\}. \tag{1.1}$$

If we have equality in (1.1) for some s, then we say that $\sigma_s(\mathbb{X})$ has the *expected dimension*, while if (1.1) is a strict inequality for some s, we say that $\sigma_s(\mathbb{X})$ is *defective* and the difference

$$\min\{s \dim \mathbb{X} + s - 1, t\} - \dim \sigma_s(\mathbb{X}),$$

is called the s-*defect* of $\mathbb{X}$.

If $\mathbb{X} \subseteq \mathbb{P}^t$ is non-degenerate then $\sigma_s(\mathbb{X}) = \mathbb{P}^t$ for some s and so the s-defect is eventually zero, for all $s \gg 0$.

The particular problem we consider in this paper is that of finding $\dim \sigma_s(\mathbb{X}_M)$ for $R = k[x_1, \dots .x_n]$ and any $M \in \mathcal{M}_{r,d}$.

The paper is organized in the following way. In Section 2 we recall Terracini's Lemma, which is our main tool in finding $\dim \sigma_s(\mathbb{X}_M)$. Terracini's Lemma needs a description of the tangent space at a general point of $\mathbb{X}_M$. This tangent space corresponds to a vector space which is the graded piece of an ideal I in R. We use information about this ideal to find the dimensions we need.

In the third section we find the dimensions of all the secant varieties of $\mathbb{X}_M$ for any $M \in \mathcal{M}_{r,d}$, for any r and any d, in case $n = 2$, i.e. for binary forms. We find that there are no defective secant varieties in this case.

In the fourth section we find the dimensions of the secant line varieties of $\mathbb{X}_M$ for any n and for any $M \in \mathcal{M}_{r,d}$ for any r and any d. In this family of cases we find exactly one defective secant line variety.

In the final section we use results of the previous sections and specialization in order to prove the non-defectiveness of new secant varieties.

2. Preliminaries

Since the cases $r = 1$ and $d = 2$ were already treated and solved (see Remark 1.2), in this paper we assume $r \geq 2$ and $d \geq 3$.

We begin by recalling the Lemma of Terracini [29].

Lemma 2.1. *Let $\mathbb{X} \subseteq \mathbb{P}^t$ be a projective variety and let $P \in \sigma_s(\mathbb{X})$ be a general point. If*

$$P \in < P_1, \dots, P_s >$$

where $P_1, \dots, P_s$ are general points of $\mathbb{X}$, then the tangent space to $\sigma_s(\mathbb{X})$ at P is the linear span of tangent spaces to $\mathbb{X}$ at $P_1, \dots, P_s$, i.e.,

$$T_P(\sigma_s(\mathbb{X})) =< T_{P_1}(\mathbb{X}), \dots, T_{P_s}(\mathbb{X}) > .$$

To apply Terracini's Lemma to our situation we first need to calculate $T_P(\mathbb{X}_M)$ for a general P in $\mathbb{X}_M$.

Proposition 2.2. *Let $R = k[x_1, \dots, x_n]$ and let $M \in \mathcal{M}_{r,d}$,*

$$M = Z_1^{d_1} \cdots Z_r^{d_r}.$$

Let L_1,...,L_r be general linear forms in R_1 so that $P = [L_1^{d_1} \cdots L_r^{d_r}]$ is a general point of $\mathbb{X}_M = \varphi_M((\mathbb{P}^{n-1})^r) \subseteq \mathbb{P}^{\binom{d+n-1}{n-1}-1}$.

If

$$F = L_1^{d_1} \cdots L_r^{d_r} \text{ and } I_P = (F/L_1, \dots, F/L_r) = \oplus_{j \geq 0}(I_P)_j$$

then

$$T_P(\mathbb{X}_M) = \mathbb{P}((I_P)_d).$$

Proof. Since $P = \varphi_M([L_1], \dots, [L_r])$, the image of a line through the point $([L_1], \dots, [L_r])$ in the direction $([\widetilde{L}_1], \dots, [\widetilde{L}_r])$ is the curve on the variety $\mathbb{X}_M$ whose points are parameterized by

$$[(L_1 + \lambda\widetilde{L}_1)^{d_1} \cdots (L_r + \lambda\widetilde{L}_r)^{d_r}].$$

The tangent vector to this curve at P is given by the coefficient of λ in this last expression, that is,

$$(F/L_1)\,\widetilde{L}_1 + \cdots + (F/L_r)\,\widetilde{L}_r.$$

These, for varying choices of the $\widetilde{L}_i$, give that the tangent vectors at P are precisely the degree d piece of the ideal generated by the F/L_i.

