

Local stationarity of graph signals: insights and experiments

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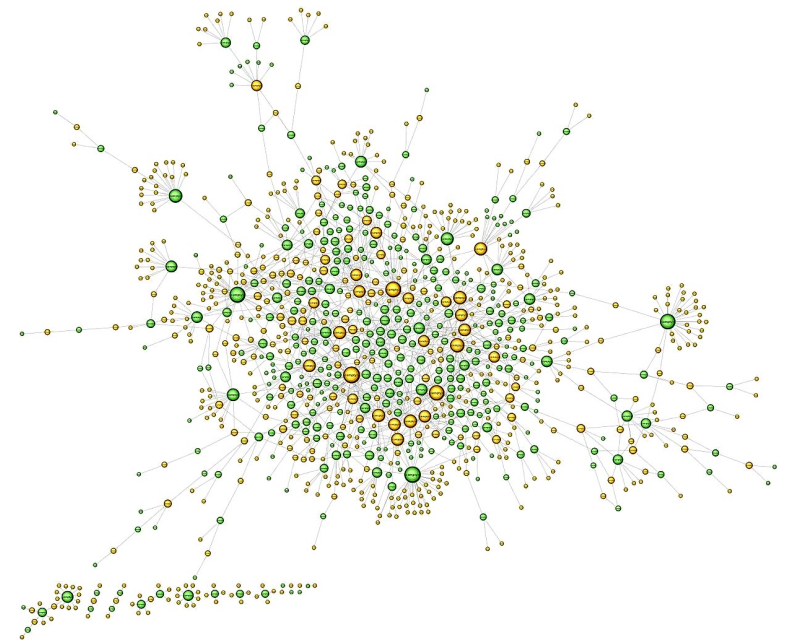
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Graph Signal Processing

Structured datasets:

- Interacting entities (graph)
- Data carried by entities (graph signal)



Source: Wikimedia.org

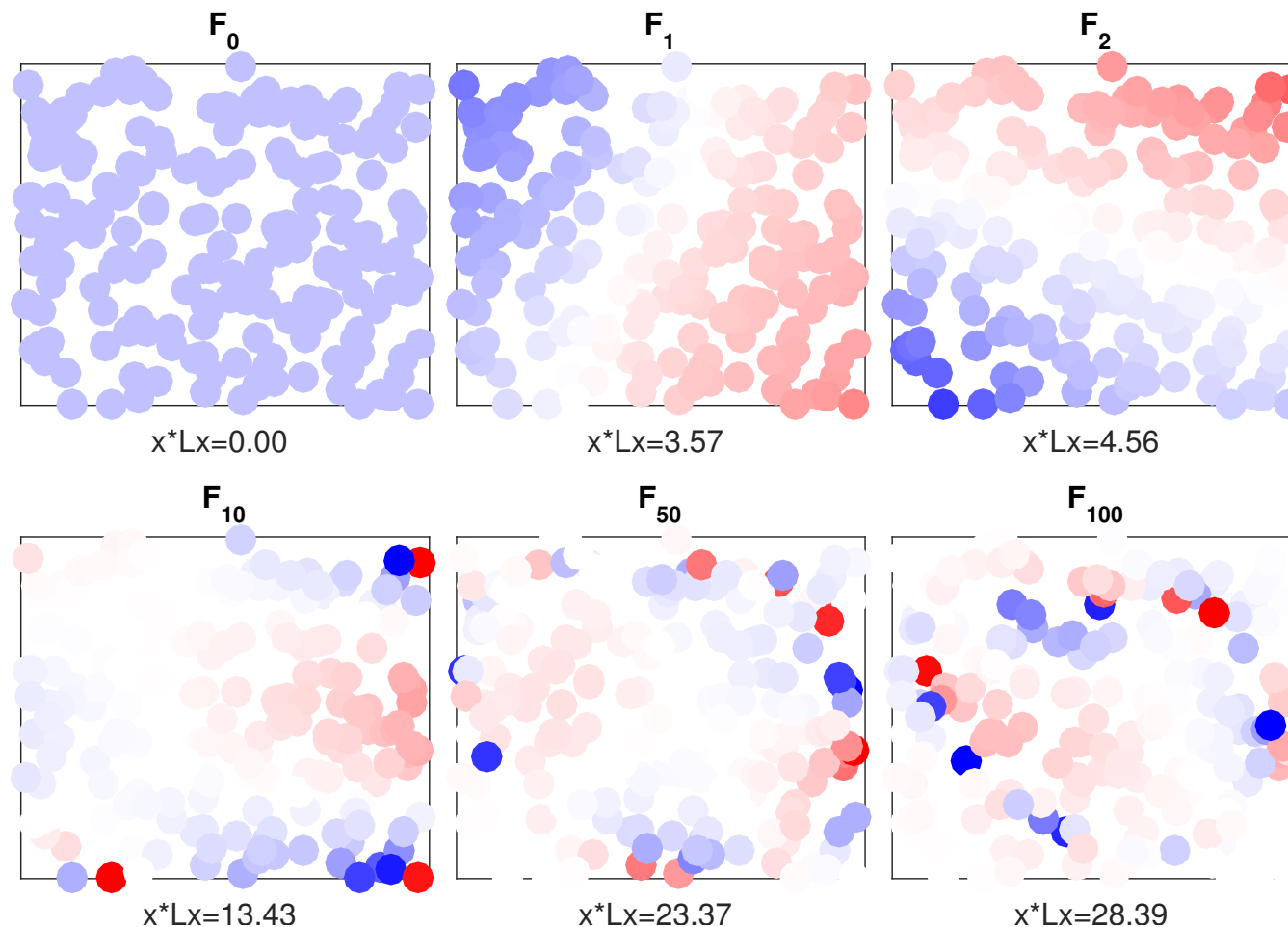
Signal processing on graphs:

