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Sparse Supernodal Solver exploiting Low-Rankness Property

September 6th, 2017 - Sparse Days

Grégoire Pichon^a, Eric Darve^b, Mathieu Faverge^a, Pierre Ramet^a, Jean Roman^a

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^bStanford University

Introduction

Current sparse direct solver for 3D problems

- $\Theta(n^2)$ time complexity
- $\Theta(n^{\frac{4}{3}})$ memory complexity
- BLAS 3 operations

Block Low-Rank solver

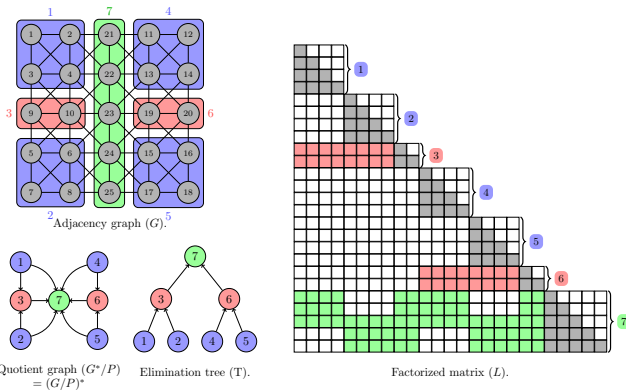
- Large blocks are compressed into a low-rank form
- Operations keep untouched the global behaviour of PASTIX
- *Minimal Memory* strategy allows to save memory
- *Just-In-Time* strategy allows to reduce time-to-solution

Objective: build an algebraic low-rank solver following the supernodal approach of PASTIX, with block-data structures

Symbolic Factorization

General approach

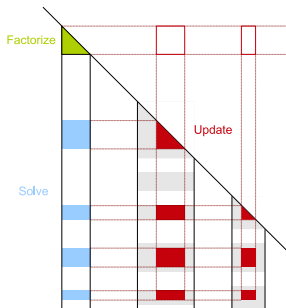
1. Build a partition with the nested dissection process
2. Compress information on data blocks
3. Compute the block elimination tree using the block quotient graph



Numerical Factorization

Algorithm to eliminate the column block k

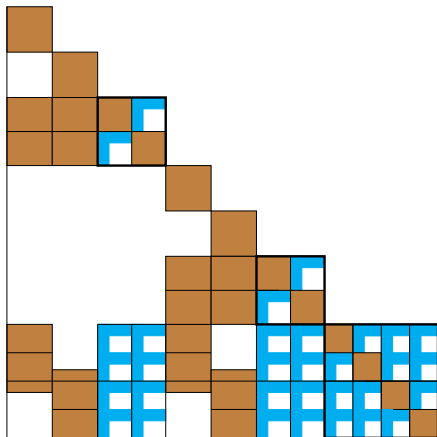
1. **Factorize** the diagonal block (POTRF/GETRF)
2. **Solve** off-diagonal blocks in the current column (TRSM)
3. **Update** the trailing matrix with the column's contribution (GEMM)



Parallelism

- Right-Looking
- Left-Looking

Block-Low-Rank Compression – Symbolic Factorization



Large off-diagonal are low-rank, in the form uv^t

Some Related Works

Block Low-Rank solvers

- BLR-MUMPS introduced an approach focused on reducing the time to solution which is close to our *Just-In-Time* strategy
- LSTC developed compression/recompression techniques in a dense context

Other approaches for sparse

- Strumpack HSS by Ghysels et al. utilizes randomized sampling to perform an efficient extend-add
- HODLR by Darve et al. uses pre-selection of rows and columns (BDLR)
- \mathcal{H} -LU by Hackbusch et al. does not fully exploit the symbolic factorization

Block-Low-Rank Algorithm

Approach

- Large supernodes are partitioned into a set of smaller supernodes
- Large off-diagonal blocks are represented as low-rank blocks

Operations

- Diagonal blocks are dense
- TRSM are performed on low-rank off-diagonal blocks
- GEMM are performed between low-rank off-diagonal blocks. It creates contributions to dense or low-rank blocks: this is the extend-add problem






Compression techniques

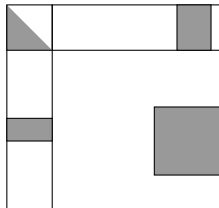
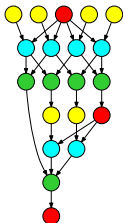
- SVD, RRQR for now
- Possible extension to any algebraic method: ACA, randomized techniques...