□

Remark 2.3. *i*) Note that if $M = Z_1^{d_1} \cdots Z_r^{d_r}$ and we are considering forms in $k[x_1, \dots, x_n]$, then $\mathbb{X}_M = \varphi_M((\mathbb{P}^{n-1})^r)$, Since the generic fibre of φ_M is finite, the dimension of $T_P(\mathbb{X}_M)$ is $r(n-1)$.

ii) It is easy to see that if F and the L_i are as above, then

$$I_P = (F/L_1, \dots, F/L_r)$$

$$= (L_1^{d_1-1} \cdots L_r^{d_r-1}) \cdot (L_2L_3 \cdots L_r, L_1L_3 \cdots L_r, \dots, L_1L_2 \cdots L_{r-1}) . \tag{2.1}$$

Corollary 2.4 (for binary forms)**.** *Let $R = k[x_1, x_2]$, $M \in \mathcal{M}_{r,d}$ $(r \geq 2)$*

$$M = Z_1^{d_1} \cdots Z_r^{d_r}.$$

If $L_1, L_2, \dots, L_r$ are general linear forms in R_1, $P = [L_1^{d_1} \dots L_r^{d_r}] \in \mathbb{X}_M$ and we set I' to be the principal ideal

$$I' := (L_1^{d_1-1} \cdots L_r^{d_r-1})$$

then we have

$$T_P(\mathbb{X}_M) = \mathbb{P}(I'_d).$$

Proof. In view of equation (2.1) above we first consider the ideal

$$J := (L_2L_3 \cdots L_r, L_1L_3 \cdots L_r, \dots, L_1L_2 \cdots L_{r-1}).$$

Claim 2.5. $J = (x_1, x_2)^{r-1}$

Proof. (of the Claim) By induction on r. Obvious for $r = 2$ so let $r > 2$. Since

$$J = (L_r \cdot (L_2L_3 \cdots L_{r-1}, L_1L_3 \cdots L_{r-1}, \dots, L_1L_2 \cdots L_{r-2}), L_1L_2 \cdots L_{r-1}),$$

we have, by the induction hypothesis, that

$$J = (L_r \cdot (x_1, x_2)^{r-2}, L_1L_2 \cdots L_{r-1}).$$

Since $R = k[x_1, x_2]$, $L_1L_2 \cdots L_{r-1}$ is a general form in R_{r-1}, hence not in the space $L_r(x_1, x_2)^{r-2}$. This last implies that $\dim J_{r-1} = r$. Since the ideal J begins in degree $r-1$ we are done with the proof of the claim. □

Now, using Claim 2.5, equation (2.1) and Proposition 2.2 we have that

$$I_P = (L_1^{d_1-1} \cdots L_r^{d_r-1})(x_1, x_2)^{r-1} \subseteq (L_1^{d_1-1} \cdots L_r^{d_r-1})$$

and since $(x_1, x_2)^{r-1} = \oplus_{j \geq r-1} R_j$, we have that

$$(I_P)_d = (L_1^{d_1-1} \cdots L_r^{d_r-1})_d$$

□

Corollary 2.6. *Let $M = Z_1^{d_1} \cdots Z_r^{d_r} \in \mathcal{M}_{r,d}$. Let $P = \varphi_M([L_1], \cdots, [L_r])$ be a general point of $\mathbb{X}_M$, where the L_i are general linear forms in $k[x_1, \dots, x_n]$ and let I_P be as in Proposition 2.2. Then,*

$$I_P = (L_1^{d_1-1} \cdots L_r^{d_r-1}) \cap (\cap_{1 \leq i < j \leq r} (L_i, L_j)^{d_i+d_j-1}).$$

Proof. By Remark 2.3 we need to prove that the two ideals

$$I_P = (L_1^{d_1-1} \cdots L_r^{d_r-1}) \cdot (L_2L_3 \cdots L_r, \dots, L_1 \cdots L_{r-1})$$

and

$$J = (L_1^{d_1-1} \cdots L_r^{d_r-1}) \cap (\cap_{1 \leq i < j \leq r} (L_i, L_j)^{d_i+d_j-1})$$

are equal.