Processing data considering a given underlying structure on data points.

Graph Fourier Transform and Signal Smoothness

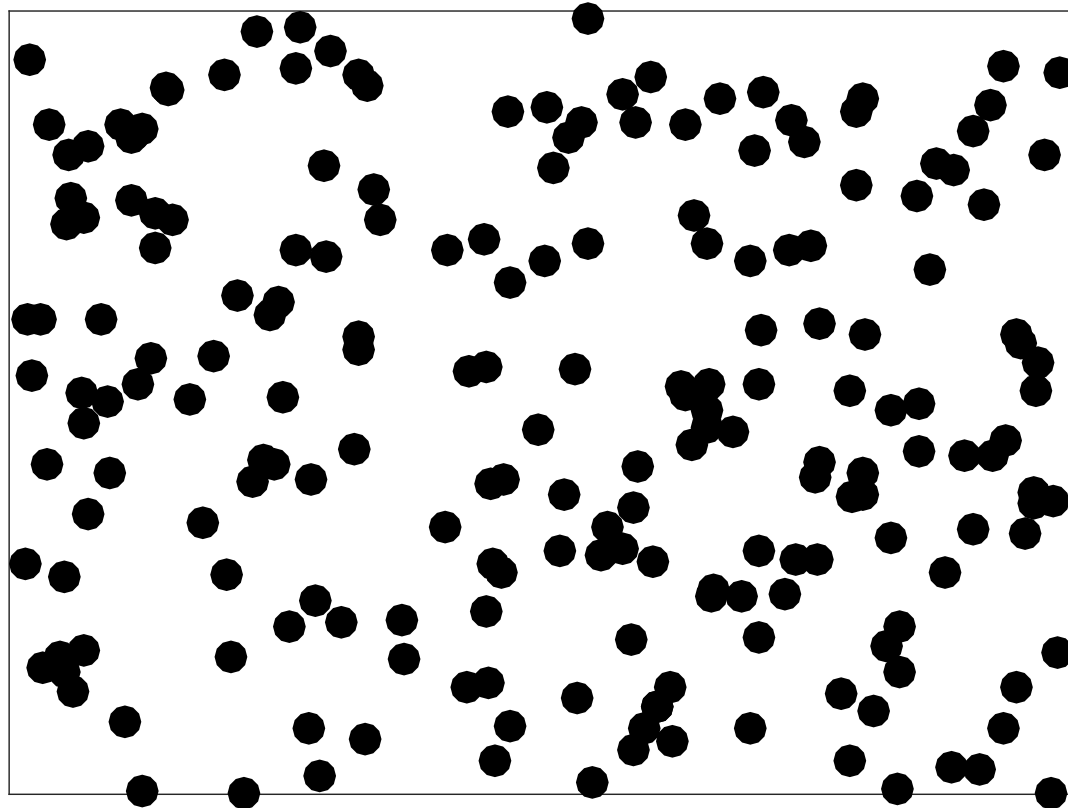
Graph signal variation: $\|x\|_L = x^* L x = \sum_{ij} w_{ij} |x_i - x_j|^2$ ($L = D - W$).