Strategy *Just-In-Time*

Compress L

1. Eliminate each column block
 - 1.1 Factorize the dense diagonal block
Compress off-diagonal blocks belonging to the supernode
 - 1.2 Apply a TRSM on LR blocks (cheaper)
 - 1.3 LR update on dense matrices (*LR2GE* extend-add)
2. Solve triangular systems with low-rank blocks

| | |
|---|----------------|
|  | Compression |
|  | GETRF (Facto) |
|  | TRSM (Solve) |
|  | LR2LR (Update) |
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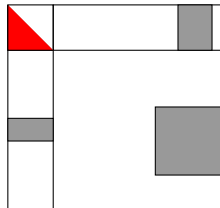
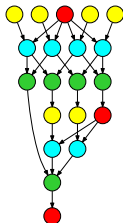


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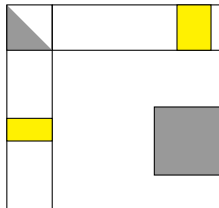
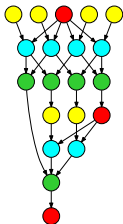


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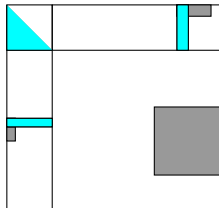
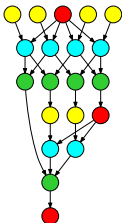


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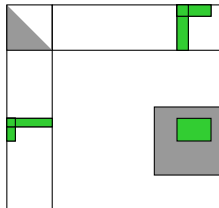
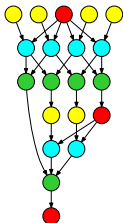


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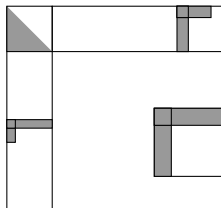
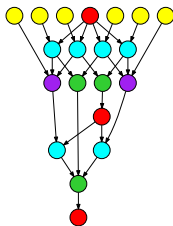


Strategy *Minimal Memory*

Compress A

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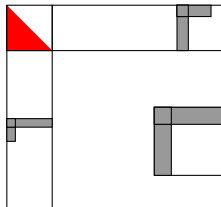
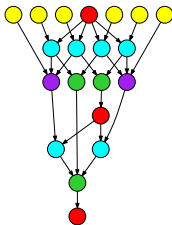


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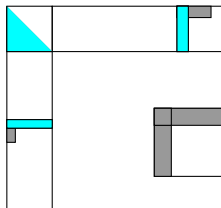
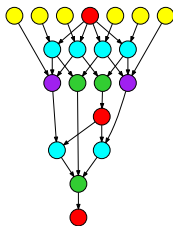


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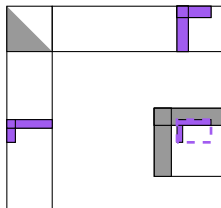
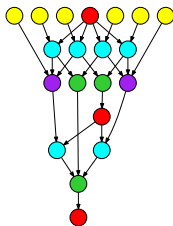


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Comparison of both strategies

Memory consumption

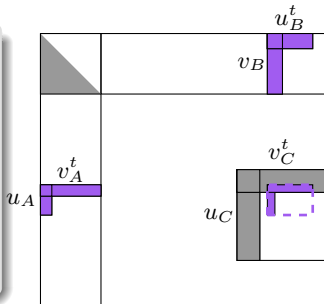
- *Minimal Memory* strategy really saves memory
- *Just-In-Time* strategy reduces the size of L' factors, but supernodes are allocated dense at the beginning: no gain in pure *right-looking*