Since each generator of I_P is in J, we have $I_P \subseteq J$.

Now let $h = L_1^{d_1-1} \cdots L_r^{d_r-1}$, and suppose that $f = hg \in J$. Since the L_i are general linear forms we have

$$h \in (L_i, L_j)^{d_i+d_j-2} \quad \text{for every } 1 \leq i < j \leq r,$$

but

$$h \notin (L_i, L_j)^{d_i+d_j-1} \quad \text{for every } 1 \le i < j \le r,$$

so $g \in rad((L_i, L_j)^{d_i+d_j-1}) = (L_i, L_j)$ for all $1 \le i < j \le r$.

Since $\bigcap_{1\le i<j\le r}(L_i, L_j) = (L_2 \cdots L_r, \dots, L_1 \cdots L_{r-1})$ we are done.

□

3. The Binary Case

For binary forms there is a simple theorem covering all cases.

Theorem 3.1. *Let $R = k[x, y] = \oplus_{j\ge0} R_j$ and let $M = Z_1^{d_1} \cdots Z_r^{d_r} \in \mathcal{M}_{r,d}$ for any r and any d with $r \le d$. Then $\sigma_s(\mathbb{X}_M)$ has the expected dimension for every s, i.e.*

$$\dim \sigma_s(\mathbb{X}_M) = \min\{s \dim \mathbb{X}_M + (s-1), d\} = \min\{sr + s - 1, d\}$$

for every s and every M.

Proof. Since every form in R of degree d splits as a product of linear forms and the general form of degree d has no repeated factors, we conlude that for $M = Z_1 \cdots Z_d$, $\mathbb{X}_M = \mathbb{P}(R_d) = \mathbb{P}^d$. This takes care of the case $r = d$. Now, for the rest of the proof, assume that $r < d$.

By Corollary 2.4 we know that if $P = [L_1^{d_1} \cdots L_r^{d_r}]$ is a general point of $\mathbb{X}_M$ (where $L_1, \dots, L_r$ are general in R_1) then

$$T_P(\mathbb{X}_M) = \mathbb{P}((I'_P)_d) \quad \text{where} \quad I'_P = (L_1^{d_1-1} \cdots L_r^{d_r-1}).$$

So, by Terracini's Lemma, if $P_1, \dots, P_s$ are a set of s general points of $\mathbb{X}_M$ then

$$\dim(\sigma_s(\mathbb{X}_M)) = \dim_k(I'_{P_1} + \cdots + I'_{P_s})_d - 1$$

where if $P_j = [L_{j1}^{d_1} \cdots L_{jr}^{d_r}]$ then $I'_{P_j} = (L_{j1}^{d_1-1} \cdots L_{jr}^{d_r-1})$.

However, [21, Cor.2.3] states that for the special points $Q_1 = [H_1^d], \dots, Q_s = [H_s^d] \in \mathbb{X}_M$ (where the H_j are general in R_1) and for the ideal $J = (H_1^{d-r}, \dots, H_s^{d-r})$ we have

$$\dim_k(J_d) = \min\{d + 1, s(r + 1)\} = \min\{d, sr + (s-1)\} + 1.$$

Since we know that $\dim \sigma_s(\mathbb{X}_M) \le \min\{d, sr + (s-1)\}$, it follows (by semicontinuity) that

$$\dim(I'_{P_1} + \cdots + I'_{P_s})_d = \min\{d, sr + (s-1)\} + 1.$$

and so $\sigma_s(\mathbb{X}_M)$ always has the expected dimension. □

4. The Secant Line Varieties to $\mathbb{X}_M$

In this section we will find the dimensions of the secant line varieties of $\mathbb{X}_M$ for every $M \in \mathcal{M}_{r,d}$ and for every polynomial ring $R = k[x_1, \dots, x_n]$.

When $r = 1$ this is one part of the complete solution to Waring's Problem solved by Alexander and Hirschowitz in [4], so we will assume that $r \ge 2$. In the previous section we solved this problem for $n = 2$, so we may now assume that $n \ge 3$. Recall that we are also assuming $d \ge 3$.