Graph Fourier modes: Orthonormal basis minimizing $\|x\|_L \Rightarrow L = F^* \Lambda F$



Sampling of a Euclidean Space

2D Example:



Choosing a Graph: Weights

Conventional approach: Gaussian weights

$$w_{ij} = \exp\left(-\frac{\|x_i - x_j\|^2}{2\sigma^2}\right)$$

Guarantee (Belkin & Niyogi, JCSS '08): For a given continuous signal \tilde{s} , its sampling $s^{(N)}$ on the graph $\mathcal{G}^{(N)}$ verifies:

$$L^{(N)} s^{(N)} \xrightarrow[N \rightarrow \infty]{} \Delta \tilde{s}$$

Question: How do the continuous and graph Fourier transforms relate?

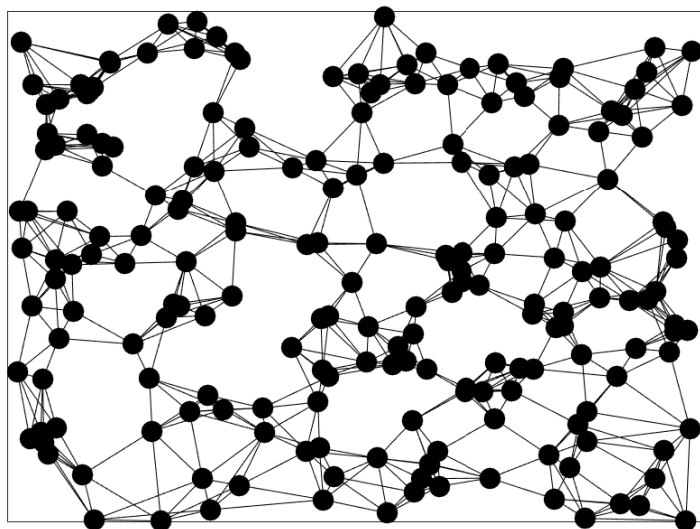
- Sampled continuous mode \neq graph mode
- continuous spectrum VS graph spectrum?

Additional Question: How do we choose σ ?

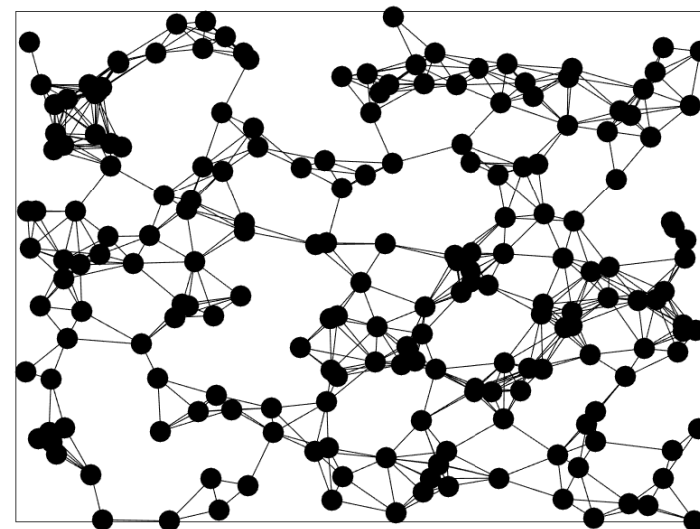
Choosing a Graph: Sparsity

GSP algorithms: *better complexity for sparse graphs.*

K -Nearest Neighbors (KNN)