Low-Rank extend-add

- *Minimal Memory* strategy requires expensive extend-add algorithms to update (recompress) low-rank structures with the *LR2LR* kernel
- *Just-In-Time* strategy continues to apply dense update at a smaller cost through the *LR2GE* kernel

Focus on the *LR2LR* kernel

- Update of C with contribution from blocs A and B
- The low-rank matrix $u_{AB}v_{AB}^t$ is added to $u_Cv_C^t$
- $u_{AB}v_{AB}^t = (u_A(v_A^t v_B))u_B^t$ or $u_{AB}v_{AB}^t = u_A((v_A^t v_B)u_B^t)$
- Eventually, recompression of $(v_A^t v_B)$



LR2LR kernel using SVD

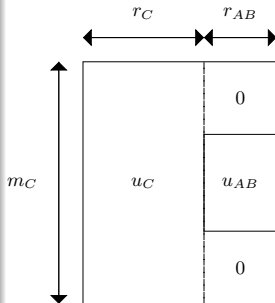
A low-rank structure $u_C v_C^t$ receives a low-rank contribution $u_{AB} v_{AB}^t$.

Algorithm

$$A = u_C v_C^t + u_{AB} v_{AB}^t = ([u_C, u_{AB}]) \times ([v_C, v_{AB}])^t$$

- QR: $[u_C, u_{AB}] = Q_1 R_1 \quad \Theta(m(r_C + r_{AB})^2)$
- QR: $[v_C, v_{AB}] = Q_2 R_2 \quad \Theta(n(r_C + r_{AB})^2)$
- SVD: $R_1 R_2^t = u \sigma v^T \quad \Theta((r_C + r_{AB})^3)$

$$A = (Q_1 u \sigma) \times (Q_2 v)^t$$



Optimizations using RRQR to reduce complexity to $\Theta(n(r_C + r_{AB})r_C^*)$

Experimental setup

Machine: 2 INTEL Xeon E5 – 2680v3 at 2.50 GHz

- 128 GB
- 24 threads

3D Matrices from The SuiteSparse Matrix Collection

- *Audi*: structural problem (943 695 dofs)
- *Atmosmodj*: atmospheric model (1 270 432 dofs)
- *Geo1438*: geomechanical model of earth (1 437 960 dofs)
- *Hook*: model of a steel hook (1 498 023 dofs)
- *Serena*: gas reservoir simulation (1 391 349 dofs)
- + Laplacian: Poisson problem (7-points stencil)

Parallelism is obtained following PASTIX static scheduling for multi-threaded architectures

Parameters

Entry parameters

- Tolerance τ : absolute parameter (normalized for each block)
- Compression method: SVD or RRQR
- Compression strategy: *Minimal Memory* or *Just-In-Time*
- Blocking sizes: between 128 and 256 in following experiments

Strategy *Minimal Memory*

- Blocks are compressed at the beginning
- Each contribution implies a recompression

Strategy *Just-In-Time*

- Blocks are compressed just before a supernode is eliminated
- Those blocks are never uncompressed