The main result of this paper is the following theorem.

Theorem 4.1. *Let $R = k[x_1, \dots, x_n]$, let $M \in \mathcal{M}_{r,d}$, $n \geq 3$, $r \geq 2$, $d \geq 3$,*
$$M = Z_1^{d_1} \cdots Z_r^{d_r}.$$
Then $\sigma_2(\mathbb{X}_M)$ is not defective, except for $M = Z_1^2 Z_2$ and $n = 3$. For this last case $\mathbb{X}_M$ has 2-defect equal to 1.

Proof. We always have $d \geq r$. The case $d = r$ is covered in [27, Theorem 4.4] so we may as well assume that $d > r$ also.

By Terracini's Lemma we need to find the vector space dimension of
$$(I_{P_1} + I_{P_2})_d,$$
where $P_1 = [L_1^{d_1} \cdots L_r^{d_r}]$ and $P_2 = [N_1^{d_1} \cdots N_r^{d_r}]$ are points of $\mathbb{X}_M$ and the
$$\{L_i, 1 \leq i \leq r\}, \{N_i, 1 \leq i \leq r\}$$
are general sets of linear forms in R.

By Corollary 2.6 we obtain
$$I_{P_1} = (L_1^{d_1-1} \cdots L_r^{d_r-1}) \cap (\cap_{1 \leq i < j \leq r} (L_i, L_j)^{d_i+d_j-1}),$$
$$I_{P_2} = (N_1^{d_1-1} \cdots N_r^{d_r-1}) \cap (\cap_{1 \leq i < j \leq r} (N_i, N_j)^{d_i+d_j-1}).$$
By the exact sequence
$$0 \longrightarrow (I_{P_1} \cap I_{P_2})_d \longrightarrow (I_{P_1} \oplus I_{P_2})_d \longrightarrow (I_{P_1} + I_{P_2})_d \longrightarrow 0, \tag{4.1}$$
and the fact that we know that $\dim(I_{P_i})_d = r(n-1) + 1$, $(i = 1, 2)$, (see Remark 2.3), it is enough to find $\dim(I_{P_1} \cap I_{P_2})_d$.

Recall that the expected dimension of $\sigma_2(\mathbb{X}_M)$ is
$$\text{exp.dim } \sigma_2(\mathbb{X}_M) = \min\{2r(n-1) + 1, \binom{d+n-1}{n-1} - 1\}.$$

Note that, if $\dim(I_{P_1} \cap I_{P_2})_d = 0$, then by (4.1) the dimension of $\sigma_2(\mathbb{X}_M)$ is as expected.

Let V be the subscheme of $\mathbb{P}^{n-1}$ defined by $I_{P_1} \cap I_{P_2}$ and let f be a form of degree d in $I_{P_1} \cap I_{P_2}$. Clearly
$$f = L_1^{d_1-1} \cdots L_r^{d_r-1} \cdot N_1^{d_1-1} \cdots N_r^{d_r-1} \cdot g$$
where g is a form of degree $d - 2(d - r) = 2r - d$.

If $2r - d < 0$, of course there are no forms of this degree, hence $(I_{P_1} \cap I_{P_2})_d = 0$ and we are done. So assume $2r - d \geq 0$.

The form g vanishes on the residual scheme W of V with respect to the $2r$ multiple hyperplanes $\{L_i = 0\}$ and $\{N_i = 0\}$ $(1 \leq i \leq r)$. It is easy to see that W is defined by the ideal
$$(\cap_{1 \leq i < j \leq r} (L_i, L_j)) \cap (\cap_{1 \leq i < j \leq r} (N_i, N_j)).$$
Now, g cannot be divisible by all the L_i and N_j because otherwise it would have degree at least $2r$.

Without loss of generality assume that g is not divisible by L_1, and let H be the hyperplane defined by L_1. The form g cuts out on H a hypersurface $\mathcal{S}$ of H having degree $2r-d$, and containing the $r-1$ hyperplanes of H cut out by $L_2, \dots, L_r$. Hence, in order for g to exist, $2r - d$ has to be at least $r - 1$, that is, $d \leq r + 1$. But we are assuming $d \geq r + 1$, so we get that $d = r + 1$.