Distance Threshold (THR)



Stochastic Graph Signal Model: Stationarity

Statistical invariance through Graph Translation.¹

Definition (Wide Sense Stationary (WSS))

$$\mu_i := \mathbb{E}[\mathbf{s}_i] = \mathbb{E}[(T_{\mathcal{G}}\mathbf{s})_i] \quad \forall i \in V \quad (1^{\text{st}} \text{ moment})$$

$$R_{ij} := \mathbb{E}[\mathbf{s}_i \mathbf{s}_j^*] = \mathbb{E}[(T_{\mathcal{G}}\mathbf{s})_i (T_{\mathcal{G}}\mathbf{s})_j^*] \quad \forall i, j \in V \quad (2^{\text{nd}} \text{ moment})$$

Theorem

s is graph Wide Sense Stationary iff

- $\mathbb{E}[\widehat{\mathbf{s}}(l)] = 0$ if $l \neq 0$,
- $S := \mathbb{E}[\widehat{\mathbf{s}}\widehat{\mathbf{s}}^*]$ is diagonal.

¹Benjamin Girault. [Stationary Graph Signals using an Isometric Graph Translation](#).
In *EUSIPCO 2015, Proceedings of. IEEE*, 2015

Graph Fourier Modes Properties

Our goal: Continuous stationarity \Rightarrow Graph stationarity

- Uncorrelated continuous frequencies \Rightarrow Uncorrelated graph frequencies
- Continuous PSD \Rightarrow Graph PSD

A Data Driven Approach to Choosing a Graph

Our Approach:

- 1 Consider a continuous stationary signal: \tilde{s}
- 2 Sample it: s
- 3 Compute its covariance: Σ_s
- 4 Learn² the Laplacian the closest to the precision matrix Σ_s^{-1}

Additional Step: Add connectivity constraints (KNN or THR).

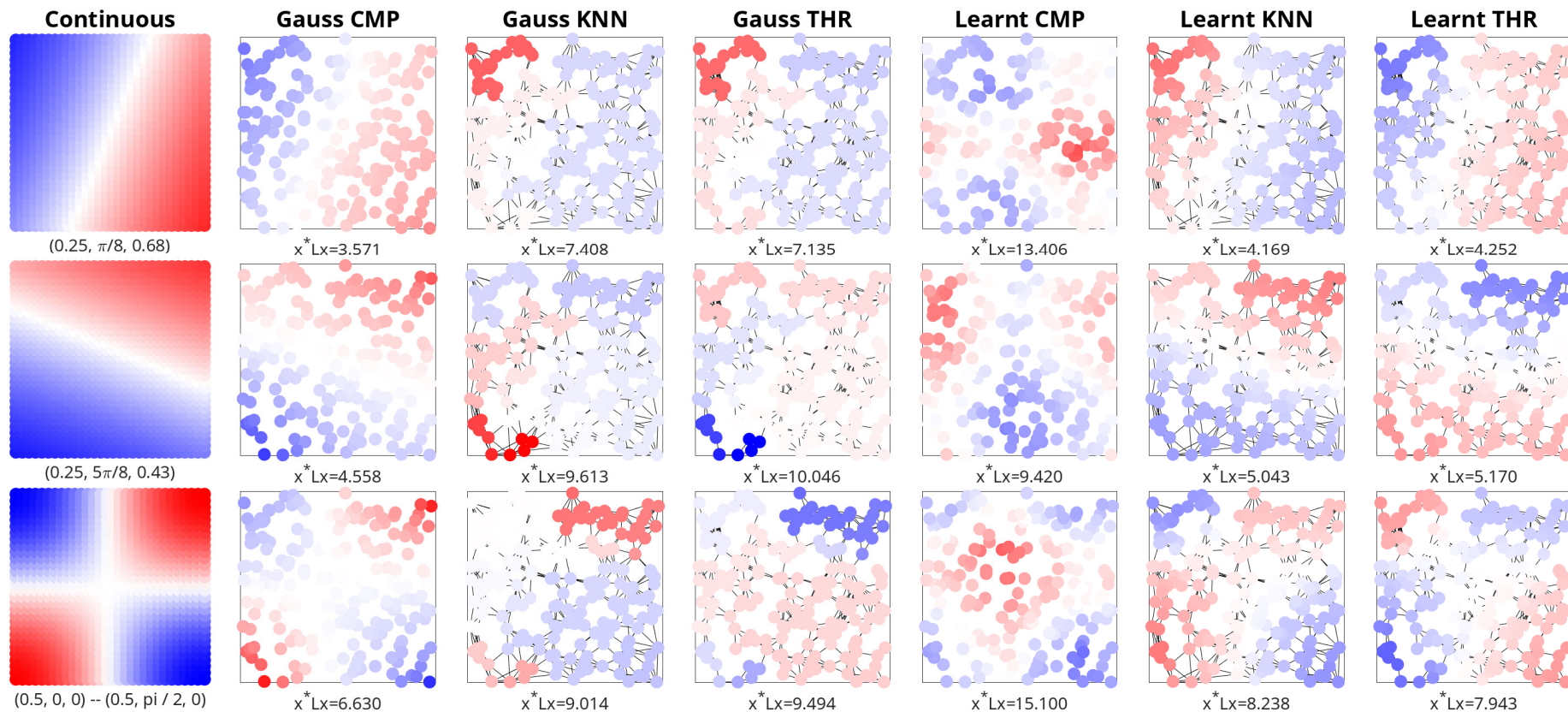
Underlying graph model: stationarity with graph PSD $1/\lambda$