Costs distribution on the Atmosmodj matrix with $\tau = 10^{-8}$

| | Dense | <i>Just-In-Time</i> | | <i>Minimal Memory</i> | |
|-----------------------------|------------|---------------------|------------|-----------------------|-------------|
| | | RRQR | SVD | RRQR | SVD |
| Factorization time (s) | | | | | |
| Compression | - | 49.53 | 418.5 | 15.20 | 180.9 |
| Block factorization (GETRF) | 0.9635 | 1.000 | 1.003 | 1.074 | 1.104 |
| Panel solve (TRSM) | 15.80 | 6.970 | 6.526 | 11.16 | 6.946 |
| Update | | | | | |
| LR product | - | 64.10 | 91.15 | 193.1 | 94.36 |
| LR addition | - | - | - | 774.6 | 6523 |
| Dense update (GEMM) | 418.7 | 47.94 | 47.03 | - | - |
| <i>Total</i> | <i>436</i> | <i>169</i> | <i>564</i> | <i>995</i> | <i>6806</i> |
| Solve time (s) | 2.43 | 1.54 | 1.8 | 2.22 | 1.29 |
| Factors final size (GB) | 15.9 | 7.4 | 6.86 | 11.4 | 6.76 |
| Memory peak (GB) | 15.9 | 15.9 | 15.9 | 11.4 | 6.76 |

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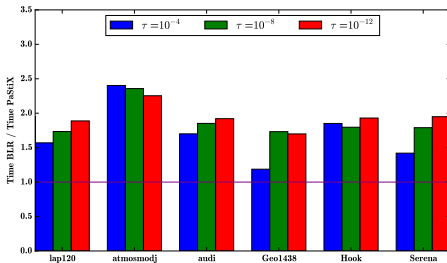
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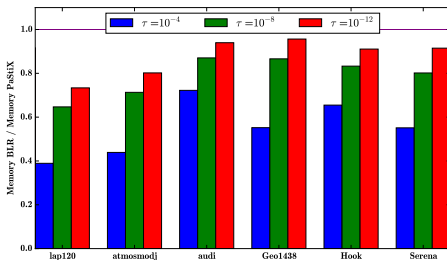
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Behaviour of RRQR/*Minimal Memory*