It follows that $\mathcal{S}$ has degree $r - 1$, contains the $r - 1$ hyperplanes of H cut out by $L_2, \dots, L_r$, and contains the trace on H of the schemes defined by the ideals (N_i, N_j). Since the N_i are generic with respect to H and to the L_i's, the only possibility for g to exist is that the schemes $Y_{i,j}$ defined by the ideals (N_i, N_j) do not intersect H. Since $H \simeq \mathbb{P}^{n-2} \subseteq \mathbb{P}^{n-1}$ and $Y_{i,j} \simeq \mathbb{P}^{n-3} \subseteq \mathbb{P}^{n-1}$, then $H \cap Y_{i,j} = \emptyset$ only for $n \leq 3$.

Therefore, we are left with the following cases:

Case 1: $n = 3$, $r \geq 3$, $d = r + 1$;

Case 2: $n = 3$, $r = 2$, $d = 3$.

In Case 1, the form g has degree $2r - d = r - 1$ and it should vanish at the scheme W which is a union of $2\binom{r}{2}$ simple points in $\mathbb{P}^2$. The dimension of the space of homogeneous polynomials of degree $r - 1$ is $\binom{r+1}{2}$, which is smaller than $2\binom{r}{2}$, for $r \geq 3$. Then, such a g cannot exist.

In Case 2, $M = Z_1^2 Z_2$, the form g has degree $2r - d = 1$ and the scheme W is the union of 2 points of $\mathbb{P}^2$. Hence g can exist and describes the line through the two points. It follows that $\dim(I_{P_1} \cap I_{P_2})_3 = 1$. So from (2.1) we get

$$\dim(I_{P_1} + I_{P_2})_3 = 9,$$

that is, $\dim \sigma_2(\mathbb{X}_M) = 8$. But exp.dim$\sigma_2(\mathbb{X}_M) = 9$, and so $\mathbb{X}_M$ has 2-defect $= 1$ and we are done. □

Remark 4.2. The exceptional case noted above was observed in [13] in connection with the study of the secant varieties of the tangential varieties to Veronese varieties.

Example. We claim that the hypersurface in $\mathbb{P}^9$ containing all those cubic forms of $k[x, y, z]$ which can be written $L_1^2 L_2 + N_1^2 N_2$ (with the L_i, N_i linear forms) is precisely the hypersurface in $\mathbb{P}^9$ containing all singular cubics. It is well-known that the (closure) of the set of cubic plane curves with a double point is a hypersurface in $\mathbb{P}^9$. It will be enough to show that every nodal cubic can be written in the desired form (since cuspidal cubics can, after a change of variables, always be written in the form $y^3 + x^2 z$).

First recall (see [20]) that every nodal cubic can, after a change of variables, be written in the form

$$xyz - x^3 - y^3.$$

With a further change of variables given by

$$x = -X - Y; \quad y = X - Y; \quad z = -Z,$$

we get

$$X^2(6Y + Z) + Y^2(2Y - Z).$$

5. Inductive results

We hope that our results above could provide a starting point for further investigations on $\mathcal{M}$-decompositions, when the number s of summands increase. The main obstruction to extend plainly our arguments resides in the fact that when r is close to n and $s > 2$, then the shape of polynomials lying in the intersection of a tangent space T_P with the *span* of two of more tangent spaces to $\mathbb{X}_M$ is not easy to control.

In this final section, in which as above we assume $n \geq 3$, we show how by Theorem 3.1 and Theorem 4.1 we can improve our knowledge on the defectivity of $\mathcal{M}$-decompositions. We will need some preliminary algebraic remarks.

Proposition 5.1. *Let I, J be ideals in the polynomial ring $R = k[x_1, \dots, x_n]$, both generated by elements of degree $d-1$, for some $d \geq 3$. Assume that $I_d \cap J_d = (0)$. Then after adding one variable y to R, we still get $IR[y]_d \cap JR[y]_d = (0)$.*

Proof. By our assumptions on the generators of I, the elements of $IR[y]_d$ are of the form $Ay + B$, with $A \in I_{d-1}$ and $B \in I_d$. Similarly the elements of $JR[y]_d$ are of the form $Cy + D$, with $C \in J_{d-1}$ and $D \in J_d$. Since y is independent mod R, the equality $Ay + B = Cy + D$ yields $A = C$, $B = D$. Hence $A = B = 0$ by assumption. □