²Hilmi E. Egilmez, Eduardo Pavez, and Antonio Ortega. [Graph Learning from Data under Laplacian and Structural Constraints](#).
IEEE Transactions on Signal Processing, 2017

Synthetic Sensor Network

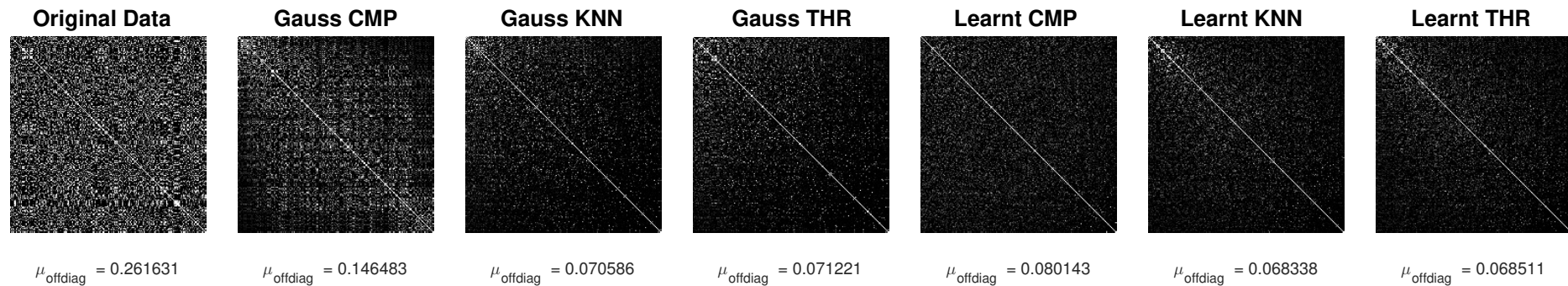
\tilde{s} : a continuous WSS signal with PSD $1/\nu^2$ (GRF model).

Goal: s is graph WSS of graph PSD $1/\lambda$.