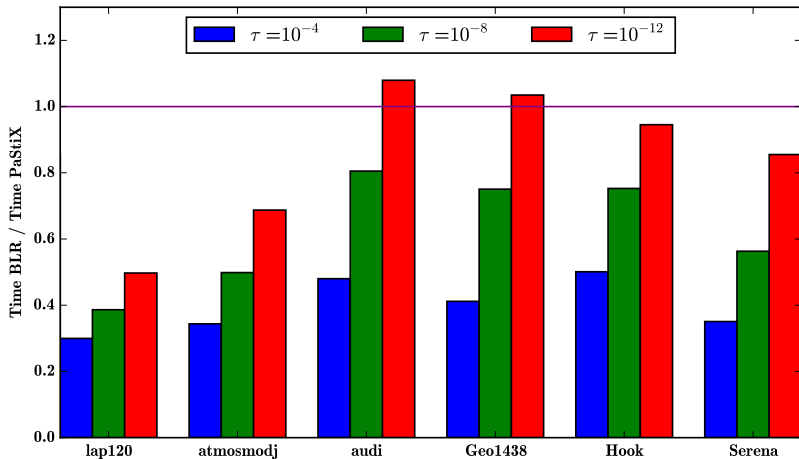


Performance

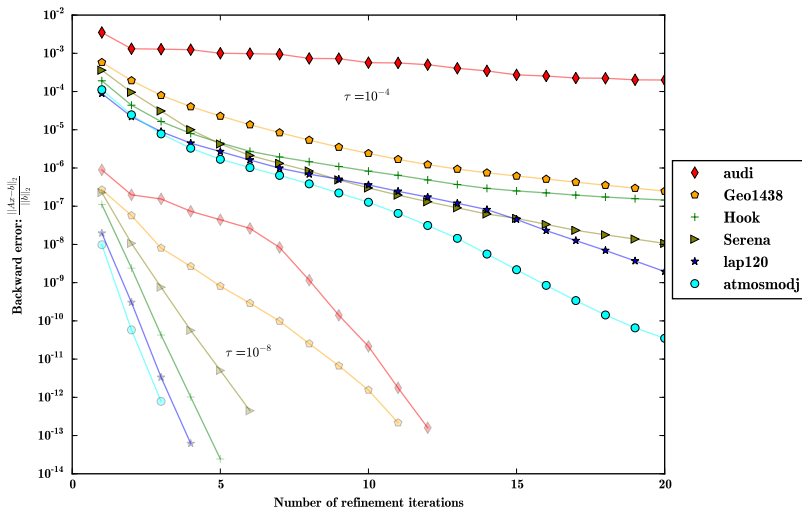


Memory footprint

Performance of RRQR/*Just-In-Time*

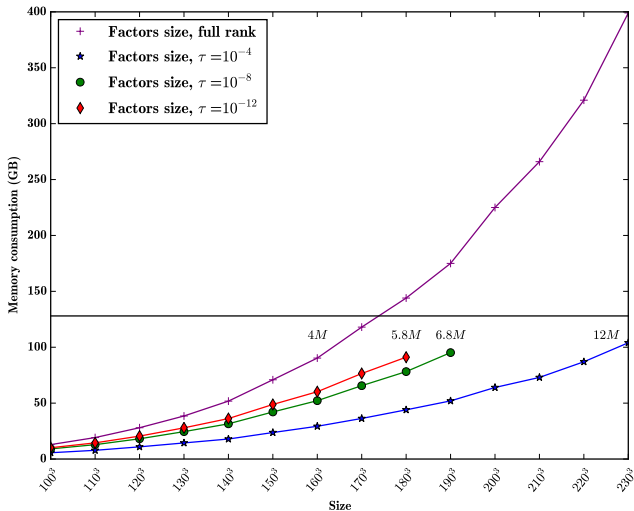


Convergence of RRQR/Minimal Memory

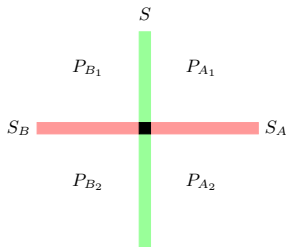


Scaling on Laplacians: Memory Consumption

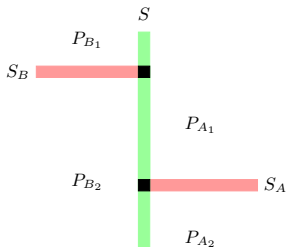
RRQR/Minimal Memory



Ongoing work with E. Esnard and M. Predari

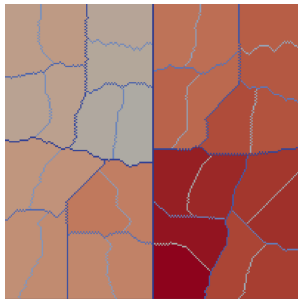


- Single interaction between S and $S_A \cup S_B$ in the original graph
- Two large interaction blocks $P_{A_1} \cup P_{B_1}$ and $P_{A_2} \cup P_{B_2}$ during the factorization

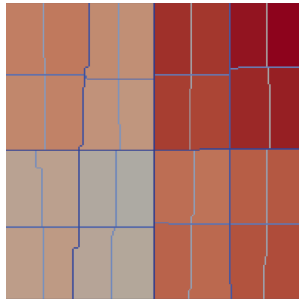


- Double interaction: between S and S_A and between S and S_B in the original graph
- Three “large” interaction blocks during the factorization

Example on a 200^2 Laplacian



Scotch ordering



Aligned ordering

Conclusion

Block Low-Rank solver

- Allows to see how to use the symbolic structure and the extend-add issues in a supernodal context
- The version presented follows the parallelism of PASTIX
- On-going work to implement efficiently this approach over the PARSEC runtime system

Ordering strategies

- Study of a nested dissection designed to increase compressibility
- Add pre-selection to isolate data which is not compressible, and thus reduce the overhead of trying to compress uncompressible blocks

PASTIX 6.0.0alpha is now available!

<http://gitlab.inria.fr/solverstack/pastix>

- Support shared memory with different schedulers:
 - ▶ sequential
 - ▶ static scheduler
 - ▶ PaRSEC runtime system
- Low-rank support
- Cholesky and LU factorizations

Thank you.