Next, fix $d \geq 3$ and go back to the ideals I_P defined in the previous sections, where P is a splitting form of fixed type:

$$M = Z_1^{d_1} \cdots Z_r^{d_r}, \qquad d = d_1 + \cdots + d_r.$$

Proposition 5.2. *Assume that for a general choice of splitting forms $P_1, \dots, P_s \in k[x_1, \dots, x_t]$ of type M in $t < n$ variables the ideal $I = I_{P_1} + \cdots + I_{P_s} \subset k[x_1, \dots, x_t]$ has the degree d piece of (maximal) dimension $s(r+1)$. Then for a general choice of splitting forms $Q_1, \dots, Q_s \in R$ of type M, the ideal $J = I_{Q_1} + \cdots + I_{Q_s} \subset R$ has a degree d piece of (maximal) dimension $sr(n-1) + s$.*

Proof. We make induction on s. Notice that it is enough to prove that the degree d piece of the extended ideal IR has dimension $sr(n-1) + s$, because then, by semicontinuity, the same will hold for a general choice of the Q_i's.

For the case $s = 1$, take $P_1 = [L_1^{d_1} \cdots L_r^{d_r}]$, where the L_i's are general linear forms in $k[x_1, \dots, x_t]$. The vector space $(IR)_d$ is spanned by the $(t-1)r+1$ independent forms in the t variables $x_1, \dots, x_t$ which are a basis for I_d, plus the $r(n-t)$ independent forms $x_i P_1 / L_j$, for all $i = t+1, \dots n$ and $j = 1, \dots r$. It follows that $\dim(IR)_d = (t-1)r + 1 + r(n-t) = r(n-1) + 1$.

For higher s, notice that our assumption implies that $I' = I_{P_1} + \cdots + I_{P_{s-1}}$ has a null intersection with I_{P_s} in degree d, inside $k[x_1, \dots, x_t]$. Thus also the extensions of I' and I_{P_s} to R have a null intersection (apply $n - t$ times the previous Proposition 5.1). Since, by the inductive hypothesis, $\dim(I'R)_d = (s-1)r(n-1) + (s-1)$, we have that

$$\dim(IR)_d = (s-1)r(n-1) + (s-1) + r(n-1) + 1 = sr(n-1) + s,$$

and the result follows.

□

Now we can apply the previous proposition to extend Theorem 3.1 to many variables.

Proposition 5.3. *Let $R = k[x_1, \dots, x_n]$ and let $M = Z_1^{d_1} \cdots Z_r^{d_r} \in \mathcal{M}_{r,d}$, $r \leq d$. Assume $s(r+1) \leq d + 1$. Then, for every M, $\sigma_s(\mathbb{X}_M)$ has the expected dimension*

$$\dim \sigma_s(\mathbb{X}_M) = s \dim \mathbb{X}_M + (s-1) = sr(n-1) + s - 1.$$

Proof. The result is true in two variables, by Theorem 3.1. In more than two variables, specialize the s points P_i's to forms in two variables: the result remains true by Proposition 5.2. □

The condition $s(r+1) \leq d + 1$ is fundamental in the previous argument, because we need that, after the specialization, each I_{P_i} has a null intersection with the span of the remaining I_{P_j} (in $k[x_1, x_2]$). We can improve the previous result with the trick of separating the variables.

Proposition 5.4. *For a fixed $j \in \{2, \dots, n-2\}$, let $R_1 = k[x_1, \dots, x_j]$ and $R_2 = k[x_{j+1}, \dots, x_n]$. Let $I \subset R_1$, $J \subset R_2$ be ideals generated by elements of degree $d-1$, for some $d \geq 3$. Then $(IR)_d \cap (JR)_d = (0)$.*

Proof. Notice that the elements of $(IR)_d$ have degree at least $d-1$ in $x_1, \dots, x_j$, while the elements of $(JR)_d$ have degree at most 1 in $x_1, \dots, x_j$. Since $d \geq 3$, the two things cannot match, and the claim follows. $\square$

Now, by dividing the points $P_1, \dots, P_s$ in groups and specializing each group to forms in two distinct variables, we get