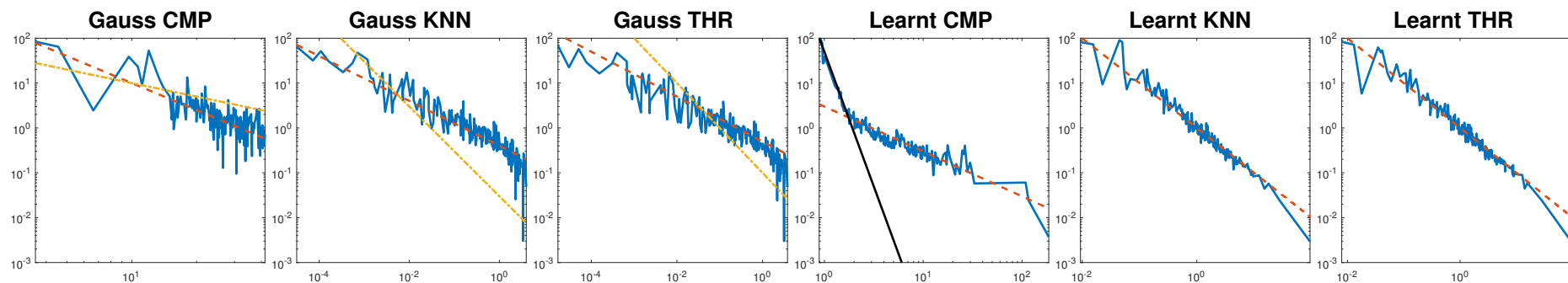


Synthetic Sensor Network: Spectral Domain

Spectral Pearson Correlations:

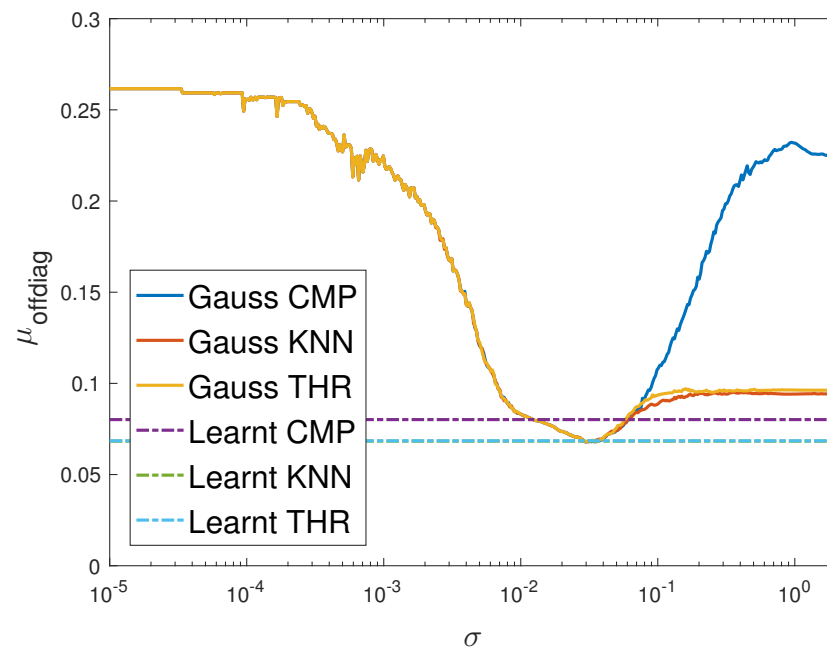


Graph Power Spectrum:

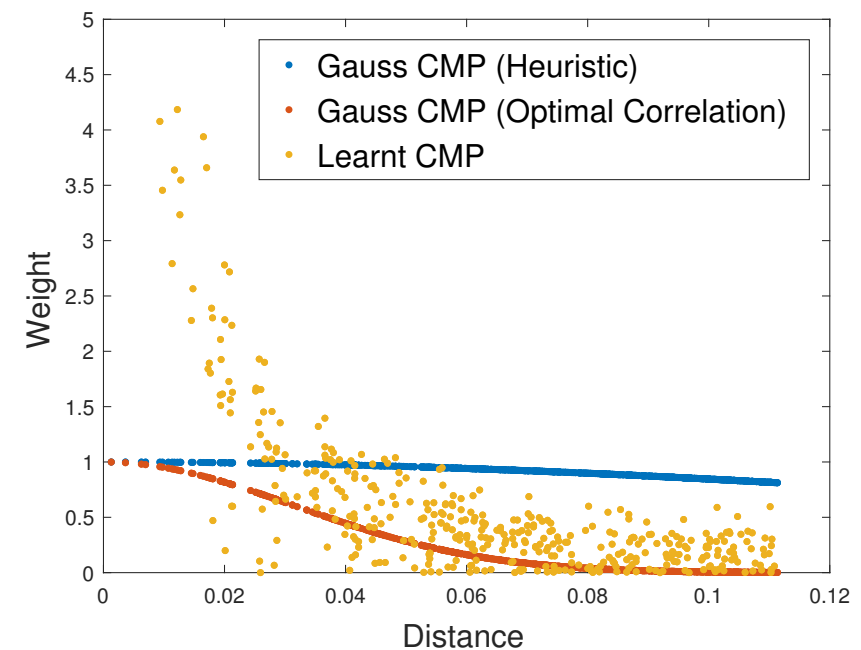


Synthetic Sensor Network: σ

Correlation VS σ :



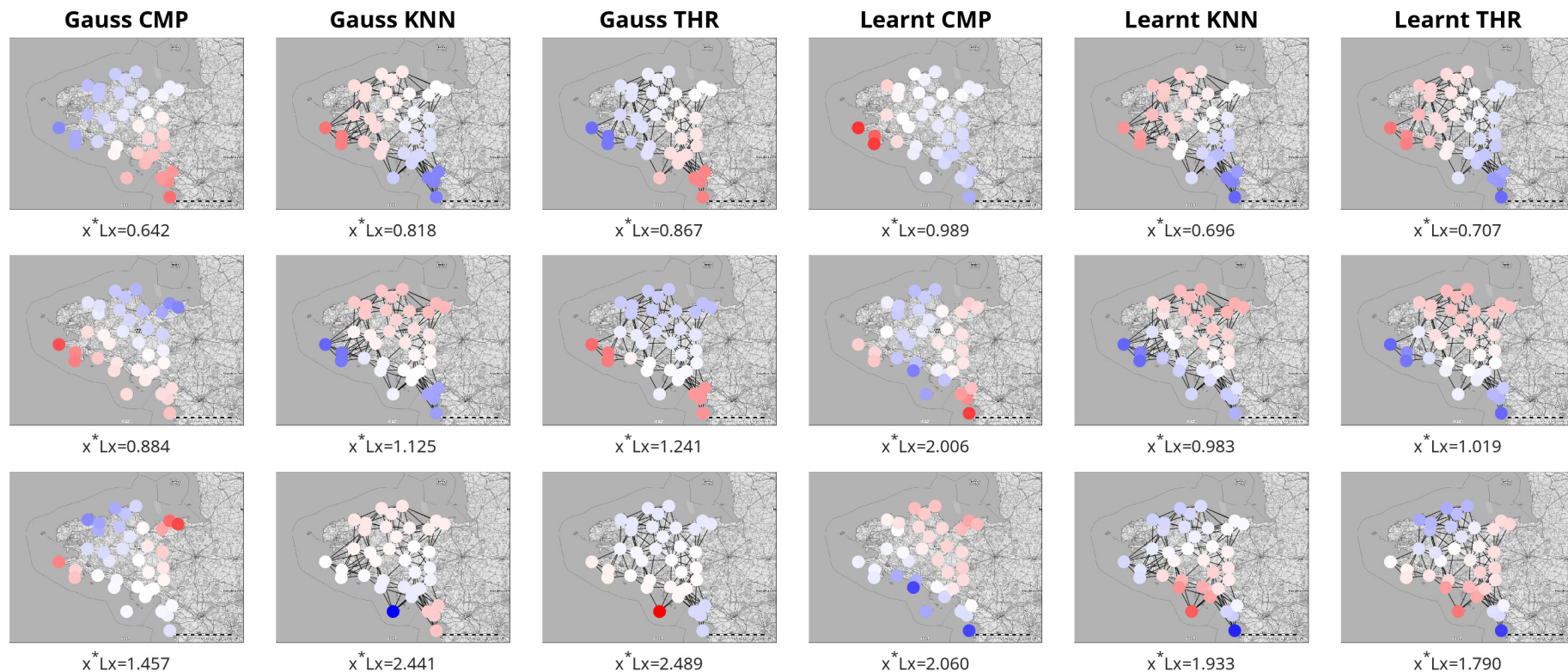
Weight VS Distance:



Temperature Data

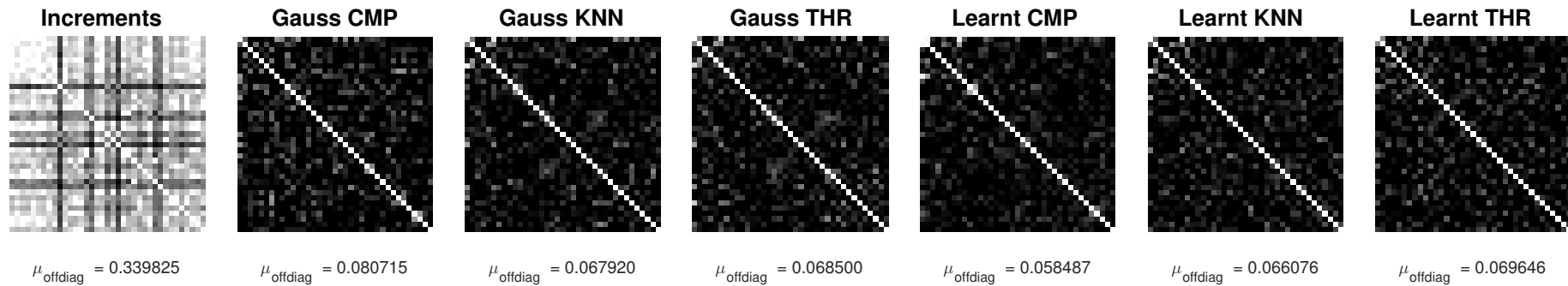
\tilde{s} are temperature increments over time.

Challenge: Does not depend only on the distance.

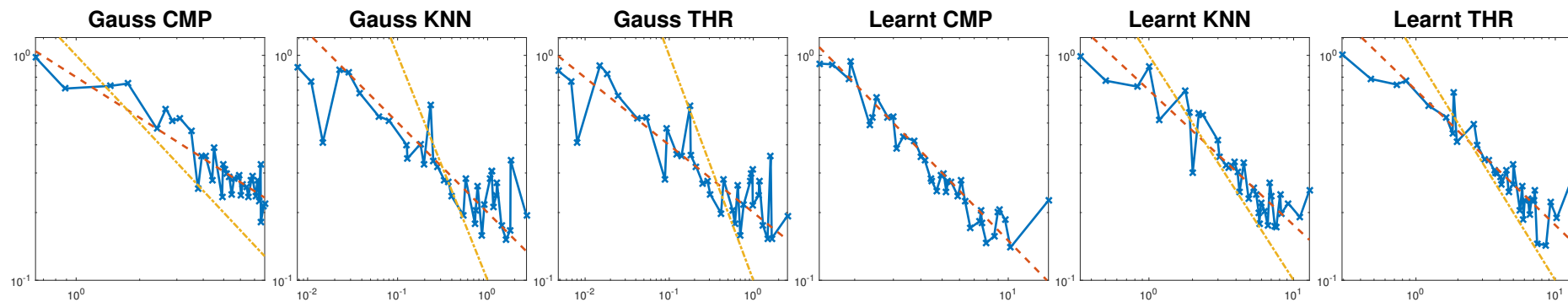


Temperature Data: Spectral Domain

Spectral Pearson Correlations:



Graph Power Spectrum:



Summary and Perspectives

Summary on Learning a Graph VS Gaussian Weights

- Less spectral correlations
- Better power spectrum
- Sparsity

Perspectives

- Formal proof?
- More complex signals?