Proposition 5.5. *Let $R = k[x_1, \dots, x_n]$ and let $M = Z_1^{d_1} \cdots Z_r^{d_r} \in \mathcal{M}_{r,d}$, $r \leq d$, and $d \geq 3$. Set $m = \left\lfloor \frac{n}{2} \right\rfloor$ and $s' = \left\lfloor \frac{d+1}{r+1} \right\rfloor$. Then, for every $s \leq s'm$ and for every M, $\sigma_s(\mathbb{X}_M)$ has the expected dimension*

$$\dim \sigma_s(\mathbb{X}_M) = s \dim \mathbb{X}_M + (s-1) = sr(n-1) + s - 1.$$

Proof. It is enough to prove the statement for $s = s'm$. For each $j = 1, \dots, m$ take s' forms $P_{1j}, \dots, P_{s'j} \in \mathbb{X}_M$ in the variables x_{2j-1}, x_{2j}. The ideal

$$I_j = (I_{P_{1j}} + \cdots + I_{P_{s'j}})$$

in $k[x_{2j-1}, x_{2j}]$ has the expected dimension $s'r + s'$ in degree d. By Proposition 5.2 with $t = 2$, the extension of I_j to R has the expected dimension $s'r(n-1) + s'$ in degree d.

Then the sum $I = I_1 + \cdots + I_m$ has the expected dimension $m(s'r(n-1) + s') = sr(n-1) + s$ in degree d, because by Proposition 5.4 every I_j has null intersection with the sum of the remaining I_k's, $k \neq j$. $\square$

We can use the idea of separating the variables also in connection with Theorem 4.1, by specializing pairs of points P_i's to forms in three distinct variables. The statement below covers some cases which do not fit with the numerical assumptions of Proposition 5.5.

Proposition 5.6. *Let $R = k[x_1, \dots, x_n]$, $r \geq 2$, $d \geq 3$. Let $M \in \mathcal{M}_{r,d}$ be different from $Z_1^2 Z_2$. Then $\sigma_s(\mathbb{X}_M)$ is not defective for $s \leq 2 \left\lfloor \frac{n}{3} \right\rfloor$.*

Proof. The case $s = 2$ is provided by Theorem 4.1. For higher s, it sufficies to prove the statement for $s = 2 \left\lfloor \frac{n}{3} \right\rfloor$. For all $i = 1, ..., \left\lfloor \frac{n}{3} \right\rfloor$ choose general points $P_{2i-1}, P_{2i} \in \mathbb{X}_M$ which involve only the variables $x_{3i-2}, x_{3i-1}, x_{3i}$. By Theorem 4.1 and its proof, we know that the ideals $I_{P_{2i+1}}, I_{P_{2i+2}}$ have null intersection in degree d, considered as ideals in $k[x_{3i+1}, x_{3i+2}, x_{3i+3}]$. Thus their sum $I_i = I_{P_{2i-1}} + I_{P_{2i}}$ has the expected dimension in degree d. The same remains true when we extend the ideals to R, by Proposition 5.2 with $t = 3$. Now it is enough to apply Proposition 5.4 to show that the sum of the I_i's has the expected dimension in degree d. $\square$

The results proved in Proposition 5.5 or Proposition 5.6 by separating the variables are clearly not optimal, in the sense that they cannot cover the full range of all possible M-ranks. In order to get a complete classification of defective varieties $\mathbb{X}_M$ for $s > 2$, new ideas should be introduced.

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(M.V. Catalisano) Dipartimento di Ingegneria Meccanica, Energetica, Gestionale e dei Trasporti, Università degli studi di Genova, Genoa, Italy.

E-mail address: `catalisano@diptem.unige.it`

(L. Chiantini) Dipartimento di Ingegneria dell'Informazione e Scienze Matematiche, Università di Siena, Italy.

E-mail address: `luca.chiantini@unisi.it`

(A.V. Geramita) Department of Mathematics and Statistics, Queen's University, Kingston, Ontario, Canada

E-mail address: `Anthony.Geramita@gmail.com, geramita@dima.unige.it`

(A. Oneto) Inria Sophia Antipolis Méditerranée, Sophia Antipolis, France.

E-mail address: `alessandro.oneto@inria.fr`